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# On finite dimensional representations of non-connected reductive groups

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**Abstract.** We extend the classification of irreducible finite dimensional representations of almost simple algebraic groups over an algebraically closed field of characteristic zero to certain non-connected groups $G$ where the component group is cyclic. We also extend some of Steinberg's results on the adjoint quotient $G \to T/W$ to these non-connected groups. These results are used to describe the geometry of $\theta$-conjugacy classes of $G^o$, where $\theta$ is an automorphism of the connected group $G^o$. As an application, we show that there is a "functorial" correspondence between virtual (finite dimensional) characters of $\theta$-invariant representations of $G$ and virtual characters of an endoscopic group $H$ of $G$.

## Contents



## 1. Introduction

Let $k$ denote an algebraically closed field of characteristic zero. Unless stated otherwise $G$ will always denote a connected semisimple (linear algebraic) group

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over $k$. In this paper, we extend some results of Steinberg to non-connected groups. Let $\theta$ denote an (algebraic) automorphism of $G$ of finite order which fixes a "splitting" of $G$ (this will be defined later). Let $G \rtimes <\theta>$ denote the semi-direct product where the multiplication is given by $(g,\sigma)(g',\sigma') = (g\sigma(g'),\sigma\sigma')$). $G$ may be identified with a normal subgroup of $\overline{G}$.

In one of our main results, we describe a twisted analog of the "adjoint quotient $G \to T/W$", where $T$ is a Cartan of $G$ and $W$ is its Weyl group, as well as prove some basic facts regarding the geometry of "twisted" conjugacy classes. As a corollary to this, we prove that a $\theta$-conjugacy class of $G$ is Zariski closed if it is $\theta$-semisimple. As a related matter, we investigate the characters of the irreducible finite dimensional representations of non-connected semisimple groups of the form $G \rtimes <\theta>$. In another of our main results, we prove that there is a natural 1-1 correspondence between the irreducible finite dimensional representations of $G \rtimes <\theta>$ and those representations on a related "endoscopic" group (one must be careful in interpreting this statement - see Remark 3.3 below).

In general, if $X$ denotes an algebraic variety defined over $k$ then we often identify $X$ with $X(k)$.

## 2. Background on conjugacy classes

Let $G$ be a closed connected (algebraic) subgroup of $GL(V)$, where $V$ is a finite dimensional $k$-vector space, and let $\theta$ be an algebraic automorphism of $G$ of finite order. Denote the semidirect product described above by $\overline{G} = G \rtimes <\theta>$. Denote the connected components of $\overline{G}$ by $G = G.1, G.\theta, G.\theta^2, \ldots$. Without loss of generality (taking $V$ to be the Lie algebra of $G$), we can (and do) assume that $\overline{G}$ is also a subgroup of $GL(V)$.

**Lemma 2.1.** *Let $\overline{G}$ be as above and let $X$ denote a variety defined over $k$ on which $\overline{G}$ acts. Then*

*(a) every orbit is open in its closure,*

*(b) for each $x \in X$, the (Zariski) closure $\overline{G \cdot x}$ of an orbit $G \cdot x$ is a union of $G \cdot x$ and other orbits of smaller dimension,*

*(c) orbits of minimal dimension are closed.*

For the proof in the connected case, see [10], §1.13.

**Proof.** Since $\theta$ acts on $X$, it is an isomorphism of $X$ to itself. In particular, the set of orbits of $G.\theta^i$ on $X$ is the same as the set of orbits of $G.1 \cong G$ on $X$. Therefore the Lemma above is a consequence of the connected case. ∎

**Proposition 2.2.** *(Jordan decomposition) Let $\overline{G}$ be as above. Each $g \in \overline{G}$ has a decomposition $g = su = us$, where $s \in \overline{G}$ is semisimple (as an element of $\overline{G}$ or, equivalently, as an endomorphism on $V$) and $u \in \overline{G}$ is unipotent (as an element of $\overline{G}$ or, equivalently, $u - 1$ is a nilpotent endomorphism on $V$). Furthermore, $g$ determines $s$ and $u$ uniquely.*

For the proof, see [10], §§2.1-2.4. As a consequence, we have the following result.

**Corollary 2.3.** *Let $G$ be as above. Let $\theta$ denote an automorphism of $G$ of finite order. Then any $g.\theta \in \overline{G}$ has a decomposition*

$$g.\theta = (s.\theta)u = u(s.\theta),$$

*for some unique $s.\theta$ semisimple in $\overline{G}$ and $u$ unipotent. Equivalently,*

$$g = s\theta(u) = us.$$

**Definition 2.4.** In the above corollary, we write $g_{\theta-ss} = s$ and $g_{\theta-un} = u$. We call $g$ *$\theta$-semisimple* if $g.\theta \in \overline{G}$ is semisimple (i.e., if $g = g_{\theta-ss}$). We call $g$ *$\theta$-unipotent* if $g_{\theta-ss} = 1$, so $g = \theta(u) = u$ for some unipotent $u$.

We call $x, y \in$G *$\theta$-conjugate* if and only if $y = g^{-1}x\theta(g)$ for some $g \in G$.

**Remark 2.5.** The map

$$g \longmapsto g_{\theta-ss}$$

is not a morphism [8], §3.3. However, later we will construct a morphism (the twisted analog of the adjoint quotient) which is closely related to this map.

A *Borel pair* of a connected reductive group $G$ is a pair $(B, T)$, where $T$ is a maximal torus contained in a Borel subgroup $B$ of $G$. An automorphism of $G$ which preserves a Borel pair $(B, T)$ will be called *quasi-semisimple*. Let $W$ denote the Weyl group of $T$ and $W^\theta$ denote the $\theta$-invariant elements of $W$.

**Lemma 2.6.** *Let $G$ denote a connected reductive group over $k$. Assume that $\theta$ is a quasi-semisimple automorphism of $G$ which is of finite order and fixes a Borel pair $(B, T)$ of $G$. Then*

$$N_G(T.\theta)/T = W^\theta,$$

*where $W$s denote the Weyl group of $T$.*

**Proof.** Write the Bruhat decomposition as $G = BWB$ (where we have temporarily identified $W$ with a complete set of coset representatives of $W$ in $N_G(T)$). The lemma follows from the claim: if $w \in W$ and $g = bwb' \in BwB$ satisfies $gT\theta(g)^{-1} = T$ then $g \in TwT$ and $w \in W^\theta$. We next prove this claim.

Let $B = NT = TN$, where $N$ is the unipotent radical of $B$. Note that since $\theta$ preserves $T$ and $B$, it preserves $N$. Suppose that $g = tnwn'$, for $n, n' \in N$ and $t \in T$. The equation $gT\theta(g)^{-1} = T$ implies $\theta(g)T\theta(g)^{-1} = \theta(g)g^{-1}T$. The theory of maximal tori implies $g\theta(g)^{-1} \in T$ and $\theta(g) \in N_G(T)$. In turn, this implies $g \in N_G(T)$ and that the image of $\theta(g)g^{-1}$ in $W$ is the identity. This (by the Bruhat decomposition) implies the claim, from which the lemma follows. ∎

For $x \in G$, let

$$\mathcal{C}_\theta(x) = \{g^{-1}x\theta(g) \mid g \in G\},$$

the *$\theta$-conjugacy class of $x$*. A *splitting* of $G$ is a triple $(B, T, \{X\})$, where $(B, T)$ is a Borel pair of $G$ and $\{X\}$ is a set of root vectors of $T$, one for each simple root of $T$ in $B$. Let $W$ and $W^\theta$ be as in the previous lemma. Let $T_\theta = T/(1-\theta)T$ denote the group of *coinvariants* of $T$.

The lemma below, based on the work of Steinberg [11], is Lemma 3.2.A in [4].

**Lemma 2.7.** *Let $G$ denote a connected reductive group over $k$ and assume that the automorphism $\theta$ is of finite order and fixes a splitting* [1] $(B, T, \{X\})$ *of $G$. If $s \in G$ is $\theta$-semisimple then*

*(a) each $\mathcal{O} = \mathcal{C}_\theta(s)$ meets $T$,*

*(b) the image of $\mathcal{O} \cap T$ in $T_\theta$ is a single $W^\theta$-orbit.*

**Remark 2.8.** This implies that there is a bijection [2]

$$G_{\theta-ss} \to T_\theta / W^\theta$$

(essentially the *abstract norm map* defined in in [4] §3.2) from the set of conjugacy classes of $\theta$-semisimple elements of $G$ to the quotient $T_\theta/W^\theta$. The construction of the above bijection should be compared with Corollary 5.4 below, which obtains this mapping from a character-theoretic perspective.

## 3. Some representation theory

Let $G$ be a connected semisimple algebraic group over $k$. Let $\theta$ be an quasi-semisimple automorphism of $G$. Let $\overline{G}$ be as in the previous section.

If $\rho$ is a finite dimensional representation of $G$, let

$$\rho^\theta(g) = \rho(\theta(g)), \qquad g \in G,$$

so $\rho^\theta$ is irreducible if and only if $\rho$ is. We call a representation $\rho$ *$\theta$-invariant* if $\rho^\theta \cong \rho$. The purpose of this section is to classify such representations.

We denote the simple roots of $(B, T)$ by $\Delta$, the root system of $(B, T)$ by $R = R(B, T)$, the root lattice of $T \subset G$ by $Q = \mathbb{Z}[\Delta] = \mathbb{Z}[R]$, the character lattice by $X = X^*(T)$, and the (abstract) weight lattice by $P = Q^\perp$ (the dual being taken with respect to the Cartan pairing), let $P^+$ denote the "cone" of dominant (abstract) weights, and let $\lambda_1, ..., \lambda_n \in P^+$ denote the fundamental (abstract) weights. We have

$$Q \subset X \subset P.$$

Let $\lambda$ denote a dominant character and let $\rho_\lambda$ denote the irreducible representation whose highest weight is $\lambda$. Let $X^+$ denote the "cone" of dominant characters. If $\rho = \rho_\lambda$, with $\lambda \in X^+$, let

$$\chi_\rho(g) = \chi_\lambda(g) = \mathrm{trace}(\rho_\lambda(g)), \quad g \in G. \tag{1}$$

Finally, let $X^\theta$ denote the subgroup of $\theta$-invariant characters in $X$ and let $X^{\theta+}$ denote the dominant ones. We call such characters *$\theta$-dominant*.

It is clear that if $\lambda \in X^\theta$ then the restriction of $\lambda$ to $(1-\theta)T$ is trivial and hence $\lambda$ is well-defined as a character of $T_\theta$. Thus, there is a natural map $\eta : X^*(T)^\theta \to X^*(T_\theta)$. The kernel of the map $t \longmapsto t/\theta(t)$, $T \to (1-\theta)T$, is

[1] As was pointed out by the referee, this condition is stronger than the notion of a quasi-central automorphism in the sense of [1]; Lemma 2.7 holds for quasi-central automorphisms but not Proposition 3.9 below.

[2] This is only a set-theoretic mapping since $G_{\theta-ss}$ is not in general a variety.

$T^\theta$. Thus $T/T^\theta \cong (1-\theta)T$. If we knew that $T^\theta$ was connected (and hence $T, T^\theta, (1-\theta)T$ were all products of $k^\times$'s) then we could conclude $T_\theta \cong T^\theta$. This connectedness is proven in [11], §8. This also proves that each element $t \in T$ can be uniquely factored $t = t_1 t_2$, where $t_1 \in T^\theta$ and $t_2 \in (1-\theta)T$ (see also Corollaire 1.33 in [1]). From this fact, one can show that $\eta$ is an isomorphism. These facts are summarized in the following lemma.

**Lemma 3.1.** *Let $G, \theta, T$ be as in Lemma 2.6. There is a natural isomorphism*

$$\eta : X^*(T)^\theta \to X^*(T_\theta).$$

*Furthermore, there is a (non-canonical) isomorphism $T_\theta \cong T^\theta$. Indeed, each element $t \in T$ can be uniquely factored $t = t_1 t_2$, where $t_1 \in T^\theta$ and $t_2 \in (1-\theta)T$.*

**Proposition 3.2.** *Let $\rho$ be a finite dimensional representation of $G$. The following are equivalent:*
*(a) $\rho$ is $\theta$-invariant,*
*(b) $\rho$ extends to a finite dimensional representation of $\overline{G}$.*

**Remark 3.3.** In the introduction, we stated that one aim of this paper is to provide a correspondence between irreducible representations of $\overline{G}$ and irreducible representations of an endoscopic group. This is not exactly what we shall do. In fact, our correspondence will between $\theta$-invariant representations of $G$ (which extend to $\overline{G}$) and irreducible representations of an endoscopic group. Be that as it may, the remaining irreducible representations of $\overline{G}$ can be provided with a similar correspondence (though with a different endoscopic group) thanks to the following well-known result.

**Lemma 3.4.**
- *If $\overline{\rho}$ denotes an irreducible finite dimensional representation of $\overline{G}$ then there is an irreducible representation $\rho$ of $G$ such that*
  - *$\rho^{\theta^m} \cong \rho$, for some $m|d$ and $m$ is chosen as small as possible,*
  - *$\rho$ extends to the non-connected group $G \rtimes <\theta^m>$,*
  - *$\overline{\rho} \cong Ind_{G\rtimes<\theta^m>}^{\overline{G}} \rho$.*
- *Let $\rho$ be an irreducible finite dimensional representation of $G$ and let $\theta$ be an automorphism of finite order $d$ of $G$. If all $d$ of the representations $\rho^{\theta^i}$, $0 \le i \le d-1$, are inequivalent then $Ind_G^{\overline{G}}\rho$ is an irreducible representation of $\overline{G}$.*

**Proof of the Lemma**. This follows from a far more general result of Mackey [6] (see also Theorem 2, Ch III, §B of [5]). ∎

**Proof of the Proposition**. Let $(\rho, V_\rho)$ denote a finite dimensional representation of $G$. If $\rho^\theta \cong \rho$ then there is an $A \in GL(V_\rho)$ of finite order dividing $d$ such that $\rho(\theta(g)) = A^{-1}\rho(g)A$, for all $g \in G$. Define

$$\overline{\rho}(g.\theta^i) = \rho(g)A^{-i}, \quad g \in G, \quad 0 \le i \le d.$$

where $d$ denotes the order of $\theta$. It may be verified that this defines an extension of $\rho$ to $\overline{G}$, so (a) implies (b).

To prove the converse, (b) implies (a), let $\overline{\rho}$ denote an extension of $\rho$ to $\overline{G}$, so the image of $\overline{G}$ under $\overline{\rho}$ is also a semi-direct product. We may write

$$\overline{\rho}(g.\theta^i) = \rho(g)A^{-i}, \quad g \in G,$$

for some $A \in GL(V)$. Calculating

$$\rho(g\theta(g'))A^{-1} = \overline{\rho}(g\theta(g').\theta) = \overline{\rho}(g.\theta)\rho(g') = \rho(g)A^{-1}\rho(g') = \rho(g)\rho(\theta(g'))A^{-1},$$

we find that $\rho(g)\rho(\theta(g'))A^{-1} = \rho(g)A^{-1}\rho(g')$. This implies (a). ■

The following lemma is, as we shall see, an immediate consequence of the Weyl character formula.

**Lemma 3.5.** *(a) If $\lambda \in X^{\theta+}$ then $\rho_\lambda \cong \rho_\lambda^\theta$. Conversely, if $\lambda \in X^+$ and $\rho_\lambda = \rho_\lambda^\theta$ then $\lambda \in X^{\theta+}$.*

*(b) Let $\lambda^\theta$ denote the highest weight of $\rho_\lambda^\theta$, $\rho_{\lambda^\theta} = \rho_\lambda^\theta$. Let $V = X \otimes \mathbb{R}$, so we may extend the action of $\theta$ on $X$ to $V$ by linearity. These two actions (the above action of $\theta$ on $V$ and the action of $\theta$ on dominant weights $\theta : \lambda \longmapsto \lambda^\theta$) are compatible.*

For $V$ as in the above lemma, let $V^\theta$ denote the subspace of $\theta$-invariants and, for any subset $S \subset V$, let $S^\theta = S \cap V^\theta$.

**Proof.** (a) This follows from the direct construction of a representation with highest weight $\lambda$, as the referee pointed out. However, we give the following simple, analytic proof.

We have

$$\chi_\lambda = \frac{A_{\lambda+\rho_0}}{A_{\rho_0}}, \tag{2}$$

where $\rho_0$ denotes half the sum of the positive roots of $(B,T)$ and

$$A_\mu = \sum_{w \in W} (-1)^{sgn(w)} w\mu$$

is the alternating sum over the equivalence class [2], Theorem 24.2. Because of this, if $\lambda \in X^{\theta+}$ then

$$\chi_{\rho_\lambda} = \chi_{\rho_{\lambda^\theta}} = \chi_{\rho_\lambda^\theta}.$$

This implies $\rho_\lambda \cong \rho_\lambda^\theta$ since the character determines the equivalence class of the representation.

On the other hand, suppose $\rho_\lambda \cong \rho_\lambda^\theta$. The character formula above implies

$$\frac{A_{\lambda+\rho_0}(\theta(t))}{A_{\rho_0}(\theta(t))} = \frac{A_{\lambda+\rho_0}(t)}{A_{\rho_0}(t)},$$

for all $t \in T$. Since $\theta$ permutes the set of positive roots (it must since it preserves $(B,T)$), we have $\theta(\rho_0) = \rho_0$. Thus,

$$\frac{A_{\lambda+\rho_0}(\theta(t))}{A_{\rho_0}(\theta(t))} = \frac{A_{\theta(\lambda)+\rho_0}(t)}{A_{\rho_0}(t)},$$

which implies that $\rho_\lambda = \rho_\lambda^\theta$ has highest weight $\theta(\lambda)$. Since the highest weight is unique, it follows that $\lambda = \theta(\lambda) = \lambda^\theta$, as desired.

(b) Part (b) is a consequence of the above proof. ■

From this it follows that the irreducible, finite dimensional, $\theta$-invariant representations are in 1-1 correspondence with the elements of $X^{\theta+}$.

For $w \in W$ and $\chi \in X$, define $(w\chi)(t) = \chi(w^{-1}tw)$ for all $t \in T$. Clearly, $w\chi \in X$. If $\chi, \chi' \in X$ then we define

$$\chi \sim \chi'$$

if there is an element of the Weyl group $w \in W$ such that $\chi' = w\chi$. This is an equivalence relation on $X$ and the set of equivalence classes, i.e. $W$-orbits, will be denoted by $X/W$. The classes in $X/W$ are in natural 1-1 correspondence with the set $X^+$ since each class in $X/W$ contains a unique dominant highest character.

If $\chi, \chi' \in X^\theta$ then we define

$$\chi \sim_\theta \chi'$$

if there is an element $w \in W$ such that $\chi' = w\chi$ (we do not know if it suffices to assume $w \in W^\theta$ in this definition). This is an equivalence relation on $X^\theta$.

Now define

$$Symm[\chi] = \oplus_{\chi' \sim \chi} \chi',$$

for any $\chi \in X$. Note $Symm[\chi] \cong Symm[\chi']$ if and only if $\chi \sim \chi'$. Let

$$Symm_\theta[\chi] = \oplus_{\chi' \sim_\theta \chi} \chi',$$

for $\chi \in X^\theta$.

The following well-known lemma describes how the restriction of a irreducible character to a Cartan subgroup decomposes. We will prove the "twisted analog" of this result.

**Lemma 3.6.** *(a) For $\lambda \in X$ dominant,*

$$\chi_\lambda|_T = \sum_{\substack{\mu \leq \lambda \\ \mu \text{ dominant}}} m(\mu) Symm[\mu],$$

*where the $m(\mu) \geq 1$ are integers satisfying $m(\lambda) = 1$. (When $G$ is simply connected then all then $m(\mu) = 1$.)*

*(b) For $\lambda \in X$ dominant, there are $\epsilon_\mu \in \{\pm 1\}$, for $\mu \leq \lambda$ a dominant character, such that*

$$Symm[\lambda] = \sum_{\substack{\mu \leq \lambda \\ \mu \text{ dominant}}} \epsilon_\mu m'(\mu) \chi_\mu|_T,$$

*and $\epsilon_\lambda = 1$. Here the $m'(\mu) \geq 1$ are integers satisfying $m'(\lambda) = 1$. (When $G$ is simply connected then all then $m'(\mu) = 1$.)*

For a proof, see [10], §3.4.

Recall $X^{\theta+} \subset X$ denotes the subset of $\theta$-invariant dominant characters. If $\rho$ is $\theta$-invariant, let $\overline{\rho}$ denote an extension of $\rho$ to $\overline{G}$. Write

$$\overline{\rho}(g.\theta) = \rho(g)\rho(\theta), \quad g \in G,$$

for some $\rho(\theta) \in GL(V_\rho)$. This is an abuse of notation since the extension is not necessarily unique. $\rho(\theta)$ is only well-defined up to a $d^{th}$ root of unity where $\theta^d = 1$. We shall fix an extension in the following definitions. First, we claim that the trace of this endomorphism is, as a function of $g$, constant on the $\theta$-conjugacy classes. Let

$$\chi_{\overline{\rho}}(g.\theta^i) = \mathrm{trace}(\rho(g)\rho(\theta)^i), \quad g \in G,$$

and, if $\lambda \in X^{\theta+}$ and $\rho = \rho_\lambda$, let

$$\chi^\theta_\lambda(g) = \chi^\theta_\rho(g) = \chi_{\overline{\rho}}(g.\theta), \quad g \in G. \tag{3}$$

This will be called the *$\theta$-character* of $\rho$. From the fact that $\overline{\rho}(g.\theta)$ is a class function on $G. < \theta >$, it follows that

$$\chi^\theta_\rho(y) = \chi^\theta_\rho(x^{-1}y\theta(x)), \qquad x, y \in G,$$

if $\rho$ is a $\theta$-invariant finite dimensional representation of $G$.

**Lemma 3.7.** *(a) For $\lambda \in X^{\theta+}$,*

$$\chi^\theta_\lambda|_{T_\theta} = \sum_{\substack{\mu \leq \lambda \\ \mu\ \theta-\text{dominant}}} m_\theta(\mu) Symm_\theta[\mu]|_{T_\theta},$$

*where the $m_\theta(\mu) \geq 1$ are integers satisfying $m_\theta(\lambda) = 1$.*

*(b) For $\lambda \in X^{\theta+}$, there are $\epsilon_\mu \in \{\pm 1\}$, for $\mu \leq \lambda$ a $\theta$-dominant character, such that*

$$Symm_\theta[\lambda]|_{T_\theta} = \sum_{\substack{\mu \leq \lambda \\ \mu\ \theta-\text{dominant}}} \epsilon_\mu m'_\theta(\mu) \chi^\theta_\mu|_{T_\theta},$$

*and $\epsilon_\lambda = 1$. Here the $m'_\theta(\mu) \geq 1$ are integers satisfying $m'_\theta(\lambda) = 1$.*

**Remark 3.8.** The "restriction" symbol $|_{T_\theta}$ used above is a slight abuse of notation (since $T_\theta$ is not a subset of $T$), which we hope the reader will pardon. Worst, it is not clear it is well-defined. It must be shown that both the left-hand side of (a) and the right-hand side of (b) above are well-defined. Let $\rho$ denote an irreducible finite dimensional (complex) representation of $G$ and let $\theta$ be a quasi-semisimple of $G$ preserving a Borel pair $(B, T)$. If $\rho \cong \rho^\theta$ then, we claim, the restriction of its "twisted character" $\chi^\theta_\rho$ to $T_\theta$ is well-defined. Indeed, if $\rho$ is $\theta$-invariant then the character $\chi_{\overline{\rho}}$ is constant on conjugacy classes. Since $\chi_{\overline{\rho}}(t'.\theta) = \chi_{\overline{\rho}}(tt'\theta(t)^{-1}.\theta) = \chi_{\overline{\rho}}(t't\theta(t)^{-1}.\theta)$, for all $t, t' \in T$.

**Proof.** The proof of part (a) is analogous to Steinberg's proof in the connected case ([10], §3.4). The restriction of $\chi_\lambda^\theta$ to $T_\theta$ is a sum of characters (weights) $\mu$ in $X^*(T_\theta)$, the multiplicity of each character occurring in the sum is the dimension of the corresponding weight space. If $\chi'$ is any weight occurring in this decomposition then there is a $w \in W$ and a dominant $\chi \in X^*(T_\theta)$ (regarded as an element in $X^*(T)^\theta$ by Lemma 3.1) such that $\chi' = w\chi$. Furthermore, these dimensions only depend on the $W^\theta$-equivalence class of $\mu$ with respect to $\sim_\theta$ above. By Lemma 3.1, only $\theta$-invariant weights can occur in this sum. The multiplicity with which (the "highest weight") $\lambda$ occurs in this sum is equal to the multiplicity of $\lambda$ in $\overline{\rho}_\lambda$. But this is equal to the multiplicity of $\lambda$ in $\rho_\lambda$, by construction (since they have the *same* representation space). Thus $m_\theta(\lambda) = 1$, as desired.

Part (b) follows by inverting the (upper triangular) system of equations given in part (a). ■

**Proposition 3.9.** *Let $\theta$ be as in Lemma 2.7. Assume that $G$ contains no irreducible component of Cartan type $A_{2n}$. Let $G^1$ denote the connected component of $G^\theta$ and let $T^1$ denote the connected component of $T^\theta$. The finite set $R|_{T^1}$ forms a root system of the connected semisimple group $G^1$ with maximal torus $T^1$.*

**Proof.** This is a consequence of results in [9], §§12.16-12.19. ■

## 4. $\theta$-conjugacy classes

Let $G$ and $\overline{G}$ be as in the previous section.

The following theorem, one of our main results, is an extension of a theorem of Steinberg to the non-connected case (see [10], §3.4, Theorem 2).

**Theorem 4.1.** *Let $G$ be a connected semi-simple group and let $\theta$ be as in Lemma 2.7.*

*(a) The restriction map*

$$k[G.\theta]^G \to k[T.\theta]^{N_G(T.\theta)}$$

*is an isomorphism.*

*(b) The "restriction" map*

$$\begin{gathered} k[T.\theta]^{N_G(T.\theta)} \to k[T_\theta]^{W^\theta} \\ f \longmapsto res(f) \end{gathered}$$

*is an isomorphism, where* $\mathrm{res}(f)(t) = f(t.\theta)$, $t \in T_\theta$ *(see Remark 3.8).*

*(c) The functions $\{\chi_\lambda^\theta|_{T_\theta} \mid \lambda \in X^{\theta+}\}$ form a basis for the $k$-vector space $k[T_\theta]^{W^\theta}$. The functions $\{\chi_\lambda \mid \lambda \in X^{\theta+}\}$ form a basis for the $k$-vector space $k[G.\theta]^G$.*

Now we begin the proof of the theorem.

**Proof.** (a): Regarding the surjectivity of the restriction map, we need to know that if two elements of $T.\theta$ are $G$-conjugate then they are $N_G(T.\theta)$-conjugate. This is a consequence of Lemma 2.7.

To prove injectivity, let $f \in k[G.\theta]^G$ be such that $f|_{T.\theta} = 0$. If $x \in G$ is $\theta$-semisimple then there is a $g \in G$ such that $g^{-1}x\theta(g) \in T$ by Lemma 2.7(a). Thus

$$f(x.\theta) = f(g^{-1}x\theta(g).\theta) = 0,$$

since $f$ is a class function. Recall $x \in G$ is $\theta$-semisimple if and only if $x.\theta \in \overline{G}$ is semisimple. Note also the image of the inclusion

$$\overline{G}_{ss} \hookrightarrow \overline{G}$$

is dense (almost all of the elements in $\overline{G}$, regarded as matrices, have distinct eigenvalues). It follows from these facts that $f$ is zero on a dense subset. This implies that the restriction map is injective, which proves (a).

(b): We have $N_G(T.\theta)/T = W^\theta$ by Lemma 2.6. Therefore,

$$k[T.\theta]^{N_G(T.\theta)} = k[T.\theta]^{W^\theta T} \cong k[T_\theta]^{W^\theta}.$$

This proves (b).

(c): The first statement is a corollary of parts (a), (b), and the proof of Theorem 2(a) in §3.4 of [10]. The second statement follows from the first and parts (a), (b).

This completes the proof of the theorem. ■

## 5. Some corollaries

We list some corollaries of Theorem 4.1 above. These are all analogs of results of Steinberg in the non-connected case.

Let $G$ and $\overline{G}$ be as in the previous section.

**Definition 5.1.** We call $G$ *$\theta$-simply connected* (resp., *$\theta$-adjoint*) and call $\overline{G}$ *simply connected* (resp., *adjoint*) if $X^\theta = P^\theta$ (resp., $X^\theta = Q^\theta$).

The following result can be proven by modifying the proof of Theorem 2, §3.4, in [10] and using the above proof.

**Corollary 5.2.** *Assume $G$ is $\theta$-simply connected and let $\lambda_1, ..., \lambda_n$ denote a set of $\theta$-fundamental weights. Then $\{\chi_{\lambda_i}|_{T_\theta} \mid 1 \leq i \leq n\}$ freely generates $k[T_\theta]^{W^\theta}$ as a $k$-algebra.*

Define $k[G_{\theta-ss}.\theta]$ to be the vector space of functions on $G.\theta$ restricted to $G_{\theta-ss}.\theta$. The result below is a component of the proof of Theorem 4.1(a).

**Corollary 5.3.** *The map*

$$\begin{gathered} k[G.\theta]^G \to k[G_{\theta-ss}.\theta]^G \\ f \longmapsto res(f) \end{gathered}$$

*is an isomorphism of vector spaces, where $res(f)$ denotes the restriction map.*

**Corollary 5.4.** *The $\theta$-semisimple conjugacy classes of $G$ are in 1-1 correspondence with the elements of $T_\theta/W^\theta$.*

**Remark 5.5.** This Corollary is the twisted analog of Corollary 2, §3.4 in [10]. See also Remark 2.8 above.

**Corollary 5.6.** *Let $x = x_0.\theta, y = y_0.\theta \in G.\theta$ be semisimple elements. The following are equivalent.*

*(a) $x, y$ are $G$-conjugate,*

*(b) $\chi_\rho(\overline{x}_0) = \chi_\rho(\overline{y}_0)$, for all $\rho = \rho_\lambda$, $\lambda \in X^{\theta+}$. Here $\overline{x}_0, \overline{y}_0 \in T_\theta$ denote the image of $\mathcal{C}_\theta(x_0) \cap T, \mathcal{C}_\theta(y_0) \cap T$ in $T_\theta$ (which exists by Lemma 2.7).*

**Proof.** [3] Follows immediately from the second statement in Theorem 4.1(c). ■

**Corollary 5.7.** *If $f \in k[G.\theta]^G$ and $x \in G$ then $f(x.\theta) = f(x_{\theta-ss}.\theta)$, for $x \in G$.*

**Proof.** This follows from the second statement in Theorem 4.1(c) and the fact that $\chi_\lambda(x.\theta) = \chi_\lambda(x_{\theta-ss}.\theta)$, for all $x \in G$ and $\lambda \in X^{\theta+}$. ■

**Corollary 5.8.** *A $G$-conjugacy class in $G.\theta$ is closed if it is semisimple.*

**Proof.** We pick a basis of the vector space $V$, in the notation of §2 where $\overline{G} \subset GL(V)$. We may identify each $g \in \overline{G}$ with a matrix in $GL(V)$.

For the proof, use the obvious twisted analog of the proof of Corollary 5 in §3.4, p. 92 of [10] (which relies on Corollary 5.6 above). In other words, fix a (semisimple) element $x_0.\theta \in G.\theta$. Let $m_0$ denote the minimal polynomial of $x_0.\theta$ and let

$$S = \{x.\theta \in G.\theta \mid \chi_\lambda^\theta(x_0) = \chi_\lambda^\theta(x), \text{ for all } \lambda \in X^{+^\theta}, \text{ and } m_0(x.\theta) = 0\}.$$

$S$ is Zariski closed and contains the conjugacy class of $x_0.\theta$. Now let $x.\theta \in S$. It is semisimple since its minimal polynomial has distinct roots (it divides $m_0$, which has distinct roots by definition). The hypothesis to Corollary 5.6 therefore holds and implies the statement of the corollary. ■

## 6. Character relations and endoscopic groups

Let $\theta$ be as in Lemma 2.7. As in Lemma 3.1, we fix an isomorphism

$$\psi : T^\theta \to T_\theta.$$

The goal of this section is to show that, if $G$ is a simple, simply connected (hence $\theta$-simply connected) group and if $\theta$ is an automorphism of order $d$ preserving a splitting $(B, T, \{X\})$ then there is a simple connected group $H$ associated to $G, \theta$ such that

[3] I thank the referee for this short proof.

1. $X^{\theta+}$ is in natural 1-1 correspondence (defined in Lemma 3.1) with the dominant characters of $H$, $X_H^+$, which we denote by

$$\lambda \leftrightarrow \lambda_H,$$

2. there are distinct $\lambda_{H,i} \in X_H^+$ such that we have a character relation

$$\chi_\lambda^\theta(\mathcal{A}_\theta(\eta(t).\theta)) = \sum_{i=1}^{N} \zeta_i \chi_{\lambda_{H,i}}(t), \tag{4}$$

for all $t \in T_H$, where $\chi_\lambda^\theta$ is as in (3) above, $T_H$ is a regular torus, and $\mathcal{A}_\theta$ is a map (to be defined below) from the semisimple conjugacy classes in $G.\theta$ to the semisimple conjugacy classes of $H$. Here the $\zeta_i \in \mathbb{C}$ (possibly all zero) depend on the extension of $\rho_\lambda$ from $G$ to $G \rtimes < \theta >$ chosen in the definition of the $\theta$-character.

As in [4], we have the following definition.

**Definition 6.1.** A group $H$ as above is called a *$\theta$-endoscopic group.*

Define $H$ to be the group $H = G^\theta = \{g \in G \mid \theta(g) = g\}$. Since $G$ is simply connected, $H$ is connected, by Steinberg's Theorem 8.1 in [11]. $H$ has maximal torus $T_H = T^\theta$, root lattice $Q_H$, character lattice $X_H = X^*(T_H)$, and weight lattice $P_H$, such that

$$Q_H = \mathbb{Z}[R(B,T)|_{T^\theta}] \subset X_H \subset P_H = Q_H^\perp,$$

by Proposition 3.9. Furthermore, the Dynkin diagram of $H$ is the "folded" Dynkin diagram of $G$ (see [9], §12.18 and Theorem 12.19). We have $T_H \cong T_\theta$, $W_H \cong W^\theta$ (see [11], §8). This implies $H$ has property (1) of an endoscopic group. Furthermore, there is a 1-1 correspondence

$$\mathcal{A}_\theta : \mathcal{C}_G(G_{\theta-ss}.\theta) \to \mathcal{C}_H(H_{ss}), \tag{5}$$

defined using the correspondences

$$\mathcal{C}_H(H_{ss}) \leftrightarrow T_H/W_H,$$

$$\mathcal{C}_G(G_{\theta-ss}.\theta) \leftrightarrow T_\theta/W^\theta,$$

and the non-canonical isomorphism

$$T_H/W_H \cong T_\theta/W^\theta. \tag{6}$$

The following theorem is our other main result.

**Theorem 6.2.** *Assume $G$ and $\theta$ are as in Theorem 4.1 above. In addition, assume $G$ is simply connected. Let $H = G^\theta$ be as defined above. There is an isomorphism of vector spaces*

$$k[G.\theta]^G \to k[H]^H,$$

*induced by the restriction maps $k[G.\theta]^G \to k[T_\theta]^{W^\theta}$, $k[H]^H \to k[T_H]^{W_H}$, and (6).*

**Proof.** This follows from Theorem 4.1 and (6):

$$k[G.\theta]^G \to k[T.\theta]^{N_G(T.\theta)} \to k[T_\theta]^{W^\theta}.$$

∎

**Remark 6.3.** The equality claimed in (4) above, which is property (2) of an endoscopic group, is a special case of Lemma 3.7, using the definition of $X_H^+$.

## 7. Irreducible representations of $\overline{G}$, $G$ simple

In this section, we use Dynkin diagrams to describe those irreducible finite dimensional representations of $G$ which extend to representations of $\overline{G}$. This forms an important component of the complete description, which may be found in Remark 3.3 above. No proofs are given in this section. All statements are either proven in [2] or may be derived by modifying arguments there.

We remark that a relationship between the automorphisms of $G$ and the automorphisms of the Dynkin diagram of $G$ is given by Proposition 1.4.1 in [7].

*$G$ simply connected of type $A_n$*

I thank the referee for pointing out that if $n$ is even the restrictions of the roots in this case do not form a restricted root system in the sense of Proposition 3.9.


Here

$$\lambda_1,\quad \text{the } 1-\text{st fundamental weight} \leftrightarrow \text{the irreducible repn } k^{n+1},$$

$$\lambda_2,\quad \text{the } 2-\text{nd fundamental weight} \leftrightarrow \text{the irreducible repn } \bigwedge^2 k^{n+1},$$

and so on. The last node on the far right end is associated to the contragrediant of the standard representation:

$$\lambda_n,\quad \text{the n}-\text{th fundamental weight} \leftrightarrow \text{the irreducible repn } \bigwedge^n k^{n+1}.$$

Let $\theta$ denote the automorphism $\theta(g) = J\ {}^tg^{-1}J$, where $J$ denotes the skew-diagonal matrix whose skew-diagonal (top right to bottom left) is given by $1, -1, ..., (-1)^{n+1}$. Then $\theta$ preserves the usual Borel pair $(B, T)$ where $B$ denotes the upper triangular subgroup and $T$ the diagonal torus. Write the simple roots

$\Delta = \{\alpha_1, ..., \alpha_n\}$ of $(B, T)$ as usual, $\alpha_1(t) = t_1/t_2, ..., \alpha_n(t) = t_n/t_{n+1}$, where $t = diag(t_1, ..., t_{n+1})$. If we write the Dynkin diagram as usual, with $\alpha_i$ in place of $\lambda_i$ above, then $\theta$ acts on the roots by

$$\theta : \alpha_i \leftrightarrow \alpha_{n+1-i},$$

and on the weights by

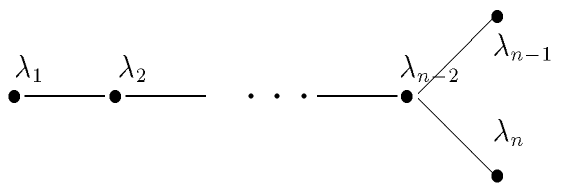
$$\theta : \lambda_i \leftrightarrow \lambda_{n+1-i},$$

for $1 \le i \le n$. The $\theta$-fundamental weights are then given by

$$\overline{\lambda}_i = \begin{cases} \lambda_i + \theta(\lambda_i) = \lambda_i + \lambda_{n+1-i}, & i < \frac{n+1}{2}, \\ \lambda_i, \quad i = \frac{n+1}{2}, \end{cases}$$

where $1 \le i \le \frac{n+1}{2}$.

*G simply connected of type* $D_n$




$$\lambda_1, \quad \text{the } 1-\text{st fundamental weight} \leftrightarrow \text{the irreducible repn } k^{2n},$$

$$\lambda_2, \quad \text{the } 2-\text{nd fundamental weight} \leftrightarrow \text{the irreducible repn } \bigwedge^2 k^{2n},$$

and so on. The third to the last node on the far right end is associated to the representation:

$$\lambda_{n-2}, \quad \text{the n} - 1 - \text{st fundamental weight} \leftrightarrow \text{the irreducible repn } \bigwedge^{n-1} k^{2n}.$$

The top node on the far right is associated to the Spin$^+$ representation:

$$\lambda_{n-1}, \quad \text{the n} - \text{th fundamental weight} \leftrightarrow \text{the irreducible repn Spin}^+.$$

The bottom node on the far right is associated to the Spin$^-$ representation:

$$\lambda_n, \quad \text{the n} - \text{th fundamental weight} \leftrightarrow \text{the irreducible repn Spin}^-.$$

Assume $n > 3$. If $n > 4$ then the only non-trivial diagram automorphism is that which exchanges the two nodes on the far right end and leaves the others fixed. Let $\theta$ denote this automorphism (even when $n = 4$). In this case, the $\theta$-fundamental weights are then given by

$$\overline{\lambda}_i = \begin{cases} \lambda_i + \theta(\lambda_i) = \lambda_i + \lambda_{n+1-i}, & i = n-1, \\ \lambda_i, \quad i < n-1, \end{cases}$$

where $1 \leq i \leq n-1$.

Now assume $n=4$ and let $\theta$ denote the diagram automorphism such that

$$\theta : \alpha_1 \longmapsto \alpha_4 \longmapsto \alpha_3, \quad \theta(\alpha_2) = \alpha_2.$$

Using the fact that

$$\begin{aligned} \lambda_1 &= \alpha_1 + \alpha_2 + \tfrac{1}{2}\alpha_3 + \tfrac{1}{2}\alpha_4, \\ \lambda_2 &= \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4, \\ \lambda_3 &= \tfrac{1}{2}\alpha_1 + \alpha_2 + \alpha_3 + \tfrac{1}{2}\alpha_4, \\ \lambda_4 &= \tfrac{1}{2}\alpha_1 + \alpha_2 + \tfrac{1}{2}\alpha_3 + \alpha_4, \end{aligned}$$

we find that

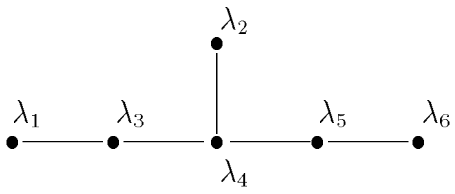
$$\theta : \lambda_1 \longmapsto \lambda_4 \longmapsto \lambda_3, \quad \theta(\lambda_2) = \lambda_2.$$

Therefore, the $\theta$-fundamental weights are then given by

$$\overline{\lambda}_i = \begin{cases} \lambda_1 + \lambda_4 + \lambda_3, & i = 1 \\ \lambda_2, & i = 2, \end{cases}$$

where $1 \leq i \leq 2$.

*$G$ simply connected of type $E_6$*




Let $\theta$ denote an automorphism of $G$ which acts on the roots by

$$\theta : \alpha_1 \leftrightarrow \alpha_6, \quad \alpha_3 \leftrightarrow \alpha_5, \quad \alpha_4 \leftrightarrow \alpha_4, \quad \alpha_2 \leftrightarrow \alpha_2.$$

Then from the table 1, [3], p. 69, we find that

$$\theta : \lambda_1 \leftrightarrow \lambda_6, \quad \lambda_3 \leftrightarrow \lambda_5, \quad \lambda_4 \leftrightarrow \lambda_4, \quad \lambda_2 \leftrightarrow \lambda_2.$$

The $\theta$-fundamental weights are then given by

$$\overline{\lambda}_i = \begin{cases} \lambda_1 + \lambda_6, & i = 1, \\ \lambda_3 + \lambda_5, & i = 3, \\ \lambda_2, & i = 2, \\ \lambda_4, & i = 4, \end{cases}$$

where $1 \leq i \leq 4$.

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