Journal of Lie Theory
Volume **10** (2000) 311–322

Harmonic analysis on SU(n,n)/SL(n,C) x $\mathbf{R}_+^*$

Nils Byrial Andersen and Jérémie M. Unterberger*

Communicated by J. Hilgert

Abstract. We find an explicit expression for the spherical functions on the ordered symmetric space $\mathcal{M}=SU(n,n)/SL(n,\mathbb{C})\times\mathbb{R}_+^*$, we formulate and prove a Paley-Wiener theorem for the spherical Laplace transform on $\mathcal{M}$ and we find an inversion formula for the Abel transform on $\mathcal{M}$.

0. Introduction

Let $\mathcal{M} = SU(n,n)/SL(n,\mathbb{C}) \times \mathbb{R}_+^*$, let $\mathfrak{a}^-$ be the negative Weyl chamber of a certain Cartan subspace $\mathfrak{a}$ for $\mathcal{M}$, let $\lambda \in \mathfrak{a}_\mathbb{C}^*$, the complex dual of $\mathfrak{a}$, and let $A^- = \exp \mathfrak{a}^-$. Let Φ_λ denote the Harish-Chandra series on the Riemannian dual $\mathcal{M}^d = SU(n,n)/S(U(n)\times U(n))$ of $\mathcal{M}$. G. Ólafsson proved in [9], §5 an expansion formula (for general ordered symmetric spaces):

$$\varphi_\lambda(a) = \sum_{w\in W_0} c(w\lambda)\Phi_{w\lambda}(a), \quad a \in A^-,$$

for the spherical functions φ_λ on $\mathcal{M}$ (see §3 for a precise definition and construction of φ_λ), where $c(\lambda)$ is the c-function for $\mathcal{M}$ and W_0 is some Weyl group.

The Berezin-Karpelevič formula for the spherical functions ψ_λ^d on $\mathcal{M}^d$ was proved by B. Hoogenboom, see [6], using the Harish-Chandra expansion of ψ_λ^d and an explicit expression for Φ_λ. We use the expansion formula above to prove a similar (explicit) formula for the spherical functions φ_λ on $\mathcal{M}$.

The spherical Laplace transform $\mathcal{L}$ on $\mathcal{M}$ is defined in terms of integrating against the spherical functions. We use the explicit formulae for the spherical functions on $\mathcal{M}$ and $\mathcal{M}^d$ to prove a Paley-Wiener Theorem for the spherical Laplace transform, generalizing results in the rank 1 case obtained by G. Ólafsson and the first author, see [1].

---

* The authors were supported by postdoc fellowships from the European Commission within the European TMR Network "Harmonic Analysis" 1998-2001 (Contract ERBFMRX-CT97-0159). The first author was also supported by The Carlsberg Foundation.

ISSN 0949–5932 / \$2.50

The Abel transform on $\mathcal{M}$ is related to the spherical Laplace transform $\mathcal{L}$ by the classical Laplace transform on the cone $c_{\max} \subset \mathfrak{a}$. We find an inversion formula for the Abel transform, using an approach similar to the method used by C. Meaney for the inversion formula for the Abel transform on $\mathcal{M}^d$, see [8].

We have tried to keep notations and proofs to a minimum in order to make the presentation as clear as possible, we refer to [3], [5] and [9] for more details on spherical functions and the spherical Laplace and Abel transforms defined on ordered symmetric spaces.

We would like to thank J. Faraut, G. Ólafsson and H. Schlichtkrull for helpful discussions and comments.

1. Notation and preliminaries

Let n be a positive integer and let $G^c = SU(n,n)$ denote the connected group of matrices with determinant 1 preserving the hermitian form

$$(x,y) = x_1\bar{y}_1 + \cdots + x_n\bar{y}_n - x_{n+1}\bar{y}_{n+1} - \cdots - x_{2n}\bar{y}_{2n},\ x,y \in \mathbb{C}^{2n}.$$

The Lie algebra $\mathfrak{g}^c = \mathfrak{su}(n,n)$ is given by $2n \times 2n$-matrices of the form

$$\mathfrak{g}^c = \left\{ \begin{pmatrix} a & c \\ c^* & b \end{pmatrix} \middle| \, a = -a^*,\, b = -b^*,\, \operatorname{tr}(a+b) = 0 \right\},$$

where a, b and c are $n \times n$-matrices. It is isomorphic (by c-duality) to

$$\mathfrak{g} = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha^* \end{pmatrix} \middle| \, \beta = \beta^*,\, \gamma = \gamma^*,\, \Im \operatorname{tr} \alpha = 0 \right\}.$$

We embed $\mathfrak{h} = \mathfrak{sl}(n,\mathbb{C}) \oplus \mathbb{R} \cong \{\alpha \in \mathfrak{gl}(n,\mathbb{C}) \mid \Im \operatorname{tr} \alpha = 0\}$ in the diagonal as follows:

$$\alpha \mapsto \begin{pmatrix} \alpha & \\ & -\alpha^* \end{pmatrix}.$$

Let G and H denote the analytic subgroups of $GL(2n,\mathbb{C})$ with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively. The involution σ on $\mathfrak{g}$ given by

$$\sigma \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & -\beta \\ -\gamma & \delta \end{pmatrix},$$

fixes $\mathfrak{h}$. The -1 eigenspace $\mathfrak{q}$ of σ is given by:

$$\mathfrak{q} = \left\{ \begin{pmatrix} & \beta \\ \gamma & \end{pmatrix} \middle| \, \beta = \beta^*,\, \gamma = \gamma^* \right\}.$$

Let $\mathcal{M} = G/H \cong SU(n,n)/SL(n,\mathbb{C}) \times \mathbb{R}_+^*$, then G/H is an ordered symmetric space of Cayley type, see [5] or [9], §1.

Let θ be the classical Cartan involution on $\mathfrak{g}$, i.e. $\theta(X) = -X^*$, $X \in \mathfrak{g}$, and let $\mathfrak{k}$ and $\mathfrak{p}$ denote the ± 1-eigenspaces of θ. Let $K \cong S(U(n) \times U(n))$

denote the maximal compact subgroup of G with Lie algebra $\mathfrak{k}$. Then G/K is isometric to the Riemannian dual $\mathcal{M}^d$ of $\mathcal{M}$, see [5] and [9], §1 for details.

We choose a Cartan subspace $\mathfrak{a} \subset \mathfrak{p} \cap \mathfrak{q}$ for $\mathcal{M}$ as follows:

$$\mathfrak{a} = \left\{ X_T = \begin{pmatrix} 0 & T \\ T & 0 \end{pmatrix} \middle| T = \mathrm{diag}(t_1/2, \ldots, t_n/2),\, t_1, \ldots, t_n \in \mathbb{R} \right\}.$$

We note that $\mathfrak{a}$ also is a Cartan subspace of $\mathfrak{p}$. We identify $\mathfrak{a}$ and $\mathbb{R}^n$ via the map $\mathbb{R}^n \ni t = (t_1, \ldots, t_n) \mapsto T = \mathrm{diag}(t_1/2, \ldots, t_n/2)$. Let $\gamma_i \in \mathfrak{a}^*$ be defined by: $\gamma_i(t) = -t_i$ for $i = 1, \ldots, n$. We identify the complexified dual $\mathfrak{a}^*_{\mathbb{C}}$ and $\mathbb{C}^n$ by the map:

$$\mathbb{C}^n \ni \lambda = (\lambda_1, \ldots, \lambda_n) \mapsto -\sum_j \lambda_j \gamma_j.$$

The root system $\Delta = \Delta(\mathfrak{g}, \mathfrak{a})$ is given by $\Delta = \{\pm\gamma_i\} \cup \left\{ \frac{\gamma_j \pm \gamma_i}{2} \right\}$, with multiplicity $m_\alpha = 2$ for the short roots $\alpha = \frac{\gamma_j \pm \gamma_i}{2}$ and $m_\alpha = 1$ for the long roots $\alpha = \pm\gamma_i$. Let $\Delta^+ = \{\gamma_i\} \cup \left\{ \frac{\gamma_j \pm \gamma_i}{2}, i < j \right\}$ be a set of positive roots. Let furthermore Δ_0 denote the root system $\Delta_0 = \left\{ \frac{\gamma_j - \gamma_i}{2} \right\}$ with positive roots $\Delta_0^+ = \left\{ \frac{\gamma_j - \gamma_i}{2}, i < j \right\}$. The negative Weyl chamber $\mathfrak{a}^-$ is given by:

$$\mathfrak{a}^- = \{ t \in \mathbb{R}^n |\, 0 < t_1 < t_2 < \cdots < t_{n-1} < t_n \}.$$

Let $W \cong \{\pm 1\}^n \times \mathfrak{S}_n$ and $W_0 \cong \mathfrak{S}_n$ (the permutation group of n elements) denote the Weyl groups of the root systems Δ and Δ_0 respectively. Let finally $\mathfrak{n} = \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$, $\bar{\mathfrak{n}} = \sum_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha}$, $A = \exp \mathfrak{a}$, $A^- = \exp \mathfrak{a}^-$, $N = \exp \mathfrak{n}$ and $\bar{N} = \exp \bar{\mathfrak{n}}$, where exp is the exponential mapping from $\mathfrak{g}$ to G.

Let $x \in \mathbb{R}^n$ and $r \in \mathbb{R}$. We will use the notation $x \geq r$ $(x > r)$ if $x_j \geq r$ $(x_j > r)$ for all j. Let $C_{\max}$ be the (unique) closed H-invariant cone in $\mathfrak{q}$ defined by $C_{\max} \cap \mathfrak{a} := c_{\max} = \{ t \in \mathbb{R}^n | t \geq 0 \}$. Let $S = \exp(C_{\max})H$ be the associated semigroup in G, and let S^o denote the interior of S. Let finally $S^o_A := S^o \cap A = \exp c^o_{\max}$.

Let $\eta : \mathbb{D}(\mathcal{M}) \to \mathbb{D}(\mathcal{M}^d)$ denote the Flensted-Jensen isomorphism between the commutative algebras of invariant differential operators on $\mathcal{M}$ and $\mathcal{M}^d$ respectively (mapping the Laplace-Beltrami operator Δ on $\mathcal{M}$ onto the Laplace-Beltrami operator $\Delta^d = \eta(\Delta)$ on $\mathcal{M}^d$). Let $\Pi(D)$ and $\Pi^d(D^d)$ denote the radial part (on A^-) of $D \in \mathbb{D}(\mathcal{M})$ and $D^d \in \mathbb{D}(\mathcal{M}^d)$ respectively. There exists a unique map $C_c^\infty(H \backslash S^o / H) \ni f \mapsto f^d \in C_c^\infty(K \backslash G / K)$ such that $f_{|A^-} = f^d_{|A^-}$ and $\Pi(D) f_{|A^-} = \Pi^d(\eta(D)) f^d_{|A^-}$, see [5] or [9], §4 for more details.

Let P_λ and Q_λ denote Legendre functions of the first and second kind. We note that

$$P_{\lambda - \frac{1}{2}}(\cosh t) = \varphi_{i\lambda}^{(0, -\frac{1}{2})}(t) = \varphi_{2i\lambda}^{(0,0)}(t/2),$$

and

$$\frac{\Gamma(\lambda + 1)}{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\lambda + \frac{1}{2}\right)} Q_{\lambda - \frac{1}{2}}(\cosh t) = \Phi_{-i\lambda}^{(0, -\frac{1}{2})}(t) = \Phi_{-2i\lambda}^{(0,0)}(t),$$

where $\varphi_\lambda^{(\alpha,\beta)}$ and $\Phi_\lambda^{(\alpha,\beta)}$ denote Jacobi functions of the first and second kind. We can furthermore view $P_{\lambda - \frac{1}{2}}(\cosh t)$ and $Q_{\lambda - \frac{1}{2}}(\cosh t)$ as spherical functions on the Riemannian symmetric space $SO_o(1,2)/SO(2)$, respectively on the ordered symmetric space $SO_o(1,2)/SO_o(1,1)$, of rank 1. From e.g. [7], §2, we get the following estimates on $P_{\lambda-\frac{1}{2}}(\cosh t)$ and $Q_{\lambda-\frac{1}{2}}(\cosh t)$:

$$\left|P_{\lambda-\frac{1}{2}}(\cosh t)\right| \leq c e^{(|\Re\lambda|-\frac{1}{2})|t|},$$

for all $t \in \mathbb{R}$ and $\lambda \in \mathbb{C}$, for some constant c; and, for any $r > 0$:

$$\left|\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\frac{1}{2})}\right| \left|Q_{\lambda-\frac{1}{2}}(\cosh t)\right| \leq c_r e^{-(\Re\lambda+\frac{1}{2})t},$$

for $\Re\lambda \geq 0$ and $t \geq r > 0$, where c_r is a constant only depending on r.

2. The spherical Fourier transform on $\mathcal{M}^d = SU(n,n)/S(U(n)\times U(n))$

In this section we recall some well-known definitions and results for the spherical Fourier transform on $\mathcal{M}^d$, see e.g. [4], Chapter 4.

Let $\lambda \in \mathbb{C}^n$. The Poisson kernel for $\mathcal{M}^d$ is defined by:

$$NAK \ni nak = x \mapsto a^{\lambda+\rho} =: p_\lambda^d(x),$$

where $\rho = \sum_{\alpha\in\Delta^+} m_\alpha \alpha$. The spherical functions on $\mathcal{M}^d$ can be written as:

$$\psi_\lambda^d(x) = \int_K p_\lambda^d(kx)dk,$$

for $x \in G$. The spherical functions are bi-K-invariant, $\psi_\lambda^d(\exp 0) = 1$ and $D\psi_\lambda^d = \gamma(D)(\lambda)\psi_\lambda^d$ for all $D \in D(\mathcal{M}^d)$ and all $\lambda \in \mathbb{C}^n$, where γ is the Harish-Chandra isomorphism. They are furthermore invariant under the action of the Weyl group W, i.e. $\psi_{w\lambda}^d = \psi_\lambda^d$ for all $w \in W$.

Let Λ denote the simple roots in Δ^+. The Harish-Chandra series:

$$\Phi_\lambda(a) = a^{\rho-\lambda} \sum_{\mu\in(\mathbb{N}\cup\{0\})\Lambda} a^\mu \Gamma_\mu(\lambda), \quad a \in A^-,$$

is a solution of the differential equation $\Delta^d\Phi_\lambda(a) = (\lambda^2-\rho^2)\Phi_\lambda(a)$ for $a \in A^-$, where $\Gamma_0(\lambda) \equiv 1$ and $\Gamma_\mu(\lambda)$, $\mu \in \mathbb{N}\Lambda$ is determined by recursion. The Harish-Chandra expansion formula states that:

$$\psi_{-\lambda}^d(a) = \psi_\lambda^d(a^{-1}) = \sum_{w\in W} c^d(w\lambda)\Phi_{w\lambda}(a), \quad a \in A^-,$$

where the Harish-Chandra c-function c^d for $\mathcal{M}^d$ is given by (modulo constants):

$$c^d(\lambda) := \int_{\bar{N}} p_\lambda^d(\bar{n})d\bar{n} = \prod_j \frac{\Gamma(-\lambda_j)}{\Gamma(-\lambda_j+\frac{1}{2})} \prod_{i<j}(\lambda_i^2-\lambda_j^2)^{-1}.$$

The Harish-Chandra series on $\mathcal{M}^d$ is given by:

$$\Phi_\lambda(\exp t) = \pi^{-n/2} \prod_i \frac{\Gamma(\lambda_i+1)}{\Gamma(\lambda_i+\frac{1}{2})} \frac{\prod_i Q_{\lambda_i-\frac{1}{2}}(\cosh t_i)}{\delta_1(t)}, \tag{1}$$

for $t>0$, where

$$\delta_1(t) = \prod_{\alpha=\frac{\gamma_j\pm\gamma_i}{2}, i<j} \sinh\langle -\alpha, t\rangle = 2^{n(n-1)/2} \prod_{i<j} (\cosh t_j - \cosh t_i),$$

see [6], Theorem 2. Using the Harish-Chandra expansion formula, this yields the Berezin-Karpelevič formula for the spherical functions on $\mathcal{M}^d$:

$$\psi_\lambda^d(\exp t) = \frac{c}{\prod_{i<j}(\lambda_j^2-\lambda_i^2)} \frac{\det\left(P_{\lambda_i-\frac{1}{2}}(\cosh t_j)\right)}{\delta_1(t)},$$

for all $t\in\mathbb{R}^n$, where c is a constant, see [6] for more details.

The spherical Fourier transform $\mathcal{F}$ on $\mathcal{M}^d$ is defined for any function $f\in C_c^\infty(K\backslash G/K)$ as:

$$\mathcal{F}(f)(\lambda) = \int_G f(x)\psi_{-\lambda}^d(x)dx = \int_{A^-} f(a)\psi_{-\lambda}^d(a)\delta(a)da,$$

where $\delta(\exp t) = \prod_{\alpha\in\Delta^+} \sinh^{m_\alpha}\langle -\alpha, t\rangle = \delta_1(t)^2 \prod_j \sinh t_j$. The inversion formula for $\mathcal{F}$ reads (after normalizing $d\lambda$ suitably):

$$f(x) = \int_{i\mathbb{R}^n} \mathcal{F}(f)(\lambda)\psi_\lambda^d(x)|c^d(\lambda)|^{-2} d\lambda,$$

for all $f\in C_c^\infty(K\backslash G/K)$ and $x\in G$.

Let $R>0$. Let $C_R^\infty(K\backslash G/K) := \{f\in C_c^\infty(K\backslash G/K) | \mathrm{supp} f \subset \exp B_R\}$, where $B_R := \{t\in\mathbb{R}^n | \, |t|\le R\}$. Define the Paley-Wiener space $\mathcal{H}_R(\mathbb{C}^n)$ as the space of W-invariant holomorphic functions g on $\mathbb{C}^n$ of exponential type R, i.e. satisfying the estimate:

$$\sup_{\lambda\in\mathbb{C}^n} e^{-R|\Re\lambda|}(1+|\lambda|)^N |g(\lambda)| < \infty,$$

for all $N\in\mathbb{N}$. Furthermore denote by $\mathcal{H}(\mathbb{C}^n)$ the union of the spaces $\mathcal{H}_R(\mathbb{C}^n)$ for all $R>0$.

Theorem 1 (The Paley-Wiener Theorem). *The Fourier transform is a bijection of $C_c^\infty(K\backslash G/K)$ onto $\mathcal{H}(\mathbb{C}^n)$. More precisely it is a bijection of $C_R^\infty(K\backslash G/K)$ onto $\mathcal{H}_R(\mathbb{C}^n)$ for all $R>0$.*

3. Spherical functions on $\mathcal{M} = SU(n,n)/SL(n,\mathbb{C})\times\mathbb{R}_+^*$

We define spherical functions on $\mathcal{M}$ according to [9], Definition 4.1:

Definition 2. An H-biinvariant continuous function $\varphi : S^o \to \mathbb{C}$ is called a spherical function if there exists a character χ of $\mathbb{D}(\mathcal{M})$ such that (in the sense of distributions) $D\varphi = \chi(D)\varphi$ for all $D \in \mathbb{D}(\mathcal{M})$.

Define the Poisson kernel for $\mathcal{M}$ (and the open orbit NAH) by:

$$NAH \ni nah = x \mapsto a^{\rho-\lambda} =: p_\lambda(x),$$

and $p_\lambda \equiv 0$ on $G\backslash NAH$. We note that $hx \in S \subset NAH$ for all $h \in H$ and $x \in S$, see [3], Theorem 4.2. We can construct spherical functions φ_λ as follows:

$$\varphi_\lambda(x) := \int_H p_\lambda(hx)dh,$$

for $x \in S^o$, and $D\varphi_\lambda = \gamma(D)(\lambda)\varphi_\lambda$ for all $D \in \mathbb{D}(\mathcal{M})$, whenever the integral exists, see [3], §5 and [9], Theorem 4.10.

The asymptotic behavior of φ_λ as $t \to \infty$, $t \in \mathfrak{a}^-$ is given by:

$$\lim_{t\to\infty} e^{(\lambda-\rho)t}\varphi_\lambda(\exp t) = c(\lambda) = c_0(\lambda)c_\Omega(\lambda),$$

see [3], §6 for details, where c is the c-function for $\mathcal{M}$ given by:

$$c(\lambda) := \int_{\bar{N}\cap NAH} p_{-\lambda}(\bar{n})d\bar{n},$$

the function c_Ω is given by (modulo constants):

$$c_\Omega(\lambda) := \int_{K\cap NAH} p_{-\lambda}(k)dk = \prod_j \frac{\Gamma(\lambda_j+\frac{1}{2})}{\Gamma(\lambda_j+1)} \prod_{i<j}(\lambda_i+\lambda_j)^{-1},$$

see [2], Corollaire 5.2, and c_0 is the c-function for a Riemannian symmetric space with root system Δ_0, given by (modulo constants):

$$c_0(\lambda) = \prod_{i<j}(\lambda_j-\lambda_i)^{-1}.$$

We note that c_Ω is W_0-invariant, i.e. $c_\Omega(w\lambda) = c_\Omega(\lambda)$ for $w \in W_0$.

Considering asymptotics of the spherical functions and the correspondence between (the radial parts of) invariant differential operators on $\mathcal{M}$, respectively on $\mathcal{M}^d$, we obtain the following expansion formula for φ_λ:

$$\varphi_\lambda(a) = c_\Omega(\lambda)\sum_{w\in W_0} c_0(w\lambda)\Phi_{w\lambda}(a), \quad a \in A^-, \tag{2}$$

for λ in a dense open subset of $\mathbb{C}^n$, see [9], Theorem 5.7. We use this expansion formula to find an explicit expression for φ_λ:

Theorem 3. *The spherical functions on $\mathcal{M}$ are given by:*

$$\varphi_\lambda(\exp t) = \frac{c}{\prod_{i<j}(\lambda_j^2 - \lambda_i^2)} \frac{\det\left(Q_{\lambda_i - \frac{1}{2}}(\cosh t_j)\right)}{\delta_1(t)},$$

for $\lambda \geq 0$ and $t > 0$, where c is a constant. The map $\lambda \to \varphi_\lambda(\exp t)$ extends (for fixed $t > 0$) to a meromorphic function with simple poles for $\lambda_i \in -\mathbb{N} + \frac{1}{2}$, $(i = 1, \ldots n)$ and $\lambda_i = -\lambda_j$ $(i \neq j)$.

Proof. The expansion formula (2) yields:

$$\varphi_\lambda(\exp t) = c_\Omega(\lambda) c_0(\lambda) \sum_{w \in W_0} \varepsilon(w) \Phi_{w\lambda}(\exp t) = c(\lambda) \sum_{w \in W_0} \varepsilon(w) \Phi_{w\lambda}(\exp t),$$

since $c_0(w\lambda) = \varepsilon(w) c_0(\lambda)$ for all $w \in W_0 = \mathfrak{S}_n$, where $\varepsilon(w)$ denotes the sign of the permutation $w \in \mathfrak{S}_n$. Inserting the explicit expression (1) of the Harish-Chandra series Φ_λ gives the result by definition of the determinant. ∎

We easily get the following estimates of the spherical functions on $\mathcal{M}$:

Lemma 4. *Let $r > 0$. There exists a constant c_r such that*

$$|\delta_1(t) \varphi_\lambda(\exp t)/c(\lambda)| \leq c_r e^{-\min_{w \in W_0} \langle w \Re\lambda, t\rangle} \leq c_r e^{-\langle \Re\lambda, r t_o\rangle},$$

for $\Re\lambda \geq 0$ and $t \geq r$, where $t_o = (1, \ldots, 1)$.

Proof. Let $r > 0$, then:

$$\begin{aligned} |\delta_1(t) \varphi_\lambda(\exp t)/c(\lambda)| &= c \left| \frac{\Gamma(\lambda_j + 1)}{\Gamma(\lambda_j + \frac{1}{2})} \det\left(Q_{\lambda_i - \frac{1}{2}}(\cosh t_j)\right) \right| \\ &\leq c e^{-\min_{w \in W_0} \langle w \Re\lambda, t\rangle}, \end{aligned}$$

for $\lambda \geq 0$ and $t \geq r$, for some constants c. ∎

From the two expansion formulae for the spherical functions we finally obtain the following correspondence between the spherical functions on $\mathcal{M}^d$ and $\mathcal{M}$:

$$\psi_\lambda^d(a^{-1}) = \psi_{-\lambda}^d(a) = \sum_{w \in W_0 \backslash W} \frac{c^d(w\lambda)}{c(w\lambda)} \varphi_{w\lambda}(a), \quad a \in A^-,$$

see also [9], Theorem 5.9. We note that the fraction $\frac{c^d(\lambda)}{c(\lambda)}$ is W_0-invariant.

4. The spherical Laplace transform on $\mathcal{M}$

We define the normalized spherical Laplace transform $\mathcal{L}^o$ on $\mathcal{M}$ as (cf. [3], §8):

$$\mathcal{L}^o(f)(\lambda) = c_\Omega(\lambda)^{-1} \int_{A^-} f(a) \varphi_\lambda(a) \delta(a) da,$$

for any $f \in C_c^\infty(H\backslash S^o/H) \cong C_c^\infty(S_A^o)^{W_0}$ (the left-W_0-invariant functions in $C_c^\infty(S_A^o)$), whenever the integral converges. From the explicit expression for φ_λ, we see that the function $\lambda \mapsto \mathcal{L}^o(f)(\lambda)$ extends to a meromorphic function on $\mathbb{C}^n$ with at most simple poles for $\lambda_i \in -\mathbb{N}\,(i = 1, \ldots, n)$.

Let $f \in C_c^\infty(S_A^o)^{W_0}$. We see that $\mathcal{L}^o f$ satisfies the following functional equation:

$$\mathcal{F}(f^d)(\lambda) = \sum_{w \in W_o\backslash W} c_1(w\lambda)\mathcal{L}^o(f)(w\lambda), \tag{3}$$

almost everywhere (and the right hand side extends to an analytic function), where

$$c_1(\lambda) := c^d(\lambda)/c_0(\lambda) = \prod_j \frac{\Gamma(-\lambda_j)}{\Gamma(-\lambda_j + \frac{1}{2})} \prod_{i<j}(-\lambda_i - \lambda_j)^{-1}.$$

The inversion formula for the normalized spherical Laplace transform is an easy consequence of (3) and the inversion formula for the spherical Fourier transform, see also [9], Theorem 6.13:

Theorem 5 (The Inversion Formula). *Let $f \in C_c^\infty(S_A^o)^{W_0}$. Then*

$$f(a) = \frac{|W|}{|W_0|} \int_{i\mathbb{R}^n} \mathcal{L}^o(f)(\lambda)\psi_\lambda^d(a) \frac{d\lambda}{c_0(\lambda)c^d(-\lambda)},$$

for all $a \in S_A^o$.

Let $R > r > 0$ and define $C_{r,R}^\infty(S_A^o)^{W_0} := \{f \in C_c^\infty(S_A^o)^{W_0} | \mathrm{supp} f \subset \exp(C_r \cap B_R)\}$, where $C_r := \{t \in \mathbb{R}^n | t \geq r\}$. Lemma 4 and (3) suggest the following definition of the Paley-Wiener space, the supposed image space of the normalized spherical Laplace transform acting on $C_c^\infty(S_A^o)^{W_0}$ (or on the subspaces $C_{r,R}^\infty(S_A^o)^{W_0}$):

Definition 6. Let $R > r > 0$. We define the Paley-Wiener space $PW_{r,R}(\mathbb{C}^n)$ as the space of W_0-invariant meromorphic functions g on $\mathbb{C}^n$, with at most simple poles for $\lambda_i \in -\mathbb{N}\,(i = 1, \ldots, n)$, such that (i)

$$\sup_{\Re\lambda \geq 0} e^{\Re\langle\lambda, rt_o\rangle}(1 + |\lambda|)^N |g(\lambda)/c_0(\lambda)| < \infty,$$

for all $N \in \mathbb{N}$, and (ii) the c_1-weighted average

$$\mathrm{P}^{\mathrm{av}} g(\lambda) = \sum_{w \in W_0\backslash W} c_1(w\lambda)g(w\lambda)$$

extends to a function in $\mathcal{H}_R(\mathbb{C}^n)$. Furthermore denote by $PW(\mathbb{C}^n)$ the union of the spaces $PW_{r,R}(\mathbb{C}^n)$ over all $R > r > 0$.

It is easily seen that $\mathcal{L}^o$ maps $C_{r,R}^\infty(S_A^o)^{W_0}$ into $PW_{r,R}(\mathbb{C}^n)$ for all $R > r > 0$ (since $\mathcal{L}^o(\Delta f)(\lambda) = (\lambda^2 - \rho^2)\mathcal{L}^o f(\lambda)$ for all $f \in C_c^\infty(S_A^o)^{W_0}$). We remark that $\mathrm{P}^{\mathrm{av}}\mathcal{L}^o$ acts injectively on $C_c^\infty(S_A^o)^{W_0}$, since $\mathrm{P}^{\mathrm{av}}\mathcal{L}^o(f) = \mathcal{F}(f^d) = 0$ implies $f = f^d = 0$ on A^- for any $f \in C_c^\infty(S_A^o)^{W_0}$ by injectivity of the spherical Fourier transform. The following lemma, due to H. Schlichtkrull in the rank 1 case, see [1], Lemma 7, shows that P^{av} is injective on $PW(\mathbb{C}^n)$:

Lemma 7. *Let g be meromorphic function on $\mathbb{C}^n$ that satisfies item (i) of Definition 6 (for some $r > 0$). Assume that $P^{\mathrm{av}} g = 0$. Then $g = 0$.*

Proof. Let $g_1(\lambda) = g(\lambda)/c^d(-\lambda)c_0(\lambda)$ and let $W_1 := \{\pm 1\}^n \cong W_0 \backslash W$. Then $\mathrm{P}^{\mathrm{av}} g(\lambda) = |W_1| c^d(\lambda) c^d(-\lambda) \mathrm{av} g_1(\lambda)$, where

$$\mathrm{av} g_1(\lambda) := \frac{1}{|W_1|} \sum_{w \in W_1} g_1(w\lambda)$$

is the average of g_1 over W_1. It follows from the assumption $\mathrm{P}^{\mathrm{av}} g = 0$ that $\mathrm{av} g_1 = 0$. The function g_1 also satisfies item (i) of Definition 6, in particular, $g_1(i\,\cdot\,) \in L^1(\mathbb{R}^n)$. Let

$$\gamma(s) = \int_{\mathbb{R}^n} g_1(i\lambda) e^{i\langle s, \lambda\rangle}\, d\lambda, \qquad s \in \mathbb{R}^n,$$

denote the Euclidean Fourier transform of $g_1(i\,\cdot\,)$. The condition (i) implies that g_1 is holomorphic in an open set containing $\{z \in \mathbb{C}^n \,|\, \Re z \geq 0\}$, and the standard argument with Cauchy's theorem gives that γ is supported on C_r. On the other hand, the average $\mathrm{av}\gamma$ of γ is the Fourier transform of $\mathrm{av} g_1(i\cdot)$, which vanishes, hence $\mathrm{av}\gamma$ vanishes as well. Hence $\gamma = 0$ by the support condition. Since the Euclidean Fourier transform is injective on $L^1(\mathbb{R}^n)$, we conclude that g_1, and hence also g, vanishes. ∎

Theorem 8 (The Paley-Wiener Theorem). *The normalized spherical Laplace transform $\mathcal{L}^o$ is a bijection of $C_c^\infty(S_A^o)^{W_0}$ onto $PW(\mathbb{C}^n)$. More precisely it is a bijection of $C_{r,R}^\infty(S_A^o)^{W_0}$ onto $PW_{r,R}(\mathbb{C}^n)$ for all $R > r > 0$.*

Proof. It only remains to show that the normalized spherical Laplace transform maps $C_{r,R}^\infty(S_A^o)^{W_0}$ onto $PW_{r,R}(\mathbb{C})$ for all $R > r > 0$.

We define an auxiliary function Ξ_λ^d by:

$$\begin{aligned}
\Xi_\lambda^d(\exp t) = \sum_{w \in W_1} c^d(w\lambda)\Phi_{w\lambda}(\exp t) &= \frac{c}{\prod_{i<j}(\lambda_j^2 - \lambda_i^2)} \frac{\prod_j P_{\lambda_j - \frac{1}{2}}(\cosh t_j)}{\delta_1(t)} \\
&= c^d(-\lambda) c \prod_j \frac{\Gamma(\lambda_j + \frac{1}{2})}{\Gamma(\lambda_j)} \frac{\prod_j P_{\lambda_j - \frac{1}{2}}(\cosh t_j)}{\delta_1(t)},
\end{aligned}$$

for $\lambda_i \neq \pm\lambda_j \, (i \neq j)$ and $t_i \neq t_j \, (i \neq j)$. Hence $\psi_\lambda^d = \sum_{w \in W_0} \Xi_{w\lambda}^d$, and we can rewrite the inversion formula as:

$$\begin{aligned}
f(a) &= \frac{|W|}{|W_0|} \int_{i\mathbb{R}^n} \mathcal{L}^o f(\lambda) \psi_\lambda^d(a) \frac{d\lambda}{c_0(\lambda) c^d(-\lambda)} \\
&= |W| \int_{i\mathbb{R}^n} \mathcal{L}^o f(\lambda) \Xi_\lambda^d(a) \frac{d\lambda}{c_0(\lambda) c^d(-\lambda)},
\end{aligned}$$

for all $a \in A^-$, by W_0-invariance of the measure $d\lambda$.

Consider the wave packet $\mathcal{I}g \in C^\infty(S_A^o)^{W_0}$ of $g \in PW_{r,R}(\mathfrak{a}_{\mathbb{C}}^*)$ defined by the inversion formula(e) (for $a \in A^-$):

$$\mathcal{I}g(a) = \frac{|W|}{|W_0|}\int_{i\mathbb{R}^n} g(\lambda)\psi_\lambda^d(a)\frac{d\lambda}{c_0(\lambda)c^d(-\lambda)}$$
$$= |W|\int_{i\mathbb{R}^n} g(\lambda)\Xi_\lambda^d(a)\frac{d\lambda}{c_0(\lambda)c^d(-\lambda)}.$$

Fix $r > 0$ and assume that $t \notin C_r$. There exists $\lambda_o > 0$ such that $\langle\lambda_o, t - rt_o\rangle = -\varepsilon < 0$ $(t_o = (1,\ldots,1))$. This yields the following estimate:

$$\left|\Xi_{\lambda+\mu\lambda_o}^d(\exp t)/c^d(-\lambda-\mu\lambda_o)\right| \le c(1+|\lambda+\mu\lambda_0|)^{n/2}e^{\mu\langle\lambda_o, rt_o\rangle}e^{-\mu\varepsilon},$$

for $\mu \ge 0$ and $\lambda \in i\mathbb{R}^n$, for some constants c not depending on λ. By Cauchy's theorem and a contour shift we get:

$$\mathcal{I}g(\exp t) = |W|\int_{i\mathbb{R}^n}\frac{g(\lambda)}{c_0(\lambda)}\frac{\Xi_\lambda^d(\exp t)}{c^d(-\lambda)}d\lambda$$
$$= |W|\int_{i\mathbb{R}^n}\frac{g(\lambda+\mu\lambda_o)}{c_0(\lambda+\mu\lambda_o)}\frac{\Xi_{\lambda+\mu\lambda_o}^d(\exp t)}{c^d(-\lambda-\mu\lambda_o)}d\lambda$$
$$\to 0 \quad \text{for} \quad \mu \to \infty.$$

By continuity and W_0-invariance this shows that $\mathcal{I}g$ is identically zero on $S_A^o\backslash \exp C_r$.

An easy calculation shows that (for $a \in A^-$):

$$\mathcal{I}g(a) = \frac{|W|}{|W_o|}\int_{i\mathbb{R}^n} g(\lambda)\psi_\lambda^d(a)\frac{d\lambda}{c_0(\lambda)c^d(-\lambda)}$$
$$= \int_{i\mathbb{R}^n} \mathrm{P}^{\mathrm{av}}g(\lambda)\psi_\lambda^d(a)\left|c^d(\lambda)\right|^{-2}d\lambda,$$

which we recognize as the inverse Fourier transform of $\mathrm{P}^{\mathrm{av}}g \in \mathcal{H}_R(\mathbb{C})$, whence $\mathcal{I}g(a) = 0$ for $a \in S_A^o\backslash\exp B_R$ by the Paley-Wiener theorem for the spherical Fourier transform on $\mathcal{M}^d$.

Since $\mathrm{P}^{\mathrm{av}}\mathcal{L}^o f = \mathcal{F}f^d$ for all $f \in C_c^\infty(S_A^o)^{W_0}$, the above also yields:

$$\mathrm{P}^{\mathrm{av}}\mathcal{L}^o\mathcal{I}g = \mathcal{F}(\mathcal{I}g)^d = \mathrm{P}^{\mathrm{av}}g,$$

for all $g \in PW(\mathbb{C}^n)$, hence Lemma 7 implies that $\mathcal{L}^o\mathcal{I}g = g$ for all $g \in PW(\mathbb{C}^n)$ and we conclude that $\mathcal{L}^o$ maps $C_{r,R}^\infty(S_A^o)^{W_0}$ onto $PW_{r,R}(\mathbb{C}^n)$ for all $R > r > 0$.■

5. The Abel transform on $\mathcal{M} = SU(n,n)/SL(n,\mathbb{C}) \times \mathbb{R}_+^*$

The Abel transform $\mathcal{A}$ of an H-invariant function f on the semigroup S is defined as (cf. [3], §8):

$$\mathcal{A}f(a) = a^{-\rho}\int_N f(na)dn,$$

for $a \in A$, whenever this integral exists (we put $f(x) \equiv 0$ for $x \in NAH \backslash S$). It has the following connection to the spherical Laplace transform (for $\lambda \gg 0$ and otherwise by analytic continuation):

$$\mathcal{L}f(\lambda) = \int_{\exp c_{\max}} a^{-\lambda} \mathcal{A}f(a)da = \mathcal{L}_A(\mathcal{A}f)(\lambda),$$

where $\mathcal{L}_A$ is the Euclidean Laplace transform on A with respect to the cone $c_{\max}$, see [3], Proposition 8.5.

Using the explicit expression of the spherical functions from Theorem 3, we get (modulo constants):

$$\prod_{i<j}(\lambda_j^2 - \lambda_i^2)\mathcal{L}(f)(\lambda) = \int_{t_n > t_{n-1} > \dots t_2 > t_1 > 0} f(\exp t) \det \left(Q_{\lambda_i - \frac{1}{2}}(\cosh t_j)\right) \frac{\delta(t)}{\delta_1(t)} dt$$
$$= \int_{t_n > t_{n-1} > \dots t_2 > t_1 > 0} f(\exp t) \det \left(Q_{\lambda_i - \frac{1}{2}}(\cosh t_j)\right) \delta_1(t) \prod_j \sinh t_j dt$$
$$= \sum_{w \in W_0} \int_{t_n > t_{n-1} > \dots t_2 > t_1 > 0} f(\exp t) \epsilon(w) \prod_j Q_{\lambda_j - \frac{1}{2}}(\cosh wt_j) \delta_1(t) \prod_j \sinh t_j dt$$
$$= \sum_{w \in W_0} \int_{t_n > t_{n-1} > \dots t_2 > t_1 > 0} f(\exp t) \prod_j Q_{\lambda_j - \frac{1}{2}}(\cosh wt_j) \delta_1(wt) \prod_j \sinh t_j dt$$
$$= \int_{c_{\max}} f(\exp t) \delta_1(t) \left\{ \prod_j Q_{\lambda_j - \frac{1}{2}}(\cosh t_j) \sinh t_j \right\} dt$$
$$= \mathcal{L}_1^{\otimes}(f(\exp \cdot) \cdot \delta_1)(\lambda) = \mathcal{L}_A \mathcal{A}_1^{\otimes}(f(\exp \cdot) \cdot \delta_1)(\lambda),$$

where $\mathcal{L}_1^{\otimes}$ is the n-fold tensor product of the Laplace transform $\mathcal{L}_1$ on the ordered symmetric space $SO_o(1,2)/SO_o(1,1)$ of rank 1:

$$\mathcal{L}_1 f(\lambda) = \int_0^\infty f(t) Q_{\lambda - \frac{1}{2}}(\cosh t) \sinh t dt,$$

for $f \in C_c^\infty(\mathbb{R}_+)$, and $\mathcal{A}_1^{\otimes}$ is the n-fold tensor product of the Abel transform $\mathcal{A}_1$ on $SO_o(1,2)/SO_o(1,1)$:

$$\mathcal{A}_1 f(t) = \int_0^t f(\tau)(2\cosh t - 2\cosh \tau)^{-1/2} \sinh \tau d\tau,$$

for $f \in C_c^\infty(\mathbb{R}_+)$, see [3], §10 for details (we have identified A^- in the rank 1 case with $\mathbb{R}_+$ via the map $a_t \mapsto t$).

We furthermore have:

$$\prod_{i<j} (\lambda_j^2 - \lambda_i^2)\, \mathcal{L}(f) = \mathcal{L}_A \left(\prod_{i<j} (\partial_j^2 - \partial_i^2) \mathcal{A}(f) \right) (\lambda),$$

which implies that:

$$\left(\prod_{i<j} (\partial_j^2 - \partial_i^2)\, \mathcal{A}(f) \right) = \mathcal{A}_1^{\otimes}(f(\exp \cdot) \cdot \delta_1),$$

by injectivity of the Laplace transform $\mathcal{L}_A$. Finally, inverting one coordinate at a time, we get by [3], §10:

Theorem 9. *Let* $f \in C_c^\infty(S_A^o)^{W_0}$. *Then:*

$$f(\exp t) = c\delta_1(t)^{-1} \prod_j \left(\frac{1}{\sinh t_j}\frac{d}{dt_j}\right) \int_0^{t_n} \cdots \int_0^{t_1} \left(\prod_{k<l}(\partial_l^2 - \partial_k^2)\mathcal{A}f\right)(\exp\tau)$$
$$\times \prod_j \left((\cosh t_j - \cosh\tau_j)^{-1/2}\sinh\tau_j\right) d\tau_1 \dots d\tau_n,$$

for $t \in \mathfrak{a}^-$, *for some constant* c.

References

[1] Andersen, N. B., and G. Ólafsson, *Paley-Wiener Theorems for the Spherical Laplace transform on Causal Symmetric Spaces of rank 1,* Proc. Amer. Math. Soc, to appear.

[2] Faraut, J., *Fonctions Sphériques sur un Espace Symétrique Ordonné de type Cayley,* Cont. Math. **191** (1995), 41–55.

[3] Faraut, J., J. Hilgert, and G. Ólafsson, *Spherical functions on ordered symmetric spaces,* Ann. Inst. Fourier **44** (1994), 927–966.

[4] Helgason, S., "Groups and Geometric Analysis," Academic Press, Orlando, 1984.

[5] Hilgert, J., and G. Ólafsson, "Causal Symmetric Spaces, Geometry and Harmonic Analysis," Perspectives in Mathematics **18**, Academic Press, San Diego, 1997.

[6] Hoogenboom, B., *Spherical functions and invariant differential operators on complex Grassmann manifolds,* Ark. Mat. **20** (1982), 69–85.

[7] Koornwinder, T. H., *A new proof of a Paley-Wiener type theorem for the Jacobi transform,* Ark. Mat. **13** (1975), 145–159.

[8] Meaney, C., *The inverse Abel transform for $SU(p,q)$,* Ark. Mat. **24** (1986), 131–140.

[9] Ólafsson, G., *Spherical Functions and Spherical Laplace Transform on Ordered Symmetric Spaces,* Preprint, 1997.

Nils Byrial Andersen
Department of Mathematics
Louisiana State University
Baton Rouge
LA 70803
USA
E-mail: byrial@math.lsu.edu

Jérémie M. Unterberger
Dipartimento di Matematica
Politecnico di Torino
Corso Duca degli Abruzzi, 24
10129 TORINO
ITALY

Received March 29, 1999
and in final form December 8, 1999