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Lie Algebras of Least Cohomology

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Abstract. We classify those finite dimensional Lie algebras, over a field k of characteristic zero, whose cohomology with trivial coefficients has dimension 2. We show that the only such algebras are the 3-dimensional simple algebras and the semi-direct products $\mathfrak{n} \rtimes_\phi \mathrm{k}$, where $\mathfrak{n}$ is a nilpotent Lie algebra and $\phi\colon \mathfrak{n} \to \mathfrak{n}$ is a derivation which induces a non-singular map in each cohomology space $H^i(\mathfrak{n})$, for $i > 0$.

1. Introduction

Let $\mathfrak{g}$ be a Lie algebra of finite dimension ≥ 2 over a field k of characteristic zero. We consider the cohomology with trivial coefficients $H(\mathfrak{g}) = H(\mathfrak{g}, \mathrm{k})$, and the total cohomology $\sigma(\mathfrak{g}) = \dim H(\mathfrak{g}) = \sum_{i=0}^{\dim \mathfrak{g}} \dim H^i(\mathfrak{g})$. Since $H^0(\mathfrak{g}) = \mathrm{k}$, one has $\sigma(\mathfrak{g}) > 0$, and it is well known that Lie algebras have trivial Euler characteristic [9], and so $\sigma(\mathfrak{g})$ is even. Thus $\sigma(\mathfrak{g}) \geq 2$. If $\mathfrak{g}$ is a 3-dimensional simple Lie algebra, then $\mathfrak{g} = \langle x, y, z \mid [x, y] = z, [y, z] = \alpha x, [z, x] = \beta y \rangle$, with $\alpha, \beta \neq 0$ (see [13]); thus $H^0(\mathfrak{g}) = H^3(\mathfrak{g}) = \mathrm{k}$ and $H^1(\mathfrak{g}) = H^2(\mathfrak{g}) = 0$, and so $\sigma(\mathfrak{g}) = 2$. For other examples of algebras $\mathfrak{g}$ with $\sigma(\mathfrak{g}) = 2$, consider a nilpotent Lie algebra $\mathfrak{n}$, let $\phi\colon \mathfrak{n} \to \mathfrak{n}$ be a derivation and consider the semi-direct product $\mathfrak{g} = \mathfrak{n} \rtimes_\phi \mathrm{k}$. Recall that ϕ induces a derivation $\bar{\phi}\colon H(\mathfrak{n}) \to H(\mathfrak{n})$ in the cohomology algebra. Let ϕ_i denote the restriction of $\bar{\phi}$ to $H^i(\mathfrak{n})$. According to [6], there is a vector space isomorphism: $H^i(\mathfrak{g}) \cong \ker \phi_i \oplus \ker \phi_{i-1}$, for all $i \geq 1$. Hence $\sigma(\mathfrak{g}) = 2$ if and only if ϕ_i is non-singular for all $i > 0$. In this case, we say that ϕ is *non-singular in cohomology.* The main aim of this present note is to prove the following:

Theorem. *If $\sigma(\mathfrak{g}) = 2$, then either $\mathfrak{g}$ is simple of dimension 3, or $\mathfrak{g} \cong \mathfrak{n} \rtimes_\phi \mathrm{k}$ for some nilpotent Lie algebra $\mathfrak{n}$ and derivation $\phi\colon \mathfrak{n} \to \mathfrak{n}$ which is non-singular in cohomology.*

The above theorem may be regarded as a classification of the simplest case of Lie algebras $\mathfrak{g}$ for which $\sigma(\mathfrak{g})$ is not a multiple of 4 (see [4]).

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Recall that the *cohomological dimension* of $\mathfrak{g}$ is the largest integer $cd(\mathfrak{g})$ such that $H^{cd(\mathfrak{g})}(\mathfrak{g}, \mathrm{k}) \neq 0$. It is well known that $cd(\mathfrak{g}) \leq \dim \mathfrak{g}$ and $cd(\mathfrak{g}) = \dim \mathfrak{g}$ if and only if $\mathfrak{g}$ is unimodular. Algebras with $cd(\mathfrak{g}) = \dim \mathfrak{g} - 1$ are classified in [1]. We have:

Corollary. *$cd(\mathfrak{g}) = 1$ if and only if $\mathfrak{g} \cong \mathfrak{n} \rtimes_\phi \mathrm{k}$ for some nilpotent Lie algebra $\mathfrak{n}$ and derivation $\phi\colon \mathfrak{n} \to \mathfrak{n}$ which is non-singular in cohomology.*

This note is organized as follows. In the next section we prove the Theorem and its Corollary. The main question left open by these results is: which nilpotent Lie algebras admit a derivation which is non-singular in cohomology? In Section 3, we give some remarks concerning this problem.

2. Proof of the Theorem and its Corollary

Proof of the Theorem. Assume that $\sigma(\mathfrak{g}) = 2$. Let $\mathfrak{r}$ denote the radial of $\mathfrak{g}$ and consider the natural projection $\pi\colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{r}$. Since $\mathfrak{g}$ has a Levi-subalgebra, π is a split extension. Let m denote the dimension of $\mathfrak{g}/\mathfrak{r}$. Note that either $m = 0$, or $m \geq 3$. As $\mathfrak{g}/\mathfrak{r}$ is unimodular, $H^m(\mathfrak{g}/\mathfrak{r}) \cong \mathrm{k}$. Consider the pull-back map $\pi^*\colon \wedge(\mathfrak{g}/\mathfrak{r}) \to \wedge(\mathfrak{g})$ and the map induced in cohomology $\pi^{\#}\colon H(\mathfrak{g}/\mathfrak{r}) \to H(\mathfrak{g})$. Since π is a split extension, $\pi^{\#}$ is injective, and so $H^m(\mathfrak{g})$ is non zero.

If $\mathfrak{g}$ is unimodular, then by Poincaré duality, $H^{\dim \mathfrak{g}}(\mathfrak{g}) = \mathrm{k}$, and hence $H^i(\mathfrak{g}) = 0$ for all $0 < i < \dim \mathfrak{g}$. Thus either $m = 0$ or $m = \dim \mathfrak{g}$; that is, $\mathfrak{g}$ is either solvable or semi-simple. If $\mathfrak{g}$ is semi-simple, then $H^3(\mathfrak{g})$ is non-trivial [14]. In this case, $\mathfrak{g}$ has dimension 3, and $\mathfrak{g}$ is simple. If $\mathfrak{g}$ is solvable, then $\mathfrak{g}$ has a codimension one ideal.

If $\mathfrak{g}$ is not unimodular, then $\{z \in g \mid \mathrm{tr}(\mathrm{ad}(z)) = 0\}$ is a codimension one ideal.

From the above, it remains to consider the case where $\mathfrak{g}$ possesses a codimension 1 ideal, $\mathfrak{h}$ say. Choose $z \in \mathfrak{g} \backslash \mathfrak{h}$. The adjoint map $\mathrm{ad}(z)\colon \mathfrak{h} \to \mathfrak{h}$ induces a derivation of the cohomology algebra $H(\mathfrak{h})$; for each $i \geq 0$, let $\phi_i\colon H^i(\mathfrak{h}) \to H^i(\mathfrak{h})$ be the resulting linear map. By [6], $\sigma(\mathfrak{g}) = 2\sum_{i \geq 0} \dim \ker \phi_i$. Hence, as $\ker \phi_0 = H^0(\mathfrak{g}) = \mathrm{k}$ and $\sigma(\mathfrak{g}) = 2$, one has $\ker \phi_i = 0$ for all $i \geq 1$; that is, ϕ is non-singular in cohomology. It remains to show that $\mathfrak{h}$ is nilpotent. We have:

Lemma. *If a Lie algebra $\mathfrak{h}$ admits a derivation ϕ which is non-singular in cohomology, then $\mathfrak{h}$ is nilpotent.*

Proof. Suppose that $\mathfrak{h}$ admits such a derivation ϕ. Let $\mathtt{r}$ denote the radial of $\mathfrak{h}$ and consider the natural projection $\pi\colon \mathfrak{h} \to \mathfrak{h}/\mathtt{r}$. Since $\mathtt{r}$ is a characteristic ideal, ϕ induces a derivation $\hat{\phi}$ on $\mathfrak{h}/\mathtt{r}$. Since $\mathfrak{h}/\mathtt{r}$ is semi-simple, $\hat{\phi}$ is an inner derivation. In particular, $\hat{\phi}$ has zero trace. Let $k = \dim(\mathfrak{h}/\mathtt{r})$. As $\hat{\phi}$ has zero trace, the map $\hat{\phi}_k$ induced by $\hat{\phi}$ in $H^k(\mathfrak{h}/\mathtt{r})$ is trivial. We have the commutative diagram:

$$\begin{array}{ccc} H^k(\mathfrak{h}) & \xrightarrow{\phi_k} & H^k(\mathfrak{h}) \\ {\scriptstyle \pi^\#}\uparrow & & \uparrow{\scriptstyle \pi^\#} \\ H^k(\mathfrak{h}/\mathtt{r}) & \xrightarrow{\hat{\phi}_k \equiv 0} & H^k(\mathfrak{h}/\mathtt{r}) \end{array}$$

Since $\pi^\#\colon H(\mathfrak{h}/\mathtt{r}) \to \mathtt{H}(\mathfrak{h})$ is injective, it follows that ϕ_k has non-trivial kernel. Hence $k = 0$; that is $\mathfrak{h}$ is solvable.

Consider the natural projection $\tau\colon \mathfrak{h} \to \mathfrak{h}/[\mathfrak{h},\mathfrak{h}]$. Recall that there is an isomorphism $f\colon H^1(\mathfrak{h}) \to \mathfrak{h}/[\mathfrak{h},\mathfrak{h}]$, such that $f\phi_1 = \phi' f$, where ϕ' is the map induced by ϕ on $\mathfrak{h}/[\mathfrak{h},\mathfrak{h}]$. Since ϕ_1 is non-singular, ϕ' is non-singular. Let n denote the nil radical of $\mathfrak{h}$; that is, $\mathfrak{n}$ is the largest nilpotent ideal of $\mathfrak{h}$. It is well known that $\mathfrak{n}$ contains the derived algebra $[\mathfrak{h},\mathfrak{h}]$ of $\mathfrak{h}$ [3]. Consider the natural projection $\rho\colon \mathfrak{h}/[\mathfrak{h},\mathfrak{h}] \to \mathfrak{h}/\mathfrak{n}$.

$$\begin{array}{ccc} \mathfrak{h} & \xrightarrow{\phi} & \mathfrak{h} \\ {\scriptstyle \tau}\downarrow & & \downarrow{\scriptstyle \tau} \\ \mathfrak{h}/[\mathfrak{h},\mathfrak{h}] & \xrightarrow{\phi'} & \mathfrak{h}/[\mathfrak{h},\mathfrak{h}] \\ {\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle \rho} \\ \mathfrak{h}/\mathfrak{n} & \xrightarrow{\phi'' \equiv 0} & \mathfrak{h}/\mathfrak{n} \end{array}$$

It is well known that every derivation of $\mathfrak{h}$ maps $\mathfrak{h}$ into $\mathfrak{n}$ [3]. Hence the map ϕ'' induced by ϕ' on $\mathfrak{h}/\mathfrak{n}$ is identically zero. Since ϕ' is surjective, it follows that $\mathfrak{n} = \mathfrak{h}$. Hence $\mathfrak{h}$ is nilpotent. ∎

Proof of the Corollary. If $cd(\mathfrak{g}) = 1$, then $\dim H^i(\mathfrak{g}) = 0$ for all $i > 1$, while $\dim H^0(\mathfrak{g}) = 1$ and $\dim H^1(\mathfrak{g}) = \sigma(\mathfrak{g}) - 1$. Thus, since the Euler characteristic of $\mathfrak{g}$ is zero, we have $\sigma(\mathfrak{g}) = 2$. The Corollary then follows from the fact that for a simple 3-dimensional Lie algebra $\mathfrak{g}$, $cd(\mathfrak{g}) = 3$. ∎

3. Remarks and Questions

Remark 1. Recall that if $f\colon \mathfrak{g} \to \mathfrak{g}$ is Lie algebra homomorphism, then f induces an algebra homomorphism $f^*\colon H(\mathfrak{g}) \to H(\mathfrak{g})$. Let f_i denote the restriction of f^* to $H^i(\mathfrak{g})$. It is well known and easy to prove that if f^* is an isomorphism, then f is non-singular [11]. In fact, this result only relies on f_1 and f_2. The following result is also elementary, but it requires that ϕ_i is non-singular for all $i > 0$. To our knowledge, it doesn't seem to have previously appeared in the literature.

Proposition. *If $\mathfrak{g}$ is a complex Lie algebra, and $\phi\colon \mathfrak{g} \to \mathfrak{g}$ is a derivation which is non-singular in cohomology, then ϕ is itself non-singular.*

Proof. Suppose that $\phi\colon \mathfrak{g} \to \mathfrak{g}$ is non-singular in cohomology, and that ϕ is singular. By replacing ϕ, if necessary, by $\lambda\phi$ for some nonzero $\lambda \in \mathbb{C}$, we may assume that for all $i > 0$, none of the eigenvalues of ϕ_i is an integer multiple of $2\pi i$. Let $f = \exp\phi$ and consider the zeta function

$$\zeta_f(T) = \prod_i \det(1 - Tf_i)^{(-1)^{i+1}}.$$

It is easy to verify that for all $i > 0$, $f_i = \exp\phi_i$ and so f_i does not have 1 as an eigenvalue. Hence $\zeta_f(T)$ has a simple pole at $T = 1$, resulting from the $i = 0$ term. According to Deninger and Singhof [5, Lemma 2.1], one has

$$\zeta_f(T) = \exp \sum_{\mu=1}^{\infty} \det(1 - f^\mu)\frac{T^\mu}{\mu}.$$

As ϕ is singular, f has 1 as an eigenvalue, and hence so too does f^μ for all μ. Hence $\exp \sum_{\mu=1}^{\infty} \det(1 - f^\mu)\frac{T^\mu}{\mu} \equiv 1$, which is a contradiction. ■

By the Borel-Serre-Jacobson theorem [12], if a Lie algebra $\mathfrak{g}$ admits a non-singular derivation, then $\mathfrak{g}$ is nilpotent (see [2] for a recent generalization). For complex Lie algebras, we could have used this, together with the Proposition, instead of the Lemma in the proof of the Theorem. However, we do not know if the Proposition holds over arbitrary fields of characteristic zero.

Remark 2. Abelian Lie algebras possess derivations which are non-singular in cohomology. Indeed, let $\mathfrak{n} = \mathrm{k}^m$. In this case $H^i(\mathfrak{n}) = \wedge^i(\mathfrak{n})$. Let $\phi\colon \mathfrak{n} \to \mathfrak{n}$ be a linear map none of whose characteristic numbers is zero. Obviously the induced map $\phi_i\colon H^i(\mathfrak{n}) \to H^i(\mathfrak{n})$ is non-singular for all $i > 0$.

Positively graded Lie algebras possess derivations which are non-singular in cohomology. Let $\mathfrak{n}$ be a graded Lie algebra; $\mathfrak{n} = \oplus_{i=1}^{k}\mathfrak{n}_i$, where $[\mathfrak{n}_i, \mathfrak{n}_j] \subseteq \mathfrak{n}_{i+j}$, for all $i, j \geq 1$, and $\mathfrak{n}_i = 0$ for all $i > k$. Consider the derivation ϕ of $\mathfrak{n}$ such that for all i, the restriction of ϕ to $\mathfrak{n}_i$ is the dilation: $\phi(x) = ix$ for all $x \in \mathfrak{n}_i$. Notice that the grading on $\mathfrak{n}$ induces a grading on $H^i(\mathfrak{n})$ and the induced map $\phi_i\colon H^i(\mathfrak{n}) \to H^i(\mathfrak{n})$ is also a direct product of non-trivial dilations. In particular, ϕ_i is non-singular for all $i > 0$.

In particular, notice that 2-step nilpotent Lie algebras are positively graded and hence they possess derivations which are non-singular in cohomology.

Remark 3. A non-singular derivation in a Lie algebra may induce a singular derivation in cohomology. For example, consider the 3-dimensional Heisenberg algebra $h_1 = \langle x, y, z \mid [x, y] = z\rangle$. The derivation $\phi\colon h_1 \to h_1$ defined by setting $\phi(x) = 2x, \phi(y) = -y, \phi(z) = z$ is non-singular, but the induced map ϕ_2 is singular.

Remark 4. A derivation and the derivation induced in cohomology may both be non-singular, while the derivation induced in the exterior algebra is singular. For example, consider the 4-dimensional algebra $\mathfrak{g} = h_1 \oplus \mathrm{k} = \langle x, y, z, w \mid [x, y] = z\rangle$. Consider the non-singular derivation $\phi\colon \mathfrak{g} \to \mathfrak{g}$ defined by setting $\phi(x) = 2x, \phi(y) = 3y, \phi(z) = 5z, \phi(w) = -5w$. It is easy to see that ϕ induces a non-singular derivation in cohomology, but the induced map in $\wedge^2\mathfrak{g}^*$ is singular.

Remark 5. Recall that Jacobson [12] originally asked whether every nilpotent Lie algebra admits a non-singular derivation and that a counter-example was provided by Dixmier and Lister [7]; their 8-dimensional algebra is characteristically nilpotent (that is, all its derivations are nilpotent and thus singular). A 7-dimensional characteristically nilpotent algebra was given by Favre [8]. It is now known that characteristically nilpotent Lie algebras are quite common (see [10,15]). By the above Proposition, characteristically nilpotent Lie algebras do not possess derivations which are non-singular in cohomology. Alternatively, one can just observe that if $\phi\colon \mathfrak{g} \to \mathfrak{g}$ is a nilpotent derivation, then for all $i \geq 0$ the induced map $\phi_i\colon H^i(\mathfrak{g}) \to H^i(\mathfrak{g})$ is nilpotent and thus singular. It is also easy to exhibit a Lie algebra which is not characteristically nilpotent, but whose derivations are all singular in cohomology. Let $\mathfrak{f}$ be Favre's algebra:

$$\begin{aligned}\mathfrak{f} = \langle x_1, \dots, x_7 \mid &[x_1, x_i] = x_{i+1}, \text{for all } 1 \leq i \leq 6,\\ &[x_2, x_3] = x_6, [x_2, x_4] = [x_2, x_5] = -[x_3, x_4] = x_7\rangle\end{aligned}$$

Consider the 8-dimensional Lie algebra $\mathfrak{g} = \mathfrak{f} \oplus \mathrm{k}$, and let $\phi\colon \mathfrak{g} \to \mathfrak{g}$ be an arbitrary derivation. Using calculations similar to those of [8], it is easy to see that the induced map $\phi_1\colon H^1(\mathfrak{g}) \to H^1(\mathfrak{g})$ is singular.

Remark 6. Every derivation $\phi\colon \mathfrak{g} \to \mathfrak{g}$ induces a derivation $\tilde{\phi}\colon \mathfrak{g}/Z(\mathfrak{g}) \to \mathfrak{g}/Z(\mathfrak{g})$, where $Z(\mathfrak{g})$ is the centre of $\mathfrak{g}$. Not surprisingly, it is possible that ϕ and $\tilde{\phi}$ are both non-singular, and $\tilde{\phi}$ is non-singular in cohomology, but ϕ is singular in cohomology. For example, let $\mathfrak{g} = h_1$ and let ϕ be the derivation considered in Remark 3. Perhaps less obvious, there are examples where ϕ is non-singular in cohomology but $\tilde{\phi}$ is singular in cohomology. Let $\mathfrak{g}$ be the 6-dimensional Lie algebra determined by the following relations:

$$[x_1, x_2] = x_3,\ [x_1, x_3] = x_4,\ [x_1, x_4] = x_5,\ [x_2, x_5] = x_6,\ [x_3, x_4] = -x_6.$$

Let $\alpha_1, \dots, \alpha_6$ denote the basis of $\mathfrak{g}^*$ dual to $x, \dots, x_6$. One finds that the cohomology of $\mathfrak{g}$ has basis

$$\begin{gathered}\alpha_1,\ \alpha_2\\ \alpha_1\alpha_5,\ \alpha_2\alpha_3\\ \alpha_1\alpha_4\alpha_5,\ \alpha_2\alpha_3\alpha_6\\ \alpha_1\alpha_4\alpha_5\alpha_6,\ \alpha_2\alpha_3\alpha_4\alpha_6\\ \alpha_1\alpha_3\alpha_4\alpha_5\alpha_6,\ \alpha_2\alpha_3\alpha_4\alpha_5\alpha_6\\ \alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6.\end{gathered}$$

Consider the derivation ϕ defined by setting $\phi(x_1) = 4x_1,\ \phi(x_2) = -7x_2$. One verifies easily that ϕ is non-singular in cohomology. Now consider $\mathfrak{g}/Z(\mathfrak{g})$. By abuse of language, we denote its basis and dual basis respectively by $x_1, \dots, x_5$ and $\alpha_1, \dots, \alpha_5$. The relations are $[x_1, x_2] = x_3,\ [x_1, x_3] = x_4,\ [x_1, x_4] = x_5$, and the cohomology of $\mathfrak{g}/Z(\mathfrak{g})$ has basis

$$\begin{gathered}\alpha_1,\ \alpha_2\\ \alpha_1\alpha_5,\ \alpha_2\alpha_3,\ \alpha_3\alpha_4-\alpha_2\alpha_5\\ \alpha_1\alpha_4\alpha_5,\ \alpha_2\alpha_3\alpha_4,\ \alpha_1\alpha_2\alpha_5-\alpha_1\alpha_3\alpha_4\\ \alpha_1\alpha_3\alpha_4\alpha_5,\ \alpha_2\alpha_3\alpha_4\alpha_5\\ \alpha_1\alpha_2\alpha_3\alpha_4\alpha_5.\end{gathered}$$

A simple calculation shows that $\tilde{\phi}(\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5)=0$.

We finish with the following question:

Question. *If a nilpotent Lie algebra $\mathfrak{g}$ possesses a non-singular derivation, then does $\mathfrak{g}$ possess a derivation which is non-singular in cohomology?*

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