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On the Chevalley restriction theorem

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Abstract. Let $\mathfrak{g}$ be a complex semisimple Lie algebra with adjoint group G. Suppose that σ is an involutive automorphism of $\mathfrak{g}$. Then σ induces an involution of G also denoted by σ. Let $K{=}G^\sigma$ be the subgroup of σ-fixed points. Consider the direct decomposition $\mathfrak{g}{=}\mathfrak{k}{\oplus}\mathfrak{p}$ of $\mathfrak{g}$ into the eigenspaces for σ, so that $\mathfrak{k}{=}\{x{\in}\mathfrak{g}\,|\,\sigma(x){=}x\}$, $\mathfrak{p}{=}\{x{\in}\mathfrak{g}\,|\,\sigma(x){=}{-}x\}$. Then $\mathfrak{p}$ is a K-module. Denote by $\mathfrak{a}{\subset}\mathfrak{p}$ a maximal abelian ad-diagonalizable subalgebra. Consider the "baby Weyl group" $W_0{=}N_K(\mathfrak{a})/Z_K(\mathfrak{a})$. The aim of this paper is to prove that the restriction map $\psi{:}\,\mathbb{C}[\mathfrak{p}{\times}\mathfrak{p}]^K{\to}\mathbb{C}[\mathfrak{a}{\times}\mathfrak{a}]^{W_0}$ is surjective.

§0. Introduction

Let $\mathfrak{g}$ be a complex semisimple Lie algebra with adjoint group G. Suppose that $\sigma \neq \mathrm{id}$ is an involutive automorphism of $\mathfrak{g}$. Then σ induces an involution of G also denoted by σ. Let $K = G^\sigma$ be the subgroup of σ-fixed points. Consider the direct decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of $\mathfrak{g}$ into the eigenspaces for σ, so that

$$\mathfrak{k} = \{x \in \mathfrak{g} \,|\, \sigma(x) = x\}, \qquad \mathfrak{p} = \{x \in \mathfrak{g} \,|\, \sigma(x) = -x\}.$$

Then, clearly, $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{p}$ is a K-module, the representation of K in $\mathfrak{g}$ is certainly nothing else but the isotropy representation of the symmetric space G/K (see e.g. [KR]). Denote by $\mathfrak{a} \subset \mathfrak{p}$ a maximal abelian ad-diagonalizable subalgebra. Consider the "baby Weyl group" $W_0 = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$. It is well-known that W_0 is a finite group generated by reflections as a linear group operating on $\mathfrak{a}$ (with respect to some real form of $\mathfrak{a}$). Let $\psi : \mathbb{C}[\mathfrak{p}]^K \to \mathbb{C}[\mathfrak{a}]^{W_0}$ be the restriction map of algebras of invariants. Then the famous Chevalley restriction theorem states that ψ is an isomorphism (see, eg., [He1]; in [Vi] a more general result is obtained in the context of so-called θ-groups attached to any periodic automorphism of $\mathfrak{g}$).

Now consider the following special case. The adjoint representation $G : \mathfrak{g}$ could be identified with the isotropy representation of the symmetric space induced by the involution σ of $G \times G$ given by $\sigma(x, y) = (y, x)$. Then the cited

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Chevalley restriction theorem is equivalent to the "usual" Chevalley restriction theorem $\mathbb{C}[\mathfrak{g}]^G \simeq \mathbb{C}[\mathfrak{h}]^W$, where $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra and W is the usual Weyl group. In the paper [J] A. Joseph obtained a remarkable "multivariable" analogue of this theorem. Namely, he proved that the restriction map

$$\mathbb{C}[\mathfrak{g} \times \mathfrak{g}]^G \to \mathbb{C}[\mathfrak{h} \times \mathfrak{h}]^W$$

is surjective. This theorem has important applications to the representation theory and the geometry of commuting varieties (cf. [Ri]), because it is essentially equivalent to the fact that the ring of G-invariant functions on the commuting variety is integrally closed (see [Hu, J]). The aim of this paper is to extend Joseph's arguments in order to prove the following generalization:

Theorem. *The restriction map* $\psi : \mathbb{C}[\mathfrak{p} \times \mathfrak{p}]^K \to \mathbb{C}[\mathfrak{a} \times \mathfrak{a}]^{W_0}$ *is surjective.*

Actually, this (and Joseph's) theorem also holds for any number of summands (a generalization of the proof is immediate), but we will prove it in this form in order to make the notation easier.

Certainly, one could try to deduce this Theorem directly from Joseph's result arguing as follows: take $f \in \mathbb{C}[\mathfrak{a} \times \mathfrak{a}]^{W_0}$ and lift it to an element of $\mathbb{C}[\mathfrak{h} \times \mathfrak{h}]^W$; then by Joseph's theorem the latter element can be lifted to an element of $\mathbb{C}[\mathfrak{g} \times \mathfrak{g}]^G$, which can be resricted to an element of $\mathbb{C}[\mathfrak{p} \times \mathfrak{p}]^K$. Unfortunately, the first step of this construction fails even in the case of one summand (see examples in [He2]). Moreover, I don't know how to prove the natural analogue of the theorem in the context of arbitrary θ-groups (see [Vi]) and this seems to be an interesting problem. I should mention also that in some particular cases these results could be proved using "generalized polarizations" (see, e.g., [Hu]), but these results do not cover our theorem (as well as Joseph's one). In [Pa1] the Theorem was proved in the case of an involution of maximal rank (and in this case the argument in the beginning of this paragraph is evidently valid).

The paper is organized as follows. In §1 we look for a proof of Chevalley restriction theorem in the case of one summand such that this proof could be generalized to the case of two summands. Another reason for doing that separately is that all the ideas distinct from the ones used by Joseph already appear in this case. In §2 we prove the theorem in full generality showing that our main tool (class 1 representations) is compatible with Joseph's idea to use the PRV Conjecture (or, better said, the Kumar-Mathieu Theorem) to "separate layers".

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§1. Surjectivity of $\psi : \mathbb{C}[\mathfrak{p}]^K \to \mathbb{C}[\mathfrak{a}]^{W_0}$

Recall, that there is 1-1 correspondence between involutions of $\mathfrak{g}$ and real forms of $\mathfrak{g}$ (see, e.g., [W, §1.1]). Namely, suppose that $\mathfrak{g}^{\mathbb{R}}$ is a real form of $\mathfrak{g}$ (that is, $\mathfrak{g} = \mathfrak{g}^{\mathbb{R}} + i\mathfrak{g}^{\mathbb{R}}$). Let $\mathfrak{g}^{\mathbb{R}} = \mathfrak{k}^{\mathbb{R}} \oplus \mathfrak{p}^{\mathbb{R}}$ be a Cartan decomposition and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ its complexification. Then the map σ defined by $\sigma(x)=x$ for $x\in\mathfrak{k}$ and $\sigma(x)=-x$ for $x\in\mathfrak{p}$ is an involution of $\mathfrak{g}$ and any involution can be obtained in this way. Now, a maximal ad-diagonalizable abelian subalgebra $\mathfrak{a}$ in $\mathfrak{p}$ (all of them are K-conjugate) can be chosen to be a complexification of some $\mathfrak{a}^{\mathbb{R}}\subset\mathfrak{p}^{\mathbb{R}}$. ¿From now on by complex conjugation we will understand the complex conjugation with respect to a given real form $\mathfrak{g}^{\mathbb{R}}$.

Let B be a Killing form on $\mathfrak{g}$. Then the restriction of B on $\mathfrak{k}^{\mathbb{R}}$ is negative-definite and on $\mathfrak{p}^{\mathbb{R}}$ is positive definite. In particular we can identify $\mathfrak{g}\simeq\mathfrak{g}^*$ and $\mathfrak{a}\simeq\mathfrak{a}^*$ by means of this scalar product. Then the restriction map $\psi:\mathbb{C}[\mathfrak{p}]^K\to\mathbb{C}[\mathfrak{a}]^{W_0}$ can be viewed as a homomorphism of symmetric algebras $\psi:(S\mathfrak{p})^K\to(S\mathfrak{a})^{W_0}$. Now, it is well-known that $K=G^\sigma$ is a reductive group, so we have a K-module decomposition $S\mathfrak{p}=(S\mathfrak{p})^K\oplus(S\mathfrak{p})_K$, where $(S\mathfrak{p})_K$ is a sum of all non-trivial K-submodules. Consider the Reynolds operator (K-equivariant projection on the first factor of this decomposition). Let $\theta:(S\mathfrak{a})^{W_0}\to(S\mathfrak{p})^K$ be its restriction to $(S\mathfrak{a})^{W_0}$. It is clear that we have a commutative diagram

$$\begin{array}{ccc} S\mathfrak{a} & \xrightarrow{\theta_1} & (S\mathfrak{p})^K \\ {\scriptstyle\theta_2}\searrow & & \nearrow{\scriptstyle\theta} \\ & (S\mathfrak{a})^{W_0} & \end{array}$$

where θ_1 is a restriction of the Reynolds operator to $S\mathfrak{a}$ and θ_2 is a W_0-equivariant projection. Now we are ready to prove the following lemma:

Lemma 1.1. *If θ is injective then ψ is surjective.*

Proof. It is clear that both maps ψ and θ are homogeneous (with respect to natural grading of $S\mathfrak{a}$ and $S\mathfrak{p}$). So the composition $\psi\theta$ is a homogeneous map of degree 0 from $(S\mathfrak{a})^{W_0}$ to $(S\mathfrak{a})^{W_0}$. Now it is clear that if θ is injective and $\operatorname{Ker}\psi\cap\operatorname{Im}\theta=0$ then $\psi\theta$ is injective, hence surjective, so ψ is surjective as well. Certainly the second assertion follows from the fact that ψ is injective (and this is an easy part of Chevalley restriction theorem), but we are looking for a proof that could be applicable in the case of two summands.

First we could extend B to the scalar product in $S\mathfrak{p}$ in the usual way. Since the restriction of B on $\mathfrak{p}^{\mathbb{R}}$ is positive definite, it is clear that the restriction of B on $S\mathfrak{p}^{\mathbb{R}}$ is also positive definite (more precisely, the restriction of B on any finite-dimensional subspace of $S\mathfrak{p}^{\mathbb{R}}$ is positive definite). In particular, if $V\subset S\mathfrak{p}$ is any subspace stable under complex conjugation, then $V\cap V^\perp=0$. Take V to be the minimal K-submodule of $S\mathfrak{p}$ generated by $S\mathfrak{a}$. Since $S\mathfrak{a}$ is stable under complex conjugation, so is V, because V is spanned by the vectors of the form $k\cdot f$, $k\in K$, $f\in S\mathfrak{a}$, and since σ commutes with complex conjugation on $\mathfrak{g}$, K is stable under complex conjugation (we are in the adjoint group!).

By definition of Reynolds operator, $\operatorname{Im}\theta=V\cap(S\mathfrak{p})^K$. By the construction of B on $S\mathfrak{p}$, it is clear that $\operatorname{Ker}\psi=(S\mathfrak{a})^\perp\cap(S\mathfrak{p})^K$ and by K-invariance of B we have $\operatorname{Ker}\psi=V^\perp\cap(S\mathfrak{p})^K$. Since $V\cap V^\perp=0$, we finally obtain $\operatorname{Ker}\psi\cap\operatorname{Im}\theta=0$. ∎

So to prove the surjectivity of ψ it remains to prove that θ is injective. This idea (as well as many others in this paper) belongs to Joseph, the only exception is that Joseph didn't use Reynolds operator directly. Instead of that he defined the "Letzer map" purely in infinitesimal terms, but it is easy to see that in the adjoint situation Joseph's "Letzer map" actually coincides with good old Reynolds operator! One could mention that in the proof of Lemma 1.1 we didn't use the fact our representation is an isotropy representation of a symmetric space. So the argument is applicable in a little bit more general setup.

Now it is time to slightly change our notations and to introduce new ones. First of all, since $S\mathfrak{p}$ is a K-submodule of $S\mathfrak{g}$, injectivity of θ will follow from the injectivity of the restriction on $(S\mathfrak{a})^{W_0}$ of the Reynolds operator $S\mathfrak{g} \to (S\mathfrak{g})^K$. Since the definition of the Reynolds operator depends only on the K-module structure of $S\mathfrak{g}$, we can substitute for $S\mathfrak{g}$ the universal enveloping algebra $U\mathfrak{g}$, since $U\mathfrak{g}$ has the same K-module structure by the Poincare-Birkhoff-Witt theorem (this K-module isomorphism could be written down explicitly as the symmetrization map $S\mathfrak{g} \to U\mathfrak{g}$). Moreover, we have an isomorphism $S\mathfrak{a} \simeq U\mathfrak{a}$. Notice that the obvious embedding $U\mathfrak{a} \subset U\mathfrak{g}$ corresponds to an embedding $S\mathfrak{a} \subset S\mathfrak{g}$ via the symmetrization map. So from now on θ will denote the map $(U\mathfrak{a})^{W_0} \to (U\mathfrak{g})^K$ induced by the Reynolds operator. We need to show that it is injective.

Take a torus $S \subset G$ such that $\mathfrak{a} = \operatorname{Lie} S$. Then S is a maximal σ-anisotropic torus in the terminology of [Vu]. Fix a Borel subgroup B and maximal torus $T \subset B$ such that KB is dense in G (so K is spherical in G and if $\mathfrak{b}$ is the Lie algebra of B then $\mathfrak{k} + \mathfrak{b} = \mathfrak{g}$) and T contains S (such B, T exist by [Vu]). Then automatically $\sigma(T) = T$. Then $\mathfrak{h} = \operatorname{Lie} T$ is a Cartan subalgebra of $\mathfrak{g}$. Let $\mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{k}$, so $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{a}$. Let $\mathfrak{X}^*(T)$ (resp. $\mathfrak{X}_*(T)$) be the character group (resp. the set of one-parameter subgroups) of T. We use the additive notation for the operation in $\mathfrak{X}^*(T)$ and sometimes identify a character with its differential. Let $P \subset \mathfrak{X}^*(T)$ be the weight lattice and $P_+ \subset P$ the semigroup of dominant weights. Set

$$Q = \{\lambda \in P \,:\, \sigma(\lambda) = -\lambda, \qquad \lambda|_S \in 2\mathfrak{X}^*(S)\}.$$

It is clear that for any $\lambda \in Q$ we have $\lambda|_{\mathfrak{h}_0} = 0$, so we may view such λ as an element $\tilde{\lambda}$ of $\mathfrak{a}^*$. Let $w \in W_0$, $\tilde{\mu} = w\tilde{\lambda}$. Then we can extend $\tilde{\mu}$ to an element μ of $\mathfrak{X}^*(T) \otimes \mathbb{Q}$. But it is clear that actually $\mu \in Q$. Moreover by the structure theory of the baby Weyl group (see e.g. [W, §1.1]), there exists $w \in W$ such that $\mu = w\lambda$. Another useful tip is that if two elements of $\mathfrak{a}^*$ are conjugate with respect to W, then they are also conjugate w.r.t. W_0 (this follows from the fact that any W_0-fundamental chamber in $\mathfrak{a}^*$ entirely lies in some W-fundamental chamber, see [loc. cit.]). Set $Q_+ = Q \cap P_+$.

Lemma 1.2. *In the identifications above, $Q = W_0 Q_+$.*

Proof. This follows from the results of T. Vust (see [Vu]). Namely, passing to dual lattices, it is sufficient to prove that if $R \in \mathfrak{X}_*(S)$ is any one-parameter subgroup then there exists $w \in W_0$ such that $R^w = wRw^{-1}$ is dominant, that is, for any $b \in B$ the limit $\lim\limits_{t\to 0} R^w(t) b R^w(t)^{-1}$ exists. Consider a subgroup

$$P(R) = \{g \in G \mid \lim_{t\to 0} R(t) g R(t)^{-1} \text{ exists}\}.$$

Then $P(R)$ is parabolic, moreover, $P(R)$ and $\sigma P(R)$ are opposite parabolics, so that $P(R)\cap\sigma P(R) = Z_G(R)$, the centralizer of R in G. Such parabolics are called σ-anisotropic. Since $Z_G(R)$ is a σ-invariant reductive subgroup and $S\subset Z_G(R)$, there exists minimal σ-anisotropic parabolic subgroup P' of $Z_G(R)$ such that $P'\cap\sigma P' = Z_{Z_G(R)}(S)$. Then $P'' = P'R_u(P(R))$ is a minimal σ-anisotropic parabolic subgroup in G and $P''\cap\sigma P'' = Z_G(S)$. Since $Z_G(R)\subset P(R)$, we have $P''\subset P(R)$. Now, it is known that $K^0SR_u(P'')$ is open in G, where K^0 is an identity component of K (algebraic analogue of Iwasawa decomposition), so if B' is a Borel subgroup of $Z_G(S)$ then $B'' = B'R_u(P'')$ is a Borel subgroup in G that contains S, is opposite to K, and is contained in $P(R)$. It remains to notice that all pairs (B,T) such that B is a Borel subgroup opposite to K, $T\subset B$ is a maximal torus that contains a maximal σ-anisotropic torus, are conjugate with respect to K (see [loc. cit.]). In particular, there exists $\tilde w\in N_K(S)$ such that $B=\tilde wB''$. Since $B''\subset P(R)$, $B\subset P(\tilde wR\tilde w^{-1})$ and hence $R^w = \tilde wR\tilde w^{-1}$ is dominant. ∎

Returning to proof of Theorem, for any spherical subgroup H of G and any irreducible representation V_λ of G with highest weight λ, it is known that $\dim V_\lambda^H\le 1$ and the set

$$\Gamma_H = \{\lambda\in P_+ \mid \dim V_\lambda^H = 1\}$$

is a submonoid in P_+ (see eg. [Pa2]). In our case by [Vu], it is known that $\Gamma_K = Q_+$. Irreducible representations V_λ such that $\lambda\in Q_+$ are known as representations of class 1. Fix $\lambda\in Q_+$. For any $v\in V_\lambda$ and $\mu\in P$, let $(v)_\mu$ denote the μ-weight component of v, and set $\operatorname{Supp} v = \{\mu\in P \mid (v)_\mu\ne 0\}$. Let v_λ be a highest weight vector in V_λ. Fix a K-invariant vector $v_K\ne 0$ in V_λ.

Lemma 1.3. *(i)* $\operatorname{Supp} v_K\subset Q$;
(ii) $\lambda\in\operatorname{Supp} v_K$;
(iii) $\operatorname{Supp} v_K\cap W\lambda = W_0\lambda$.

Proof. Fix $\mu\in\operatorname{Supp} v_K$. Since $\mathfrak{h}_0 v_K = 0$ we have $\mu(\mathfrak{h}_0) = 0$, and so $\sigma\mu = -\mu$. Moreover, if $\tau\in S$ has order 2, then $\tau\in K$, so $\tau v_K = v_K$ and $\mu\in 2\mathfrak{X}^*(S)$. This proves (i). To prove (ii) let us notice that if B^0 is a Borel subgroup, opposite to B (so that $B\cap B^0 = T$), then KB^0 is dense in G (this follows from the description of B as in proof of Lemma 1.2). Then $\mathfrak{g} = \mathfrak{b}^0 + \mathfrak{k}$, so $U\mathfrak{g} = U\mathfrak{b}^0 U\mathfrak{k}$. Then

$$V_\lambda = (U\mathfrak{g})v_K = (U\mathfrak{b}^0)v_K.$$

But for any $x\in U\mathfrak{b}^0$, $\operatorname{Supp} xv_K < \operatorname{Supp} v_K$ (with respect to a usual order relation on P), so $\lambda\in\operatorname{Supp} v_K$. The last assertion follows from (i), (ii), $N_K(S)$-invariance of v_K and the discussion before Lemma 1.2. ∎

Now we can finish the proof of the theorem in the case of one summand. Suppose that $f\in\operatorname{Ker}\theta$, $f\ne 0$. To obtain a contradiction, it is sufficient to prove that f (viewed as an element of $S\mathfrak{a}$) vanishes at $\tilde\lambda$ for all $\lambda\in Q$. Since f is W_0-invariant we need to prove that $f(\tilde\lambda) = 0$ for all $\lambda\in Q_+$ (by Lemma

1.2). Arguing by induction on the standard order relation in P_+, we may assume that $f(\tilde{\mu}) = 0$ for all $\mu \in Q_+$, $\mu < \lambda$. Suppose that $f(\tilde{\lambda}) \neq 0$. Let π denote the K-invariant projection $V_\lambda \to V_\lambda^K$. Let F denote the operator on V_λ of left multiplication by f (viewed as an element of $U\mathfrak{g}$). Then, clearly, $F(v_\lambda) = f(\tilde{\lambda})v_\lambda$. Since $\theta f = 0$ and since the Reynolds operator commutes with K-module homomorphisms, we have $\pi F \pi = 0$ (because $\pi F \pi$ is clearly K-invariant, so Reynolds operator maps it onto itself, on the other hand, the Reynolds operator commutes with left (and right) multiplication on (K-invariant!) element π). Hence $\pi F v_K = 0$. By Lemma 1.3 and by induction we have

$$Fv_K = \rho f(\tilde{\lambda}) \sum_{w \in W_0} \tilde{w} v_\lambda,$$

where $\tilde{w}$ is some representative of w in $N_K(S)$ and $\rho \neq 0$. Since π is K-invariant, we get $\pi(v_\lambda) = 0$. Therefore $(U\mathfrak{k})v_\lambda \neq V_\lambda$. But

$$V_\lambda = (U\mathfrak{g})v_\lambda = (U(\mathfrak{k} + \mathfrak{b}))v_\lambda = (U\mathfrak{k})(U\mathfrak{b})v_\lambda = (U\mathfrak{k})v_\lambda.$$

Contradiction.

§2. Surjectivity of $\psi : \mathbb{C}[\mathfrak{p} \times \mathfrak{p}]^K \to \mathbb{C}[\mathfrak{a} \times \mathfrak{a}]^{W_0}$

First, we can identify $\mathbb{C}[\mathfrak{p} \times \mathfrak{p}]$ with $S(\mathfrak{p} \times \mathfrak{p})$, and $\mathbb{C}[\mathfrak{a} \times \mathfrak{a}]$ with $S(\mathfrak{a} \times \mathfrak{a})$ by means of the Killing form. Then we can identify $S(\mathfrak{a} \times \mathfrak{a})$ with $U(\mathfrak{a} \times \mathfrak{a})$. Let θ denote the restriction of the Reynolds operator $U(\mathfrak{g} \times \mathfrak{g}) \to U(\mathfrak{g} \times \mathfrak{g})^K$ on $U(\mathfrak{a} \times \mathfrak{a})^{W_0}$ (with respect to the diagonal action).

Lemma 2.1. *If θ is injective then ψ is surjective.*

Proof. The proof is the same as that of Lemma 1.1 taking into account the discussion after it. ∎

Now suppose that $f \in \operatorname{Ker} \theta$. In order to show that $f = 0$ it is sufficient to show that f (viewed as an element of $S(\mathfrak{a} \times \mathfrak{a})$) vanishes at all $(\lambda, \mu) \in Q \times Q$. We will proceed by the induction on the standard order relation. Take $(\lambda, \mu) \in Q_+ \times Q_+$ and suppose that f vanishes at all $(\lambda', \mu') \in Q \times Q$ such that there exist $w_1, w_2 \in W_0$ with $(w_1\lambda', w_2\mu') \in Q_+ \times Q_+$ and $(w_1\lambda', w_2\mu') < (\lambda, \mu)$. We need to show that $f(w_1\lambda, w_2\mu) = 0$ for any $w_1, w_2 \in W_0$.

Consider the $G \times G$ module $W = V_\lambda \otimes V_\mu$. Let π denote the K-invariant projection $W \to W^K$ (with respect to the diagonal action). Let F denote the operator of left multiplication by f (viewed as an element of $U(\mathfrak{g} \times \mathfrak{g})$). Since $\theta f = 0$, we get $\pi F \pi = 0$ as in the proof for a case of one summand. Let v_K^λ (resp. v_K^μ) denote a non-zero K-invariant vector in V_λ (resp. V_μ). Then $\pi F(v_K^\lambda \otimes v_K^\mu) = 0$. Let v_λ (resp. v_μ) denote the highest weight vector of V_λ (resp. V_μ). By Lemma 1.3

$$v_K^\lambda \otimes v_K^\mu = \sum_{\substack{w' \in W_0/(W_0)_\lambda \\ w'' \in W_0/(W_0)_\mu}} (\tilde{w}' v_\lambda) \otimes (\tilde{w}'' v_\mu) + U,$$

where $\operatorname{Supp} U$ lies in $Q \times Q$ and for any $(\lambda', \mu') \in \operatorname{Supp} U$ by induction (and Lemma 1.2) we have $f(\lambda', \mu') = 0$. So we obtain that

$$\sum_{\substack{w' \in W_0/(W_0)_\lambda \\ w'' \in W_0/(W_0)_\mu}} f(w'\lambda, w''\mu)(\tilde{w}'v_\lambda) \otimes (\tilde{w}''v_\mu)$$

lies in the kernel of π. Now for any $w' \in W_0/(W_0)_\lambda$, $w'' \in W_0/(W_0)_\mu$ we can find an element $(w_1(w', w''), w_2(w', w'')) \in W_0 \times W_0$ such that $w_1 = yw'$, $w_2 = yw''$ for some $y \in W_0$ and such that $w_1\lambda + w_2\mu \in Q_+$. Denote by w_1^i, w_2^i, $i = 1, \dots, r$ the complete set of such pairs obtained for all $w' \in W_0/(W_0)_\lambda$, $w'' \in W_0/(W_0)_\mu$. Now by W_0-invariance of f and by K-invariance of π we proved that an element

$$\sum_{i=1,\dots r} a_i f(w_1^i\lambda, w_2^i\mu)(\tilde{w}_1^i v_\lambda) \otimes (\tilde{w}_2^i v_\mu)$$

lies in the kernel of π for some non-zero integers a_i. It is sufficient to prove that all $f(w_1^i\lambda, w_2^i\mu)$ are equal to zero. So we reduced our proof to the following claim:

Lemma. *Suppose that an element*

$$A = \sum_{i=1,\dots r} b_i(\tilde{w}_1^i v_\lambda) \otimes (\tilde{w}_2^i v_\mu)$$

lies in the kernel of π. Then all b_i are equal to zero.

Suppose that some $b_i \neq 0$. By Kostant's refinement of PRV conjecture (see [J] or [Ku1]) for any i the $\mathfrak{g}$-module $(U\mathfrak{g})\left[\tilde{w}_1^i v_\lambda) \otimes (\tilde{w}_2^i v_\mu)\right]$ containes a unique $\mathfrak{g}$-submodule of highest weight $w_1^i\lambda + w_2^i\mu$. By Kumar's refinement of PRV conjecture (see [J] or [Ku2]), these $\mathfrak{g}$ submodules form a direct sum $S = \bigoplus_{i=1}^{r} S_i$. Take i such that the weight $\rho = w_1^i\lambda + w_2^i\mu$ is minimal and $b_i \neq 0$. Let π_0 denote a G-invariant projection $W \to W_\rho$ on the ρ-isotipical component. Then it is clear by checking the weights that $\pi_0(A)$ is equal either to zero or to the highest weight vector v of some irreducible submodule S_0 of highest weight ρ. But the first case is impossible by the previous discussion of PRV, because vectors $\pi_0\left((\tilde{w}_1^i v_\lambda) \otimes (\tilde{w}_2^i v_\mu)\right)$ (for various i such that $b_i \neq 0$ and $w_1^i\lambda + w_2^i\mu = \rho$) are not equal to zero by Kostant's form of PRV and are linearly independent by Kumar's form of PRV. Denote by π_1 a K-invariant projection $W_\rho \to W_\rho^K$. Then arguing as in proof for one summand we see that $\pi_1(v) \neq 0$, so that $\pi_1\pi_0(A) \neq 0$. But then $\pi(A) \neq 0$ as well, because $\pi_1\pi_0$ is K-invariant and hence factors through $\pi|_{(U\mathfrak{g})A}$. This completes the proof.

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