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Capelli elements for the orthogonal Lie algebras

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Abstract. For the central elements in the universal enveloping algebra of the orthogonal Lie algebra given by R. Howe and T. Umeda, Newton type and Cayley-Hamilton type identities are given, and their images under the Harish-Chandra homomorphism are determined. These results show that these central elements can be regarded as the counterparts of the Capelli elements. The calculation of the images under the Harish-Chandra homomorphism clearly reveals the fact that these central elements coincide with those given with the Sklyanin determinant.

Introduction

The Capelli elements are well-known central elements in the universal enveloping algebra of the Lie algebra $\mathfrak{gl}_n$, which played an important role in the classical invariant theory. They are defined as sums of determinants of the generators of $\mathfrak{gl}_n$. Also in the universal enveloping algebra $U(\mathfrak{o}_n)$ of the orthogonal Lie algebra $\mathfrak{o}_n$, a set of central elements is given with determinant in a similar way as the Capelli elements (the Appendix of [6]). Their properties have not been studied so minutely as the Capelli elements so far. In this paper, we give the following results for these central elements: (i) Newton type and Cayley-Hamilton type identities, and (ii) the calculation of the images under the Harish-Chandra homomorphism. These results show that these central elements have strong similarities to the Capelli elements not only in their form but properties. From (ii), we also clearly see that they are essentially identical to the central elements given with the Sklyanin determinant in [10].

We work over a fixed field $\mathbb{K}$ of characteristic 0. We realize the orthogonal Lie algebra $\mathfrak{o}_n$ as the Lie algebra consisting of the alternating matrices. Put $A_{ij} = E_{ij} - E_{ji}$, and arrange a matrix $\boldsymbol{A} = (A_{ij})_{1\le i,j\le n} \in \mathrm{Mat}(n, U(\mathfrak{o}_n))$ from these generators of $\mathfrak{o}_n$. Furthermore, we consider the submatrix $\boldsymbol{A}_I = (A_{ij})_{i,j\in I}$ for an index set $I \subseteq \{1, 2, \dots, n\}$. The sum of determinants of submatrices

$$C_{2k} = \sum_{|I|=2k} \det(\boldsymbol{A}_I + \mathrm{diag}(k, k-1, \dots, -k+1))$$

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is known to be invariant under the adjoint action of the orthogonal Lie group O_n, hence central in $U(\mathfrak{o}_n)$ ([6], [9]). More precisely, the set $\{C_{2k} \mid 0 \leq 2k \leq n\}$ generates the subalgebra $U(\mathfrak{o}_n)^{O_n}$ consisting of the invariants under the adjoint action of O_n. This definition is very similar to that of the Capelli elements. Here we define the determinant of a matrix $\Phi = (\Phi_{ij})_{1\leq i,j\leq n}$ whose entries are not necessarily commutative by the following alternating sum, which is often called the "column-determinant":

$$\det(\Phi) = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sign}(\sigma) \Phi_{\sigma(1)1} \Phi_{\sigma(2)2} \cdots \Phi_{\sigma(n)n}. \tag{1}$$

Another set of generators of $U(\mathfrak{o}_n)^{O_n}$ is given with the trace. Take the trace of the power of the matrix $\boldsymbol{A}$:

$$\operatorname{tr}(\boldsymbol{A}^r) = \sum_{i_1, i_2, \ldots, i_r} A_{i_1 i_2} A_{i_2 i_3} \cdots A_{i_r i_1}.$$

This $\operatorname{tr}(\boldsymbol{A}^r)$ is known to be invariant under the adjoint action of O_n ([3]). Furthermore the set $\{\operatorname{tr}(\boldsymbol{A}^r) \mid 0 \leq r \leq n\}$ generates $U(\mathfrak{o}_n)^{O_n}$.

The first result of this paper is the following relation between these two sets of generators of $U(\mathfrak{o}_n)^{O_n}$:

Theorem A. **(Newton's formula for $\mathfrak{o}_n$)** *The following equality holds as $U(\mathfrak{o}_n)^{O_n}$-coefficient formal power series in λ:*

$$\frac{N(\lambda)}{D(\lambda)} = \sum_{r=0}^{\infty} \lambda^{-1-r} \operatorname{tr}\left(\left(\boldsymbol{A} - \left(\frac{n}{2} - 1 \right) I \right)^r \right).$$

Here $D(\lambda)$ and $N(\lambda)$ are defined by the following polynomials in λ with coefficients C_{2k}:

$$\begin{aligned} D(\lambda) &= \sum_{k=0}^{m} C_{2m-2k} \lambda^{\overline{k}} \lambda^{\underline{k}}, \\ N(\lambda) &= \sum_{k=0}^{m-1} (2k+2) C_{2m-2k-2} \lambda^{\overline{k}} \lambda^{\underline{k+1}} \end{aligned} \tag{2}$$

for $n = 2m$, and

$$\begin{aligned} D(\lambda) &= \sum_{k=0}^{m} C_{2m-2k} (\lambda - 1/2)^{\overline{k+1}} (\lambda - 1/2)^{\underline{k}}, \\ N(\lambda) &= \sum_{k=0}^{m} (2k+1) C_{2m-2k} (\lambda - 1/2)^{\overline{k}} (\lambda - 1/2)^{\underline{k}} \end{aligned} \tag{3}$$

for $n = 2m+1$. Here the symbols $\lambda^{\overline{k}}$ and $\lambda^{\underline{k}}$ mean respectively the rising and falling factorial functions:

$$\lambda^{\overline{k}} = \lambda(\lambda+1)\cdots(\lambda+k-1), \quad \lambda^{\underline{k}} = \lambda(\lambda-1)\cdots(\lambda-k+1).$$

Theorem A is a counterpart of Newton's formula for symmetric functions and its analogues for $\mathfrak{gl}_n$ given in [21] (see also [15], [14]). In fact, considering the case that the matrix in question is replaced by a matrix with commutative entries, we can regard our C_{2k}'s and $\mathrm{tr}(\boldsymbol{A}^r)$'s as analogues of the elementary symmetric functions and the power sum symmetric functions in the eigenvalues respectively. In classical Newton's formula, the denominator on the left-hand side is the characteristic polynomial of a matrix, and the numerator is its differentiation. In the $\mathfrak{gl}_n$ case, these are given respectively as a Capelli type determinant with parameter λ, and its backward difference. In our case of $\mathfrak{o}_n$, the relation between $N(\lambda)$ and $D(\lambda)$ is a bit more complicated:

$$N(\lambda) = \frac{\lambda}{\lambda - 1/2}(D(\lambda) - D(\lambda - 1))$$

for $n = 2m$, and

$$N(\lambda) = \tilde{D}(\lambda) - \tilde{D}(\lambda - 1)$$

for $n = 2m + 1$ with $\tilde{D}(\lambda) = \frac{\lambda}{\lambda - 1/2} D(\lambda)$. More explicit relations between C_{2k}'s and $\mathrm{tr}(\boldsymbol{A}^r)$'s are given in Theorem 5.2 and Corollary 5.3.

In the case of $\mathfrak{gl}_n$, an analogue of the Cayley-Hamilton theorem is known ([10], [21], [17]). Also in our case of $\mathfrak{o}_n$, we have a similar result:

Theorem B. **(The Cayley-Hamilton theorem for $\mathfrak{o}_n$)** *The following equality holds in* $\mathrm{Mat}(n, U(\mathfrak{o}_n))$:

$$D\left(\boldsymbol{A} - \left(\frac{n}{2} - 1\right) I\right) = 0.$$

These results imply that $D(\lambda)$ is the "characteristic polynomial" of the matrix $\boldsymbol{A}$. It is interesting that $D(\lambda)$ is expressed as a "symmetrized determinant" of the matrix $\lambda I - \boldsymbol{A}$ (Proposition 2.2).

The proofs of Theorems A and B are similar to those in the $\mathfrak{gl}_n$ case given in [21]. Namely our main subject is to construct the "cofactor matrix" $\Delta(\lambda)$, which satisfies the following relation (Proposition 4.2):

$$\Delta(\lambda)\left(\boldsymbol{A} - \left(\frac{n}{2} - 1\right) I - \lambda I\right) = D(\lambda) I. \tag{4}$$

Theorems A and B are immediately shown from this relation (§5).

The cofactor matrix $\Delta(\lambda)$ is explicitly described with the Pfaffian by progressing the exterior calculus in [9]. This calculation is based on the following two results: the first is the expression

$$C_{2k} = \sum_{|I| = 2k} \mathrm{Pf}(\boldsymbol{A}_I)^2 \tag{5}$$

given in [9], and the second is an explicit description of the cofactor matrix for $\mathrm{Pf}(\boldsymbol{A})$ (§3). Such a computation utilizing the Pfaffian is one of the key points of this paper.

Theorems A and B can be extended to the orthogonal Lie algebra realized with any non-degenerate symmetric matrix (§6). In fact, we can naturally define the counterparts of $\boldsymbol{A}$ and $D(\lambda)$, which satisfy Theorems A and B. This generalization is based on the fact that our C_{2k}'s and $D(\lambda)$ are written with the "symmetrized determinant", which is invariant under the conjugation (§§1 and 2).

For the characteristic polynomial $D^{\dagger}(\lambda)$ for the split realization of the orthogonal Lie algebra, we have the following theorem as another application of the cofactor matrix:

Theorem C. *The image of $D^{\dagger}(\lambda)$ under the Harish-Chandra homomorphism $\overline{\gamma}$ is given by*

$$\overline{\gamma}(D^{\dagger}(\lambda)) = \begin{cases} (\lambda^2 - F_{11}^2)\cdots(\lambda^2 - F_{mm}^2), & (n = 2m), \\ (\lambda - \frac{1}{2})\,(\lambda^2 - F_{11}^2)\cdots(\lambda^2 - F_{mm}^2), & (n = 2m+1). \end{cases}$$

Here, $\{F_{11}, \ldots, F_{mm}\}$ is the canonical basis of the Cartan subalgebra.

From these Theorems A, B and C, we clearly see that our $D^{\dagger}(\lambda)$ is identical to a central element $C(\lambda)$ given with the Sklyanin determinant in [10]. In fact, the counterparts of these theorems in terms of $C(\lambda)$ are given in the framework of the twisted Yangians (see [14] and [10]). It is interesting that such similar results are deduced from two different motives.

It should be noted that this equality $D^{\dagger}(\lambda) = C(\lambda)$ is also seen from the expression (5) of C_{2k} above and the recent result in [13]. Once we have this identity, our theorems can be deduced from the known results in [14] and [10]. However, our method of cofactors is quite elementary and produces these theorems directly from a unified point of view. We also remark that the techneque in this paper can be applied to obtain a new Cayley-Hamilton type formula for differential operators related with the Skew Capelli elements in $U(\mathfrak{gl}_n)$ [8].

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1. Expressions of Pfaffians and determinants with the exterior algebra

We start with the expressions of the Pfaffians and the determinants in the exterior calculus, which are valid for matrices with non-commutative entries. From these expressions, some fundamental properties of the Pfaffians and the determinants are naturally deduced. Most of contents of the present and the next sections are discussed in [9].

Consider an $n \times n$ matrix Φ whose entries are elements of an associative algebra $\mathcal{A}$. The exterior algebra Λ_n is the associative algebra generated by the n elements $e_1, e_2, \ldots, e_n$ subject to the relations $e_i e_j + e_j e_i = 0$. Hereafter, we calculate in the extended algebra $\Lambda_n \otimes \mathcal{A}$, in which the two subalgebras Λ_n and $\mathcal{A}$ commute with each other. The elements $e_1, e_2, \ldots, e_n$ are thus considered to be formal (anti-commuting) variables to make "generating functions" with coefficients in $\mathcal{A}$.

We often use the symbol of the divided power: $x^{(k)} = \frac{1}{k!}x^k$. Note that the binomial expansion is simply expressed with this symbol: $(x+y)^{(n)} = \sum_{k=0}^{n} x^{(k)} y^{(n-k)}$.

1.1. For an alternating matrix $\Phi = (\Phi_{ij})_{1\le i,j\le 2m}$ of size $n = 2m$, the Pfaffian $\mathrm{Pf}(\Phi)$ is defined by

$$\mathrm{Pf}(\Phi) = \frac{1}{2^m m!} \sum_{\sigma\in\mathfrak{S}_{2m}} \mathrm{sign}(\sigma)\Phi_{\sigma(1)\sigma(2)}\Phi_{\sigma(3)\sigma(4)}\cdots\Phi_{\sigma(2m-1)\sigma(2m)}.$$

To express this in the exterior calculus, we form an element θ_Φ in $\Lambda_n \otimes \mathcal{A}$ as

$$\theta_\Phi = \frac{1}{2}\boldsymbol{e}\Phi{}^t\boldsymbol{e} = \frac{1}{2}\sum_{i,j=1}^{n} e_i e_j \Phi_{ij}$$

with $\boldsymbol{e} = (e_1, \ldots, e_n)$. The Pfaffian $\mathrm{Pf}(\Phi)$ is expressed with this θ_Φ as follows:

$$e_1 \cdots e_n \,\mathrm{Pf}(\Phi) = \theta_\Phi^{(m)}. \tag{1.1}$$

This relation is easy to see from the definition of the Pfaffian.

1.2. The determinant defined by (1) is often called "column-determinant" compared with the name "row-determinant" ([9], [19]). This can be expressed with the element $\psi_j = \sum_{i=1}^{n} e_i \Phi_{ij}$ as follows:

$$e_1 \cdots e_n \det(\Phi) = \psi_1\psi_2\cdots\psi_n. \tag{1.2}$$

We introduce another kind of determinant, which should be called the "symmetrized determinant" or the "double-determinant":

$$\mathrm{Det}(\Phi) = \frac{1}{n!}\sum_{(\sigma,\sigma')\in\mathfrak{S}_n\times\mathfrak{S}_n} \mathrm{sign}(\sigma)\,\mathrm{sign}(\sigma')\Phi_{\sigma(1)\sigma'(1)}\Phi_{\sigma(2)\sigma'(2)}\cdots\Phi_{\sigma(n)\sigma'(n)}.$$

Furthermore consider the following quantity with scalar parameters $u_1, \ldots, u_n$:

$$\begin{aligned}&\mathrm{Det}(\Phi\,;\, u_1, u_2, \ldots, u_n)\\ &\quad = \frac{1}{n!}\sum_{(\sigma,\sigma')\in\mathfrak{S}_n\times\mathfrak{S}_n} \mathrm{sign}(\sigma)\,\mathrm{sign}(\sigma')\Phi_{\sigma(1)\sigma'(1)}(u_1)\cdots\Phi_{\sigma(n)\sigma'(n)}(u_n),\end{aligned}$$

where $\Phi_{ij}(u) = \Phi_{ij} + \delta_{ij}u$. To express this new determinant in terms of the exterior calculus, we must double the anti-commutative variables. Let Λ_{2n} be the exterior algebra generated by the anti-commutative variables e_i, e'_i $(i = 1, \ldots, n)$. Form the elements Ξ_Φ and τ in $\Lambda_{2n}\otimes\mathcal{A}$ as

$$\Xi_\Phi = \boldsymbol{e}\Phi{}^t\boldsymbol{e}' = \sum_{i,j=1}^{n} e_i e'_j \Phi_{ij}, \quad \tau = \boldsymbol{e}{}^t\boldsymbol{e}' = \sum_{i=1}^{n} e_i e'_i$$

with $\boldsymbol{e}' = (e'_1, \ldots, e'_n)$. Then we can express the symmetrized determinant as

$$e_1 e'_1 \cdots e_n e'_n \operatorname{Det}(\Phi) = \Xi_\Phi^{(n)}, \tag{1.3}$$

$$e_1 e'_1 \cdots e_n e'_n \operatorname{Det}(\Phi; u_1, u_2, \ldots, u_n) = \frac{1}{n!}(\Xi_\Phi + u_1\tau)(\Xi_\Phi + u_2\tau)\cdots(\Xi_\Phi + u_n\tau). \tag{1.4}$$

Since the factors $(\Xi_\Phi + u_i\tau)$'s are commutative, we see that $\operatorname{Det}(\Phi\,;\, u_1, \ldots, u_n)$ does not depend on the order of the parameters. Note that $\det(\Phi) = \operatorname{Det}(\Phi)$, when the entries of Φ are commutative.

Remark. The expressions of the Pfaffians and the determinants above are generalized to the following expressions of the subpfaffians and the principal minors. For an alternating matrix Ψ and an $n \times n$ matrix Φ, we have

$$\sum_{|I|=2k} e_{i_1} \cdots e_{i_{2k}} \operatorname{Pf}(\Psi_I) = \theta_\Psi^{(k)} \tag{1.5}$$

with $I = \{i_1, \ldots, i_{2k}\}$, and

$$\begin{aligned} e_1 e'_1 \cdots e_n e'_n \sum_{|I|=\nu} \operatorname{Det}(\Phi_I\,;\, u_1, u_2, \ldots, u_\nu) \\ = \frac{1}{\nu!}(\Xi_\Phi + u_1\tau)(\Xi_\Phi + u_2\tau)\cdots(\Xi_\Phi + u_\nu\tau)\tau^{(n-\nu)}. \end{aligned} \tag{1.6}$$

1.3. The Pfaffians and the symmetrized determinant have the following invariances under the actions of $GL_n = GL_n(\mathbb{K})$:

Proposition 1.1. *The following equality holds for $g \in GL_n$:*

$$\operatorname{Pf}(g\Phi\,{}^t g) = \det(g)\operatorname{Pf}(\Phi).$$

Proposition 1.2. *The symmetrized determinant is invariant under the conjugation by $g \in GL_n$:*

$$\operatorname{Det}(g\Phi g^{-1}\,;\, u_1, \ldots, u_n) = \operatorname{Det}(\Phi\,;\, u_1, \ldots, u_n).$$

More generally we have

$$\sum_{|I|=\nu} \operatorname{Det}((g\Phi g^{-1})_I\,;\, u_1, \ldots, u_\nu) = \sum_{|I|=\nu} \operatorname{Det}(\Phi_I\,;\, u_1, \ldots, u_\nu).$$

Proposition 1.3. *The following equality holds:*

$$\operatorname{Det}({}^t\Phi\,;\, u_1, \ldots, u_n) = \operatorname{Det}(\Phi\,;\, u_1, \ldots, u_n).$$

These propositions are naturally deduced from the expressions of the Pfaffian and the determinant in the exterior calculus by using the invariance of the exterior algebra given in Lemma 1.4 below.

Consider the natural action of $g \in GL_n$ on the vector space $\mathbb{K}^n$ spanned by the anti-commutative variables $e_1, \ldots, e_n$. This action is naturally extended to an automorphism g_* of the exterior algebra Λ_n, and $\Lambda_n \otimes \mathcal{A}$. Similarly, the action of $\alpha \in GL_{2n}$ on the vector space $\mathbb{K}^{2n}$ spanned by $\{e_i, e'_i \,|\, 1 \le i \le n\}$ is extended to an automorphism α_* of the exterior algebra Λ_{2n} and $\Lambda_{2n} \otimes \mathcal{A}$. Note that the exterior algebras Λ_n and Λ_{2n} have standard graded structures: $\Lambda_n = \bigoplus_{k=0}^n \Lambda_n^{(k)}$, $\Lambda_{2n} = \bigoplus_{k=0}^{2n} \Lambda_{2n}^{(k)}$. The following is an elementary fact for the exterior algebras:

Lemma 1.4.

(i) *For* $\varphi \in \Lambda_n^{(n)} \otimes \mathcal{A}$ *and* $g \in GL_n$, *the equality* $g_*(\varphi) = \det(g)\varphi$ *holds.*

(ii) *For* $\varphi \in \Lambda_{2n}^{(2n)} \otimes \mathcal{A}$ *and* $\alpha \in GL_{2n}$, *the equality* $\alpha_*(\varphi) = \det(\alpha)\varphi$ *holds.*

Proof of Proposition 1.1. By definition, the following relation holds:

$$\theta_{g\Phi^t g} = \boldsymbol{e} g\Phi^t g^t \boldsymbol{e} = (\boldsymbol{e}g)\Phi^t(\boldsymbol{e}g) = g_*\boldsymbol{e}\Phi^t\boldsymbol{e} = g_*\theta_\Phi.$$

In particular, since g_* is an automorphism, we have

$$\theta^{(m)}_{g\Phi^t g} = g_*\theta^{(m)}_\Phi = \det(g)\theta^{(m)}_\Phi.$$

Here, the second equality is seen from Lemma 1.4, because $\theta^{(m)}_\Phi$ is of top-degree in $\Lambda_n \otimes \mathcal{A}$. Then our formula is immediate from the expression (1.1) of the Pfaffian. ∎

Propositions 1.2 and 1.3 are similarly proved by considering the following two types of elements of GL_{2n}. We put

$$\alpha_g = \mathrm{diag}(g, {}^t g^{-1}) = \begin{pmatrix} g & 0 \\ 0 & {}^t g^{-1} \end{pmatrix} \in GL_{2n}$$

for $g \in GL_n$, and define an involution $\iota \in GL_{2n}$ by $\iota(e_i) = e_i'$ and $\iota(e_i') = e_i$. We often write the action of this ι on $\varphi \in \Lambda_{2n} \otimes \mathcal{A}$ as $\iota_*(\varphi) = \varphi'$ simply. Note that $\det(\alpha_g) = 1$ and $\det(\iota) = (-)^n$.

Proof of Proposition 1.2. A direct calculation leads us to the following relations:

$$\Xi_{g\Phi g^{-1}} = (\alpha_g)_*\Xi_\Phi, \qquad (\alpha_g)_*\tau = \tau.$$

Since $(\alpha_g)_*$ is an automorphism, we have

$$\begin{aligned}(\Xi_{g\Phi g^{-1}} + u_1\tau)\cdots(\Xi_{g\Phi g^{-1}} + u_\nu\tau)\tau^{n-\nu} &= (\alpha_g)_*(\Xi_\Phi + u_1\tau)\cdots(\Xi_\Phi + u_\nu\tau)\tau^{n-\nu} \\ &= (\Xi_\Phi + u_1\tau)\cdots(\Xi_\Phi + u_\nu\tau)\tau^{n-\nu}.\end{aligned}$$

Here, the second equality is seen from Lemma 1.4, because $\det(\alpha_g) = 1$. This implies our formula by (1.6). ∎

Proof of Proposition 1.3. The following relations are easily seen:

$$\Xi'_\Phi = -\Xi_{{}^t\Phi}, \qquad \tau' = -\tau.$$

Noting these, we apply the involution $' = \iota_*$ to $\varphi = (\Xi_\Phi + u_1\tau)\cdots(\Xi_\Phi + u_n\tau)$. Then we have on one hand $\varphi' = (-)^n(\Xi_{{}^t\Phi} + u_1\tau)\cdots(\Xi_{{}^t\Phi} + u_n\tau)$ by a direct calculation. On the other hand, since φ is of top-degree, we see $\varphi' = \det(\iota)\varphi = (-)^n\varphi$ from Lemma 1.4. Thus our formula holds by the equality (1.4). ∎

2. Determinants for the matrix A

In this section, we apply the consideration in the previous section to the matrix $\boldsymbol{A}$, and give two expressions of $D(\lambda)$ respectively with the symmetrized determinant and the Pfaffian.

2.1. The following relation holds between the column-determinant and the symmetrized determinant for the matrix $\boldsymbol{A}$:

Proposition 2.1. ([9, Proposition 3.2]) *The following equality holds:*

$$\det(\boldsymbol{A} + \mathrm{diag}(u, u-1, \dots, u-n+1)) = \mathrm{Det}(\boldsymbol{A}\,;\, u, u-1, \dots, u-n+1).$$

This is easily checked by using the following commutation relation [9, Lemma 3.1]:

$$\psi_i(u+1)\psi_j(u) + \psi_j(u+1)\psi_i(u) = -\delta_{ij}\cdot 2\theta_{\boldsymbol{A}}. \tag{2.1}$$

Here we put $\psi_j(u) = \sum_{i=1}^n e_i A_{ij}(u) = \psi_j + u e_j$.

Applying Propositions 1.3 and 2.1 to the submatrices $\boldsymbol{A}_I$, we obtain the following expressions of C_{2k}, because $\boldsymbol{A}_I$ is alternating:

$$\begin{aligned} C_{2k} &= \sum_{|I|=2k} \det(\boldsymbol{A}_I + \mathrm{diag}(k, k-1, \dots, -k+1)) \\ &= \sum_{|I|=2k} \det(\boldsymbol{A}_I + \mathrm{diag}(k-1, k-2, \dots, -k)) \\ &= \sum_{|I|=2k} \mathrm{Det}(\boldsymbol{A}_I\,;\, k, k-1, \dots, -k+1) \\ &= \sum_{|I|=2k} \mathrm{Det}(\boldsymbol{A}_I\,;\, k-1, k-2, \dots, -k). \end{aligned} \tag{2.2}$$

It is reasonable to put $C_{2k+1} = 0$. In fact, we have

$$\sum_{|I|=2k+1} \det(\boldsymbol{A}_I + \mathrm{diag}(k, k-1, \dots, -k)) = \sum_{|I|=2k+1} \mathrm{Det}(\boldsymbol{A}_I\,;\, k, k-1, \dots, -k) = 0.$$

Our $D(\lambda)$ defined as a linear combination of C_{2k}'s in (2) and (3) can be expressed simply with the symmetrized determinant just like a characteristic polynomial:

Proposition 2.2. *The following equality holds:*

$$D(\lambda) = \begin{cases} \mathrm{Det}(\lambda I - \boldsymbol{A}\,;\, m-1, m-2, \dots, -m+1, 0), & (n = 2m), \\ \dfrac{\lambda - 1/2}{\lambda} \mathrm{Det}\big(\lambda I - \boldsymbol{A}; m - \frac{1}{2}, m - \frac{3}{2}, \dots, -m + \frac{1}{2}, 0\big), & (n = 2m+1). \end{cases}$$

Here, the dots indicate a sequence of numbers descending by 1.

Proof. Suppose that $n = 2m$. By induction, we see the binomial expansion

$$f^{\pm}_{2m}(u+w; h) = \sum_{\nu=0}^{2m} \binom{2m}{\nu} f^{\pm}_{2m-\nu}(u; h) f^{\pm}_{\nu}(w; h)$$

for the polynomials $f^{\pm}_{\nu}(u; h)$ defined by

$$\begin{aligned} f^{+}_{2k}(u; h) &= (u + kh)(u + (k-1)h)\cdots(u - (k-1)h), \\ f^{-}_{2k}(u; h) &= (u + (k-1)h)(u + (k-2)h)\cdots(u - kh), \\ f^{\pm}_{2k+1}(u; h) &= (u + kh)(u + (k-1)h)\cdots(u - kh). \end{aligned}$$

Using this, we have

$$
\begin{aligned}
(2.3)\qquad f^{\pm}_{2m}(\varXi+\lambda\tau;\tau) &= \sum_{\nu=0}^{2m}\binom{2m}{\nu} f^{\pm}_{2m-\nu}(\varXi;\tau) f^{\pm}_{\nu}(\lambda\tau;\tau)\\
&= \sum_{\nu=0}^{2m}\binom{2m}{\nu} f^{\pm}_{2m-\nu}(\varXi;\tau)\tau^{\nu} f^{\pm}_{\nu}(\lambda;1)\\
&= (2m)!\sum_{k=0}^{m} e_1e_1'\cdots e_ne_n' C_{2m-2k}\cdot f^{\pm}_{2k}(\lambda;1).
\end{aligned}
$$

Here, we used the relation

$$
f^{\pm}_{\nu}(\varXi;\tau)\tau^{2m-\nu} = \nu!(2m-\nu)! e_1e_1'\cdots e_ne_n' C_{\nu},
$$

which follows from (1.6) and (2.2). Sum up the both sides of this (2.3) over the indices $+,-$, and divide by 2. Then, we obtain

$$
\begin{aligned}
&(\varXi+(\lambda+m-1)\tau)(\varXi+(\lambda+m-2)\tau)\cdots(\varXi+(\lambda-m+1)\tau)\cdot(\varXi+\lambda\tau)\\
&\qquad\qquad = (2m)!\sum_{k=0}^{m} e_1e_1'\cdots e_ne_n' C_{2m-2k}\cdot \lambda^{\overline{k}}\lambda^{\underline{k}}.
\end{aligned}
$$

This means our formula by (1.6). In the case of $n=2m+1$, since the proof is almost the same, we omit it. ■

From the proof, we see another expression of $D(\lambda)$ with two column-determinant:

Corollary 2.3. *The following equality holds:*

$$
2D(\lambda) = \begin{cases} H(\lambda)+H(\lambda-1), & (n=2m),\\ \dfrac{\lambda-1/2}{\lambda}\{H(\lambda)+H(\lambda-1)\}, & (n=2m+1).\end{cases}
$$

Here we put

$$
H(\lambda) = \det\bigl(\boldsymbol{A}+\mathrm{diag}(\tfrac{n}{2},\tfrac{n}{2}-1,\dots,-\tfrac{n}{2}+1)+\lambda I\bigr).
$$

Remark. Theorem A can be written with this $H(\lambda)$ in a form which does not depend on the parity of n (§5).

Remark. It is known that $\mathrm{Pf}(\boldsymbol{A})$ and $\sum_{|I|=\nu}\mathrm{Det}(\boldsymbol{A}_I\,;\,u_1,u_2,\dots,u_\nu)$ are invariant under the adjoint actions of SO_n and O_n respectively. This fact is easily seen from Propositions 1.1 and 1.2 by using the following relation for the adjoint action of the orthogonal Lie group O_n on the matrix $\boldsymbol{A}$ [9]:

$$
\mathrm{Ad}(g)\boldsymbol{A} = (gA_{ij}g^{-1})_{1\le i,j\le n} = {}^t g\boldsymbol{A}{}^t g^{-1} = {}^t g\boldsymbol{A} g,
$$

where $g \in O_n$. This follows from the relation

$$\operatorname{Ad}(g)\boldsymbol{E} = (gE_{ij}g^{-1})_{1\leq i,j\leq n} = {}^t g\boldsymbol{E}{}^t g^{-1} \tag{2.4}$$

for $\boldsymbol{E} = (E_{ij})_{1\leq i,j\leq n} \in \operatorname{Mat}(n, U(\mathfrak{gl}_n))$ and $g \in GL_n$, because $\boldsymbol{A} = \boldsymbol{E} - {}^t\boldsymbol{E}$.

2.2. The following expression of C_{2k} is given in [9]:

$$C_{2k} = \sum_{|I|=2k} \operatorname{Pf}(\boldsymbol{A}_I)^2. \tag{2.5}$$

Using the expression (1.5) of the subpfaffians, we can rewrite this as the expression with $\theta = \theta_{\boldsymbol{A}}$ and $\theta' = \theta'_{\boldsymbol{A}} = \frac{1}{2}\boldsymbol{e}'\boldsymbol{A}{}^t\boldsymbol{e}' = \frac{1}{2}\sum_{i,j=1}^n e'_i e'_j A_{ij}$:

$$e_1 e'_1 \cdots e_n e'_n C_{2k} = (-)^k \theta^{(k)}\theta'^{(k)}\tau^{(n-2k)}. \tag{2.6}$$

The cofactor matrix for $D(\lambda)$ is constructed in §4 by combining this expression of C_{2k} and the cofactors for $\operatorname{Pf}(\boldsymbol{A})$ given in the next section.

3. The cofactor matrix for the Pfaffian $\operatorname{Pf}(\boldsymbol{A})$

In this section, we introduce the cofactor matrix for the Pfaffian $\operatorname{Pf}(\boldsymbol{A})$ as a clue to the construction of the cofactors of the characteristic polynomial $D(\lambda)$.

We first recall the cofactor matrix for the Pfaffian in the classical situation [4]. For an alternating matrix Φ of size $n = 2m$ whose entries are commutative, we define a matrix Γ_Φ by

$$e_1 \cdots e_{2m}(\Gamma_\Phi)_{ij} = -\theta_\Phi^{(m-1)} e_i e_j. \tag{3.1}$$

Here $(\Gamma_\Phi)_{ij}$ is the (i,j)-entry of the matrix Γ_Φ. From (1.5), we see that $(\Gamma_\Phi)_{ij}$ is equal to the subpfaffian $\operatorname{Pf}(\Phi_{(i,j)})$ up to sign. Here $\Phi_{(i,j)}$ is the alternating submatrix obtained by deleting the ith and jth rows and ith and jth columns from Φ. We note that this definition is rewritten as

$$e_1 \cdots e_{2m}\Gamma_\Phi = -{}^t\boldsymbol{e}\left(\frac{1}{2}\boldsymbol{e}\Phi{}^t\boldsymbol{e}\right)^{(m-1)}\boldsymbol{e}.$$

We call this matrix Γ_Φ the cofactor matrix for the Pfaffian $\operatorname{Pf}(\Phi)$. In fact, the product of Γ_Φ and Φ is known to equal to the "scalar matrix" $\operatorname{Pf}(\Phi)I$:

$$\Gamma_\Phi\Phi = \Phi\Gamma_\Phi = \operatorname{Pf}(\Phi)I. \tag{3.2}$$

For the Pfaffian $\operatorname{Pf}(\boldsymbol{A})$ of our matrix $\boldsymbol{A} = (A_{ij})_{1\leq i,j\leq 2m}$, whose entries are non-commutative, we also define its cofactor matrix $\Gamma_{\boldsymbol{A}}$ by the same formula (3.1). Then the following relation holds corresponding to (3.2):

Proposition 3.1. *The following equality holds in* $\mathrm{Mat}(2m, U(\mathfrak{o}_{2m}))$:

$$\Gamma_{\boldsymbol{A}} \boldsymbol{A}' = \boldsymbol{A}' \Gamma_{\boldsymbol{A}} = \mathrm{Pf}(\boldsymbol{A}) I,$$

where $\boldsymbol{A}' = \boldsymbol{A} - (m-1)I$. *This relation is rewritten in the exterior calculus as*

$$-\theta^{(m-1)} e_\beta \psi_\alpha(-m+1) = \delta_{\alpha\beta} \theta^{(m)},$$

where $\psi_\alpha(u) = \sum_{i=1}^n e_i A_{i\alpha}(u) = \psi_\alpha + u e_\alpha$.

Proof. (i) The case of $\alpha = \beta$. We put $\theta_{(\alpha)} = \frac{1}{2} \sum_{i,j \neq \alpha} e_i e_j A_{ij}$, so that $\theta = \theta_{(\alpha)} - e_\alpha \psi_\alpha$. By a direct calculation we see that $\theta_{(\alpha)}$ and $e_\alpha \psi_\alpha(u)$ commute. Our assertion is deduced from the binomial expansion of $\theta^{(m)} = (\theta_{(\alpha)} - e_\alpha \psi_\alpha)^{(m)}$. In fact, since $\theta_{(\alpha)}^m = 0$ and $(e_\alpha \psi_\alpha)^2 = 0$, we have

$$\theta^{(m)} = (\theta_{(\alpha)} - e_\alpha \psi_\alpha)^{(m)} = -\theta_{(\alpha)}^{(m-1)} e_\alpha \psi_\alpha = -\theta^{(m-1)} e_\alpha \psi_\alpha(-m+1)$$

as desired. The last equality is easily seen, because the influence of the terms of θ and $\psi_\alpha(u)$ containing the variable e_α is killed by the factor e_α:

$$\theta e_\alpha = \theta_{(\alpha)} e_\alpha, \qquad e_\alpha \psi_\alpha = e_\alpha \psi_\alpha(u) \qquad \text{for any } u. \tag{3.3}$$

(ii) The case of $\alpha \neq \beta$. The following expansion holds in a similar way as in (i):

$$\theta^{(m-1)} = (\theta_{(\alpha)} - e_\alpha \psi_\alpha)^{(m-1)} = \theta_{(\alpha)}^{(m-1)} - \theta_{(\alpha)}^{(m-2)} e_\alpha \psi_\alpha.$$

Hence, we have

$$\theta^{(m-1)} e_\beta \psi_\alpha(u) = \theta_{(\alpha)}^{(m-1)} e_\beta \psi_\alpha(u) - \theta_{(\alpha)}^{(m-2)} e_\alpha \psi_\alpha e_\beta \psi_\alpha(u). \tag{3.4}$$

Here, on one hand, the first term on the right-hand side is equal to $u \theta_{(\alpha)}^{(m-1)} e_\beta e_\alpha$. In fact, since this is of top-degree, and the factor $\theta_{(\alpha)}^{(m-1)} e_\beta$ does not contain the variable e_α, we see that, in the factor $\psi_\alpha(u) = \sum_{p=1}^n e_p A_{p\alpha}(u)$, only the term $e_\alpha A_{\alpha\alpha}(u) = u e_\alpha$ survives. On the other hand, the second term is computed as follows. Putting $i = j = \alpha$ in the formula (2.1), we have

$$\psi_\alpha(u+1) \psi_\alpha(u) = -\theta,$$

and in particular

$$e_\alpha \psi_\alpha \psi_\alpha(u) = -e_\alpha \theta_{(\alpha)}$$

by the relations (3.3). Hence the second term on the right-hand side of (3.4) is equal to $\theta_{(\alpha)}^{(m-2)} \theta_{(\alpha)} e_\beta e_\alpha = (m-1) \theta_{(\alpha)}^{(m-1)} e_\beta e_\alpha$. Thus, we have

$$\theta^{(m-1)} e_\beta \psi_\alpha(u) = (u + m - 1) \theta_{(\alpha)}^{(m-1)} e_\beta e_\alpha,$$

and our goal $\theta^{(m-1)} e_\beta \psi_\alpha(-m+1) = 0$ in particular. ∎

The second equality in Proposition 3.1 is generalized to the following formula for arbitrary natural number n not necessarily even:

Corollary 3.2. *The following formula holds as an equality in $\Lambda_n \otimes U(\mathfrak{o}_n)$ for distinct indices $\beta_1, \dots, \beta_{n-2k+1}$:*

$$\theta^{(k-1)} e_{\beta_1} e_{\beta_2} \cdots e_{\beta_{n-2k+1}} \psi_\alpha(-k+1) = \begin{cases} (-)^{n+i}\theta^{(k)} e_{\beta_1} \cdots \widehat{e_{\beta_i}} \cdots e_{\beta_{n-2k+1}}, & \alpha = \beta_i, \\ 0, & \alpha \notin \{\beta_1, \beta_2, \dots, \beta_{n-2k+1}\}. \end{cases}$$

Here $\widehat{e_{\beta_i}}$ means that e_{β_i} is omitted.

Proof. We can assume that $\alpha \neq \beta_p$ for $p \neq i$. Applying the second equality in Proposition 3.1 to the submatrix $\boldsymbol{A}_I$ with

$$I = \{1, 2, \dots, n\} \setminus \{\beta_1, \dots, \widehat{\beta_i}, \dots, \beta_{n-2k+1}\},$$

we have

$$\tilde{\theta}^{(k-1)} e_{\beta_i} \tilde{\psi}_\alpha(-k+1) = -\delta_{\alpha\beta_i} \tilde{\theta}^{(k)}$$

with $\tilde{\theta} = \sum_{p,q \in I} e_p e_q A_{pq}$ and $\tilde{\psi}_\alpha(u) = \sum_{p \in I} e_p A_{p\alpha}(u)$. Then, we obtain our formula multiplying the both sides of this equality by $e_{\beta_1} \cdots \widehat{e_{\beta_i}} \cdots e_{\beta_{n-2k+1}}$. In fact, we have

$$\tilde{\theta} e_{\beta_1} \cdots \widehat{e_{\beta_i}} \cdots e_{\beta_{n-2k+1}} = \theta e_{\beta_1} \cdots \widehat{e_{\beta_i}} \cdots e_{\beta_{n-2k+1}},$$
$$\tilde{\psi}_\alpha(u) e_{\beta_1} \cdots \widehat{e_{\beta_i}} \cdots e_{\beta_{n-2k+1}} = \psi_\alpha(u) e_{\beta_1} \cdots \widehat{e_{\beta_i}} \cdots e_{\beta_{n-2k+1}}. \qquad \blacksquare$$

4. The cofactor matrix for $D(\lambda)$

In this section, we construct the cofactor matrix $\Delta(\lambda)$ for $D(\lambda)$ using the cofactors for $\mathrm{Pf}(\boldsymbol{A})$ given in the previous section. The main results in this paper are obtained as applications of this cofactor matrix.

Put the matrices B_ν's in $\mathrm{Mat}(n, U(\mathfrak{o}_n))$ for $0 \leq \nu \leq n-1$ as

$$e_1 e'_1 \cdots e_n e'_n (B_\nu)_{ij} = \begin{cases} (-)^{k+1} \theta^{(k)} \theta'^{(k)} \tau^{(n-2k-1)} e_i e'_j, & (\nu = 2k), \\ (-)^k \theta^{(k+1)} \theta'^{(k)} \tau^{(n-2k-2)} e'_i e'_j, & (\nu = 2k+1). \end{cases}$$

Here $(B_\nu)_{ij}$ is the (i,j)-entry of the matrix B_ν. Moreover we put $B_n = 0$. Note that this definition is rewritten as

$$(B_{2k})_{ij} = \sum_{|I| = 2k+1,\, i,j \in I} \pm \mathrm{Pf}(\boldsymbol{A}_{I \setminus \{i\}}) \mathrm{Pf}(\boldsymbol{A}_{I \setminus \{j\}}),$$

$$(B_{2k+1})_{ij} = \begin{cases} \displaystyle\sum_{|I| = 2k+2,\, i,j \in I} \pm \mathrm{Pf}(\boldsymbol{A}_I) \mathrm{Pf}(\boldsymbol{A}_{I \setminus \{i,j\}}), & (i \neq j) \\ 0, & (i = j). \end{cases}$$

Here, the sign in each term is determined by i, j and I. We define the matrix $\Delta(\lambda)$ by the following linear combination of these B_ν's:

$$\Delta(\lambda) = \begin{cases} \displaystyle\sum_{\nu=0}^{n-1} B_{n-\nu-1} h^-_\nu(\lambda), & (n = 2m), \\ \displaystyle\sum_{\nu=0}^{n-1} B_{n-\nu-1} h^+_\nu(\lambda - 1/2), & (n = 2m+1). \end{cases}$$

Here $h_\nu^{\pm}(\lambda)$ are polynomials of degree ν defined by

$$h_\nu^{+}(\lambda)=\begin{cases}\lambda^{\overline{k}}\lambda^{\underline{k}}, & (\nu=2k),\\ \lambda^{\overline{k+1}}\lambda^{\underline{k}}, & (\nu=2k+1),\end{cases}\qquad h_\nu^{-}(\lambda)=\begin{cases}\lambda^{\overline{k}}\lambda^{\underline{k}}, & (\nu=2k),\\ \lambda^{\overline{k}}\lambda^{\underline{k+1}}, & (\nu=2k+1).\end{cases}$$

We easily see the following relations from the expression (2.6) of C_{2k}:

Proposition 4.1. *The following relations hold:*

$$\mathrm{tr}(B_{2k})=-(n-2k)C_{2k},\qquad \mathrm{tr}(B_{2k+1})=0.$$

In particular, the trace of the matrix $\Delta(\lambda)$ is equal to $-N(\lambda)$:

$$\mathrm{tr}(\Delta(\lambda))=-N(\lambda).$$

We call this $\Delta(\lambda)$ the cofactor matrix for $D(\lambda)$. In fact, the product of $\Delta(\lambda)$ and $\boldsymbol{A}-(n/2-1)I-\lambda I$ is equal to the "scalar matrix" $D(\lambda)I$ as is seen in the following proposition. Newton's formulas and the Cayley-Hamilton theorem are obtained as corollaries of this relation (§5).

Proposition 4.2. *The following equality holds:*

$$\Delta(\lambda)(\boldsymbol{A}'-\lambda I)=(\boldsymbol{A}'-\lambda I)\Delta(\lambda)=D(\lambda)I,$$

where $\boldsymbol{A}'=\boldsymbol{A}-(n/2-1)I$.

For the proof of Proposition 4.2, we use the following three lemmas. In these, Lemmas 4.3 and 4.5 are based on the cofactors for $\mathrm{Pf}(\boldsymbol{A})$.

Lemma 4.3. *We have*

$$\begin{aligned}\theta'^{(k-1)}\tau^{(n-2k+1)}\psi'_j(-k+1)&=-\theta'^{(k)}\tau^{(n-2k)}e_j,\\ \theta'^{(k-1)}\tau^{(n-2k)}e'_i\psi'_j(-k+1)&=-\theta'^{(k)}\tau^{(n-2k)}\delta_{ij}-\theta'^{(k)}\tau^{(n-2k-1)}e'_ie_j.\end{aligned}$$

where $\psi'_j(u)=\sum_{i=1}^n e'_iA_{ij}(u)=\psi'_j+ue'_j$.

Lemma 4.4. *We have*

$$\theta^{(k)}\theta'^{(k-1)}\tau^{(n-2k)}e'_ie'_j=\theta^{(k-1)}\theta'^{(k)}\tau^{(n-2k)}e_ie_j.$$

Lemma 4.5. *We have*

$$\theta^{(k)}\theta'^{(k)}\tau^{(n-2k-1)}(e_ie'_j+e'_ie_j)+(n-2k)\theta^{(k)}\theta'^{(k-1)}\tau^{(n-2k)}e'_ie'_j=0.$$

Proof of Lemma 4.3. We can reduce this lemma to Corollary 3.2 by applying the involution $\iota_*={}'$ and expanding the powers of $\tau=\sum_{p=1}^n e_pe'_p$. ■

Proof of Lemma 4.4. Our assertion is deduced from the special case of $n = 2k$ by applying to the submatrices $\boldsymbol{A}_I$ of $\boldsymbol{A}$ for $|I| = 2k$. In fact, we have

$$\theta^{(k)}\theta'^{(k-1)}\tau^{(n-2k)}e'_ie'_j = \sum_{|I|=2k,\ i,j\in I} \theta_I^{(k)}\theta_I'^{(k-1)}e'_ie'_j \cdot \tau_{I^c}^{(n-2k)},$$
$$\theta^{(k-1)}\theta'^{(k)}\tau^{(n-2k)}e_ie_j = \sum_{|I|=2k,\ i,j\in I} \theta_I^{(k-1)}\theta_I'^{(k)}e_ie_j \cdot \tau_{I^c}^{(n-2k)},$$

where θ_I, θ'_I and τ_{I^c} are defined by

$$\theta_I = \sum_{p,q\in I} e_pe_qA_{pq}, \quad \theta'_I = \sum_{p,q\in I} e'_pe'_qA_{pq}, \quad \tau_{I^c} = \sum_{p\notin I} e_pe'_p.$$

Therefore, we may assume that $n = 2k$ without loss of generality. We start with the application of the involution $\iota_* = {}'$ to $\varphi = \theta^{(k)}\theta'^{(k-1)}e'_ie'_j$. On one hand, a direct calculation leads us to $\varphi' = \theta'^{(k)}\theta^{(k-1)}e_ie_j$. On the other hand we see $\varphi' = (-)^n\varphi = \varphi$ from Lemma 1.4. Thus we have

$$\theta^{(k)}\theta'^{(k-1)}e'_ie'_j = \theta'^{(k)}\theta^{(k-1)}e_ie_j. \tag{4.1}$$

Here, the factor $\theta'^{(k)}$ on the right-hand side is central in $\Lambda_{2n}\otimes U(\mathfrak{o}_n)$. In fact, as is seen in (1.1), the coefficient of $\theta'^{(k)}$ is equal to the Pfaffian $\mathrm{Pf}(\boldsymbol{A})$, which is central in $U(\mathfrak{o}_n)$. Hence, we can exchange $\theta'^{(k)}$ and $\theta^{(k-1)}$ on the right-hand side of (4.1). Thus our assertion has been proved. ∎

Proof of Lemma 4.5. We assume that $i \neq j$, because our assertion is obviously true in the case of $i = j$. Moreover we can assume that $n = 2k+1$ as in the proof of Lemma 4.4. In this case, our formula is equivalent to

$$\theta^{(k)}\theta'^{(k)}e_ie'_j = \theta'^{(k)}\theta^{(k)}e_ie'_j - \theta^{(k)}\theta'^{(k-1)}\tau e'_ie'_j. \tag{4.2}$$

In fact, we see that $\theta^{(k)}\theta'^{(k)}e'_ie_j = -\theta'^{(k)}\theta^{(k)}e_ie'_j$ by calculating the action of the involution $\iota_* = {}'$ to $\theta^{(k)}\theta'^{(k)}e'_ie_j$ in two ways as in the proof of Lemma 4.4. This (4.2) is proved by computing the element $\varphi = \theta^{(k)}\theta'^{(k)}e_ie'_j$ as follows. Note the relations

$$\theta^{(k)}e_j = -\theta^{(k-1)}e_i\psi_ie_j, \quad \theta'^{(k)}e'_j = -\theta'^{(k-1)}e'_i\psi'_ie'_j. \tag{4.3}$$

These are immediate from Corollary 3.2, because we have $e_i\psi_i(u) = e_i\psi_i$ for any u. Using this (4.3), we have

$$\varphi = \theta^{(k)}\theta'^{(k)}e_ie'_j = \theta^{(k)}e_i\theta'^{(k)}e'_j = -\theta^{(k)}e_i\theta'^{(k-1)}e'_i\psi'_ie'_j.$$

Here, the factors $\theta^{(k)}e_i$ and $\theta'^{(k-1)}e'_i$ are anti-commutative. This is seen from the fact that the coefficient of $\theta^{(k)}e_i$ is equal to the Pfaffian $\mathrm{Pf}(\boldsymbol{A}_{(i)})$ for the submatrix $\boldsymbol{A}_{(i)} = (A_{pq})_{p,q\neq i}$, which commutes with A_{pq} for $p, q \neq i$. In fact, the coefficients of $\theta'^{(k-1)}e'_i$ are generated by $\{A_{pq}\,|\,p,q\neq i\}$. Hence we have

$$\varphi = \theta'^{(k-1)}e'_i\theta^{(k)}e_i\psi'_ie'_j.$$

Furthermore we consider the commutation relation between $\theta^{(k)}$ and $e_i\psi_i'$:

$$[\theta^{(k)}, e_i\psi_i'] = \theta^{(k-1)}e_i\psi_i\tau.$$

This is easily seen from the direct calculations $[\theta, e_i\psi_i'] = e_i\psi_i\tau$ and $[\theta, e_i\psi_i] = 0$. Thus, we have

$$\begin{aligned}\varphi &= \theta'^{(k-1)}e_i'\theta^{(k)}e_i\psi_i'e_j' = \theta'^{(k-1)}e_i'e_i\psi_i'\theta^{(k)}e_j' + \theta'^{(k-1)}e_i'\theta^{(k-1)}e_i\psi_i\tau e_j' \\ &= \theta'^{(k)}\theta^{(k)}e_ie_j' - \theta'^{(k)}\theta^{(k-1)}\tau e_i'e_j'.\end{aligned}$$

On the last equality, we used (4.3) again. This implies (4.2), hence our formula.∎

Now we are ready to prove Proposition 4.2. Combining the three lemmas above, we have the following relation among B_ν's, C_ν's and $\boldsymbol{A}$:

Proposition 4.6. *The following equalities hold in* $\mathrm{Mat}(n, U(\mathfrak{o}_n))$:

$$\begin{aligned}&B_{2k}(\boldsymbol{A} - kI) = B_{2k}(\boldsymbol{A}' + (n/2 - k - 1)I) = B_{2k+1}, \\ &B_{2k+1}(\boldsymbol{A} - (n - k - 2)I) = B_{2k+1}(\boldsymbol{A}' - (n/2 - k - 1)I) = B_{2k+2} + C_{2k+2}I.\end{aligned}$$

Proof. Using Lemma 4.3 and then Lemma 4.4, we have the following equality:

$$\begin{aligned}(-)^{k+1}\theta^{(k)}\theta'^{(k)}\tau^{(n-2k-1)}e_i\psi_j'(-k) &= (-)^k\theta^{(k)}\theta'^{(k+1)}\tau^{(n-2k-2)}e_ie_j \\ &= (-)^k\theta^{(k+1)}\theta'^{(k)}\tau^{(n-2k-2)}e_i'e_j'.\end{aligned}$$

Our first formula is immediate from this together with the definition of B_ν.

To show the second formula, we calculate the quantity

$$\varphi = (-)^k\theta^{(k+1)}\theta'^{(k)}\tau^{(n-2k-2)}e_i'\psi_j'(-k).$$

In the case of $i = j$, we have the following by Lemma 4.3:

$$\varphi = (-)^{k+1}\theta^{(k+1)}\theta'^{(k+1)}\tau^{(n-2k-2)} + (-)^k\theta^{(k+1)}\theta'^{(k+1)}\tau^{(n-2k-3)}e_ie_i'.$$

In the case of $i \neq j$, applying Lemma 4.3 and then Lemma 4.5, we have

$$\begin{aligned}\varphi &= (-)^{k+1}\theta^{(k+1)}\theta'^{(k+1)}\tau^{(n-2k-3)}e_i'e_j \\ &= (-)^k\theta^{(k+1)}\theta'^{(k+1)}\tau^{(n-2k-3)}e_ie_j' + (-)^k(n-2k-2)\theta^{(k+1)}\theta'^{(k)}\tau^{(n-2k-2)}e_i'e_j'.\end{aligned}$$

By the definition of B_ν and the expression (2.6) of C_{2k}, these equalities imply

$$B_{2k+1}(\boldsymbol{A} - kI) = B_{2k+2} + C_{2k+2}I + (n - 2k - 2)B_{2k+1}.$$

Our second formula is immediate from this. ∎

Proof of Proposition 4.2. The proposition is obtained as a linear combination of the relations in Proposition 4.6. Assume that $n = 2m$. By the definition of $\Delta(\lambda)$ and Proposition 4.6, we have

$$
\begin{aligned}
&\Delta(\lambda)(\boldsymbol{A}' - \lambda I) \\
&\quad = \left\{ \sum_{k=0}^{m-1} B_{2m-2k-2}\lambda^{\overline{k}}\lambda^{\underline{k+1}} + \sum_{k=0}^{m-1} B_{2m-2k-1}\lambda^{\overline{k}}\lambda^{\underline{k}} \right\} (\boldsymbol{A} - (m-1)I - \lambda I) \\
&\quad = \sum_{k=0}^{m-1} \{B_{2m-2k-2}(\boldsymbol{A} - (m-k-1)I) + B_{2m-2k-2}\cdot(-k-\lambda)\}\lambda^{\overline{k}}\lambda^{\underline{k+1}} \\
&\qquad + \sum_{k=0}^{m-1} \{B_{2m-2k-1}(\boldsymbol{A} - (n-(m-k-1)-2)I) + B_{2m-2k-1}\cdot(k-\lambda)\}\lambda^{\overline{k}}\lambda^{\underline{k}} \\
&\quad = \sum_{k=0}^{m-1} \left\{ B_{2m-2k-1}\lambda^{\overline{k}}\lambda^{\underline{k+1}} - B_{2m-2k-2}\lambda^{\overline{k+1}}\lambda^{\underline{k+1}} \right\} \\
&\qquad + \sum_{k=0}^{m-1} \left\{ (B_{2m-2k} + C_{2m-2k}I)\lambda^{\overline{k}}\lambda^{\underline{k}} - B_{2m-2k-1}\lambda^{\overline{k}}\lambda^{\underline{k+1}} \right\} \\
&\quad = \sum_{k=0}^{m-1} C_{2m-2k} I \lambda^{\overline{k}}\lambda^{\underline{k}} - B_0\lambda^{\overline{m}}\lambda^{\underline{m}} + B_{2m}.
\end{aligned}
$$

This coincides with $D(\lambda)I$ as desired, because $B_0 = -I$ and $B_{2m} = 0$. The proof for the case of $n = 2m+1$ is similar. ∎

In the remainder of this section, we present some supplementary formulas for the cofactor matrix $\Delta(\lambda)$.

In the case that $n = 2m$ is even, we can still factorize the cofactor matrix $\Delta(\lambda)$. Define a matrix $Y(\lambda)$ by

$$Y(\lambda) = \sum_{k=0}^{m-1} B_{2m-2k-2}\lambda^{\overline{k}}\lambda^{\underline{k}}.$$

Then we have the following relation:

Proposition 4.7. *When $n = 2m$, we have*

$$Y(\lambda)(\boldsymbol{A}' + \lambda I) = \Delta(\lambda),$$

and in particular

$$Y(\lambda)(\boldsymbol{A}' + \lambda I)(\boldsymbol{A}' - \lambda I) = D(\lambda)I.$$

This can be checked in a similar way as Proposition 4.2 by using Proposition 4.6.

For odd $n = 2m+1$, such a factorization of the cofactor matrix $\Delta(\lambda)$ is not available. We have the following proposition instead of Proposition 4.7. Define a matrix $Y(\lambda)$ by

$$Y(\lambda) = \sum_{k=0}^{m-1} B_{2m-2k-2}\,(\lambda - 1/2)^{\overline{k+1}}\,(\lambda - 1/2)^{\underline{k}}.$$

Proposition 4.8. *When $n = 2m+1$, the following equalities hold:*

$$Y(\lambda)(\boldsymbol{A}' + \lambda I) + \Delta(1/2) = \Delta(\lambda),$$

$$(\boldsymbol{A} - mI)Y(\lambda)(\boldsymbol{A}' + \lambda I)(\boldsymbol{A}' - \lambda I) = D(\lambda)(\boldsymbol{A} - mI).$$

Here, the second equality is immediately follows from the first, because we have $(\boldsymbol{A} - mI)\Delta(1/2) = D(1/2)I = 0$.

5. Newton's formulas and the Cayley-Hamilton theorem for $\mathfrak{o}_n$

As corollaries of Proposition 4.2 for the cofactor matrix $\Delta(\lambda)$ for $D(\lambda)$, we have Newton's formulas and the Cayley-Hamilton theorem for $\mathfrak{o}_n$.

Theorem 5.1. **(Newton's formula for $\mathfrak{o}_n$ (1))** *The following equality holds as $U(\mathfrak{o}_n)$-coefficient formal power series in λ:*

$$\frac{N(\lambda)}{D(\lambda)} = \sum_{r=0}^{\infty} \lambda^{-1-r} \operatorname{tr}(\boldsymbol{A}'^r).$$

Here, we put $\boldsymbol{A}' = \boldsymbol{A} - (n/2 - 1)I$.

Remark. This Theorem 5.1 is rewritten with a column-determinant $H(\lambda)$ by using Corollary 2.3:

$$\frac{\lambda}{\lambda - 1/2} \cdot \frac{H(\lambda) - H(\lambda-2)}{H(\lambda) + H(\lambda-1)} = \sum_{r=0}^{\infty} \lambda^{-1-r} \operatorname{tr}(\boldsymbol{A}'^r).$$

This expression does not depend on the parity of n.

Theorem 5.2. **(Newton's formula for $\mathfrak{o}_n$ (2))** *A relation between C_{2k}'s and $\operatorname{tr}(\boldsymbol{A}^r)$'s is given in the form*

$$\sum_{i=0}^{k} C_{2k-2i} \operatorname{tr}\left(\left(\boldsymbol{A}' + \left(\frac{n}{2} - k\right) I\right)^{\overline{i}} \left(\boldsymbol{A}' - \left(\frac{n}{2} - k\right) I\right)^{\underline{i}}\right) = (n-2k)C_{2k},$$

$$\sum_{i=0}^{k} C_{2k-2i} \operatorname{tr}\left(\left(\boldsymbol{A}' + \left(\frac{n}{2} - k - 1\right) I\right)^{\overline{i+1}} \left(\boldsymbol{A}' - \left(\frac{n}{2} - k\right) I\right)^{\underline{i}}\right) = 0.$$

From this Theorem 5.2, we can deduce an explicit expression of C_{2k} with $\operatorname{tr}(\boldsymbol{A}^r)$'s by induction:

Corollary 5.3. *The central element* C_{2k} *is expressed as the following determinant of degree* k*:*

$$(-)^k 2^k k! C_{2k} = \det \begin{pmatrix} Q_{1,1} & 2 & & & \mathbf{0} \\ Q_{2,2} & Q_{2,1} & 4 & & \\ Q_{3,3} & Q_{3,2} & Q_{3,1} & \ddots & \\ \vdots & \vdots & \vdots & \ddots & 2k-2 \\ Q_{k,k} & Q_{k,k-1} & Q_{k,k-2} & \cdots & Q_{k,1} \end{pmatrix}.$$

Here, we put $Q_{\alpha,\beta} = \mathrm{tr}((\boldsymbol{A}' + (n/2-\alpha)I)^{\overline{\beta}}(\boldsymbol{A}' - (n/2-\alpha)I)^{\underline{\beta}})$.

Proof of Theorem 5.1. Multiplying the both sides of the equality in Proposition 4.2 by $(\lambda I - \boldsymbol{A}')^{-1}$ from right, we have

$$-\Delta(\lambda) = D(\lambda)(\lambda I - \boldsymbol{A}')^{-1} = D(\lambda)\sum_{r=0}^{\infty} \lambda^{-1-r}\boldsymbol{A}'^r.$$

Then, taking the trace of the both sides, we obtain the theorem. In fact, the trace of $\Delta(\lambda)$ is equal to $-N(\lambda)$ as is seen in Proposition 4.1. ∎

Proof of Theorem 5.2. Define $\widetilde{B}_\nu \in \mathrm{Mat}(n, U(\mathfrak{o}_n))$ by

$$\widetilde{B}_{2k} = B_{2k} + \sum_{i=0}^{k} C_{2k-2i}(\boldsymbol{A}' + (n/2-k)I)^{\overline{i}}(\boldsymbol{A}' - (n/2-k)I)^{\underline{i}},$$

$$\widetilde{B}_{2k+1} = B_{2k+1} + \sum_{i=0}^{k} C_{2k-2i}(\boldsymbol{A}' + (n/2-k-1)I)^{\overline{i+1}}(\boldsymbol{A}' - (n/2-k)I)^{\underline{i}}.$$

Using Proposition 4.6, we see the relations

$$\widetilde{B}_{2k}(\boldsymbol{A}' + (n/2-k-1)I) = \widetilde{B}_{2k+1},$$

$$\widetilde{B}_{2k+1}(\boldsymbol{A}' - (n/2-k-1)I) = \widetilde{B}_{2k+2}.$$

Then, since $\widetilde{B}_0 = B_0 + I = 0$, we see that $\widetilde{B}_\nu = 0$ by induction. Taking the trace, we come to the theorem by Proposition 4.1. ∎

Since $B_n = 0$, we see that $\widetilde{B}_n = D(\boldsymbol{A}')$ from the definition of $\widetilde{B}_\nu$. Thus, the equality $\widetilde{B}_n = 0$ in the proof of Theorem 5.2 implies the following theorem:

Theorem 5.4. **(The Cayley-Hamilton theorem for $\mathfrak{o}_n$)** *The following equality holds in* $\mathrm{Mat}(n, U(\mathfrak{o}_n))$:

$$D(\boldsymbol{A}') = 0.$$

Remarks. (1) Theorems 5.1 and 5.2 are equivalent. In fact, Theorem 5.2 is directly deduced from Theorem 5.1 by comparing the coefficients of the powers in λ.

(2) Another Newton type identity in $U(\mathfrak{o}_n)$ is given in [12]. This has the same form as Theorem 5.1 except that $N(\lambda)$ and $D(\lambda)$ are given with another certain "single" determinant $\widetilde{\det}$ instead of the column-determinant. This means that our C_{2k}'s are equal to this determinant $\widetilde{\det}$. This fact will be also deduced more clearly in §7 by calculating the image of $D(\lambda)$ under the Harish-Chandra homomorphism.

6. Relation to other realizations of the orthogonal Lie algebras

We have discussed on the orthogonal Lie algebra $\mathfrak{o}_n$ realized as the Lie algebra consisting of the alternating matrices, and constructed some identities in its universal enveloping algebra. In this section, we extend these results to the realization of the orthogonal Lie algebra with any non-degenerate symmetric matrix. For this general realization, we can naturally arrange a matrix $\boldsymbol{F}$ in a similar way as $\boldsymbol{A}$, and will see that this $\boldsymbol{F}$ satisfies Newton type and Cayley-Hamilton type identities, which are described with the symmetrized determinant.

Fix a non-degenerate symmetric matrix $S \in \mathrm{Mat}_n(\mathbb{K})$, and consider the orthogonal Lie algebra defined with this S:

$$\mathfrak{o}(S) = \{X \in \mathfrak{gl}_n \mid {}^tXS + SX = 0\}.$$

According to the embedding of the Lie algebras $\mathfrak{o}(S)$ and $\mathfrak{o}_n$ into $\mathfrak{gl}_n$, we regard the universal enveloping algebras $U(\mathfrak{o}(S))$ and $U(\mathfrak{o}_n)$ as subalgebras of $U(\mathfrak{gl}_n)$. Consider an involution $i_S : X \mapsto S^{-1}{}^tXS$ of $\mathfrak{gl}_n$, so that $X - i_S(X) \in \mathfrak{o}(S)$ for any $X \in \mathfrak{gl}_n$. Then the elements $F_{ij} = E_{ij} - i_S(E_{ij})$ span $\mathfrak{o}(S)$. We arrange a matrix $\boldsymbol{F} = (F_{ij})_{1\le i,j\le n}$ from these. This $\boldsymbol{F}$ is expressed with $\boldsymbol{E} = (E_{ij})_{1\le i,j\le n}$ as

$$\boldsymbol{F} = \boldsymbol{E} - \mathrm{Ad}(S^{-1}){}^t\boldsymbol{E} = \boldsymbol{E} - S{}^t\boldsymbol{E}S^{-1}.$$

Here we used the relation (2.4). In particular, the matrices $\boldsymbol{F}S$ and $S^{-1}\boldsymbol{F}$ are alternating.

The two orthogonal Lie algebras $\mathfrak{o}_n$ and $\mathfrak{o}(S)$ are related with each other as follows. Fix a matrix $s \in \mathrm{Mat}_n(\mathbb{K})$ such that $S = {}^tss$. (Such an s can be taken by extending the ground field $\mathbb{K}$.) Then the restriction of the automorphism $\mathrm{Ad}(s^{-1})\colon X \mapsto s^{-1}Xs$ of $\mathfrak{gl}_n$ gives a natural isomorphism $\mathfrak{o}_n \xrightarrow{\sim} \mathfrak{o}(S)$. By this isomorphism, the matrix $\boldsymbol{A}$ is mapped to $\mathrm{Ad}(s^{-1})\boldsymbol{A} = {}^ts^{-1}\boldsymbol{F}{}^ts$ as is seen from the relation (2.4). Then, we obtain the images of $\mathrm{Pf}(\boldsymbol{A})$, C_{2k} and $\mathrm{tr}(\boldsymbol{A}^r)$:

(6.1)
$$\mathrm{Ad}(s^{-1})\,\mathrm{Pf}(\boldsymbol{A}) = \det(s)^{-1}\,\mathrm{Pf}(\boldsymbol{F}S) = \det(s)\,\mathrm{Pf}(S^{-1}\boldsymbol{F}),$$
$$\mathrm{Ad}(s^{-1})C_{2k} = \sum_{|I|=2k} \mathrm{Det}(\boldsymbol{F}_I\,;\,k,k-1,\dots,-k+1), \quad \mathrm{Ad}(s^{-1})\,\mathrm{tr}(\boldsymbol{A}^r) = \mathrm{tr}(\boldsymbol{F}^r).$$

These are easily checked by using Propositions 1.1, 1.2 and the invariance of the trace under the conjugation. We can express $D_{\boldsymbol{F}}(\lambda) = \mathrm{Ad}(s^{-1})D(\lambda)$ and $N_{\boldsymbol{F}}(\lambda) = \mathrm{Ad}(s^{-1})N(\lambda)$ as linear combinations of the symmetrized determinants in similar ways as (2) and (3). Moreover $D_{\boldsymbol{F}}(\lambda)$ is written as follows by Propositions 1.2 and 2.2:

$$D_{\boldsymbol{F}}(\lambda) = \begin{cases} \mathrm{Det}(\lambda I - \boldsymbol{F}\,;\,m-1,m-2,\dots,-m+1,0), & (n=2m), \\ \dfrac{\lambda-1/2}{\lambda}\mathrm{Det}\big(\lambda I - \boldsymbol{F}; m-\dfrac{1}{2}, m-\dfrac{3}{2},\dots,-m+\dfrac{1}{2},0\big), & (n=2m+1). \end{cases}$$

Newton's formulas and the Cayley-Hamilton theorem given in §5 still hold by replacing $D(\lambda)$, $N(\lambda)$ and $\boldsymbol{A}$ respectively by $D_{\boldsymbol{F}}(\lambda)$, $N_{\boldsymbol{F}}(\lambda)$ and $\boldsymbol{F}$:

Theorem 6.1. **(Newton's formula for $\mathfrak{o}(S)$)** *The following equality holds as $U(\mathfrak{o}(S))$-coefficient formal power series in λ:*

$$\frac{N_{\boldsymbol{F}}(\lambda)}{D_{\boldsymbol{F}}(\lambda)} = \sum_{r=0}^{\infty} \lambda^{-1-r} \operatorname{tr}(\boldsymbol{F}'^r),$$

where $\boldsymbol{F}' = \boldsymbol{F} - (n/2-1)I$.

Theorem 6.2. **(The Cayley-Hamilton theorem for $\mathfrak{o}(S)$)** *The following equality holds in* $\mathrm{Mat}(n, U(\mathfrak{o}(S)))$:

$$D_{\boldsymbol{F}}(\boldsymbol{F}') = 0.$$

In the remainder of this section, we rewrite the cofactor matrices for $\mathrm{Pf}(\boldsymbol{A})$ and $D(\lambda)$ and their equalities in terms of the matrix $\boldsymbol{F}$. First, let us consider the cofactor matrix for $\mathrm{Pf}(S^{-1}\boldsymbol{F})$ in the case of $n = 2m$. We define the matrix $\Gamma_{S^{-1}\boldsymbol{F}}$ by

$$e_1 \cdots e_n (\Gamma_{S^{-1}\boldsymbol{F}})_{ij} = -\theta_{S^{-1}\boldsymbol{F}}^{(m-1)} e_i e_j$$

with $\theta_{S^{-1}\boldsymbol{F}} = \frac{1}{2}\boldsymbol{e}S^{-1}\boldsymbol{F}{}^t\boldsymbol{e} = \frac{1}{2}\sum_{i,j=1}^n e_i e_j (S^{-1}\boldsymbol{F})_{ij}$. We have the following relation corresponding to Proposition 3.1:

Proposition 6.3. *The following equality holds:*

$$\Gamma_{S^{-1}\boldsymbol{F}} S^{-1}\boldsymbol{F}' = \mathrm{Pf}(S^{-1}\boldsymbol{F}) I.$$

Proof. We apply the isomorphism $\mathrm{Ad}(s^{-1})\colon \mathfrak{o}_n \simeq \mathfrak{o}(S)$ to the following expression of Proposition 3.1 in the exterior calculus:

$$-{}^t\boldsymbol{e} \left(\frac{1}{2}\boldsymbol{e}\boldsymbol{A}{}^t\boldsymbol{e}\right)^{(m-1)} \boldsymbol{e} \cdot \boldsymbol{A}' = e_1 \cdots e_n \,\mathrm{Pf}(\boldsymbol{A}) I. \tag{6.2}$$

Then, since $\mathrm{Ad}(s^{-1})\boldsymbol{A} = sS^{-1}\boldsymbol{F}{}^ts$, the left-hand side is mapped to

$$-{}^ts^{-1}\,{}^t(\boldsymbol{e}s) \left(\frac{1}{2}(\boldsymbol{e}s)S^{-1}\boldsymbol{F}{}^t(\boldsymbol{e}s)\right)^{(m-1)} (\boldsymbol{e}s) \cdot S^{-1}\boldsymbol{F}'{}^ts.$$

This is equal to the following by Lemma 1.4:

$$\begin{aligned} -\det(s){}^ts^{-1}\,{}^t\boldsymbol{e} \left(\frac{1}{2}\boldsymbol{e}S^{-1}\boldsymbol{F}{}^t\boldsymbol{e}\right)^{(m-1)} &\boldsymbol{e} \cdot S^{-1}\boldsymbol{F}'{}^ts \\ &= -e_1 \cdots e_n \det(s){}^ts^{-1}\Gamma_{S^{-1}\boldsymbol{F}} S^{-1}\boldsymbol{F}'{}^ts. \end{aligned}$$

This implies our assertion. In fact, the isomorphism $\mathrm{Ad}(s^{-1})$ maps the right-hand side of (6.2) to $e_1 \cdots e_n \det(s)\,\mathrm{Pf}(S^{-1}\boldsymbol{F})$ as is seen in (6.1). ∎

We can similarly rewrite the cofactor matrix for $D(\lambda)$ in terms of $\boldsymbol{F}$ using Lemma 1.4. We put the matrices R_ν's for $0 \le \nu \le n-1$ as

(6.3)
$$e_1e_1' \cdots e_ne_n'(R_\nu)_{ij} = \begin{cases} (-)^{k+1}\theta^{(k)}_{S^{-1}\boldsymbol{F}}\theta'^{(k)}_{S^{-1}\boldsymbol{F}}\tau_{S^{-1}}^{(n-2k-1)}e_ie_j', & (\nu = 2k), \\ (-)^k\theta^{(k+1)}_{S^{-1}\boldsymbol{F}}\theta'^{(k)}_{S^{-1}\boldsymbol{F}}\tau_{S^{-1}}^{(n-2k-2)}e_i'e_j', & (\nu = 2k+1), \end{cases}$$

and define $\Delta_{S^{-1}\boldsymbol{F}}(\lambda)$ by

$$\Delta_{S^{-1}\boldsymbol{F}}(\lambda) = \begin{cases} \sum_{\nu=0}^{n-1} R_{n-\nu-1}h_\nu^-(\lambda), & (n = 2m), \\ \sum_{\nu=0}^{n-1} R_{n-\nu-1}h_\nu^+(\lambda - 1/2), & (n = 2m+1), \end{cases}$$

where $\theta'_{S^{-1}\boldsymbol{F}} = \frac{1}{2}\boldsymbol{e}'S^{-1}\boldsymbol{F}{}^t\boldsymbol{e}'$, and $\tau_{S^{-1}} = \boldsymbol{e}S^{-1}{}^t\boldsymbol{e}'$. Then, we have the following proposition corresponding to Proposition 4.2. This is deduced from Proposition 4.2 in a similar way as Proposition 6.3.

Proposition 6.4. *The following equality holds:*

$$\Delta_{S^{-1}\boldsymbol{F}}(\lambda)S^{-1}(\boldsymbol{F}' - \lambda I) = \det(S^{-1})D_{\boldsymbol{F}}(\lambda)I.$$

Moreover we define $Y_{S^{-1}\boldsymbol{F}}(\lambda)$ by

$$(6.4)\quad Y_{S^{-1}\boldsymbol{F}}(\lambda) = \begin{cases} \sum_{k=0}^{m-1} R_{2m-2k-2}\lambda^{\overline{k}}\lambda^{\underline{k}}, & (n = 2m), \\ \sum_{k=0}^{m-1} R_{2m-2k-2}\,(\lambda - 1/2)^{\overline{k+1}}\,(\lambda - 1/2)^{\underline{k}}, & (n = 2m+1). \end{cases}$$

We have following propositions respectively corresponding to Propositions 4.7 and 4.8:

Proposition 6.5. *When $n = 2m$, the following equality holds:*

$$Y_{S^{-1}\boldsymbol{F}}(\lambda)S^{-1}(\boldsymbol{F}' + \lambda I)(\boldsymbol{F}' - \lambda I) = \det(S^{-1})D_{\boldsymbol{F}}(\lambda)I.$$

Proposition 6.6. *When $n = 2m+1$, the following equality holds:*

$$(\boldsymbol{F} - mI)Y_{S^{-1}\boldsymbol{F}}(\lambda)S^{-1}(\boldsymbol{F}' + \lambda I)(\boldsymbol{F}' - \lambda I) = \det(S^{-1})D_{\boldsymbol{F}}(\lambda)(\boldsymbol{F} - mI).$$

Finally, we note the counterpart of (2.6) in terms of $\boldsymbol{F}$, which we use in §7:

Proposition 6.7. *The following equality holds:*

$$e_1e_1'\cdots e_ne_n'\det(S^{-1})\sum_{|I|=2k}\mathrm{Det}(\boldsymbol{F}_I\,;\,k,k-1,\ldots,-k+1)$$
$$=(-)^k\theta^{(k)}_{S^{-1}\boldsymbol{F}}\theta'^{(k)}_{S^{-1}\boldsymbol{F}}\tau^{(n-2k)}_{S^{-1}}.$$

7. Image of $D(\lambda)$ under the Harish-Chandra homomorphism

In this section, we work on the split realization of the orthogonal Lie algebra, and consider the image of the characteristic polynomial $D^\dagger(\lambda)=D_{\boldsymbol{F}}(\lambda)$ under the Harish-Chandra homomorphism. From this result, it turns out that our $D^\dagger(\lambda)=D_{\boldsymbol{F}}(\lambda)$ is identical to a central element $C(\lambda)$ given with the Sklyanin determinant in [10].

For a reductive Lie algebra $\mathfrak{g}$, the Harish-Chandra homomorphism is defined as follows. Fix a triangular decomposition of $\mathfrak{g}$

$$\mathfrak{g}=\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}_+.$$

This leads us to the following direct sum decomposition of $U(\mathfrak{g})$ as a vector space by the Poincaré-Birkhoff-Witt theorem:

$$U(\mathfrak{g})=U(\mathfrak{h})\oplus(\mathfrak{n}_-U(\mathfrak{g})+U(\mathfrak{g})\mathfrak{n}_+).$$

Denote by γ the projection of $U(\mathfrak{g})$ onto $U(\mathfrak{h})$ along this decomposition. The restriction of γ to the centralizer $U(\mathfrak{g})^{\mathfrak{h}}$ of $\mathfrak{h}$ gives an algebra homomorphism. Moreover we consider the rho-shift $T_{-\rho}$, the automorphism of $U(\mathfrak{h})$ determined by $\mathfrak{h}\ni H\mapsto H-\rho(H)$. Here ρ is the element of $\mathfrak{h}^*$ defined by

$$\rho(H)=\frac{1}{2}\mathrm{tr}_{\mathfrak{n}_+}(\mathrm{ad}\,H),\qquad \text{for } H\in\mathfrak{h}.$$

The composition $\overline{\gamma}=T_{-\rho}\circ\gamma$ gives an algebra homomorphism on $U(\mathfrak{g})^{\mathfrak{h}}$. Moreover this $\overline{\gamma}$ is known to give an algebra isomorphism $ZU(\mathfrak{g})\simeq U(\mathfrak{h})^W$, where $ZU(\mathfrak{g})$ is the center of $U(\mathfrak{g})$, and $U(\mathfrak{h})^W$ is the subalgebra of $U(\mathfrak{h})$ consisting of the invariants under the action of the Weyl group W. We call this $\overline{\gamma}$ the Harish-Chandra homomorphism.

Hereafter we work on the orthogonal Lie algebra realized with the symmetric matrix $S_0=(\delta_{i,n+1-j})_{1\le i,j\le n}$:

$$\mathfrak{o}(S_0)=\{X\in\mathfrak{gl}_n\,|\,{}^tXS_0+S_0X=0\}.$$

This $\mathfrak{o}(S_0)$ is spanned by $F_{ij}=E_{ij}-E_{n+1-j,n+1-i}$. Let $\mathfrak{n}_-,\mathfrak{h}$ and $\mathfrak{n}_+$ be the Lie subalgebras spanned by the elements F_{ij} such that $i>j$, $i=j$, $i<j$ respectively. Then these give a triangular decomposition of $\mathfrak{o}(S_0)$:

$$\mathfrak{o}(S_0)=\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}_+.$$

The rho-shift $T_{-\rho}$ acts on the basis $\{F_{ii}\,|\,1\le i\le n/2\}$ of $\mathfrak{h}$ as

$$T_{-\rho}F_{ii}=F_{ii}-\frac{n}{2}+i.$$

The images of $\mathrm{Pf}(S_0^{-1}\boldsymbol{F})$ and $D_{\boldsymbol{F}}(\lambda)$ under $\overline{\gamma}$ are given as follows. We note that Proposition 7.1 has been known. (For example see [25, §127] or [10, p. 942].)

Proposition 7.1. *When $n = 2m$ is even, the image of* $\mathrm{Pf}(S_0^{-1}\boldsymbol{F})$ *under the Harish-Chandra homomorphism is given by*

$$\overline{\gamma}(\mathrm{Pf}(S_0^{-1}\boldsymbol{F})) = (-)^m F_{11}F_{22}\cdots F_{mm}.$$

Theorem 7.2. *The image of* $D_{\boldsymbol{F}}(\lambda)$ *under the Harish-Chandra homomorphism is given by*

$$\overline{\gamma}(D_{\boldsymbol{F}}(\lambda)) = \begin{cases} (\lambda^2 - F_{11}^2)\cdots(\lambda^2 - F_{mm}^2), & (n = 2m), \\ \left(\lambda - \frac{1}{2}\right)(\lambda^2 - F_{11}^2)\cdots(\lambda^2 - F_{mm}^2), & (n = 2m+1). \end{cases}$$

These can be proved by induction in the following way. The Lie subalgebra generated by $\{F_{ij} \,|\, 2 \le i,j \le n-1\}$ is regarded as the orthogonal Lie algebra $\mathfrak{o}(\tilde{S}_0)$ realized with the symmetric matrix $\tilde{S}_0 = (S_0)_{(1n)} = (\delta_{i,n+1-j})_{2\le i,j\le n-1}$. For this $\mathfrak{o}(\tilde{S}_0)$, we can consider the matrix $\tilde{\boldsymbol{F}} = \boldsymbol{F}_{(1n)} = (F_{ij})_{2\le i,j\le n-1}$ and its characteristic polynomial $D_{\tilde{\boldsymbol{F}}}(\lambda)$, or the Pfaffian $\mathrm{Pf}(\tilde{S}_0^{-1}\tilde{\boldsymbol{F}})$ as for $\mathfrak{o}(S_0)$. The triangular decomposition of $\mathfrak{o}(\tilde{S}_0)$ is naturally given by restricting that of $\mathfrak{o}(S_0)$. We will prove Proposition 7.1 and Theorem 7.2 reducing their assertions to the case of $\mathfrak{o}(\tilde{S}_0)$. For this reduction, we use the cofactors for $\mathrm{Pf}(S_0^{-1}\boldsymbol{F})$ and $D_{\boldsymbol{F}}(\lambda)$.

Proof of Proposition 7.1. Apply γ to the (n,n)-entry of the both sides of the equality in Proposition 6.3. Then, we have

$$\gamma(\mathrm{Pf}(S_0^{-1}\boldsymbol{F})) = \sum_{i=1}^{n} \gamma((\Gamma_{S_0^{-1}\boldsymbol{F}})_{ni}(\boldsymbol{F}')_{n-i+1,n}) = \gamma((\Gamma_{S_0^{-1}\boldsymbol{F}})_{n1})(-F_{11} - (m-1)).$$

In fact, $(\boldsymbol{F}')_{n-i+1,n}$ belongs to $\mathfrak{n}_+$ for $i \ne 1$. Here, $(\Gamma_{S_0^{-1}\boldsymbol{F}})_{n1}$ is equal to the Pfaffian $\mathrm{Pf}(\tilde{S}_0^{-1}\tilde{\boldsymbol{F}})$. This fact is easily seen from the following expressions in the exterior calculus by noting that $\theta_{S_0^{-1}\boldsymbol{F}} e_n e_1 = \tilde{\theta}_{S_0^{-1}\boldsymbol{F}} e_n e_1$:

$$e_1\cdots e_n(\Gamma_{S_0^{-1}\boldsymbol{F}})_{n1} = -\theta^{(m-1)}_{S_0^{-1}\boldsymbol{F}} e_n e_1, \qquad e_2\cdots e_{n-1}\,\mathrm{Pf}(\tilde{S}_0^{-1}\tilde{\boldsymbol{F}}) = \tilde{\theta}^{(m-1)}_{S_0^{-1}\boldsymbol{F}},$$

where $\tilde{\theta}_{S_0^{-1}\boldsymbol{F}} = \frac{1}{2}\sum_{2\le i,j\le n-1} e_i e_j F_{n+1-i,j}$. Hence, applying $T_{-\rho}$, we have

$$\overline{\gamma}(\mathrm{Pf}(S_0^{-1}\boldsymbol{F})) = \overline{\gamma}(\mathrm{Pf}(\tilde{S}_0^{-1}\tilde{\boldsymbol{F}}))\cdot(-F_{11}).$$

Thus our task has been reduced to the calculation of the image of the Pfaffian $\mathrm{Pf}(\tilde{S}_0^{-1}\tilde{\boldsymbol{F}})$ of degree $n-2$. The proposition is proved by induction. ∎

To prove Theorem 7.2 similarly, we need the following relation between the $(n,1)$-entry of the matrix $Y_{S_0^{-1}\boldsymbol{F}}(\lambda)$ given in (6.4) and the characteristic polynomial $D_{\tilde{\boldsymbol{F}}}(\lambda)$ for the submatrix $\tilde{\boldsymbol{F}}$:

Lemma 7.3. *The following equality holds:*

$$\gamma((R_{2k})_{n1}) = \det(\tilde{S}_0^{-1})\gamma\Big(\sum_{|I|=2k} \mathrm{Det}(\tilde{\boldsymbol{F}}_I\,;\, k, k-1, \ldots, -k+1)\Big).$$

In particular, we have

$$\gamma((Y_{S_0^{-1}\boldsymbol{F}}(\lambda))_{n1}) = \det(\tilde{S}_0^{-1})\gamma(D_{\tilde{\boldsymbol{F}}}(\lambda)).$$

Proof. By (6.3) and Proposition 6.7, the lemma is rewritten in terms of the exterior calculus as

$$\gamma(\theta^{(k)}_{S_0^{-1}\boldsymbol{F}}\theta'^{(k)}_{S_0^{-1}\boldsymbol{F}}\tau^{(n-2k-1)}_{S_0^{-1}}e_n e'_1) = \gamma(\tilde{\theta}^{(k)}_{S_0^{-1}\boldsymbol{F}}\tilde{\theta}'^{(k)}_{S_0^{-1}\boldsymbol{F}}\tilde{\tau}^{(n-2k-2)}_{S_0^{-1}}e_1 e'_n e_n e'_1), \tag{7.1}$$

where

$$\tilde{\theta}'_{S_0^{-1}\boldsymbol{F}} = \frac{1}{2}\sum_{2\le i,j\le n-1} e'_i e'_j F_{n+1-i,j}, \qquad \tilde{\tau}_{S_0^{-1}} = \sum_{2\le i\le n-1} e_i e'_{n+1-i}.$$

We put $\omega'_n = \sum_{i=1}^n e'_i F_{n+1-i,n}$, so that $\theta'_{S_0^{-1}\boldsymbol{F}}e'_1 = (\tilde{\theta}'_{S_0^{-1}\boldsymbol{F}} - e'_n\omega'_n)e'_1$. By a direct calculation, we see that $\tilde{\theta}'_{S_0^{-1}\boldsymbol{F}}$ and $e'_n\omega'_n$ are commutative. Since $(e'_n\omega'_n)^2 = 0$, we have the binomial expansion

$$\theta'^{(k)}_{S_0^{-1}\boldsymbol{F}}e_n e'_1 = (\tilde{\theta}'_{S_0^{-1}\boldsymbol{F}} - e'_n\omega'_n)^{(k)}e_n e'_1 = \tilde{\theta}'^{(k)}_{S_0^{-1}\boldsymbol{F}}e_n e'_1 - \tilde{\theta}'^{(k-1)}_{S_0^{-1}\boldsymbol{F}}e'_n\omega'_n e_n e'_1.$$

Here, the second term belongs to the subspace $\Lambda_{2n}\otimes U(\mathfrak{g})\mathfrak{n}_+$, so that we have

$$\gamma(\theta^{(k)}_{S_0^{-1}\boldsymbol{F}}\theta'^{(k)}_{S_0^{-1}\boldsymbol{F}}\tau^{(n-2k-1)}_{S_0^{-1}}e_n e'_1) = \gamma(\theta^{(k)}_{S_0^{-1}\boldsymbol{F}}\tilde{\theta}'^{(k)}_{S_0^{-1}\boldsymbol{F}}\tau^{(n-2k-1)}_{S_0^{-1}}e_n e'_1). \tag{7.2}$$

To complete the proof, we consider another binomial expansion

$$\begin{aligned}\tilde{\theta}'^{(k)}_{S_0^{-1}\boldsymbol{F}}\tau^{(n-2k-1)}_{S_0^{-1}}e_n e'_1 &= \tilde{\theta}'^{(k)}_{S_0^{-1}\boldsymbol{F}}(\tilde{\tau}_{S_0^{-1}} + e_1 e'_n)^{(n-2k-1)}e_n e'_1 \\ &= \tilde{\theta}'^{(k)}_{S_0^{-1}\boldsymbol{F}}\tilde{\tau}^{(n-2k-1)}_{S_0^{-1}}e_n e'_1 + \tilde{\theta}'^{(k)}_{S_0^{-1}\boldsymbol{F}}\tilde{\tau}^{(n-2k-2)}_{S_0^{-1}}e_1 e'_n e_n e'_1.\end{aligned}$$

Here, the first term in the last line vanishes, because this is of degree n in the $n-1$ variables $e'_1,\ldots,e'_{n-1}$. Then, since $\theta_{S_0^{-1}\boldsymbol{F}}e_1 e_n = \tilde{\theta}_{S_0^{-1}\boldsymbol{F}}e_1 e_n$, we have

$$\theta^{(k)}_{S_0^{-1}\boldsymbol{F}}\tilde{\theta}'^{(k)}_{S_0^{-1}\boldsymbol{F}}\tau^{(n-2k-1)}_{S_0^{-1}}e_n e'_1 = \tilde{\theta}^{(k)}_{S_0^{-1}\boldsymbol{F}}\tilde{\theta}'^{(k)}_{S_0^{-1}\boldsymbol{F}}\tilde{\tau}^{(n-2k-2)}_{S_0^{-1}}e_1 e'_n e_n e'_1.$$

From this and (7.2), we have (7.1) as desired. ∎

Proof of Theorem 7.2. The following is the key equality of the proof:

$$\begin{aligned}&\det(S_0^{-1})\gamma(D_{\boldsymbol{F}}(\lambda)) \\ &\quad = \gamma((Y_{S_0^{-1}\boldsymbol{F}}(\lambda))_{n1})\left(-F_{11} - \left(\frac{n}{2}-1\right) + \lambda\right)\left(-F_{11} - \left(\frac{n}{2}-1\right) - \lambda\right).\end{aligned} \tag{7.3}$$

In the case of $n = 2m$, this is directly obtained by applying γ to the (n,n)-entry of the both sides of the equality in Proposition 6.5 as in the proof of

Proposition 7.1. In the case of $n = 2m+1$, we can similarly deduce the following from Proposition 6.6:

$$\begin{aligned}&\det(S_0^{-1})(-F_{11}-m)\gamma(D_{\boldsymbol{F}}(\lambda))\\&=(-F_{11}-m)\gamma((Y_{S_0^{-1}\boldsymbol{F}}(\lambda))_{n1})\left(-F_{11}-\left(\frac{n}{2}-1\right)+\lambda\right)\left(-F_{11}-\left(\frac{n}{2}-1\right)-\lambda\right).\end{aligned}$$

This leads us to (7.3) again. In fact, we can remove the factor $(-F_{11}-m)$ from the both sides, because $U(\mathfrak{o}(S_0))$ is an integral domain.

Since $\gamma((Y_{S_0^{-1}\boldsymbol{F}}(\lambda))_{n1})$ is equal to $\det(\tilde{S}_0^{-1})\gamma(D_{\tilde{\boldsymbol{F}}}(\lambda))$ by Lemma 7.3, applying $T_{-\rho}$ to (7.3), we have

$$\det(S_0^{-1})\overline{\gamma}(D_{\boldsymbol{F}}(\lambda)) = \det(\tilde{S}_0^{-1})\overline{\gamma}(D_{\tilde{\boldsymbol{F}}}(\lambda))(-F_{11}+\lambda)(-F_{11}-\lambda).$$

Thus our task is reduced to the calculation of $\overline{\gamma}(D_{\tilde{\boldsymbol{F}}}(\lambda))$. The theorem is proved by induction. ∎

Finally, we comment the relation between our $D_{\boldsymbol{F}}(\lambda)$ and the Sklyanin determinant in the twisted Yangians. In [10], a central element $C(\lambda)$ in $U(\mathfrak{o}(S_0))$ is given with the Sklyanin determinant. The image of this $C(\lambda)$ under the Harish-Chandra homomorphism is equal to the image of $D_{\boldsymbol{F}}(\lambda)$ given in Theorem 7.2, as is calculated in [10]. This implies that our $D_{\boldsymbol{F}}(\lambda) = \mathrm{Ad}(s^{-1})D(\lambda)$ is identical to this $C(\lambda)$. Thus, we obtain relations among the three type of determinants, i.e., the column-, symmetrized and Sklyanin determinants.

Remark. The fact that our $D_{\boldsymbol{F}}(\lambda)$ is identical to $C(\lambda)$ is seen also from the following equality, which is obtained by applying the isomorphism $\mathrm{Ad}(s^{-1})$ to (2.5):

$$\det(S_0^{-1})\sum_{|I|=2k}\mathrm{Det}(\boldsymbol{F}_I\,;\,k,k-1,\dots,-k+1) = \sum_{|I|=2k}\mathrm{Pf}((S_0^{-1}\boldsymbol{F})_I)^2.$$

In fact, the coefficients of $C(\lambda)$ have the same expressions [13].

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