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# Geodesic loops

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**Abstract.** Using new tangential objects, which we call $\Lambda$-algebras, we obtain for geodesic loops $(L, \nabla, e)$ statements that are analogous to Lie's theorems. We prove that any geodesic loop $L$ defines in the tangential space $T_eL$ a unique $\Lambda$-algebra, and that to any finite dimensional real $\Lambda$-algebra $F$ there exists a geodesic loop $L$, the $\Lambda$-algebra of which is isomorphic to $F$. Geodesic loops which have isomorphic $\Lambda$-algebras, are locally isomorphic.

If the local loop $L$ is diassociative then to the $\Lambda$-algebra of $L$ there corresponds a subseries of the Hausdorff-Campbell formula with respect to the binary Lie algebra belonging to $L$.

## Introduction

A.I. Malcev perceived that the fundamental ideas of Sophus Lie to assign to any local Lie group $G$ a tangential object -its Lie algebra- which determines $G$ in a unique way can be extended to non-associative structures. In particular with any differentiable Moufang loop $L$ is associated a Malcev algebra which determines $L$ uniquely up to coverings (cf. [9] Theorem 4.4). Moreover, in the mean time the theory of Lie Moufang loops reached the same level as the theory of Lie groups (cf. [3], [5], [8]). The most general class of local analytical loops for which we know that they are determined by their tangential objects -by their Bol algebras- in a unique way are the Bol loops (cf. [7] pp. 424-425 ). In general, the differentiable loops have as tangential objects the Akivis algebras, but an Akivis algebra corresponds to a variety of local differentiable loops the algebraic properties of which show greate differences (cf. [2] pp. 240-241).

The aim of this paper is to show that there exist suitable tangential objects for very wide classes of local analytical loops $L$ which determines $L$ in a unique way; concretely we demonstrate this for the class of geodesic loops which respect to linear connections the curvature of which is zero; these tangential objects have in general infinitely many binary operations. Since a differentiable local loop is strongly left alternative if and only if it can be represented as the opposite loop of a geodesic loop with respect to a linear connection with curvature zero, the local

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analytic Bol loops are a proper subclass of geodesic loops considered in our paper (cf. [11] Proposition 4.7).

The tangential object $T_eL$ of a geodesic loop $L$ with respect to a linear connection with vanishing curvature is called a $\Lambda$-algebra. The infinitely many binary operations $\Lambda_{(p,1)}(x,y) : T_eL \times T_eL \to T_eL$ are linear in $y$ and $p$-homogeneous in $x$; moreover, they satisfy certain symmetry conditions. In the case that the local loop $L$ is diassociative to the $\Lambda$-algebra of $L$ there corresponds a subseries of the Hausdorff-Campbell formula with respect to the binary Lie algebra belonging to $L$.

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## 1. Geodesic loops and their $\Lambda$-algebras

**Definition 1.1.** Let $L$ be an analytic manifold and $B$ an open neighbourhood of the identity $e$ of $L$. Let $*, /, \backslash : B \times B \to L$ be analytic mappings. The manifold $L$ is called a local analytic loop if the analytic mappings $*, /, \backslash : B \times B \to L$ satisfy the following identities:

1. $x * e = e * x = x/e = e \backslash x = x$ for all $x \in B$.
2. $(x/y) * y = x, \quad y * (y \backslash x) = x,$
   $(x * y)/y = x, \quad y \backslash (y * x) = x$ for all $x, y \in B$.

provided the left sides of these equations are defined.

**Definition 1.2.** Let $L$ be a differentiable manifold equipped with an affine connection $\nabla$ and let $e$ be a point in $L$. A local analytic loop $(L, \nabla, e)$ is called a geodesic loop if in a normal neighbourhood $U$ of $e \in L$ the multiplication of $L$ is given by

$$x * y = exp_y \circ \tau_{e,y} \circ exp_e^{-1}(x)$$

for all $x, y \in U$, where $\tau_{e,y}$ denotes the parallel translation $T_eL \to T_yL$ along the unique geodesic segment, contained in $U$, between $e$ and $y$ (cf. [4], p. 160 and [7], p. 369).

**Definition 1.3.** Let $V$ be an $n$-dimensional real or complex vector space. $V$ is called a $\Lambda$-algebra if there exists for every natural number $p$ an algebraic map $\Lambda_{(p,1)} : V \times V \to V$ such that the following properties are satisfied:

1. $\Lambda_{(p,1)}(x, \lambda y) = \lambda \Lambda_{(p,1)}(x,y)$ for all $\lambda \in K$ and $x, y \in V$.
2. $\Lambda_{(p,1)}(\lambda x, y) = \lambda^p \Lambda_{(p,1)}(x,y)$ for all $\lambda \in K$ and $x, y \in V$.
3. $\sum_{p=1}^{\infty} \Lambda_{(p,1)}(x,y)$ converges on a neigbourhood $N$ of 0 in $V$.

**Remark 1.4.** The coordinate functions $\Lambda^i_{(p,1)}(x,y)$, $i = 1, \dots, n$, of $\Lambda_{(p,1)}(x,y)$ are homogeneous polynomials of degree $p+1$ which are linear in the coordinates of $y$ and homogeneous polynomials of degree $p$ in the coordinates of $x$.

The $\Lambda$-algebras are suitable tangential objects for geodesic loops $(L, \nabla, e)$ for which the curvature tensor is 0, in order to obtain analogous statements to Lie's theorems.

**Theorem 1.5.** (Lie's First Theorem) *Any geodesic loop $L$ defines in the tangential space $T_eL$ a unique $\Lambda$-algebra.*

**Proof.** The 1-parameter subgroups of a geodesic loop $(L, \nabla, e)$ are geodesic lines through the identity $e$ of $L$ with respect to the linear connection $\nabla$. Hence the canonical coordinate system for $L$ (which is determined by the 1-parameter subgroups) is a normal coordinate system of the connection $\nabla$ in a normal neighbourhood $U \subset \mathbb{R}^n$. Since the curvature of $\nabla$ is zero every parallel translation with respect to $\nabla$ can be given in $U$ by the independent parallel vector fields $A_1(x), \dots, A_n(x)$, $(i = 1, \dots, n)$, where $A_i(0) = e_i$ yield the canonical basis of $\mathbb{R}^n$. The parallel translation $\tau_{e,y}$ is determined by $e_i \mapsto A_i(y)$ $(i = 1, \dots, n)$. Hence the map $\sum\limits_{i=1}^{n} \xi^i e_i \mapsto \sum_{i=1}^n A_i(y)\xi^i$ gives the tangent map $T_e\rho_y$ for the right translation $\rho_y : x \to x * y$. The differential equations of the coordinate functions $y^i(t)$ for the geodesic line $y(t)$ with the initial values $y(0) = y_0$ and $y'(0) = \sum\limits_{j=1}^{n} A_j(y_0)\xi^j$ have the form

$$y'^i(t) = \sum_{j=1}^{n} A_j^i(y(t))\xi^j, \tag{1}$$

where $A_j^i$ denote the coordinate functions of the vector fields $A_j$ and $\xi^j$ are the coordinates with respect to the basis $A_j(y_0)$. Since in the coordinate system of $U$ the geodesic lines are the euclidean lines $(\xi^1, \dots, \xi^n)t$, $t \in \mathbb{R}$, equation (1) implies that the coordinate functions $A_j^i(x)$ of $A_j$ satisfy the identities

$$\xi^i = \sum_{j=1}^{n} A_j^i((\xi^1, \dots, \xi^n)t)\xi^j.$$

The power series of the coordinate functions $A_j^i(x)$ through the point $(0, \dots, 0)$ can be written as

$$\xi^i = \sum_{j=1}^{n} [A_j^i(0) + \sum_{p=1}^{\infty} \frac{1}{p!} \sum_{j_1, \dots, j_p = 1}^{n} \frac{\partial^p A_j^i}{\partial x^{j_1} \dots \partial x^{j_p}}(0)\xi^{j_1} \dots \xi^{j_p}]\xi^j.$$

Since $A_j^i(0) = \delta_j^i$ we obtain

$$\xi^i = \xi^i + \sum_{p=1}^{\infty} \frac{1}{p!} \sum_{j_1, \dots, j_p = 1}^{n} \frac{\partial^p A_j^i}{\partial x^{j_1} \dots \partial x^{j_p}}(0)\xi^{j_1} \dots \xi^{j_p}\xi^{j_{p+1}}.$$

Consequently the coefficients of this expansion satisfy

$$\sum_{\sigma \in \mathbb{Z}_{p+1}} \frac{\partial^p A^i_{j(p+1)\sigma}}{\partial x^{j_{1\sigma}} \dots \partial x^{j_{p\sigma}}}(0) = 0, \tag{2}$$

where $\mathbb{Z}_{p+1}$ is the cyclic permutation group of order $p+1$.

In the tangential space of the identity $e \in L$ we define now the binary operations $\Lambda_{(p,1)}$ as follows:

$$\Lambda_{(1,1)}(x, y) = (\sum_{j=1, h_1=1}^{n} \frac{\partial A_j^i}{\partial x^{h_1}}(0)x^{h_1}y^j)_{i=1}^n$$

$$\Lambda_{(2,1)}(x,y) = (\sum_{j=1,h_1=1,h_2=1}^{n} \frac{1}{2!}\frac{\partial^2 A^i_j}{\partial x^{h_1}\partial x^{h_2}}(0)x^{h_1}x^{h_2}y^j)^n_{i=1}$$

$$\Lambda_{(p,1)}(x,y) = (\sum_{j=1,h_1=1,\dots,h_p=1}^{n} \frac{1}{p!}\frac{\partial^p A^i_j}{\partial x^{h_1}\dots\partial x^{h_p}}(0)x^{h_1}\dots x^{h_p}y^j)^n_{i=1}.$$

The $i$-th coordinate of the vector $\Lambda_{(p,1)}(x,y)$ is

$$\Lambda^i_{(p,1)}(x,y) = \sum_{j=1,h_1=1,\dots,h_p=1}^{n} \frac{1}{p!}\frac{\partial^p A^i_j}{\partial x^{h_1}\dots\partial x^{h_p}}(0)x^{h_1}\dots x^{h_p}y^j.$$

This function is linear in the coordinates of $y$ since it is depending only of the sum of the coordinates of $y$. Hence $\Lambda_{(p,1)}(x,y)$ is linear in $y$. We have

$$\Lambda^i_{(p,1)}(\lambda x,y) = \sum_{j=1,h_1=1,\dots,h_p=1}^{n} \frac{1}{p!}\frac{\partial^p A^i_j}{\partial x^{h_1}\dots\partial x^{h_p}}(0)\lambda x^{h_1}\dots\lambda x^{h_p}y^j =$$

$$\lambda^p\Lambda^i_{(p,1)}(x,y).$$

Hence the second condition for a $\Lambda$-algebra is satisfied. The coordinate functions of the product $(x^1,\dots,x^n)*(y^1,\dots,y^n)$ are given by $A^i_j(y)$ and its partial derivaties (cf.[10] pp.309-312). Since $L$ is analytic loop and the series $\sum\limits_{p=1}^{\infty}\Lambda_{(p,1)}(x,y)$ is a subseries of the power expansion of the product $x*y$ the condition 3. in the definition of a $\Lambda$-algebra is valid. ∎

**Theorem 1.6.** (Lie's Third Theorem) *To any finite dimensional real $\Lambda$-algebra $F$ there exists a geodesic loop $L$, the $\Lambda$-algebra of which is isomorphic to $F$. The geodesic loops which have isomorphic $\Lambda$-algebras, are locally isomorphic.*

**Proof.** Let $F$ be an $n$-dimensional real $\Lambda$-algebra. The convergent series $\sum\limits_{p=1}^{\infty}\Lambda_{(p,1)}(x,y)$ determines a function $f=(f^1,\dots,f^n)$ from $F\times F$ into $F$. The function $f$ is linear in $y$ since the operations $\Lambda_{(p,1)}(x,y)$ are linear in $y$. Hence the $i$-th coordinate function $f^i(x,y)$ has the form

$$f^i(x^1,\dots,x^n,y^1,\dots,y^n) = \sum_{j=1}^{n} f^i_j(x)y^j$$

with suitable real analytic functions $f^i_j$. If we develop the functions $f^i_j$ into Taylor series in the point zero we obtain

$$f^i_j(x) = \sum_{p=1}^{\infty}\frac{1}{p!}\sum_{h_1=1,\dots,h_p=1}^{n}\frac{\partial^p f^i_j}{\partial x^{h_1}\dots\partial x^{h_p}}(0)x^{h_1}\dots x^{h_p}.$$

Since the real analytic function $f$ is described uniquely by a convergent series we can write

$$\Lambda_{(p,1)}(x,y) = (\sum_{j=1,h_1=1,\dots,h_p=1}^{n} \frac{1}{p!}\frac{\partial^p A^i_j}{\partial x^{h_1}\dots\partial x^{h_p}}(0)x^{h_1}\dots x^{h_p}y^j)^n_{i=1},$$

where $A^i_j = f^i_j$. The coordinate functions $A^i_j$ determine the vector fields $A_j$ which are defined as $A_j : y \to (A^1_j(y), \dots, A^n_j(y))$. Considering $F$ as the affine space $\mathbb{R}^n$ we can assign to any point $y$ of $F$ a tangent space $T_yF = \sum_{i=1}^{n} \eta^i A_i(y), \eta_i \in \mathbb{R}$ and form the tangent bundle $TF$. Any vector $v \in T_0F$, $v = \sum_{i=1}^{n} \eta^i e_i$, determines a vector field $Y_v$ putting $Y_v(y) = \sum_{i=1}^{n} \eta^i A_i(y)$.

We define now a covariant derivation $\nabla$, such that the vector fields $A_i$ become parallel vector fields with respect to $\nabla$. Hence in the expression

$$\nabla_X Y = \nabla_X(\sum_{i=1}^{n} Y^i A_i) = \sum_{i=1}^{n}(X(Y)^i)A_i + \sum_{i=1}^{n} Y^i \nabla_X A_i$$

we put $\sum_{i=1}^{n} Y^i \nabla_X A_i = 0$ for any vector field $X$. For the curvature tensor $R(X,Z)Y$ with respect to $\nabla$ we have

$$R(X,Z)Y = \nabla_X \nabla_Z(\sum_{i=1}^{n} Y^i A_i) - \nabla_Z \nabla_X(\sum_{i=1}^{n} Y^i A_i) - \nabla_{[X,Z]}(\sum_{i=1}^{n} Y^i A_i) =$$

$$\nabla_X(\sum_{i=1}^{n}(Z(Y^i))A_i) - \nabla_Z(\sum_{i=1}^{n}(X(Y^i))A_i) - \sum_{i=1}^{n}([X,Z](Y^i))A_i =$$

$$X(\sum_{i=1}^{n}(Z(Y^i))A_i) - Z(\sum_{i=1}^{n}(X(Y^i))A_i) - \sum_{i=1}^{n}([X,Z](Y^i))A_i.$$

Since $[X,Z] = XZ - ZX$ we can write $[X,Z](Y^i) = X(Z(Y^i) - Z(X(Y^i))$. It follows

$$R(X,Z)Y = X(\sum_{i=1}^{n}(Z(Y^i))A_i) - Z(\sum_{i=1}^{n}(X(Y^i))A_i) - \sum_{i=1}^{n}([X,Z](Y^i))A_i = 0.$$

The parallel translations $\tau_{0,y}$ are determined by $\tau_{0,y}(e_i) = A_i(y)$. Since the curvature tensor $R$ with respect to $\nabla$ is zero the parallel translations do not depend on the path. Let $exp$ be the exponential map belonging to the linear connection $\nabla$. We have $exp_y : T_yF \to F : exp_y(v_y) = \gamma(1)$ where $\gamma$ is the geodesic curve on $F$ with respect to $\nabla$ satisfying the conditions $\gamma(0) = y$ and $\gamma'(0) = v_y$. The multiplication $x * y = exp_y \circ \tau_{e,y} \circ exp_e^{-1}(x)$ for $x, y \in F$ defines on $F$ a geodesic loop $L$ with $0$ as the identity. The $\Lambda$-algebra of $L$ is uniquely determined by the parallel translations $\tau_{0,y}$ and hence by the vector fields $A_i$, $i = 1, \dots, n$, as well as by the derivaties of their coordinate functions. ∎

A real algebra $(C, [.,.])$ is called a binary Lie algebra if any two elements of $C$ generate a Lie subalgebra.

To the $\Lambda$-algebras of Lie algebras and of binary Lie algebras there correspond subseries of the Campbell-Hausdorff-series. These subseries are determined by the Lie algebras or binary Lie algebras.

**Theorem 1.7.** *Let $C$ a binary Lie algebra and $(b_{2n})_{n\in\mathbb{N}}$ the sequence of the even Bernoulli numbers. The operations $\Lambda_{(p,1)} : C^2 \to C$ given by*

$$\begin{aligned}
\Lambda_{(1,1)}(X,Y) &= \frac{1}{2}[X,Y] \\
\Lambda_{(2,1)}(X,Y) &= \frac{1}{12}[X,[X,Y]] \\
\Lambda_{(2n,1)}(X,Y) &= \frac{1}{(2n)!}b_{2n}[X,\dots,[X,Y],\dots] \\
\Lambda_{(2n+1,1)}(X,Y) &= 0
\end{aligned}$$

*define on the vector space $C$ a $\Lambda$-algebra $\Lambda(C)$.*

**Proof.** The operations $\Lambda_{(p,1)}$ are linear in $Y$. Moreover, we have

$$\Lambda_{(p,1)}(\lambda X,Y) = \lambda^p \Lambda_{(p,1)}(X,Y)$$

for any $\lambda \in \mathbb{R}$. The series $\sum_{p=1}^{\infty} \Lambda_{(p,1)}(X,Y)$ converges since it is a subseries of the Hausdorff-Campbell series (cf. [1] pp. 201-202). ∎

An analytic loop $L$ is called diassociative if every two elements generate a subgroup of $L$.

**Theorem 1.8.** *The $\Lambda$-algebra $F$ of a local diassociative analytic loop $L = exp(F)$ coincides with the $\Lambda$-algebra $\Lambda(C)$, where $C$ is the binary Lie algebra belonging to $L$.*

**Proof.** According to Malcev, theorems 1 and 2 (cf. [6] pp. 570-571), the tangential algebra of $L$ is a binary Lie algebra $C$. The power series of the multiplication of $L$ is the classical Hausdorff-Campbell series. Let $\Lambda_{(p,1)}, p \in \mathbb{N}$, be the operations of the $\Lambda$-algebra of $L$ and $\Lambda^C_{(p,1)}, p \in \mathbb{N}$, be the operations for the $\Lambda$-algebra $\Lambda(C)$. Since $\sum_{p=1}^{\infty} \Lambda_{(p,1)}(x,y)$ is a subseries of the loop multiplication and

$$\sum_{p=1}^{\infty} \Lambda_{(p,1)}(x,y) = \sum_{p=1}^{\infty} \Lambda^C_{(p,1)}(x,y)$$

we have $\Lambda_{(p,1)} = \Lambda^C_{(p,1)}$. ∎

**Theorem 1.9.** *The local analytic loop $L$ the $\Lambda$-algebra of which is $\Lambda(C)$ for a binary Lie algebra $C$ is diassociative and $C$ is the tangential algebra of $L$ in sense of Malcev.*

**Proof.** Since $\Lambda(C)$ is the $\Lambda$-algebra of $L$, the series

$$\sum_{p=1}^{\infty} \Lambda^C_{(p,1)}(x,y) \tag{3}$$

is a subseries of the series of the multiplication $(x,y) \mapsto x*y : L \times L \to L$ such that $x*y = x+y+\sum_{p,q=1}^{\infty} \Lambda^C_{(p,q)}(x,y)$ and the operations $\Lambda^C_{(p,q)}$ are uniquely determined by $\Lambda^C_{(p,1)}$ through a recursive formula (G) in (cf. [10] p. 311). Since (3) is a subseries of the classical Hausdorff-Campbell series too, hence the multiplication $(x,y) \mapsto x*y$ is described by the classical Hausdorff-Campbell formula and there exists a diassociative loop having the same subseries (3) as $L$. The multiplication of this loop coincides with the multiplication of $L$ and $C$ is the binary Lie algebra in sense of Malcev (cf. [6] pp. 570-571). ■

**Remark 1.10.** Theorems 4 and 5 remain valid also for monoalternative analytic loops since any such loop is diassociative (cf. [12] ).

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