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# Compactification structure and conformal compressions of symmetric cones

**Jimmie D. Lawson and Yongdo Lim**

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**Abstract.** In this paper we show that the boundary of a symmetric cone $\Omega$ in the standard real conformal compactification $\mathcal{M}$ of its containing euclidean Jordan algebra $V$ has the structure of a double cone, with the points at infinity forming one of the cones. We further show that $\overline{\Omega}^{\mathcal{M}}$ admits a natural partial order extending that of $\Omega$. Each element of the compression semigroup for $\Omega$ is shown to act in an order-preserving way on $\overline{\Omega}^{\mathcal{M}}$ and carries it into an order interval contained in $\overline{\Omega}^{\mathcal{M}}$.

## 1. Introduction

Let $V$ be a euclidean Jordan algebra with identity $e$ and let $\Omega = \exp V$. Then $\Omega$ is a symmetric cone and $T_\Omega := V + i\Omega$ is a symmetric tube domain with symmetry $j(z) = -z^{-1}$ at $ie$. The biholomorphic automorphism group $G(T_\Omega)$ of $T_\Omega$ is generated by $N^+, H$ and $j$, where $N^+$ is the group of all real translations $t_x$ and $H$ is the automorphism group of the symmetric cone $\Omega$. There exists a conformal compactification $\mathcal{M} := G(T_\Omega)/P$ with respect to the parabolic subgroup $P = HN^-$, where $N^- = jN^+j$, and the embedding of $V$ into $\mathcal{M}$ given by $x \in V \to t_x \cdot P \in \mathcal{M}$ is an open dense embedding. A Lie semigroup that is naturally related to the symmetric cone $\Omega$ occurs as the semigroup of compressions of $\Omega$ in $G(T_\Omega)$:

$$\Gamma_\Omega := \{g \in G(T_\Omega) \mid g \cdot \Omega \subseteq \Omega\}.$$

In this paper we investigate the structure of $\overline{\Omega}^{\mathcal{M}}$, particularly its boundary and the points at infinity, and we establish a natural order structure on this compactification of $\Omega$. We show that the compression semigroup acts in an order-preserving way on $\overline{\Omega}^{\mathcal{M}}$ and carries it to an internal order interval.

This paper is organized as follows. In section 2 we realize the *ideal boundary* (boundary points in $\mathcal{M} \setminus V$) of $\Omega$ as a cone in $V$, and in section 3 we show how to endow $\overline{\Omega}^{\mathcal{M}}$ with a natural partial order. In section 4, we represent the compressed domain $g(\Omega)$ by $g$ in $\Gamma_\Omega$ as an order interval which is determined by the points $g(0)$ and $gj(0)$.

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## 2. Boundary structure of symmetric cones

We recall certain basic notions and well-known facts concerning Jordan algebras (see, for example, [3]). A commutative algebra $V$ over the field $\mathbb{R}$ or $\mathbb{C}$ with product $xy$ is said to be a *Jordan algebra* if for all elements $x, y$ in $V$, $x(x^2y) = x^2(xy)$. For $x \in V$, let $L(x)$ be the linear map of $V$ defined by $L(x)y = xy$, and let $P(x) = 2L(x)^2 - L(x^2)$. A finite dimensional real Jordan algebra $V$ is called a *euclidean Jordan algebra* if it carries an inner product $\langle\cdot|\cdot\rangle$ such that $\langle xy|\, z\rangle = \langle y|\, xz\rangle$, for all $x, y, z \in V$.

Let $V$ be a euclidean Jordan algebra with identity $e$, and let $Q = \{x^2 \mid x \in V\}$ be the set of squares. Then the set $Q$ is a self dual cone and the interior $\Omega$ of $Q$ is a symmetric cone, i.e., $\Omega$ is a self-dual cone (with respect to the inner product) and the group $G(\Omega) := \{h \in GL(V) \mid h(\Omega) = \Omega\}$ acts on it transitively. The closure $\overline{\Omega}$ is a closed pointed generating cone in $V$ with interior $\Omega$.

A *Jordan frame* is a family $\mathbf{c} = \{c_1, \cdots, c_r\}$ of primitive idempotents $c_i$ with

$$\begin{aligned} c_i c_j &= 0, i \neq j, \\ \sum_{i=1}^{r} c_i &= e. \end{aligned}$$

The spectral theorem states that for $x \in V$, there is a Jordan frame $\mathbf{c} = \{c_1, \cdots, c_r\}$ ($r$ fixed) and real numbers $\lambda_1, \cdots, \lambda_r$ such that $x = \sum_{i=1}^{r} \lambda_i c_i$. The real numbers $\lambda_i$ (with their multiplicities) are uniquely determined by $x$. Furthermore, $x \in \Omega$ (resp. $x \in \overline{\Omega}$) if and only if $\lambda_i > 0$ (resp. $\lambda_i \geq 0$) for each $i$.

The set $T_\Omega := V + i\Omega \subseteq V + iV$ is a symmetric tube domain. Let $G(T_\Omega)$ be the Lie group of all biholomorphic automorphisms on the tube domain $T_\Omega$. The group $G(T_\Omega)$ can be described in the following way: an element in $G(\Omega)$ acts on the tube domain $T_\Omega$ by $z = x + iy \longrightarrow g(z) = g(x) + ig(y)$. For $x$ in $V$, the translation by $x$, $t_x : z \longrightarrow z + x$ is a holomorphic automorphism of $T_\Omega$ and the group of all real translations is an abelian group $N^+$ isomorphic to the vector group $V$. The map (the symmetry of $T_\Omega$ at $ie$) $j : z \longrightarrow -z^{-1}$ is in $G(T_\Omega)$. We set $\tilde{t}_x = j \circ t_x \circ j$ and $N^- = j \circ N^+ \circ j$. Then $G(T_\Omega)$ is generated by $N^+, G(\Omega)$ and $j$ [3].

Let $G$ be the identity component of $G(T_\Omega)$ and let $H := G(\Omega)_0$. Let $P = HN^-$. Then $P$ is a maximal parabolic subgroup of $G$. Then it is known that the homogeneous space $\mathcal{M} := G/P$ is a compact real manifold containing $V$ as an open dense subset, i.e., a real conformal compactification of the Jordan algebra $V$ [1]. The embedding is given by

$$V \longrightarrow \mathcal{M}, \quad x \to t_x P,$$

and we henceforth identify $V$ with its image in $\mathcal{M}$. The space $N^+HN^-$ can be characterized by the elements $g \in G$ such that $g(0) \in V$ (in this case, $g(0) = g \cdot 0$, see Theorem 2.14 [1] and Corollary 2.20 [5]). One computes that $j(x)(= jt_xP) = -x^{-1}$ for $x$ an invertible element of $V$.

For a subset $A$ of $V$, let $\overline{A}$ denote its closure in $V$ and let $\overline{A}^{\mathcal{M}}$ denote its closure in $\mathcal{M}$. We are interested in studying the structure of the boundary $\overline{\Omega}^{\mathcal{M}} \setminus \Omega$

of $\Omega$, which consists of the set of "infinite" points in the closure $\partial_\infty\Omega := \overline{\Omega}^{\mathcal{M}} \setminus \overline{\Omega}$ and the set of "finite" points in the closure given by $\partial\Omega = \overline{\Omega} \setminus \Omega$. This study is facilitated by considering an equivalent bounded symmetric domain that arises by applying the "real" version of the Cayley transform to $\Omega$ resp. $\overline{\Omega}^{\mathcal{M}}$. We recall that this Cayley transform is given by

$$c(x) = (x-e)(x+e)^{-1} = e - 2(x+e)^{-1} = t_e \circ 2\mathrm{id}_V \circ j \circ t_e(x);$$

note that the first description has singular points, but that the last extends it and is defined on all of $\mathcal{M}$. The Cayley transform has a global inverse given by

$$d(x) = -(x+e)(x-e)^{-1} = -e + 2(e-x)^{-1} = t_{-e} \circ j \circ (1/2)\mathrm{id}_V \circ t_{-e}(x).$$

We define two useful relations on $V$, the partial order on $V$ defined by $x \le y$ if and only if $y - x \in \overline{\Omega}$, and $x \ll y$ if and only if $y - x \in \Omega$. The relation $\le$ is a closed partial order (i.e., closed as a relation in $V \times V$) and is the closure of $\ll$. We have $x \le w \le y$ (resp. $x \ll w \ll y$) if and only if $w \in (x + \overline{\Omega}) \cap (y - \overline{\Omega})$ (resp. $w \in (x + \Omega) \cap (y - \Omega)$). We note that if $x \ll y$, then the closure of $x + \Omega \cap y - \Omega$ is equal to $x + \overline{\Omega} \cap y - \overline{\Omega}$ (if $x \le w \le y$ for $x \ll y$, then the sequence $w_n = (1/n)((x+y)/2)) + ((n-1)/n)w \in x + \Omega \cap y - \Omega$ and converges to $w$). For $x \le y$, we define the order interval

$$[x,y] = \{w : x \le w \le y\} = y - \overline{\Omega} \cap x + \overline{\Omega}.$$

If $x = \sum_{i=1}^r \lambda_i c_i$ and $y = \sum_{i=1}^r \mu_i c_i$ are spectral decompositions for the Jordan frame $\{c_1, \ldots, c_r\}$, then $x \le y$ (resp. $x \ll y$) if and only if $\lambda_i \le \mu_i$ (resp. $\lambda_i < \mu_i$) for all $i$. In particular if $e \ll x = \sum_{i=1}^r \lambda_i c_i$, then $1 < \lambda_i$ for all $i$, and thus $x^{-1} = \sum_{i=1}^r (1/\lambda_i) c_i \ll e$.

The following is a basic fact about the Cayley transform applied to $\Omega$ (see, for example, [2] for observation (1)).

**Proposition 2.1.** *Let $V$ be a euclidean Jordan algebra with symmetric cone $\Omega$. Then we have the following:*

*(1)* $c(\Omega) = e - \Omega \cap \Omega - e$.

*(2)* $c(\overline{\Omega}^{\mathcal{M}}) = e - \overline{\Omega} \cap \overline{\Omega} - e = [-e, e]$.

*(3)* $x \in c(\partial_\infty(\Omega))$ *if and only if* $x$ *has a spectral decomposition of the form* $\sum_{i=1}^r \lambda_i c_i$ *such that* $-1 \le \lambda_i \le 1$ *for all* $i$ *and at least one* $\lambda_j = 1$.

*(4)* $x \in c(\partial(\Omega))$ *if and only if* $x$ *has a spectral decomposition of the form* $\sum_{i=1}^r \lambda_i c_i$ *such that* $-1 \le \lambda_i < 1$ *for all* $i$ *and at least one* $\lambda_j = -1$.

**Proof.** Let $x \in \Omega$. Then $e \ll e + x = t_e(x)$. It follows that $(e+x)^{-1} \ll e$ and thus $-e \ll jt_e(x)$. Thus $-e = e + 2(-e) \ll e + 2jt_e(x) = c(x)$. From $jt_e(x) \ll 0$ we derive $c(x) \ll e$, and thus $c(\Omega) \subseteq e - \Omega \cap (-e + \Omega)$. Similar arguments applied to $d$ and $e - \Omega \cap (-e + \Omega)$ yield $d(e - \Omega \cap (-e + \Omega)) \subseteq \Omega$. Since $c$ and $d$ are inverses on $\mathcal{M}$, we conclude that $c(\Omega) = e - \Omega \cap (-e + \Omega)$. This completes (1).

Assertion (2) follows from taking closure of both sides of (1) (note that the right-hand side is compact, hence closed in $\mathcal{M}$). Assertion (4) follows from the straightforward observation that $x \in \partial\Omega$ if and only if all $\lambda_i \geq 0$ and some $\lambda_j = 0$ in some spectral decomposition of $x$ if and only if all $\mu_i$ satisfy $-1 \leq \mu_i < 1$ and some $\mu_j = -1$ in some spectral decomposition of $c(x)$. Assertion (3) then follows from the other three. □

We give a more explicit characterization of the boundary of $[-e, e]$ and hence implicitly of the boundary in $\mathcal{M}$ of $\Omega$. We set

$$\begin{aligned} \mathcal{E} &= c(\partial_\infty\Omega) \cap c(\overline{\partial\Omega}^{\mathcal{M}}) = c(\partial_\infty\Omega) \cap \overline{c(\partial\Omega)} \\ &= \{x = \sum_{i=1}^{r} \lambda_i c_i : \max\{\lambda_i\} = 1,\ \min\{\lambda_i\} = -1 \\ &\qquad \text{for some Jordan frame}\} \end{aligned}$$

We note that second and third sets are equal since $c$ is a homeomorphsim. To see that the third and fourth sets are equal, let $x = \sum_{i=1}^{r} \lambda_i c_i$ with $\max\{\lambda_i\} = 1$ and $\min\{\lambda_i\} = -1$, then by the preceding proposition $x \in c(\partial_\infty\Omega)$. Let $x_n$ have $i-th$ coefficient the larger of $-1$ and $\lambda_i - (1/n)$. Then $x_n \in c(\partial\Omega)$ and converges to $x$, so $x \in c(\partial_\infty\Omega) \cap \overline{c(\partial\Omega)}^{\mathcal{M}}$. Conversely let $y \in c(\partial_\infty\Omega) \cap \overline{c(\partial\Omega)}^{\mathcal{M}}$. Then it must have an eigenvalue of 1 among the coefficients in a spectral decomposition with respect to a Jordan frame. Pick a sequence $x_n \in c(\partial\Omega)$ converging to $x$, and choose a spectral decompostion for each $x_n$. Then for each $n$ the $2n$-tuple consisting of the $i$-th coefficient $\lambda_i$ in the coordinate $i$ and of $c_i$ in coordinate $r + i$ lies in the compact space $[-1, 1]^r \times [0, e]^r$, and thus admits a convergence subsequence. Since each $x_n$ has a coefficient of $-1$, there must be a entry of $-1$ in the limit, and the Jordan algebra components must converge to another Jordan frame. Thus $y$ must also be in the terminal set.

**Theorem 2.2.** *Let $V$ be a euclidean Jordan algebra with symmetric cone $\Omega$. Then we have the following:*

*(1)* $c(\partial_\infty\Omega) = \{(1-t)e + tx : x \in \mathcal{E},\ t \in [0, 1]\}$*, the cone over* $\mathcal{E}$ *with vertex* $e$*.*

*(2)* $c(\overline{\partial\Omega}^{\mathcal{M}}) = \{(1-t)(-e) + tx : x \in \mathcal{E},\ t \in [0, 1]\}$*, the cone over* $\mathcal{E}$ *with vertex* $-e$*.*

*Thus the boundary* $c(\partial_\infty(\Omega) \cup \partial\Omega)$ *of* $c(\overline{\Omega}^{\mathcal{M}})$ *is the double cone over* $\mathcal{E}$ *with vertices* $e$ *and* $-e$*.*

**Proof.** We use the characterization of $c(\partial_\infty\Omega)$ given in Proposition 2.1. For any point $x$ besides the cone point $e$ in this set, the spectral decomposition $\sum_{i=1}^{r} \lambda_i c_i$ must have some $\lambda_i = 1$ and $\min\{\lambda_i\} = \gamma < 1$. Then elementary calculations yield that the one and only possibility for obtaining $x$ in the right-hand side of (1) is to choose $y = \sum_{i=1}^{r} \mu_i c_i \in \mathcal{E}$, where $\mu_i = (2\lambda_i - \gamma - 1)/(1 - \gamma)$, and $t = 2/(1 - \gamma)$. Similar calculations hold for part (2), and the last assertion then follows directly from these. □

## 3. The orders $\ll$ and $\leq$

We continue in the same setting as the previous section: a euclidean Jordan algebra $V$ with symmetric cone $\Omega$. We consider for $a \ll b$ the order interval $[a,b] = a + \overline{\Omega} \cap b - \overline{\Omega}$. We note that the translation $t_{-a}$ is order-preserving for both $\leq$ and $\ll$ and carries the interval to $[0,c]$, where $c = b - a \in \Omega$. The linear transformation $P(c^{-\frac{1}{2}})$ preseves $\Omega$, hence is order-preserving, and carries $c$ to $e$ and $0$ to $0$. Thus we see that any interval $[a,b]$ with $a \ll b$ can be carried onto $[0,e]$ by an affine isomorphism that preserves both the orders $\ll$ and $\leq$. Thus we can without loss of generality restrict our attention to the study of one such interval.

We choose for consideration the order interval $[-e,e] = e - \overline{\Omega} \cap (-e + \overline{\Omega})$. This set has dense interior $c(\Omega) = \{w : -e \ll w \ll e\} = e - \Omega \cap \Omega - e$ (Proposition 2.1). Let $x, y \in [-e,e]$ with $x \leq y$. Then for $0 < t < 1$, we have

$$-e \ll -te \leq tx \leq ty \ll ty + \frac{1}{2}(1-t)e \ll ty + (1-t)e \leq e.$$

Since as $t \to 1$, $tx \to x$ and $ty + (1/2)(1-t)e \to y$, we conclude that there exist $x_n \to x$ and $y_n \to y$ such that $-e \ll x_n \ll y_n \ll e$ for each $n$. We have thus proved:

**Proposition 3.1.** *Let $V$ be a euclidean Jordan algebra with symmetric cone $\Omega$. If $a \ll b$, then the order relation $\leq$ on $[a,b]$ is the closure of the order relation $\ll$ on $\{w : a \ll w \ll b\} = a + \Omega \cap b - \Omega$, the interior of $[a,b]$.*

The next lemma is a known result (see, for example, Exercise 7 of Chapter III of [3]).

**Lemma 3.2.** *Let $a, b \in \Omega$. Then $a \ll b$ if and only if $b^{-1} \ll a^{-1}$.*

**Proof.** Suppose that $a, b \in \Omega$ with $a \ll b$. Since $P(a^{-\frac{1}{2}}) \in H$, $P(a^{-\frac{1}{2}})(b-a) \in \Omega$. This implies that $P(a^{-\frac{1}{2}})(a) = e \ll P(a^{-\frac{1}{2}})b$. Since $a^{\frac{1}{2}}, b \in \Omega$, $(P(a^{-\frac{1}{2}})b)^{-1} = P(a^{\frac{1}{2}})b^{-1} \ll e$ (see Section II.3 of [3]). This implies that $b^{-1} \ll P(a^{-\frac{1}{2}})e = a^{-1}$. $\square$

**Corollary 3.3.** *Each of the following mappings preserve $\ll$.*

*(1) The mapping $j : \Omega \to -\Omega$ and $j : -\Omega \to \Omega$;*

*(2) The Cayley transform $c(x) = (x-e)(x+e)^{-1} = t_e \circ 2id_V \circ j \circ t_e(x)$ from $\Omega$ to $e - \Omega \cap \Omega - e$;*

*(3) The inverse Cayley transform $d(x) = -(x+e)(x-e)^{-1} = -e + 2(e-x)^{-1} = t_{-e} \circ j \circ (1/2)id_V \circ t_{-e}(x)$ from $e - \Omega \cap \Omega - e$ to $\Omega$.*

**Proof.** Item (1) is an immediate consequence of Lemma 3.2, since $j(x) = -x^{-1}$ is the composition of two order-reversing transformations. It is immediate that the other mappings besides $j$ in the right-most definitions of $c$ and $d$ are order-preserving for $\ll$, and hence $c$ and $d$ are. $\square$

**Corollary 3.4.** *The closure of the relation* $\{(x,y)\in\Omega\times\Omega : x\ll y\}$ *in* $\overline{\Omega}^{\mathcal{M}}\times\overline{\Omega}^{\mathcal{M}}$ *is a closed partial order on* $\overline{\Omega}^{\mathcal{M}}$ *which extends the partial order* $\leq$ *on* $\Omega$. *The mappings* $c$ *and* $d$ *are order-preserving inverses between* $\overline{\Omega}^{\mathcal{M}}$ *and* $[-e,e]$ *with respect to this order. We can then denote* $\overline{\Omega}^{\mathcal{M}}$ *as the order interval* $[0,\infty]$, *where* $\infty := j(0)$.

**Proof.** By Proposition 3.1 the closed partial order $\leq$ on $[-e,e]$ is the closure of $\ll$ on the interior. We pull this closed order back to $\overline{\Omega}^{\mathcal{M}}$ via the homeomorphism $c^{-1}=d$. Since by Corollary 3.3 $d$ is an order isomorphism from the interior of $[-e,e]$ to $\Omega$ with respect to $\ll$, it follows immediately that the pulled back order is the closure of $\ll$ on $\Omega$. Since $d$ is also an order isomorphism on the interior of $[-e,e]$ with respect to $\leq$, it follows that the closure agrees with $\leq$ on $\Omega$. We note that the largest point of $\overline{\Omega}^{\mathcal{M}}$ will be $d(e)=t_{-e}\circ j\circ(1/2)\mathrm{id}_V\circ t_{-e}(e)=t_{-e}j(0)$. Note also that $-(1/n)e\to 0$ implies that $ne=j(-(1/n)e)\to j(0)=\infty$, so that $t_{-e}(\infty)=\lim t_{-e}(ne)=\lim(n-1)e=\infty$. Thus $d(e)=\infty$ is the largest element of $\overline{\Omega}^{\mathcal{M}}$. □

## 4. The compression semigroup

Let $V$ be a euclidean Jordan algebra with the corresponding symmetric cone $\Omega$. In the action of $G$ on $\mathcal{M}=G/P$, we consider the compression semigroup of $\Omega\subset\mathcal{M}$

$$\Gamma_\Omega=\{g\in G \mid g\cdot\Omega\subseteq\Omega\}.$$

Since the closure $\overline{\Omega}^{\mathcal{M}}$ of $\Omega$ in $\mathcal{M}$ is compact with $\Omega$ as its interior, the compression semigroup $\Gamma_\Omega$ is a closed subsemigroup of $G$ [4].

Now let

$$\Gamma^+=\{t_x\in N^+ \mid x\in\overline{\Omega}\},\qquad \Gamma^+_\circ=\{t_x \mid x\in\Omega\},$$

$$\Gamma^-=\{\tilde{t}_{-x}\in N^- \mid x\in\overline{\Omega}\},\qquad \Gamma^-_\circ=\{\tilde{t}_x \mid x\in-\Omega\}.$$

Then $\Gamma^+$ and $\Gamma^-$ are closed subsemigroups of $N^+$ and $N^-$, respectively.

The following appear as Theorem 4.9 in [6] and Corollary 7.7 in [7]:

**Theorem 4.1.** *We have* $\Gamma_\Omega=\Gamma^+H\Gamma^-$. *Furthermore, the interior* $\Gamma_\Omega^\circ$ *of* $\Gamma_\Omega$ *is given by* $\Gamma^\circ_\Omega=\Gamma^+_\circ H\Gamma^-_\circ$. *In particular,* $\Gamma^\circ_\Omega=\{g\in G \mid g\cdot\overline{\Omega}^{\mathcal{M}}\subset\Omega\}$.

**Theorem 4.2.** *Let* $g\in\Gamma_\Omega$. *Then* $g(\overline{\Omega}^{\mathcal{M}})=[g(0),g(\infty)]$, *and the mapping* $g$ *is an order isomorphism from the first to the second set.*

**Proof.** Let $g=t_xh\tilde{t}_{-y}\in\Gamma_\Omega$ (by Theorem 4.1). Since $j$ is an order isomorphism for $\ll$ from $\Omega$ to $-\Omega$, $t_{-y}$ is order isomorphism from $-\Omega$ into $-\Omega$, and $j$ is again an order isomorphism from $-\Omega$ to $\Omega$, we conclude that $\tilde{t}_{-y}$ is an order isomorphism into $\Omega$ for $\ll$ on $\Omega$. Since $h$ is a linear mapping on $V$ preserving $\Omega$, it is an order isomorphism for $\ll$ on $\Omega$, and $t_x$ obviously is. Thus the composition $g$ is an order isomorphism for $\ll$ from $\Omega$ onto the image of $\Omega$, and then by continuity for it is also one for $\leq$ on $\overline{\Omega}^{\mathcal{M}}$ (see Corollary 3.4). Thus $g(\overline{\Omega}^{\mathcal{M}}) \subseteq [g(0), g(\infty)]$, where $g(0) = x$ and $g(\infty) = x + h(y^{-1})$ if $y \in \Omega$.

Consider the case that $y \in \Omega$. Suppose that $z$ in in the interior of $[x, x + h(y^{-1})]$. Then $z = x + a = x + h(y^{-1}) - b$ for some $a, b \in \Omega$. Note that $a = h(y^{-1}) - b$, so $a \ll h(y^{-1})$. Since the inversion $-j$ is order decreasing on $\Omega$ (Lemma 3.2), $w := ((h^{-1}(a))^{-1} - y)^{-1} \in \Omega$. This implies that $z = x + a = g(w) \in g(\Omega)$. Since the interior of $[g(0), g(\infty)]$ is dense, it follows that $g(\overline{\Omega}^{\mathcal{M}}) = [g(0), g(\infty)]$.

Finally consider the case that $y \in \partial\Omega$. Let $z \in [g(0), g(\infty)]$. Pick a sequence $h_n$ in $\Gamma_{\Omega}^{\circ}$ converging to the identity (for example, $t_{\epsilon e} \circ \tilde{t}_{-\epsilon e}$ for $\epsilon = 1/n$). Then $g_n := h_n g \in \Gamma_{\Omega}^{\circ}$, since the latter is a semigroup ideal, and thus by the previous paragraph there exists $w_n \in \overline{\Omega}^{\mathcal{M}}$ such that $g_n(w_n) = h_n(z)$. By compactness and continuity the sequence $w_n$ converges to $w \in \overline{\Omega}^{\mathcal{M}}$ such that $g(w) = z$.

□

## References


[1] Bertram, W., *Un théorème de Liouville pour les algèbres de Jordan*, Bulletin de la Soc. Math. Française **124** (1996), 299–327.

[2] Faraut, J., *Fonctions de Legendre sur une algè bre de Jordan*, CWI Quarterly **5** (1992), 309–320.

[3] Faraut, J., and A. Koranyi, "Analysis on symmetric cones," Oxford University Press, Oxford, 1994.

[4] Hilgert, J., and G. Ólafsson, "Causal symmetric spaces," Academic Press, San Diego, 1997.

[5] Kaup, W., *A Riemann mapping theorem for bounded symmetric domains in complex Banach spaces*, Math. Z. **183** (1983), 503–529.

[6] Koufany, K., *Semi-groupe de Lie associé á un cône symmétrique*, Ann. Inst. Fourier, Grenoble **45** (1995),1–29.

[7] J. D. Lawson and Y. Lim, *Lie semigroups with triple decompositions*, Pacific J. of Math., to appear.



Department of Mathematics
Louisiana State University
Baton Rouge La 70803, U.S.A.

Topology and Geometry
Research Center
Kyungpook National University
Taegu 702-701, Korea



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