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# On some degenerate principal series representations of O(p, 2)

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**Abstract.** We consider representations of $O(p,2)$ ($p > 4$) induced from one-dimensional representations of a maximal parabolic subgroup. We first decompose them into K-types using Stiefel harmonics theory, then write down the actions of the noncompact part. Now the reducibility and the unitarizability of the irreducible constituents are deduced.

## 1. Introduction

The importance of parabolic induction in representation theory is widely acknowledged. Representations induced from one-dimensional representations of maximal parabolic subgroups are sometimes called "degenerate principal series". This paper treats degenerate principal series of $O(p,2)$ in a manner analogous to those used in [9]. That is, we first decompose the representations into irreducibles of the maximal compact subgroup, and then calculate the infinitesimal action of the non-compact part. We get informations such as reducibility and unitarizability. The computations are done using Euler operators, thus avoiding the use of explicit formulae of the vectors.

For other settings (non $p$-adic), see for example [18] for $SO(p,q)$, [13] for $U(p,q)$, [14] and [26] for $Sp(p,q)$, [24] for $SU(2,2)$, [15] for $Mp(n,\mathbb{R})$, [10] for $Spin_0(n,n)$, $SU(n,n)$ and $Sp(n,n)$, [4] for $SU(n,n,\mathbb{F})$, [1] for $GL(2n,\mathbb{R})$, [9] for $SO(p,q)$, $U(p,q)$ and $Sp(p,q)$, [11], [22], [23], [20] and [27] for Hermitian symmetric spaces of tube type, [19] and [16] for $U(n,n)$, [17] for $Sp(2n,\mathbb{R})$, [8] for $GL(n,\mathbb{R})$ and $GL(n,\mathbb{C})$, and [3] for rank one case, suggesting further applications.

A special case of [21] gives a sufficient condition for our representations to be complementary series. Our result implies that the condition is also necessary.

Now we describe our result in more details. Let $G = O(p,2)$ where $p > 4$. We consider the maximal parabolic subgroup $S$ of $G$ which has Levi part isomorphic to $GL(2,\mathbb{R}) \times O(p-2)$. $G$ has only two maximal parabolic subgroups, namely, this one and the one treated by [9].

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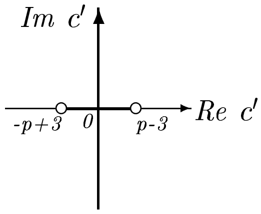
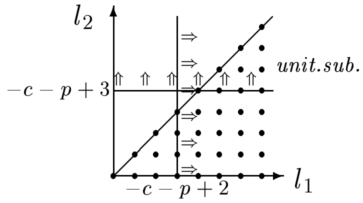
In the Langlands decomposition $S = MAN$, $A$ is one-dimensional, and its character is parametrized by a complex number $c$. (The precise definition will be given later.) We regard this character as a representation of $S$ where $M$ and $N$ act trivially. We induce this representation to $G$, and this is what we will work on. We denote it by $S^c(\mathbf{X}^+)$. Its Harish-Chandra module is decomposed into the sum of blocks, each block invariant under the action of the maximal compact subgroup $K = O(p) \times O(2)$ of $G$. The blocks are parametrized by two non-negative integers $l_1, l_2$ satisfying $l_1 \geq l_2$, which we identify with points $(l_1, l_2)$ in a plane. Here is the main result.

**Theorem.** *The Harish-Chandra module of $S^c(\mathbf{X}^+)$ is irreducible and unitary if and only if $c' = -2c - p + 1 \in i\mathbb{R}$ or $-p + 2 < c < -1$.*

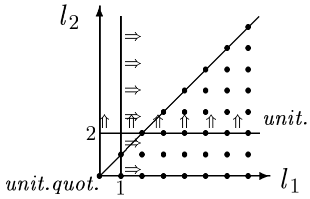



*When it is reducible(i.e. $c \in \mathbb{Z}$), its irreducible constituents are as follows.*

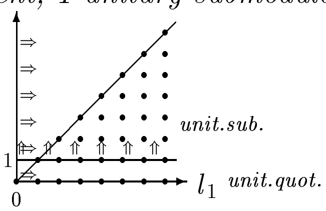
• $c \leq -p$ : *1 non-unitary finite-dimensional quotient, 1 non-unitary subquotient, 1 unitary submodule.*



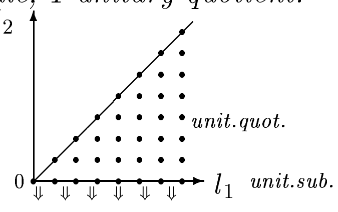

• $c = -p+1$ : *1 unitary 1-dimensional quotient, 1 non-unitary subquotient, 1 unitary submodule.*




• $c = -p + 2$ : *1 unitary quotient, 1 unitary submodule.*

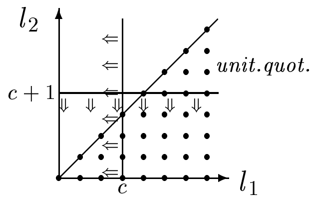



• $c = -1$ : *1 unitary submodule, 1 unitary quotient.*

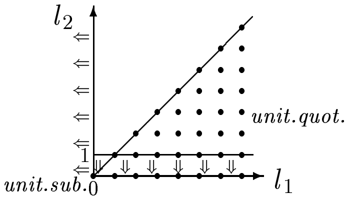



• $c \geq 1$ : *1 non-unitary finite-dimensional submodule, 1 non-unitary sub-*

*quotient, 1 unitary quotient.*




• $c = 0$ : *1 unitary 1-dimensional submodule, 1 non-unitary subquotient, 1 unitary quotient.*




*This completes the statement of the theorem.*

In Section 2, we define the representation in more details, and realize it concretely on the space of functions satisfying some relations.

In Section 3, there is a quick review on the special case of Stiefel harmonics theory (which is generalized spherical harmonics theory).

In Section 4, the representation is identified with the space of functions on a Stiefel manifold. Using Section 3, we decompose the representation into the sum of irreducible $K$-modules, or "$K$-types", where $K$ is the maximal compact subgroup of $G$.

In Section 5, the action of $\mathfrak{p}$ is computed in terms of differential operators. Here, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is the Cartan decomposition of the complexified Lie algebra. This is done essentially by writing explicitly a projection operator which extracts the harmonic polynomial part from an arbitrary polynomial, described in Section 6.

In Section 7, we deduce the reducibility. On points where it is reducible, the composition series is determined. We see that the representation is built up from "blocks", forming a lattice in a plane. These blocks are separated by lines, each territory corresponding to an irreducible constituent.

Finally in Section 8, we investigate unitarizability. The complementary series is determined. We fix a standard inner product on each $K$-type. The ratios of an inner product restricted to $K$-types compared with the standard ones are related to each other by recursive relations. The existence of these ratios tells us whether or not the irreducible constituent is unitarizable.

I would like to thank my advisor Prof. H. Matumoto for many helpful suggestions. I was also informed that he had recently generalized my results.

## 2. Construction of the representation

Write $I_k$ for the identity matrix of size $k$. Let

$$
\begin{aligned}
G = O(p,2) &= \left\{ g \in GL(p+2,\mathbb{R}) \;\middle|\; {}^t g \begin{pmatrix} I_p & 0 \\ 0 & -I_2 \end{pmatrix} g = \begin{pmatrix} I_p & 0 \\ 0 & -I_2 \end{pmatrix} \right\}, \\
K &= \left\{ \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} \;\middle|\; A \in O(p), C \in O(2) \right\}, \\
\mathfrak{p}_\circ &= \left\{ \begin{pmatrix} 0 & B \\ {}^t B & 0 \end{pmatrix} \;\middle|\; B \in M(p,2;\mathbb{R}) \right\}.
\end{aligned}
$$

The Cartan decomposition is $G = K \exp \mathfrak{p}_\circ$. We use German letters $\mathfrak{g}_\circ, \mathfrak{k}_\circ, \ldots$ for Lie algebras corresponding to $G, K, \ldots$, and $\mathfrak{g}, \mathfrak{k}, \ldots$ for their complexifications.

$G$ acts by left multiplication on $\mathbb{R}^{p+2} \oplus \mathbb{R}^{p+2}$ (the standard representation), and we define:

$$
\begin{aligned}
v_0 &= \begin{pmatrix} I_2 \\ 0_{p-2,2} \\ I_2 \end{pmatrix} \in \mathbb{R}^{p+2} \oplus \mathbb{R}^{p+2}, \\
\mathbf{X} &= G v_0, \\
S &= \left\{ p \in G \;\middle|\; \exists X \in GL(2,\mathbb{R}) \quad s.t. \quad p v_0 = \begin{pmatrix} X \\ 0 \\ X \end{pmatrix} \right\}.
\end{aligned}
$$

$S$ is a parabolic subgroup of $G$. (It is the stabilizer of a two-dimensional subspace in $\mathbb{R}^{p+2}$.)

We define a map $\chi : G \to \mathbb{R}^\times$ by

$$
\chi(g) = \det C \quad \text{if} \quad g v_0 = \begin{pmatrix} A_1 \\ C \end{pmatrix} \begin{matrix} \}p \\ \}2 \end{matrix}.
$$

The restriction of $\chi$ to $S$ is a group homomorphism.

For any complex number $c$, we define characters $\chi_c^\pm$ of $S$:

$$
\begin{aligned}
\chi_c^+(p) &= |\det \chi(p)|^c, \\
\chi_c^-(p) &= \operatorname{sign}(\chi(p)) |\det \chi(p)|^c \quad (\forall p \in S).
\end{aligned}
$$

We study the Harish-Chandra module of:

$$
S^{c\pm}(\mathbf{X}) = \left\{ F \in C^\infty(\mathbf{X}) \;\middle|\; F(gpv_0) = \chi_c^\pm(p) F(gv_0) \quad (\forall g \in G, \forall p \in S) \right\}.
$$

These are $G$-modules by left translation. We can interpret these as (generalized) principal series representations as follows.

In the Langlands decomposition $S = MAN$, we can take as $A$:

$$
A = \left\{ a = \exp H(h) \;\middle|\; H(h) = \left( \begin{array}{cc|c|cc} & & & h & \\ & & & & h \\ \hline & & & & \\ \hline h & & & & \\ & h & & & \end{array} \right), \; h \in \mathbb{R}, \text{zeros elsewhere} \right\},
$$

which is one dimensional. For any complex number $c'$, we define a linear functional $\nu_{c'}$ on the Lie algebra of $A$:

$$\nu_{c'}(H(h)) = c'h \quad (\forall h \in \mathbb{R}).$$

Since

$$\begin{gathered} \chi_c^+(man) = \exp(2ch), \\ \chi_c^-(man) = \operatorname{sign}(\chi(m)) \exp(2ch) \\ (\forall m \in M, \forall a = \exp H(h) \in A, \forall n \in N), \end{gathered}$$

we get

$$\begin{aligned} S^{c+}(\mathbf{X}) &\cong \operatorname{Ind}_{MAN}^G(\mathbf{1}_M \otimes \exp \nu_{c'} \otimes \mathbf{1}_N), \\ S^{c-}(\mathbf{X}) &\cong \operatorname{Ind}_{MAN}^G(\operatorname{sign} \chi|_M \otimes \exp \nu_{c'} \otimes \mathbf{1}_N). \end{aligned}$$

Here we take $2c = -(c' + p - 1)$ and $\mathbf{1}_M, \mathbf{1}_N$ denote the trivial representations. The induction on the right hand side is shifted by $\rho = \nu_{p-1}$ to preserve unitarity.

Next, we pay attention to the connectivity. Let

$$\begin{aligned} G^+ &= O^+(p,2) \\ &= (O(p) \times SO(2)) \exp \mathfrak{p}_\circ \\ &= \{g \in G | \chi(g) > 0\}, \\ k_C &= \begin{pmatrix} I_p & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix} \in O(p,2) - O^+(p,2), \\ \mathbf{X}^+ &= O^+(p,2) v_0 \\ &= \{g v_0 \in \mathbf{X} | \chi(g) > 0\}, \\ \mathbf{X}^- &= k_C O^+(p,2) v_0 \\ &= \{g v_0 \in \mathbf{X} | \chi(g) < 0\}. \end{aligned}$$

$O^+(p,2)$ acts on $\mathbf{X}^\pm$. Since $\mathbf{X} = \mathbf{X}^+ \sqcup \mathbf{X}^-$ (disjoint union), we can define

$$\operatorname{sign}_{\mathbf{X}}(x) = \begin{cases} 1 & (x \in \mathbf{X}^+) \\ -1 & (x \in \mathbf{X}^-) \end{cases}$$

which is an $O^+(p,2)$-invariant function on $\mathbf{X}$.

Elements in $S^{c\pm}(\mathbf{X})$ are determined by their restrictions on $\mathbf{X}^+$:

$$S^c(\mathbf{X}^+) = \left\{ F \in C^\infty(\mathbf{X}^+) \;\middle|\; \begin{array}{c} F\left(\begin{pmatrix} A_1 X \\ CX \end{pmatrix}\right) = (\det X)^c F\left(\begin{pmatrix} A_1 \\ C \end{pmatrix}\right) \\ \forall X \in GL^+(2,\mathbb{R}),\ \forall \begin{pmatrix} A_1 \\ C \end{pmatrix} \in \mathbf{X}^+ \end{array} \right\}.$$

$F \in S^c(\mathbf{X}^+)$ is extended to $\mathbf{X}^-$ by

$$F\left(\begin{pmatrix} A_1 \\ C \end{pmatrix} \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\right) = \pm F\left(\begin{pmatrix} A_1 \\ C \end{pmatrix}\right).$$

Thus, multiplication by $\operatorname{sign}_{\mathbf{X}}$ is an $O^+(p,2)$-module isomorphism between $S^{c+}(\mathbf{X})$ and $S^{c-}(\mathbf{X})$.

By $O(p,2) = O^+(p,2) \cup k_C O^+(p,2)$, we first consider the decomposition of $S^c(\mathbf{X}^+)$ as an $O^+(p,2)$-module, and then see the effects of $k_C$.

## 3. Stiefel harmonics theory

In Howe-Tan's case [9], the module is identified with a space of functions on $SO(p)/SO(p-1) \simeq S^{p-1}$, and the theory of spherical harmonics is used for further analysis. In our case we have instead $SO(p)/SO(p-2)$, a Stiefel manifold. But we have at hand the Stiefel version of harmonics theory by Gelbart [6] and Ton-That [25]. We quote the results on $SO(p,\mathbb{C})/SO(p-2,\mathbb{C})$.

**Theorem 3.1.** *We identify* $\mathbf{X}_p = SO(p,\mathbb{C})\begin{pmatrix} I_2 \\ 0_{p-2,2}\end{pmatrix}$ *with* $SO(p,\mathbb{C})/SO(p-2,\mathbb{C})$. *The space of functions on* $\mathbf{X}_p$ *is an* $SO(p,\mathbb{C})$*-module by left translation.*

*We denote by* $\mathcal{P}$ *the space of polynomials on* $\mathbb{C}^p \oplus \mathbb{C}^p$, *with coordinates* $x_i, y_i (1 \le i \le p)$. *Then,*
*(a) The space of* $SO(p,\mathbb{C})$*-finite vectors in* $C^\infty(\mathbf{X}_p)$ *is*

$$A(\mathbf{X}_p) = \{\textit{restriction to } \mathbf{X}_p \textit{ of a polynomial} \in \mathcal{P}\}.$$

*(b)* $A(\mathbf{X}_p) \cong \mathcal{H}$ *by restriction. Here,*

$$\mathcal{H} = \left\{h \in \mathcal{P} \;\middle|\; \Delta_{x^2} h = 0, \Delta_{y^2} h = 0, \Delta_{xy} h = 0\right\},$$

$$\Delta_{x^2} = \sum_{i=1}^p \frac{\partial^2}{\partial x_i^2}, \quad \Delta_{y^2} = \sum_{i=1}^p \frac{\partial^2}{\partial y_i^2}, \quad \Delta_{xy} = \sum_{i=1}^p \frac{\partial^2}{\partial x_i \partial y_i}.$$

*(c) From now on, assume* $p > 4$. *If we denote by* $\mathcal{J}$ *the space of* $SO(p,\mathbb{C})$ *invariant polynomials, then*

$$\mathcal{P} \cong \mathcal{J} \otimes \mathcal{H}.$$

*(d)* $\mathcal{J}$ *is generated by* $\xi_{x^2}, \xi_{y^2}, \xi_{xy}$. *Here,*

$$\xi_{x^2} = \sum_{i=1}^p x_i^2, \quad \xi_{y^2} = \sum_{i=1}^p y_i^2, \quad \xi_{xy} = \sum_{i=1}^p x_i y_i.$$

*(e)* $\mathcal{H}$ *has also a* $GL(2,\mathbb{C})$*-action by right translation. Then, we have the decomposition into irreducibles as* $SO(p,\mathbb{C}) \times GL(2,\mathbb{C})$*-modules (multiplicity free):*

$$\mathcal{H} \cong \bigoplus_{\substack{l_1, l_2 \in \mathbb{Z} \\ l_1 \ge l_2 \ge 0}} \mathcal{H}_{(l_1,l_2)},$$

$$\mathcal{H}_{(l_1,l_2)} \cong \sigma^p_{(l_1,l_2,0,\dots,0)} \otimes \rho^2_{(l_1,l_2)},$$

*where* $\sigma^p_{(l_1,l_2,0,\dots,0)}$ *is the irreducible representation of* $SO(p,\mathbb{C})$ *with highest weight* $\overbrace{(l_1, l_2, 0, \dots, 0)}^{[p/2]}$ *and* $\rho^2_{(l_1,l_2)}$ *is the irreducible representation of* $GL(2,\mathbb{C})$ *with highest weight* $(l_1, l_2)$.

In fact, their results are stated more generally for $SO(p)/SO(p-n)$ $(n < [p/2])$. Part (a) is due to Helgason [7] and Part (e) to Kashiwara and Vergne [12]. For general $n$, see [5].

## 4. Decomposition into $K$-types

Let $K_0 = SO(p) \times SO(2)$, $K_1 = O(p) \times SO(2)$. We first decompose $S^c(\mathbf{X}^+)$ into the sum of irreducible $K_0$-modules.

**Lemma 4.1.** *We have an isomorphism of $K_0$-modules:*

$$S^c(\mathbf{X}^+) \cong C^\infty(SO(p)/SO(p-2)).$$

*The action of $K_0$ on the right hand side is given by*

$$((k_1, k_2)\, h)\,(x) = h(k_1^{-1} x k_2)$$

*for*

$$\begin{gathered}\forall (k_1, k_2) \in SO(p) \times SO(2), \forall h \in C^\infty(SO(p)/SO(p-2)),\\ \forall x \in SO(p)/SO(p-2) = SO(p) \left(\begin{smallmatrix} I_2 \\ 0 \end{smallmatrix}\right).\end{gathered}$$

*In particular, the Harish-Chandra module of $S^c(\mathbf{X}^+)$ is*

$$\begin{aligned} S^c(\mathbf{X}^+)_{K-finite} &\cong A(SO(p)/SO(p-2)) \\ &\cong \mathcal{H} \\ &= \bigoplus_{\substack{l_1, l_2 \in \mathbb{Z} \\ l_1 \geq l_2 \geq 0}} \mathcal{H}_{(l_1, l_2)}. \end{aligned}$$

*We denote this isomorphism by $j : \mathcal{H} \xrightarrow{\cong} S^c(\mathbf{X}^+)$.*

**Proof.** Since $G^+ = K_1 MAN$, the elements of $S^c(\mathbf{X}^+)$ are determined by their values on $K_1 v_0$. Thus,

$$\begin{aligned} S^c(\mathbf{X}^+) &= \left\{ F \in C^\infty(G^+ v_0) \;\middle|\; F(gpv_0) = \chi_c(p) F(gv_0) \quad (\forall g \in G^+, \forall p \in S \cap G^+) \right\} \\ &= \left\{ F \in C^\infty(K v_0) \;\middle|\; F(kpv_0) = F(kv_0) \quad (\forall k \in K_1, \forall p \in S \cap K_1) \right\} \\ &\cong \left\{ F \in C^\infty(O(p) \left(\begin{smallmatrix} I_2 \\ 0 \end{smallmatrix}\right) \times SO(2)) \;\middle|\; \begin{matrix} F((A \left(\begin{smallmatrix} X \\ 0 \end{smallmatrix}\right), CX)) = F((A \left(\begin{smallmatrix} I_2 \\ 0 \end{smallmatrix}\right), C)) \\ (\forall A \in O(p), \forall C \in SO(2), \forall X \in SO(2)) \end{matrix} \right\} \\ &= \left\{ F \in C^\infty(O(p) \left(\begin{smallmatrix} I_2 \\ 0 \end{smallmatrix}\right) \times SO(2)) \;\middle|\; \begin{matrix} F((A \left(\begin{smallmatrix} I_2 \\ 0 \end{smallmatrix}\right), C)) = F((AC^{-1}, I_2)) \\ (\forall A \in O(p), \forall C \in SO(2)) \end{matrix} \right\} \\ &\cong C^\infty(O(p) \left(\begin{smallmatrix} I_2 \\ 0 \end{smallmatrix}\right)). \end{aligned}$$

Here, we used the fact that $\chi_c$ is trivial on

$$S \cap K_1 = \left\{ \begin{pmatrix} X & & \\ & Y & \\ & & X \end{pmatrix} \;\middle|\; X \in SO(2), Y \in O(p-2) \right\}.$$

The rest is Thm.3.1. ∎

We want to decompose the irreducible $SO(p) \times GL(2, \mathbb{R})$-module $\mathcal{H}_{(l_1, l_2)}$ into the sum of irreducible $SO(p) \times SO(2)$-modules. For this purpose, we complexify the actions and apply the Caley transform.

**Lemma 4.2.** *Let* $J_C = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$. *The map* $\epsilon : \mathcal{P} \to \mathcal{P}$ *defined by*

$$(\epsilon(f))(x) = f(xJ_C^{-1})$$

*restricts to an isomorphism of* $SO(p, \mathbb{C}) \times GL(2, \mathbb{C})$*-modules:*

$$\mathcal{H}_{(l_1,l_2)} \underset{\cong}{\xrightarrow{\epsilon}} \mathcal{H}_{(l_1,l_2)}$$

*if we define the* $GL(2, \mathbb{C})$*-action on the image* $\epsilon(\mathcal{H})$ *by*

$$(gf)(x) = f(xJ_C^{-1}gJ_C)$$
$$(g \in SO(2, \mathbb{C}), f \in \epsilon(\mathcal{H}), x \in \mathbb{C}^p \oplus \mathbb{C}^p).$$

**Proof.** That $\epsilon$ preserves the space of harmonic functions follows from the chain rule in differentiation. Since left translations and right translations commute, the rest is obvious. ∎

The set of diagonal matrices $T \subseteq GL(2, \mathbb{C})$ is a Cartan subgroup. We decompose $\mathcal{H}_{(l_1,l_2)}$ into weight spaces:

$$\mathcal{H}_{(l_1,l_2)} = \bigoplus_{m=0}^{l_1-l_2} \mathcal{H}_{(l_1,l_2),m}.$$

$\mathcal{H}_{(l_1,l_2),m}$ has weight $(l_1 - m, l_2 + m)$ with respect to $T$.

Note that $\mathcal{H}_{(l_1,l_2),m}$ consists of harmonic polynomials which are

homogeneous of degree $l_1 - m$ with respect to $x_1, \ldots, x_p$,
homogeneous of degree $l_2 + m$ with respect to $y_1, \ldots, y_p$.

**Lemma 4.3.**

$$S^c(\mathbf{X}^+) \cong \epsilon(\mathcal{H}) = \bigoplus_{\substack{l_1,l_2 \in \mathbb{Z} \\ l_1 \geq l_2 \geq 0}} \bigoplus_{m=0}^{l_1-l_2} \mathcal{H}_{(l_1,l_2),m}$$

*is the irreducible decomposition of* $S^c(\mathbf{X}^+)$ *as an* $SO(p, \mathbb{C}) \times SO(2, \mathbb{C})$*-module. In particular, it is multiplicity free.*

**Proof.** Since $J_C^{-1}SO(2, \mathbb{C})J_C = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \,\middle|\, a \in \mathbb{C} \right\}$, the action of $SO(2, \mathbb{C})$ on $\epsilon(\mathcal{H})$ is reduced to the action of $T$, which is determined by weights. The lemma follows by Lemma4.1 and Lemma 4.2. ∎

**Lemma 4.4.**

$$
\begin{aligned}
S^c(\mathbf{X}^+) \cong & \epsilon(\mathcal{H}) \\
= & \bigoplus_{\substack{l_1,l_2\in\mathbb{Z}\\ l_1\geq l_2\geq 0\\ l_1-l_2\, odd}} \left( \bigoplus_{m=0}^{\frac{l_1-l_2-1}{2}} \left(\mathcal{H}_{(l_1,l_2),m} \oplus \mathcal{H}_{(l_1,l_2),l_1-l_2-m}\right)\right) \\
& \oplus \bigoplus_{\substack{l_1,l_2\in\mathbb{Z}\\ l_1\geq l_2\geq 0\\ l_1-l_2\, even}} \left( \bigoplus_{m=0}^{\frac{l_1-l_2}{2}-1} \left(\mathcal{H}_{(l_1,l_2),m} \oplus \mathcal{H}_{(l_1,l_2),l_1-l_2-m}\right) \oplus \mathcal{H}_{(l_1,l_2),\frac{l_1-l_2}{2}}\right)
\end{aligned}
$$

*is the irreducible decomposition of $S^c(\mathbf{X}^+)$ as an $O(p,\mathbb{C}) \times O(2,\mathbb{C})$-module. In particular, it is multiplicity free.*

**Proof.** This follows from the fact that the action of $k_C (\in G - G^+)$ is substitution of the coordinates between the $x_i$'s and $y_i$'s. ■

## 5. $\mathfrak{p}$-action

The next thing to do is to calculate the action of the non compact part.
We transfer harmonic polynomials to $S^c(\mathbf{X}^+)$ by the $K$-isomorphism $\Phi = j \circ \epsilon^{-1}$, apply the Lie algebra actions, and pull-back again to $\epsilon(\mathcal{H})$.

**Proposition 5.1.** *For $(s_1, s_2, s_3) \in \{\pm(1,0,0), \pm(1,0,1), \pm(0,1,-1), \pm(0,1,0)\}$, there exist linear maps*

$$T^{s_1,s_2,s_3}_{(l_1,l_2),m} : \mathfrak{p} \otimes \mathcal{H}_{(l_1,l_2),m} \to \mathcal{H}_{(l_1+s_1,l_2+s_2),m+s_3},$$

*satisfying the followings:*

- *They do not depend on $c$,*
- *They are not zero maps (when the target space is not zero), and*
- *For $\forall X \in \mathfrak{p}, \forall h \in \mathcal{H}_{(l_1,l_2),m}$,*

$$
\begin{aligned}
X(\Phi(h)) = & (c-l_1)\left(\Phi(T^{(1,0,0)}_{(l_1,l_2),m}(X\otimes h)) + \Phi(T^{(1,0,1)}_{(l_1,l_2),m}(X\otimes h))\right) \\
& + (c-l_2+1)\left(\Phi(T^{(0,1,-1)}_{(l_1,l_2),m}(X\otimes h)) + \Phi(T^{(0,1,0)}_{(l_1,l_2),m}(X\otimes h))\right) \\
& + (c+l_1+p-2)\left(\Phi(T^{(-1,0,-1)}_{(l_1,l_2),m}(X\otimes h)) + \Phi(T^{(-1,0,0)}_{(l_1,l_2),m}(X\otimes h))\right) \\
& + (c+l_2+p-3)\left(\Phi(T^{(0,-1,0)}_{(l_1,l_2),m}(X\otimes h)) + \Phi(T^{(0,-1,1)}_{(l_1,l_2),m}(X\otimes h))\right).
\end{aligned}
$$

We need several lemmas to prove this. Anyone who is willing to admit this proposition may safely skip the rest of this section and the next one. The proof itself is given in the end of the next section.

We fix some notations:

$$E_x^x = \sum_{i=1}^{p} x_i \frac{\partial}{\partial x_i}, \qquad E_y^y = \sum_{i=1}^{p} y_i \frac{\partial}{\partial y_i},$$
$$E_y^x = \sum_{i=1}^{p} x_i \frac{\partial}{\partial y_i}, \qquad E_x^y = \sum_{i=1}^{p} y_i \frac{\partial}{\partial x_i}.$$

These are infinitesimal translations corresponding to the $GL(2,\mathbb{C})$-action on $\mathcal{H}$. Under these notations,

$$\mathcal{H}_{(l_1,l_2),0} = \left\{ h_0 \in \mathcal{H} \;\middle|\; E_y^x h_0 = 0 \right\},$$
$$\mathcal{H}_{(l_1,l_2),m} = \left\{ (E_x^y)^m h_0 \;\middle|\; h_0 \in \mathcal{H}_{(l_1,l_2),0} \right\}.$$

For a basis of $\mathfrak{p} = \left\{ \begin{pmatrix} 0 & B \\ {}^tB & 0 \end{pmatrix} \right\}$, we may choose

$$\{X_{i,k} = -E_{i,k} - E_{k,i} | i = 1,2,\ k = 1,2,\dots,p\}$$

where $E_{i,k}$ are matrix units.

**Lemma 5.2.** *The explicit formula for $j$ is*

$$j(h)\left(\begin{pmatrix} A_1 \\ C \end{pmatrix}\right) = (\det C)^c h(A_1 C^{-1}) \tag{$*$}$$

*for $h \in \mathcal{H}$, $\begin{pmatrix} A_1 \\ C \end{pmatrix} \in O^+(p,q)\begin{pmatrix} I_2 \\ 0 \\ I_2 \end{pmatrix}$. We extend $j$ to a map:*

$$j : \mathcal{P} \to C^\infty\left(\left\{ \left(\begin{smallmatrix} A_1 \\ C \end{smallmatrix}\right) \in \mathbb{R}^{p+2} \oplus \mathbb{R}^{p+2} \;\middle|\; \det C \neq 0 \right\}\right)$$

*by the same formula $(*)$. Then, for $1 \le k \le p$, $h \in \mathcal{H}$,*

$$X_{1,k} j(h) = j\left( \frac{\partial h}{\partial x_k} + c x_k h - x_k E_x^x h - y_k E_y^x h \right),$$
$$X_{2,k} j(h) = j\left( \frac{\partial h}{\partial y_k} + c y_k h - y_k E_y^y h - x_k E_x^y h \right).$$

*(In general, the polynomials in the parentheses of $j$ in the right hand sides are no longer harmonic.)*

**Proof.** Let $\begin{pmatrix} x_1 & y_1 \\ \vdots & \vdots \\ x_p & y_p \\ z_1 & w_1 \\ z_2 & w_2 \end{pmatrix}$ be the coordinates for $\mathbb{R}^{p+2} \oplus \mathbb{R}^{p+2}$. Then, for a function $F$ on $\mathbb{R}^{p+2} \oplus \mathbb{R}^{p+2}$, the action of $X_{i,k}$ in terms of differential operators is

$$X_{i,k} F = z_i \frac{\partial F}{\partial x_k} + x_k \frac{\partial F}{\partial z_i} + w_i \frac{\partial F}{\partial y_k} + y_k \frac{\partial F}{\partial w_i}.$$

We want to compute this for $F = j(h)$. In terms of coordinates,

$$F\left(\begin{pmatrix} x_1 & y_1 \\ \vdots & \vdots \\ x_p & y_p \\ z_1 & w_1 \\ z_2 & w_2 \end{pmatrix}\right) = D^c h\left(\begin{pmatrix} \frac{x_1 w_2 - y_1 z_2}{D} & \frac{-x_1 w_1 + y_1 z_1}{D} \\ \vdots & \vdots \\ \frac{x_p w_2 - y_p z_2}{D} & \frac{-x_p w_1 + y_p z_1}{D} \end{pmatrix}\right),$$

where $D = \det\begin{pmatrix} z_1 & w_1 \\ z_2 & w_2 \end{pmatrix} = z_1 w_2 - z_2 w_1$.
Using the chain rule,

$$\begin{aligned}
\frac{\partial F}{\partial x_k}\left(\begin{pmatrix} A_1 \\ C \end{pmatrix}\right) =& D^c\left(\frac{w_2}{D}\frac{\partial h}{\partial x_k}(A_1 C^{-1}) - \frac{w_1}{D}\frac{\partial h}{\partial y_k}(A_1 C^{-1})\right), \\
\frac{\partial F}{\partial y_k}\left(\begin{pmatrix} A_1 \\ C \end{pmatrix}\right) =& D^c\left(-\frac{z_2}{D}\frac{\partial h}{\partial x_k}(A_1 C^{-1}) + \frac{z_1}{D}\frac{\partial h}{\partial y_k}(A_1 C^{-1})\right),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial F}{\partial z_1}\left(\begin{pmatrix} A_1 \\ C \end{pmatrix}\right) =& c w_2 D^{c-1} h(A_1 C^{-1}) \\
&+ D^c \sum_{i=1}^{p} \frac{-(x_i w_2 - y_i z_2) w_2}{D^2}\frac{\partial h}{\partial x_i}(A_1 C^{-1}) \\
&+ D^c \sum_{i=1}^{p} \frac{(x_i w_2 - y_i z_2) w_2}{D^2}\frac{\partial h}{\partial y_i}(A_1 C^{-1}), \\
\frac{\partial F}{\partial w_1}\left(\begin{pmatrix} A_1 \\ C \end{pmatrix}\right) =& c(-z_2) D^{c-1} h(A_1 C^{-1}) \\
&+ D^c \sum_{i=1}^{p} \frac{-(x_i w_2 - y_i z_2)(-z_2)}{D^2}\frac{\partial h}{\partial x_i}(A_1 C^{-1}) \\
&+ D^c \sum_{i=1}^{p} \frac{-(x_i w_2 - y_i z_2) z_1}{D^2}\frac{\partial h}{\partial y_i}(A_1 C^{-1}),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial F}{\partial z_2}\left(\begin{pmatrix} A_1 \\ C \end{pmatrix}\right) =& c(-w_1) D^{c-1} h(A_1 C^{-1}) \\
&+ D^c \sum_{i=1}^{p} \frac{(x_i w_1 - y_i z_1) w_2}{D^2}\frac{\partial h}{\partial x_i}(A_1 C^{-1}) \\
&+ D^c \sum_{i=1}^{p} \frac{-(-x_i w_1 + y_i z_1)(-w_1)}{D^2}\frac{\partial h}{\partial y_i}(A_1 C^{-1}), \\
\frac{\partial F}{\partial w_2}\left(\begin{pmatrix} A_1 \\ C \end{pmatrix}\right) =& c z_1 D^{c-1} h(A_1 C^{-1}) \\
&+ D^c \sum_{i=1}^{p} \frac{-(x_i w_1 - y_i z_1) z_2}{D^2}\frac{\partial h}{\partial x_i}(A_1 C^{-1}) \\
&+ \sum_{i=1}^{p} \frac{-(-x_i w_1 + y_i z_1) z_1}{D^2}\frac{\partial h}{\partial y_i}(A_1 C^{-1}).
\end{aligned}$$

Thus,

$$
\begin{aligned}
&(X_{1,k}F)\left(\begin{pmatrix} A_1 \\ C \end{pmatrix}\right) \\
=&\left(z_1\frac{\partial F}{\partial x_k}+x_k\frac{\partial F}{\partial z_1}+w_1\frac{\partial F}{\partial y_k}+y_k\frac{\partial F}{\partial w_1}\right)\left(\begin{pmatrix} A_1 \\ C \end{pmatrix}\right) \\
=&D^c\frac{z_1w_2-z_2w_1}{D}\frac{\partial h}{\partial x_k}(A_1C^{-1})+cD^{c-1}(x_kw_2-y_kz_2)h(A_1C^{-1}) \\
&+D^c\sum_{i=1}^{p}\frac{-(x_iw_2-y_iz_2)(x_kw_2-y_kz_2)}{D^2}\frac{\partial h}{\partial x_i}(A_1C^{-1}) \\
&+D^c\sum_{i=1}^{p}\frac{(x_iw_2-y_iz_2)(x_kw_1-y_kz_1)}{D^2}\frac{\partial h}{\partial y_i}(A_1C^{-1}) \\
=&D^c\Bigg(\frac{\partial h}{\partial x_k}(A_1C^{-1})+c\frac{x_kw_2-y_kz_2}{D}\left(h(A_1C^{-1})\right) \\
&\quad-\sum_{i=1}^{p}\frac{x_iw_2-y_iz_2}{D}\frac{x_kw_2-y_kz_2}{D}\left(\frac{\partial h}{\partial x_i}(A_1C^{-1})\right) \\
&\quad+\sum_{i=1}^{p}\frac{x_iw_2-y_iz_2}{D}\frac{x_kw_1-y_kz_1}{D}\left(\frac{\partial h}{\partial y_i}(A_1C^{-1})\right)\Bigg) \\
=&D^c\Bigg(\frac{\partial h}{\partial x_k}(A_1C^{-1})+c\,(x_kh)\,(A_1C^{-1}) \\
&\quad-\sum_{i=1}^{p}\left(x_ix_k\frac{\partial h}{\partial x_i}\right)(A_1C^{-1})+\sum_{i=1}^{p}\left(x_i(-y_k)\frac{\partial h}{\partial y_i}\right)(A_1C^{-1})\Bigg) \\
=&j\left(\frac{\partial h}{\partial x_k}+cx_kh-x_kE_x^xh-y_kE_y^xh\right).
\end{aligned}
$$

$X_{2,k}F$ is similar. ∎

**Lemma 5.3.** *We extend $\Phi=j\circ\epsilon^{-1}$ to a map from $\mathcal{P}$. For $h\in\mathcal{H}_{(l_1,l_2),m}\subseteq\epsilon(\mathcal{H})$, $1\le k\le p$,*

$$
\begin{aligned}
X_{1,k}\Phi(h)=\Phi\Big(\Big(&\frac{1}{\sqrt{2}}\frac{\partial}{\partial x_k}+\frac{i}{\sqrt{2}}\frac{\partial}{\partial y_k}+\frac{1}{\sqrt{2}}(c-l_1+m)x_k \\
&-\frac{i}{\sqrt{2}}(c-l_2-m)y_k+\frac{i}{\sqrt{2}}x_kE_x^y-\frac{1}{\sqrt{2}}y_kE_y^x\Big)h\Big), \\
X_{2,k}\Phi(h)=\Phi\Big(\Big(&\frac{i}{\sqrt{2}}\frac{\partial}{\partial x_k}+\frac{1}{\sqrt{2}}\frac{\partial}{\partial y_k}-\frac{i}{\sqrt{2}}(c-l_1+m)x_k \\
&+\frac{1}{\sqrt{2}}(c-l_2-m)y_k-\frac{1}{\sqrt{2}}x_kE_x^y+\frac{i}{\sqrt{2}}y_kE_y^x\Big)h\Big).
\end{aligned}
$$

*(In general, the polynomials in the parentheses of $\Phi$ in the right hand sides are no longer harmonic.)*

**Proof.** By the definition of $\epsilon$, for $\forall f \in \mathcal{P}$,

$$\frac{\partial}{\partial x_k}\epsilon^{-1} f = \epsilon^{-1}\left(\left(\frac{1}{\sqrt{2}}\frac{\partial}{\partial x_k} + \frac{i}{\sqrt{2}}\frac{\partial}{\partial y_k}\right) f\right),$$
$$\frac{\partial}{\partial y_k}\epsilon^{-1} f = \epsilon^{-1}\left(\left(\frac{i}{\sqrt{2}}\frac{\partial}{\partial x_k} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial y_k}\right) f\right),$$

$$x_k\epsilon^{-1} f = \epsilon^{-1}\left(\left(\frac{1}{\sqrt{2}}x_k - \frac{i}{\sqrt{2}}y_k\right) f\right),$$
$$y_k\epsilon^{-1} f = \epsilon^{-1}\left(\left(-\frac{i}{\sqrt{2}}x_k + \frac{1}{\sqrt{2}}y_k\right) f\right),$$

$$E_x^x\epsilon^{-1} f = \epsilon^{-1}\left(\left(\frac{1}{2}E_x^x + \frac{1}{2}E_y^y + \frac{i}{2}E_y^x - \frac{i}{2}E_x^y\right) f\right),$$
$$E_y^y\epsilon^{-1} f = \epsilon^{-1}\left(\left(\frac{1}{2}E_x^x + \frac{1}{2}E_y^y - \frac{i}{2}E_y^x + \frac{i}{2}E_x^y\right) f\right),$$

$$E_y^x\epsilon^{-1} f = \epsilon^{-1}\left(\left(\frac{i}{2}E_x^x - \frac{i}{2}E_y^y + \frac{1}{2}E_y^x + \frac{1}{2}E_x^y\right) f\right),$$
$$E_x^y\epsilon^{-1} f = \epsilon^{-1}\left(\left(-\frac{i}{2}E_x^x + \frac{i}{2}E_y^y + \frac{1}{2}E_y^x + \frac{1}{2}E_x^y\right) f\right).$$

Since $E_x^x$ and $E_y^y$ are Euler operators which tell us the degree of homogeneity,

$$E_x^x h = (l_1 - m)h, \quad E_y^y h = (l_2 + m)h$$
$$(h \in \mathcal{H}_{(l_1,l_2),m}.)$$

By Lemma 5.2,

$$\begin{aligned}
X_{1,k}\Phi(h) =& X_{1,k} j(\epsilon^{-1}(h)) \\
=& j\left(\left(\frac{\partial}{\partial x_k} + c x_k - x_k E_x^x - y_k E_y^x\right)(\epsilon^{-1}(h))\right) \\
=& j\circ\epsilon^{-1}\Bigg(\Bigg(\left(\frac{1}{\sqrt{2}}\frac{\partial}{\partial x_k} + \frac{i}{\sqrt{2}}\frac{\partial}{\partial y_k}\right) + c\left(\frac{1}{\sqrt{2}}x_k - \frac{i}{\sqrt{2}}y_k\right) \\
&- \left(\frac{1}{\sqrt{2}}x_k - \frac{i}{\sqrt{2}}y_k\right)\left(\frac{1}{2}(l_1 - m) + \frac{1}{2}(l_2 + m) + \frac{i}{2}E_y^x - \frac{i}{2}E_x^y\right) \\
&- \left(-\frac{i}{\sqrt{2}}x_k + \frac{1}{\sqrt{2}}y_k\right)\left(\frac{i}{2}(l_1 - m) - \frac{i}{2}(l_2 + m) + \frac{1}{2}E_y^x + \frac{1}{2}E_x^y\right)\Bigg) h\Bigg) \\
=& \Phi\Big(\Big(\frac{1}{\sqrt{2}}\frac{\partial}{\partial x_k} + \frac{i}{\sqrt{2}}\frac{\partial}{\partial y_k} + \frac{1}{\sqrt{2}}(c - l_1 + m)x_k \\
&- \frac{i}{\sqrt{2}}(c - l_2 - m)y_k + \frac{i}{\sqrt{2}}x_k E_x^y - \frac{1}{\sqrt{2}}y_k E_y^x\Big) h\Big).
\end{aligned}$$

$X_{2,k}\Phi(h)$ is similar. ∎

In order to pull-back the $\mathfrak{p}$-action by $\Phi$, we have to modify the polynomials occuring in this lemma to harmonic ones. As will be shown, it is possible to decompose them according to $\mathcal{P} \cong \mathcal{J} \otimes \mathcal{H}$ (c.f.Thm.3.1), and furthermore to determine to which $\mathcal{H}_{(l_1,l_2),m}$ they belong.
This is done in the next section.

## 6. Decomposition of polynomials

We fix $k \quad (1 \leq k \leq p)$ during this section.

**Lemma 6.1.** *Let $P \in \mathcal{P}$ be homogeneous of degree $m_1$ in $x_1, \ldots, x_p$, homogeneous of degree $m_2$ in $y_1, \ldots, y_p$.*
*Suppose $(E_y^x)^2 P = 0$. Then, in the decomposition*

$$P = \left(P - \frac{1}{m_1 - m_2 + 2} E_x^y E_y^x P\right) + \frac{1}{m_1 - m_2 + 2} E_x^y (E_y^x P),$$

*we have*

$$E_y^x \left(P - \frac{1}{m_1 - m_2 + 2} E_x^y E_y^x P\right) = 0.$$

**Proof.**

$$\begin{aligned} E_y^x E_x^y E_y^x P &= E_x^y E_y^x E_y^x P + (E_x^x - E_y^y) E_y^x P \\ &= E_x^y 0 + ((m_1 + 1) - (m_2 - 1)) E_y^x P = (m_1 - m_2 + 2) E_y^x P \end{aligned}$$

gives us

$$E_y^x \left(P - \frac{1}{m_1 - m_2 + 2} E_x^y E_y^x P\right) = 0. \qquad \blacksquare$$

**Lemma 6.2.** *For $h_0 \in \mathcal{H}_{(l_1,l_2),0}$,*

$$\frac{\partial h_0}{\partial y_k} \in \mathcal{H}_{(l_1,l_2-1),0},$$

$$\frac{\partial h_0}{\partial x_k} = \left(\frac{\partial h_0}{\partial x_k} + \frac{1}{l_1 - l_2 + 1} E_x^y \frac{\partial h_0}{\partial y_k}\right) - \frac{1}{l_1 - l_2 + 1} E_x^y \frac{\partial h_0}{\partial y_k},$$

*with*

$$\frac{1}{l_1 - l_2 + 1} E_x^y \frac{\partial h_0}{\partial y_k} \in \mathcal{H}_{(l_1-1,l_2),0}.$$

**Proof.** Recall that $h_0 \in \mathcal{H}_{(l_1,l_2),0}$ implies $E_y^x h_0 = 0$.

For $\frac{\partial h_0}{\partial y_k}$,

$$\Delta \frac{\partial h_0}{\partial y_k} = \frac{\partial}{\partial y_k} \Delta h_0 = 0 \quad (\text{for} \quad \Delta = \Delta_{x^2}, \Delta_{xy}, \Delta_{y^2}),$$

that is, $\frac{\partial h_0}{\partial y_k} \in \mathcal{H}$. It is homogeneous of degree $l_1$ w.r.t. $x_1, \ldots, x_p$ and of degree $l_2 - 1$ w.r.t. $y_1, \ldots, y_p$.

$$E_y^x \frac{\partial h_0}{\partial y_k} = \frac{\partial}{\partial y_k} (E_y^x h_0) = 0$$

shows that
$$\frac{\partial h_0}{\partial y_k} \in \mathcal{H}_{(l_1,l_2-1),0}.$$

For $\frac{\partial h_0}{\partial x_k}$,
$$\Delta \frac{\partial h_0}{\partial x_k} = \frac{\partial}{\partial x_k}\Delta h_0 = 0 \quad (\text{for} \quad \Delta = \Delta_{x^2}, \Delta_{xy}, \Delta_{y^2}),$$
that is, $\frac{\partial h_0}{\partial x_k} \in \mathcal{H}$. Since
$$(E^x_y)^2 \frac{\partial h_0}{\partial x_k} = E^x_y \frac{\partial}{\partial x_k}(E^x_y h_0) - E^x_y \frac{\partial h_0}{\partial y_k} = 0,$$
we can apply Lemma 6.1 with $P = \frac{\partial h_0}{\partial x_k}$, $m_1 = l_1 - 1$, $m_2 = l_2$. ∎

**Lemma 6.3.** *For $h_0 \in \mathcal{H}_{(l_1,l_2),0}$, define*
$$\begin{aligned}(x_k h_0)\tilde{} =& x_k h_0 - \xi_{xy}\left(\frac{1}{l_1+l_2+p-3}\frac{\partial h_0}{\partial y_k}\right)\\ &- \xi_{x^2}\left(\frac{1}{2l_1+p-2}\left(\frac{\partial h_0}{\partial x_k} + \frac{1}{l_1-l_2+1}E^y_x\frac{\partial h_0}{\partial y_k}\right)\right.\\ &\left. - \frac{1}{(l_1-l_2+1)(l_1+l_2+p-3)}E^y_x\frac{\partial h_0}{\partial y_k}\right),\end{aligned}$$

$$\begin{aligned}(y_k h_0)\tilde{} =& y_k h_0 - \xi_{xy}\left(\frac{2l_1+p-4}{(l_1+l_2+p-3)(2l_1+p-2)}\frac{\partial h_0}{\partial x_k}\right.\\ &\left. + \frac{-2(2l_1+p-4)}{(l_1+l_2+p-3)(2l_1+p-2)(2l_2+p-4)}E^y_x\frac{\partial h_0}{\partial y_k}\right)\\ &- \xi_{x^2}\left(\frac{-1}{(l_1+l_2+p-3)(2l_1+p-2)}E^y_x\frac{\partial h_0}{\partial x_k}\right.\\ &\left. + \frac{2}{(l_1+l_2+p-3)(2l_1+p-2)(2l_2+p-4)}(E^y_x)^2\frac{\partial h_0}{\partial y_k}\right)\\ &- \xi_{y^2}\frac{l_1+l_2+p-4}{(l_1+l_2+p-3)(2l_2+p-4)}\frac{\partial h_0}{\partial y_k}.\end{aligned}$$

*Then,*
$$(x_k h_0)\tilde{} \in \mathcal{H}, \quad (y_k h_0)\tilde{} \in \mathcal{H}.$$
*A fortiori, $E^x_y (y_k h_0)\tilde{} = (x_k h_0)\tilde{}$.*

**Proof.** We first deal with
$$\begin{aligned}(x_k h_0)\tilde{} =& x_k h_0 - \frac{1}{l_1+l_2+p-3}\xi_{xy}\frac{\partial h_0}{\partial y_k} - \frac{1}{2l_1+p-2}\xi_{x^2}\frac{\partial h_0}{\partial x_k}\\ &+ \frac{1}{(2l_1+p-2)(l_1+l_2+p-3)}E^y_x\frac{\partial h_0}{\partial y_k}.\end{aligned}$$

$$\Delta_{x^2} x_k h_0 = x_k(\Delta_{x^2} h_0) + 2\frac{\partial h_0}{\partial x_k} = 2\frac{\partial h_0}{\partial x_k},$$

$$\begin{aligned}\Delta_{x^2}\xi_{x^2}\frac{\partial h_0}{\partial x_k} &= \xi_{x^2}(\Delta_{x^2}\frac{\partial h_0}{\partial x_k}) + (2p+4E^x_x)\frac{\partial h_0}{\partial x_k} = (2p+4l_1-4)\frac{\partial h_0}{\partial x_k},\\ \Delta_{x^2}\xi_{x^2}E^y_x\frac{\partial h_0}{\partial y_k} &= \xi_{x^2}(\Delta_{x^2}E^y_x\frac{\partial h_0}{\partial y_k}) + (2p+4E^x_x)E^y_x\frac{\partial h_0}{\partial y_k}\\ &= (2p+4l_1-4)E^y_x\frac{\partial h_0}{\partial y_k},\end{aligned}$$

and

$$\Delta_{x^2}\xi_{xy}\frac{\partial h_0}{\partial y_k} = \xi_{xy}(\Delta_{x^2}\frac{\partial h_0}{\partial y_k}) + 2E^y_x\frac{\partial h_0}{\partial y_k} = 2E^y_x\frac{\partial h_0}{\partial y_k}$$

give

$$\begin{aligned}\Delta_{x^2}(x_kh_0)\tilde{} =&\Delta_{x^2}x_kh_0 - \frac{1}{l_1+l_2+p-3}\Delta_{x^2}\xi_{xy}\frac{\partial h_0}{\partial y_k} - \frac{1}{2l_1+p-2}\Delta_{x^2}\xi_{x^2}\frac{\partial h_0}{\partial x_k}\\ &+ \frac{1}{(2l_1+p-2)(l_1+l_2+p-3)}\Delta_{x^2}E^y_x\frac{\partial h_0}{\partial y_k}\\ =&2\frac{\partial h_0}{\partial x_k} - \frac{1}{l_1+l_2+p-3}2E^y_x\frac{\partial h_0}{\partial y_k} - \frac{1}{2l_1+p-2}(2p+4l_1-4)\frac{\partial h_0}{\partial x_k}\\ &+ \frac{1}{(2l_1+p-2)(l_1+l_2+p-3)}(2p+4l_1-4)E^y_x\frac{\partial h_0}{\partial y_k}\\ =&0.\end{aligned}$$

$$\begin{aligned}\Delta_{y^2}x_kh_0 &= x_k(\Delta_{y^2}h_0) = 0,\\ \Delta_{y^2}\xi_{x^2}\frac{\partial h_0}{\partial x_k} &= \xi_{x^2}(\Delta_{y^2}\frac{\partial h_0}{\partial x_k}) = 0,\\ \Delta_{y^2}\xi_{x^2}E^y_x\frac{\partial h_0}{\partial y_k} &= \xi_{x^2}(\Delta_{y^2}E^y_x\frac{\partial h_0}{\partial y_k}) = 0,\end{aligned}$$

and

$$\Delta_{y^2}\xi_{xy}\frac{\partial h_0}{\partial y_k} = \xi_{xy}(\Delta_{y^2}\frac{\partial h_0}{\partial y_k}) + 2E^x_y\frac{\partial h_0}{\partial y_k} = 2\frac{\partial}{\partial y_k}(E^x_yh_0) = 0$$

give

$$\Delta_{y^2}(x_kh_0)\tilde{} = 0.$$

$$\begin{aligned}\Delta_{xy}x_kh_0 &= x_k(\Delta_{xy}h_0) + \frac{\partial}{\partial y_k} = \frac{\partial h_0}{\partial y_k},\\ \Delta_{xy}\xi_{x^2}\frac{\partial h_0}{\partial x_k} &= \xi_{x^2}(\Delta_{xy}\frac{\partial h_0}{\partial x_k}) + 2E^x_y\frac{\partial h_0}{\partial x_k}\\ &= 2\left(\frac{\partial}{\partial x_k}(E^x_yh_0) - \frac{\partial}{\partial y_k}h_0\right) = -2\frac{\partial h_0}{\partial y_k},\\ \Delta_{xy}\xi_{x^2}E^y_x\frac{\partial h_0}{\partial y_k} &= \xi_{x^2}(\Delta_{xy}E^y_x\frac{\partial h_0}{\partial y_k}) + 2E^x_yE^y_x\frac{\partial h_0}{\partial y_k}\\ &= 2\left(E^y_x(E^x_y\frac{\partial h_0}{\partial y_k}) + (E^x_x - E^y_y)\frac{\partial h_0}{\partial y_k}\right) = 2(l_1-l_2+1)\frac{\partial h_0}{\partial y_k},\end{aligned}$$

and

$$\Delta_{xy}\xi_{xy}\frac{\partial h_0}{\partial y_k} = \xi_{xy}(\Delta_{xy}\frac{\partial h_0}{\partial y_k}) + (p + E^x_x + E^y_y)\frac{\partial h_0}{\partial y_k} = (p + l_1 + l_2 - 1)\frac{\partial h_0}{\partial y_k}$$

give

$$\begin{aligned}\Delta_{xy}(x_k h_0)\tilde{} =& \Delta_{xy} x_k h_0 - \frac{1}{l_1 + l_2 + p - 3}\Delta_{xy}\xi_{xy}\frac{\partial h_0}{\partial y_k} - \frac{1}{2l_1 + p - 2}\Delta_{xy}\xi_{x^2}\frac{\partial h_0}{\partial x_k}\\ &+ \frac{1}{(2l_1 + p - 2)(l_1 + l_2 + p - 3)}\Delta_{xy}E^y_x\frac{\partial h_0}{\partial y_k}\\ =& \frac{\partial h_0}{\partial y_k} - \frac{1}{l_1 + l_2 + p - 3}(p + l_1 + l_2 - 1)\frac{\partial h_0}{\partial y_k} - \frac{1}{2l_1 + p - 2}\left(-2\frac{\partial h_0}{\partial y_k}\right)\\ &+ \frac{1}{(2l_1 + p - 2)(l_1 + l_2 + p - 3)}2(l_1 - l_2 + 1)\frac{\partial h_0}{\partial y_k}\\ =& 0.\end{aligned}$$

Thus,

$$\Delta(x_k h_0)\tilde{} = 0 \,(\text{for} \Delta = \Delta_{x^2}, \Delta_{xy}, \Delta_{y^2}),$$

that is, $(x_k h_0)\tilde{} \in \mathcal{H}$.

We turn to

$$\begin{aligned}(y_k h_0)\tilde{} =& y_k h_0 - \frac{2l_1 + p - 4}{(l_1 + l_2 + p - 3)(2l_1 + p - 2)}\xi_{xy}\frac{\partial h_0}{\partial x_k}\\ &+ \frac{2(2l_1 + p - 4)}{(l_1 + l_2 + p - 3)(2l_1 + p - 2)(2l_2 + p - 4)}\xi_{xy}E^y_x\frac{\partial h_0}{\partial y_k}\\ &+ \frac{1}{(l_1 + l_2 + p - 3)(2l_1 + p - 2)}\xi_{x^2}E^y_x\frac{\partial h_0}{\partial x_k}\\ &- \frac{2}{(l_1 + l_2 + p - 3)(2l_1 + p - 2)(2l_2 + p - 4)}\xi_{x^2}(E^y_x)^2\frac{\partial h_0}{\partial y_k}\\ &- \frac{l_1 + l_2 + p - 4}{(l_1 + l_2 + p - 3)(2l_2 + p - 4)}\xi_{y^2}\frac{\partial h_0}{\partial y_k}.\end{aligned}$$

$$\begin{aligned}\Delta_{x^2} y_k h_0 &= 0,\\ \Delta_{x^2}\xi_{x^2}E^y_x\frac{\partial h_0}{\partial x_k} &= 2(2l_1 + p - 4)E^y_x\frac{\partial h_0}{\partial x_k},\\ \Delta_{x^2}\xi_{x^2}(E^y_x)^2\frac{\partial h_0}{\partial y_k} &= 2(2l_1 + p - 4)(E^y_x)^2\frac{\partial h_0}{\partial y_k},\\ \Delta_{x^2}\xi_{y^2}\frac{\partial h_0}{\partial y_k} &= 0,\\ \Delta_{x^2}\xi_{xy}\frac{\partial h_0}{\partial x_k} &= 2E^y_x\frac{\partial h_0}{\partial x_k},\end{aligned}$$

and

$$\Delta_{x^2}\xi_{xy}E^y_x\frac{\partial h_0}{\partial y_k} = 2(E^y_x)^2\frac{\partial h_0}{\partial y_k}$$

give $\Delta_{x^2}(y_k h_0)\tilde{} = 0$.

$$\Delta_{y^2} y_k h_0 = 2\frac{\partial h_0}{\partial y_k},$$

$$
\begin{aligned}
\Delta_{y^2}\xi_{x^2}E_x^y\frac{\partial h_0}{\partial x_k} &= 0,\\
\Delta_{y^2}\xi_{x^2}(E_x^y)^2\frac{\partial h_0}{\partial y_k} &= 0,\\
\Delta_{y^2}\xi_{y^2}\frac{\partial h_0}{\partial y_k} &= 2(2l_2+p-2)\frac{\partial h_0}{\partial y_k},\\
\Delta_{y^2}\xi_{xy}\frac{\partial h_0}{\partial x_k} &= -2\frac{\partial h_0}{\partial y_k},
\end{aligned}
$$

and

$$
\Delta_{y^2}\xi_{xy}E_x^y\frac{\partial h_0}{\partial y_k} = 2(l_1-l_2+1)\frac{\partial h_0}{\partial y_k}
$$

give $\Delta_{y^2}(y_kh_0)\tilde{} = 0$.

$$
\begin{aligned}
\Delta_{xy}y_kh_0 &= \frac{\partial h_0}{\partial x_k},\\
\Delta_{xy}\xi_{x^2}E_x^y\frac{\partial h_0}{\partial x_k} &= -2E_x^y\frac{\partial h_0}{\partial y_k}+2(l_1-l_2-1)\frac{\partial h_0}{\partial x_k},\\
\Delta_{xy}\xi_{x^2}(E_x^y)^2\frac{\partial h_0}{\partial y_k} &= 0,\\
\Delta_{xy}\xi_{y^2}\frac{\partial h_0}{\partial y_k} &= 4(l_1-l_2)E_x^y\frac{\partial h_0}{\partial y_k},\\
\Delta_{xy}\xi_{xy}\frac{\partial h_0}{\partial x_k} &= 2E_x^y\frac{\partial h_0}{\partial y_k},
\end{aligned}
$$

and

$$
\Delta_{xy}\xi_{xy}E_x^y\frac{\partial h_0}{\partial y_k} = (l_1+l_2+p-1)E_x^y\frac{\partial h_0}{\partial y_k}
$$

give $\Delta_{xy}(y_kh_0)\tilde{} = 0$.
Thus, $(y_kh_0)\tilde{} \in \mathcal{H}$.

$$
\begin{aligned}
E_y^xy_kh_0 &= x_kh_0,\\
E_y^x\xi_{x^2}E_x^y\frac{\partial h_0}{\partial x_k} &= -\xi_{x^2}E_x^y\frac{\partial h_0}{\partial y_k}+(l_1-l_2-1)\xi_{x^2}\frac{\partial h_0}{\partial x_k},\\
E_y^x\xi_{x^2}(E_x^y)^2\frac{\partial h_0}{\partial y_k} &= 2(l_1-l_2)\xi_{x^2}E_x^y\frac{\partial h_0}{\partial y_k},\\
E_y^x\xi_{y^2}\frac{\partial h_0}{\partial y_k} &= 2\xi_{xy}\frac{\partial h_0}{\partial y_k},\\
E_y^x\xi_{xy}\frac{\partial h_0}{\partial x_k} &= -\xi_{xy}\frac{\partial h_0}{\partial y_k}+\xi_{x^2}\frac{\partial h_0}{\partial x_k},
\end{aligned}
$$

and

$$
E_y^x\xi_{xy}E_x^y\frac{\partial h_0}{\partial y_k} = (l_1-l_2+1)\xi_{xy}\frac{\partial h_0}{\partial y_k}+\xi_{x^2}E_x^y\frac{\partial h_0}{\partial y_k}
$$

give

$$
\begin{aligned}
&E_y^x(y_kh_0)^{\sim}\\
=&x_kh_0\\
&-\frac{2l_1+p-4}{(l_1+l_2+p-3)(2l_1+p-2)}\left(-\xi_{xy}\frac{\partial h_0}{\partial y_k}+\xi_{x^2}\frac{\partial h_0}{\partial x_k}\right)\\
&+\frac{2(2l_1+p-4)}{(l_1+l_2+p-3)(2l_1+p-2)(2l_2+p-4)}\left((l_1-l_2+1)\xi_{xy}\frac{\partial h_0}{\partial y_k}+\xi_{x^2}E_x^y\frac{\partial h_0}{\partial y_k}\right)\\
&+\frac{1}{(l_1+l_2+p-3)(2l_1+p-2)}\left(-\xi_{x^2}E_x^y\frac{\partial h_0}{\partial y_k}+(l_1-l_2-1)\xi_{x^2}\frac{\partial h_0}{\partial y_k}\right)\\
&-\frac{2}{(l_1+l_2+p-3)(2l_1+p-2)(2l_2+p-4)}2(l_1-l_2)\xi_{x^2}E_x^y\frac{\partial h_0}{\partial y_k}\\
&-\frac{l_1+l_2+p-4}{(l_1+l_2+p-3)(2l_2+p-4)}2\xi_{xy}\frac{\partial h_0}{\partial y_k}\\
=&x_kh_0-\frac{1}{l_1+l_2+p-3}\xi_{xy}\frac{\partial h_0}{\partial y_k}-\frac{1}{2l_1+p-2}\xi_{x^2}\frac{\partial h_0}{\partial x_k}\\
&+\frac{1}{(2l_1+p-2)(l_1+l_2+p-3)}E_x^y\frac{\partial h_0}{\partial y_k}\\
=&(x_kh_0)^{\sim}.
\end{aligned}
$$

Thus, $E_y^x(y_kh_0)^{\sim}=(x_kh_0)^{\sim}$. ■

**Lemma 6.4.** *For* $h_0\in\mathcal{H}_{(l_1,l_2),0}$, *let*

$$
\begin{aligned}
h_{0,k}^{\partial x}&=\frac{1}{(l_1+l_2+p-3)(2l_1+p-2)}\left(\frac{\partial h_0}{\partial x_k}+\frac{1}{l_1-l_2+1}E_x^y\frac{\partial h_0}{\partial y_k}\right),\\
h_{0,k}^{\partial y}&=\frac{1}{(l_1+l_2+p-3)(2l_2+p-4)(l_1-l_2+1)}\frac{\partial h_0}{\partial y_k},\\
h_{0,k}^{x}&=\frac{1}{l_1-l_2+1}(x_kh_0)^{\sim},\\
h_{0,k}^{y}&=(y_kh_0)^{\sim}-\frac{1}{l_1-l_2+1}E_x^y(x_kh_0)^{\sim}.
\end{aligned}
$$

*Then,*

$$
\begin{aligned}
&h_{0,k}^{\partial x}\in\mathcal{H}_{(l_1-1,l_2),0},\quad h_{0,k}^{\partial y}\in\mathcal{H}_{(l_1,l_2-1),0},\\
&h_{0,k}^{x}\in\mathcal{H}_{(l_1+1,l_2),0},\quad h_{0,k}^{y}\in\mathcal{H}_{(l_1,l_2+1),0},
\end{aligned}
$$

*and*

$$
\begin{aligned}
\frac{\partial h_0}{\partial x_k} &= (l_1+l_2+p-3)(2l_1+p-2)h^{\partial x}_{0,k} \\
&\quad - (l_1+l_2+p-3)(2l_2+p-4)E^y_x h^{\partial y}_{0,k}, \\
\frac{\partial h_0}{\partial y_k} &= (l_1+l_2+p-3)(2l_2+p-4)(l_1-l_2+1)h^{\partial y}_{0,k}, \\
x_k h_0 =& (l_1-l_2+1)h^x_{0,k} + \xi_{xy}\left((2l_2+p-4)(l_1-l_2+1)h^{\partial y}_{0,k}\right) \\
&+ \xi_{x^2}\left((l_1+l_2+p-3)h^{\partial x}_{0,k} - (2l_2+p-4)E^y_x h^{\partial y}_{0,k}\right), \\
y_k h_0 =& h^y_{0,k} + E^y_x h^x_{0,k} + \xi_{xy}\left((2l_1+p-4)h^{\partial x}_{0,k} - (2l_1+p-4)E^y_x h^{\partial y}_{0,k}\right) \\
&+ \xi_{x^2}\left(-E^y_x h^{\partial x}_{0,k} + (E^y_x)^2 h^{\partial y}_{0,k}\right) + \xi_{y^2}(l_1-l_2+1)(l_1+l_2+p-4)h^{\partial y}_{0,k}.
\end{aligned}
$$

**Proof.** The latter part is a reformulation of Lemma 6.2 and Lemma 6.3. $h^{\partial x}_{0,k} \in \mathcal{H}_{(l_1-1,l_2),0}$ and $h^{\partial y}_{0,k} \in \mathcal{H}_{(l_1,l_2-1),0}$ are also in Lemma 6.2. $h^x_{0,k} \in \mathcal{H}_{(l_1+1,l_2),0}$ follows from $E^x_y(x_k h_0)\tilde{} = 0$:

$$
\begin{aligned}
E^x_y x_k h_0 &= x_k(E^x_y h) = 0, \\
E^x_y \xi_{x^2}\frac{\partial h_0}{\partial x_k} &= \xi_{x^2}E^x_y\frac{\partial h_0}{\partial x_k} = \xi_{x^2}\frac{\partial}{\partial x_k}(E^x_y h_0) - \xi_{x^2}\frac{\partial}{\partial y_k}h = -\xi_{x^2}\frac{\partial h_0}{\partial y_k}, \\
E^x_y \xi_{x^2} E^y_x\frac{\partial h_0}{\partial y_k} &= \xi_{x^2}E^x_y E^y_x\frac{\partial h_0}{\partial y_k} = \xi_{x^2}E^y_x(E^x_y\frac{\partial h_0}{\partial y_k}) + \xi_{x^2}(E^x_x - E^y_y)\frac{\partial h_0}{\partial y_k} \\
&= (l_1-l_2+1)\xi_{x^2}\frac{\partial h_0}{\partial y_k},
\end{aligned}
$$

and

$$
E^x_y \xi_{xy}\frac{\partial h_0}{\partial y_k} = \xi_{xy}(E^x_y\frac{\partial h_0}{\partial y_k}) + \xi_{x^2}\frac{\partial h_0}{\partial y_k} = \xi_{x^2}\frac{\partial h_0}{\partial y_k}
$$

give

$$
\begin{aligned}
E^x_y h^x_{0,k} &= \frac{1}{l_1-l_2+1}\Bigg((E^x_y x_k h_0) - E^x_y\xi_{xy}\left(\frac{1}{l_1+l_2+p-3}\frac{\partial h_0}{\partial y_k}\right) \\
&\qquad - E^x_y\xi_{x^2}\Bigg(\frac{1}{2l_1+p-2}\left(\frac{\partial h_0}{\partial x_k} + \frac{1}{l_1-l_2+1}E^y_x\frac{\partial h_0}{\partial y_k}\right) \\
&\qquad\qquad - \frac{1}{(l_1-l_2+1)(l_1+l_2+p-3)}E^y_x\frac{\partial h_0}{\partial y_k}\Bigg)\Bigg) \\
&= \frac{1}{l_1-l_2+1}\Bigg(-\frac{1}{l_1+l_2+p-3}\xi_{x^2}\frac{\partial h_0}{\partial y_k} - \frac{1}{2l_1+p-2}\left(-\xi_{x^2}\frac{\partial h_0}{\partial y_k}\right) \\
&\qquad - \frac{1}{(2l_1+p-2)(l_1-l_2+1)}(l_1-l_2+1)\xi_{x^2}\frac{\partial h_0}{\partial y_k} \\
&\qquad + \frac{1}{(l_1-l_2+1)(l_1+l_2+p-3)}(l_1-l_2+1)\xi_{x^2}\frac{\partial h_0}{\partial y_k}\Bigg) \\
&= 0.
\end{aligned}
$$

Finally, $h^y_{0,k} \in \mathcal{H}_{(l_1,l_2+1),0}$ is obtained by Lemma 6.1 with $P = E^x_y(y_k h_0)\tilde{}$, $m_1 = l_1$ and $m_2 = l_2 + 1$, using $(E^x_y)^2(y_k h_0)\tilde{} = E^x_y(x_k h_0)\tilde{} = 0$. ■

**Lemma 6.5.**

$$\begin{aligned}
y_k(E^y_x)^m &= (E^y_x)^m y_k,\\
x_k(E^y_x)^m &= (E^y_x)^m x_k - m(E^y_x)^{m-1} y_k,\\
(E^y_x)^m \xi_{xy} &= \xi_{xy}(E^y_x)^m + m\xi_{y^2}(E^y_x)^{m-1},\\
(E^y_x)^m \xi_{y^2} &= \xi_{y^2}(E^y_x)^m,\\
(E^y_x)^m \xi_{x^2} &= \xi_{x^2}(E^y_x)^m + 2m\xi_{xy}(E^y_x)^{m-1} + m(m-1)\xi_{y^2}(E^y_x)^{m-2},\\
\frac{\partial}{\partial x_k}(E^y_x)^m &= (E^y_x)^m \frac{\partial}{\partial x_k},\\
\frac{\partial}{\partial y_k}(E^y_x)^m &= (E^y_x)^m \frac{\partial}{\partial y_k} + m\frac{\partial}{\partial x_k}(E^y_x)^{m-1},\\
E^x_y(E^y_x)^m &= (E^y_x)^m E^x_y + (E^y_x)^{m-1} m(E^x_x - E^y_y - m + 1).
\end{aligned}$$

**Proof.** The case $m = 1$ being easy, this is proved by induction on $m$. ■

**Proposition 6.6.** *For $h_0 \in \mathcal{H}_{(l_1,l_2),0}$, put $h = (E^y_x)^m h_0 \in \mathcal{H}_{(l_1,l_2),m}$. (Recall that any element of $\mathcal{H}_{(l_1,l_2),m}$ is of this form.) Then,*

$$\begin{aligned}
\frac{\partial h}{\partial x_k} =&(l_1 + l_2 + p - 3)(2l_1 + p - 2)(E^y_x)^m h^{\partial x}_{0,k}\\
&- (l_1 + l_2 + p - 3)(2l_2 + p - 4)(E^y_x)^{m+1} h^{\partial y}_{0,k},\\
\frac{\partial h}{\partial y_k} =&m(l_1 + l_2 + p - 3)(2l_1 + p - 2)(E^y_x)^{m-1} h^{\partial x}_{0,k}\\
&+ (l_1 + l_2 - m + 1)(l_1 + l_2 + p - 3)(2l_2 + p - 4)(E^y_x)^m h^{\partial y}_{0,k},\\
x_k h =&(l_1 - l_2 - m + 1)(E^y_x)^m h^x_{0,k} - m(E^y_x)^{m-1} h^y_{0,k}\\
&+\xi_{xy}\big(m(2l_2 + 2m + p - 4)(E^y_x)^{m-1} h^{\partial x}_{0,k}\\
&\quad + (l_1 - l_2 - m + 1)(2l_2 + 2m + p - 4)(E^y_x)^m h^{\partial y}_{0,k}\big)\\
&+\xi_{x^2}\left((l_1 + l_2 + m + p - 3)(E^y_x)^m h^{\partial x}_{0,k} - (2l_2 + m + p - 4)(E^y_x)^{m+1} h^{\partial y}_{0,k}\right),\\
y_k h =&(E^y_x)^{m+1} h^x_{0,k} + (E^y_x)^m h^y_{0,k}\\
&+\xi_{xy}\left((2l_1 - 2m + p - 4)(E^y_x)^m h^{\partial x}_{0,k} - (2l_1 - 2m + p - 4)(E^y_x)^{m+1} h^{\partial y}_{0,k}\right)\\
&+\xi_{x^2}\left(-(E^y_x)^{m+1} h^{\partial x}_{0,k} + (E^y_x)^{m+2} h^{\partial y}_{0,k}\right)\\
&+\xi_{y^2}\big(m(2l_1 - m + p - 3)(E^y_x)^{m-1} h^{\partial x}_{0,k}\\
&\quad + (l_1 - l_2 - m + 1)(l_1 + l_2 - m + p - 4)(E^y_x)^{m+1} h^{\partial y}_{0,k}\big).
\end{aligned}$$

**Proof.** This follows from Lemma 6.4 and Lemma 6.5. ■

Now the proof of Prop. 5.1.

**Proof.** For $P \in \mathcal{P}$,

$$j(\xi_{x^2}P) = j(\xi_{y^2}P) = j(P), \quad j(\xi_{xy}P) = 0$$

by the definition of $j$. Since

$$\begin{aligned}
\epsilon^{-1}\xi_{xy} &= \left(\frac{i}{2}\xi_{x^2} + \frac{i}{2}\xi_{y^2}\right)\epsilon^{-1}(P), \\
\epsilon^{-1}\xi_{x^2} &= \left(i\xi_{xy} + \frac{1}{2}\xi_{x^2} - \frac{1}{2}\xi_{y^2}\right)\epsilon^{-1}(P), \\
\epsilon^{-1}\xi_{y^2} &= \left(i\xi_{xy} - \frac{1}{2}\xi_{x^2} + \frac{1}{2}\xi_{y^2}\right)\epsilon^{-1}(P),
\end{aligned}$$

we have

$$\Phi(\xi_{x^2}P) = \Phi(\xi_{y^2}P) = 0, \quad \Phi(\xi_{xy}P) = \Phi(iP). \tag{$**$}$$

Recall Lemma 5.3 for $h = (E_x^y)^m h_0 \in \mathcal{H}_{(l_1,l_2),m}$:

$$\begin{aligned}
X_{1,k}\Phi(h) = \Phi\Bigg(\Big(&\frac{1}{\sqrt{2}}\frac{\partial}{\partial x_k} + \frac{i}{\sqrt{2}}\frac{\partial}{\partial y_k} + \frac{1}{\sqrt{2}}(c - l_1 + m)x_k \\
&- \frac{i}{\sqrt{2}}(c - l_2 - m)y_k + \frac{i}{\sqrt{2}}x_k E_x^y - \frac{1}{\sqrt{2}}y_k E_y^x\Big)h\Bigg).
\end{aligned}$$

By Prop. 6.6,

$$\begin{aligned}
X_{1,k}\Phi(h) = \Phi\Bigg(&\frac{1}{\sqrt{2}}\Big((l_1 + l_2 + p - 3)(2l_1 + p - 2)(E_x^y)^m h_{0,k}^{\partial x} \\
&\quad - (l_1 + l_2 + p - 3)(2l_2 + p - 4)(E_x^y)^{m+1} h_{0,k}^{\partial y}\Big) \\
&+ \frac{i}{\sqrt{2}}\Big(m(l_1 + l_2 + p - 3)(2l_1 + p - 2)(E_x^y)^{m-1} h_{0,k}^{\partial x} \\
&\quad + (l_1 + l_2 - m + 1)(l_1 + l_2 + p - 3)(2l_2 + p - 4)(E_x^y)^m h_{0,k}^{\partial y}\Big) \\
&+ \frac{1}{\sqrt{2}}(c - l_1 + m)\Big((l_1 - l_2 - m + 1)(E_x^y)^m h_{0,k}^{x} - m(E_x^y)^{m-1} h_{0,k}^{y} \\
&\quad + \xi_{xy}\big(m(2l_2 + 2m + p - 4)(E_x^y)^{m-1} h_{0,k}^{\partial x} \\
&\qquad + (l_1 - l_2 - m + 1)(2l_2 + 2m + p - 4)(E_x^y)^m h_{0,k}^{\partial y}\big) \\
&\quad + \xi_{x^2}(\cdots)\Big)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{i}{\sqrt{2}}(c - l_2 - m)\Big((E_x^y)^{m+1} h^x_{0,k} + (E_x^y)^m h^y_{0,k} \\
&\quad + \xi_{xy}\big((2l_1 - 2m + p - 4)(E_x^y)^m h^{\partial x}_{0,k} \\
&\qquad\quad - (2l_1 - 2m + p - 4)(E_x^y)^{m+1} h^{\partial y}_{0,k}\big) \\
&\quad + \xi_{x^2}(\cdots) + \xi_{y^2}(\cdots)\Big) \\
&+ \frac{i}{\sqrt{2}}\Big((l_1 - l_2 - m)(E_x^y)^{m+1} h^x_{0,k} - (m+1)(E_x^y)^m h^y_{0,k} \\
&\quad + \xi_{xy}\big((m+1)(2l_2 + 2m + p - 2)(E_x^y)^m h^{\partial x}_{0,k} \\
&\qquad\quad + (l_1 - l_2 - m)(2l_2 + 2m + p - 2)(E_x^y)^{m+1} h^{\partial y}_{0,k}\big) \\
&\quad + \xi_{x^2}(\cdots)\Big) \\
&- \frac{1}{\sqrt{2}} m(l_1 - l_2 - m + 1)\Big((E_x^y)^m h^x_{0,k} + (E_x^y)^{m-1} h^y_{0,k} \\
&\quad + \xi_{xy}\big((2l_1 - 2m + p - 2)(E_x^y)^{m-1} h^{\partial x}_{0,k} \\
&\qquad\quad - (2l_1 - 2m + p - 2)(E_x^y)^m h^{\partial y}_{0,k}\big) \\
&\quad + \xi_{x^2}(\cdots) + \xi_{y^2}(\cdots)\Big)\Bigg).
\end{aligned}
$$

Here we used

$$
\begin{aligned}
E_y^x (E_x^y)^m h_0 &= (E_x^y)^m E_y^x h_0 + (E_x^y)^{m-1} m (E_x^x - E_y^y - m + 1) h_0 \\
&= m(l_1 - l_2 - m + 1)(E_x^y)^{m-1} h_0.
\end{aligned}
$$

By $(**)$,

$$
\begin{aligned}
X_{1,k}\Phi(h) =\Phi\Bigg(&\frac{1}{\sqrt{2}}\Big((l_1 + l_2 + p - 3)(2l_1 + p - 2)(E_x^y)^m h^{\partial x}_{0,k} \\
&\quad - (l_1 + l_2 + p - 3)(2l_2 + p - 4)(E_x^y)^{m+1} h^{\partial y}_{0,k}\Big) \\
&+ \frac{i}{\sqrt{2}}\Big(m(l_1 + l_2 + p - 3)(2l_1 + p - 2)(E_x^y)^{m-1} h^{\partial x}_{0,k} \\
&\quad + (l_1 + l_2 - m + 1)(l_1 + l_2 + p - 3)(2l_2 + p - 4)(E_x^y)^m h^{\partial y}_{0,k}\Big) \\
&+ \frac{1}{\sqrt{2}}(c - l_1 + m)\Big((l_1 - l_2 - m + 1)(E_x^y)^m h^x_{0,k} - m(E_x^y)^{m-1} h^y_{0,k} \\
&\quad + i\big(m(2l_2 + 2m + p - 4)(E_x^y)^{m-1} h^{\partial x}_{0,k} \\
&\quad + (l_1 - l_2 - m + 1)(2l_2 + 2m + p - 4)(E_x^y)^m h^{\partial y}_{0,k}\big)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{i}{\sqrt{2}}(c-l_2-m)\Big((E_x^y)^{m+1}h_{0,k}^x+(E_x^y)^m h_{0,k}^y\\
&\quad+i\big((2l_1-2m+p-4)(E_x^y)^m h_{0,k}^{\partial x}\\
&\qquad-(2l_1-2m+p-4)(E_x^y)^{m+1}h_{0,k}^{\partial y}\big)\\
&\quad+(l_1-l_2-m+1)(l_1+l_2-m+p-4)(E_x^y)^{m+1}h_{0,k}^{\partial y})\Big)\\
&+\frac{i}{\sqrt{2}}\Big((l_1-l_2-m)(E_x^y)^{m+1}h_{0,k}^x-(m+1)(E_x^y)^m h_{0,k}^y\\
&\quad+i\big((m+1)(2l_2+2m+p-2)(E_x^y)^m h_{0,k}^{\partial x}\\
&\quad+(l_1-l_2-m)(2l_2+2m+p-2)(E_x^y)^{m+1}h_{0,k}^{\partial y}\big)\Big)\\
&-\frac{1}{\sqrt{2}}m(l_1-l_2-m+1)\Big((E_x^y)^m h_{0,k}^x+(E_x^y)^{m-1}h_{0,k}^y\\
&\quad+i\big((2l_1-2m+p-2)(E_x^y)^{m-1}h_{0,k}^{\partial x}\\
&\qquad-(2l_1-2m+p-2)(E_x^y)^m h_{0,k}^{\partial y}\big)\Big)\Bigg)\\
=\Phi\Bigg(&\frac{1}{\sqrt{2}}(c-l_1)(l_1-l_2-m+1)(E_x^y)^m h_{0,k}^x\\
&-\frac{i}{\sqrt{2}}(c-l_1)(E_x^y)^{m+1}h_{0,k}^x\\
&-\frac{1}{\sqrt{2}}(c-l_2+1)m(E_x^y)^{m-1}h_{0,k}^y\\
&-\frac{i}{\sqrt{2}}(c-l_2+1)(E_x^y)^m h_{0,k}^y\\
&+\frac{i}{\sqrt{2}}(c+l_1+p-2)m(2m+2l_2+p-4)(E_x^y)^{m-1}h_{0,k}^{\partial x}\\
&+\frac{1}{\sqrt{2}}(c+l_1+p-2)(2l_1-2m+p-4)(E_x^y)^m h_{0,k}^{\partial x}\\
&+\frac{i}{\sqrt{2}}(c+l_2+p-3)(l_1-l_2-m+1)(2m+2l_2+p-4)(E_x^y)^m h_{0,k}^{\partial y}\\
&-\frac{1}{\sqrt{2}}(c+l_2+p-3)(2l_1-2m+p-4)(E_x^y)^{m+1}h_{0,k}^{\partial y}\Bigg).
\end{aligned}
$$

Define

$$
\begin{aligned}
T^{(1,0,0)}_{(l_1,l_2),m}(X_{1,k}\otimes h) &= \frac{1}{\sqrt{2}}(l_1-l_2-m+1)(E^y_x)^m h^x_{0,k},\\
T^{(1,0,1)}_{(l_1,l_2),m}(X_{1,k}\otimes h) &= -\frac{i}{\sqrt{2}}(E^y_x)^{m+1} h^x_{0,k},\\
T^{(0,1,-1)}_{(l_1,l_2),m}(X_{1,k}\otimes h) &= -\frac{1}{\sqrt{2}}m(E^y_x)^{m-1} h^y_{0,k},\\
T^{(0,1,0)}_{(l_1,l_2),m}(X_{1,k}\otimes h) &= -\frac{i}{\sqrt{2}}(E^y_x)^{m} h^y_{0,k},\\
T^{(-1,0,-1)}_{(l_1,l_2),m}(X_{1,k}\otimes h) &= \frac{i}{\sqrt{2}}m(2m+2l_2+p-4)(E^y_x)^{m-1} h^{\partial x}_{0,k},\\
T^{(-1,0,0)}_{(l_1,l_2),m}(X_{1,k}\otimes h) &= \frac{1}{\sqrt{2}}(2l_1-2m+p-4)(E^y_x)^{m} h^{\partial x}_{0,k},\\
T^{(0,-1,0)}_{(l_1,l_2),m}(X_{1,k}\otimes h) &= \frac{i}{\sqrt{2}}(l_1-l_2-m+1)(2m+2l_2+p-4)(E^y_x)^{m} h^{\partial y}_{0,k},\\
T^{(0,-1,1)}_{(l_1,l_2),m}(X_{1,k}\otimes h) &= -\frac{1}{\sqrt{2}}(2l_1-2m+p-4)(E^y_x)^{m+1} h^{\partial y}_{0,k}.
\end{aligned}
$$

$X_{2,k}$ is similar. ■

## 7. Reducibility

Since the decomposition of $S^c(\mathbf{X}^+)$ into irreducible $K$-modules is multiplicity free, Prop. 5.1 is a criterion to see whether or not we can pass from a $K_0$-type (which is of the form $\mathcal{H}_{(l_1,l_2),m}$) to another. That is, starting from a $K_0$-type, we can go to an adjacent $K_0$-type with an action of $\mathfrak{p}$ if and only if the coefficients in Prop. 5.1 for that direction is not zero.

The first observation is:

**Lemma 7.1.** *If an irreducible constituent of $S^c(\mathbf{X}^+)$ contains $\mathcal{H}_{(l_1,l_2),m_0}$ for some $l_1$, $l_2$ and $m_0$, then it contains all of $\mathcal{H}_{(l_1,l_2)} = \bigoplus_{m=0}^{l_1-l_2}\mathcal{H}_{(l_1,l_2),m}$.*

**Proof.** If $l_1=0$, then $l_2=m_0=0$, $\mathcal{H}_{(0,0),0}=\mathcal{H}_{(0,0)}$ and we have nothing to prove.

We may assume $l_1\geq 1$. To go from $\mathcal{H}_{(l_1,l_2),m_0}$ to $\mathcal{H}_{(l_1,l_2),m_0+1}$, consider two different paths:

$$
\begin{aligned}
\mathcal{H}_{(l_1,l_2),m_0} &\to \mathcal{H}_{(l_1+1,l_2),m_0+1} \to \mathcal{H}_{(l_1,l_2),m_0+1},\\
\mathcal{H}_{(l_1,l_2),m_0} &\to \mathcal{H}_{(l_1-1,l_2),m_0} \to \mathcal{H}_{(l_1,l_2),m_0+1}.
\end{aligned}
$$

The first path is available iff the coefficients of $T^{(1,0,1)}_{(l_1,l_2),m}$ and of $T^{(-1,0,0)}_{(l_1+1,l_2),m+1}$ are both non-zero, namely,

$$c-l_1\neq 0,\quad c+l_1+p-1\neq 0.$$

The second path is available iff

$$c+l_1+p-2\neq 0,\quad c-l_1+1\neq 0.$$

Since these do not occur at the same time, we conclude that we can always go from $\mathcal{H}_{(l_1,l_2),m_0}$ to $\mathcal{H}_{(l_1,l_2),m_0+1}$. Conversely, we can also prove that we can always go from $\mathcal{H}_{(l_1,l_2),m_0}$ to $\mathcal{H}_{(l_1,l_2),m_0-1}$, proving the assertion of the lemma. ∎

This lemma says that each $\mathcal{H}_{(l_1,l_2)}$ is a sort of "building block" for irreducible constituents, and in consequence that we can forget the parameter $m$. The next thing to do is to determine which of these blocks are tied together.

The formula in Prop. 5.1 is rewritten as:

$$\begin{aligned} X(\Phi(h)) =& A^{+0}(l_1,l_2)\Phi(h_X^{+0}) + A^{-0}(l_1,l_2)\Phi(h_X^{-0}) \\ &+ A^{0+}(l_1,l_2)\Phi(h_X^{0+}) + A^{0-}(l_1,l_2)\Phi(h_X^{0-}) \end{aligned}$$

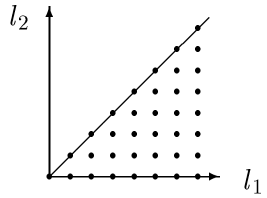
with

$$\begin{aligned} A^{+0}(l_1,l_2) &= c - l_1, & A^{-0}(l_1,l_2) &= c + l_1 + p - 2, \\ A^{0+}(l_1,l_2) &= c - l_2 + 1, & A^{0-}(l_1,l_2) &= c + l_2 + p - 3, \end{aligned}$$

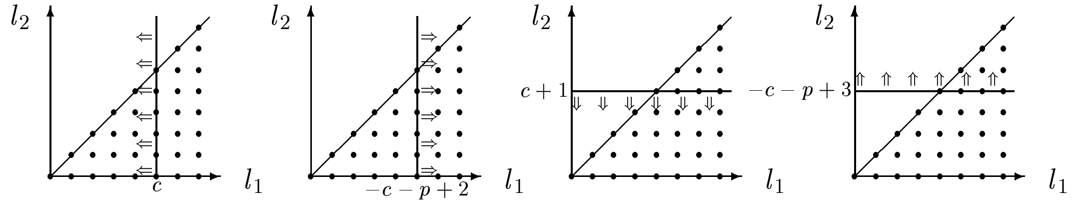
$$\begin{aligned} h_X^{+0} &\in \mathcal{H}_{(l_1+1,l_2)}, & h_X^{-0} &\in \mathcal{H}_{(l_1-1,l_2)}, \\ h_X^{0+} &\in \mathcal{H}_{(l_1,l_2+1)}, & h_X^{0-} &\in \mathcal{H}_{(l_1,l_2-1)}. \end{aligned}$$

To understand better what is going on, we translate the ingredients into diagrams.

The block $\mathcal{H}_{(l_1,l_2)}$ is identified with the point $(l_1,l_2) \in \mathbb{R}^2$. The blocks occuring in $\mathcal{H}$ are $\{(l_1,l_2) \in \mathbb{Z}^2 | l_1 \geq l_2 \geq 0\}$, with multiplicity one.




The four lines $A^{\pm 0}(l_1,l_2) = 0$ and $A^{0\pm}(l_1,l_2) = 0$ are "potential barriers" which are obstructions to the transitions.




$A^{+0} = 0$ $A^{-0} = 0$ $A^{0+} = 0$ $A^{0-} = 0$

In general, starting from a point $(l_1,l_2)$ corresponding to a block, we can go to four directions ($(l_1 \pm 1, l_2), (l_1, l_2 \pm 1)$). If a barrier meets a block, the transition crossing the barrier is not permitted, and this yields a submodule and a quotient module.

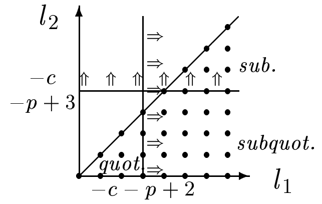
All that remains is to determine the positions of the barriers according to the value of $c$.

**Theorem 7.2.** *The decomposition of the Harish-Chandra module of $S^c(\mathbf{X}^+)$ is*

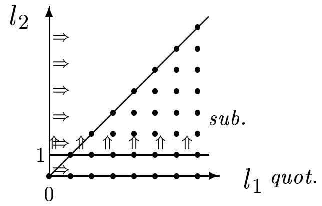
(0) $c \notin \mathbb{Z}$ : *The barriers never cross the $K$-types, so it is irreducible.*

*From now on, assume $c \in \mathbb{Z}$.*

(1) $c \leq -p+1$ : *1 finite-dimensional quotient, 1 subquotient, 1 submodule.*

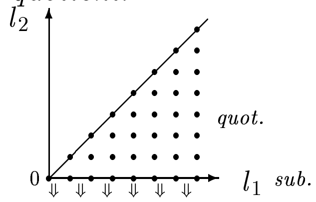



(2) $c = -p+2$ : *1 quotient, 1 submodule.*

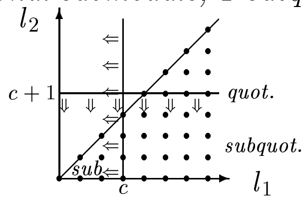



(3) $-p+3 \leq c \leq -2$ : *irreducible.*

(4) $c = -1$ : *1 submodule, 1 quotient.*




(5) $c \geq 0$ : *1 finite-dimensional submodule, 1 subquotient, 1 quotient.*




## 8. Unitarizability

We determine which of the irreducible constituents are unitarizable, that is, admit a $G$-invariant positive definite Hermitian form.

We define an inner product $\langle \cdot, \cdot \rangle_{(l_1,l_2),m}$ on $\mathcal{H}_{(l_1,l_2),m}$ by

$$\langle h_1, h_2 \rangle_{(l_1,l_2),m} = \int_{O(p)\times SO(2)} \Phi(h_1)(k)\overline{\Phi(h_2)(k)}dk$$
$$(h_1, h_2 \in \mathcal{H}_{(l_1,l_2),m})$$

where $dk$ is the normalized Haar measure on $K_1$. Then, this is a $K_1$-invariant hermitian inner product, called "standard inner product".

**Lemma 8.1.** *On each $K_1$-type $\mathcal{H}_{(l_1,l_2),m}$, $K_1$-invariant hermitian inner products are unique up to scalar multiple.*

**Proof.** Suppose we have $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle'$.
The map $A$ defined by $\langle h_1, h_2 \rangle = \langle A(h_1), h_2 \rangle'$ intertwines the $K_1$-action, and hence is a scalar operator by Schur's Lemma. ■

If an irreducible constituent of $S^c(\mathbf{X}^+)$ admits a $K_1$-invariant hermitian inner product $\langle\cdot,\cdot\rangle$, its restriction to $\Phi(\mathcal{H}_{(l_1,l_2),m})$ must be a constant multiple of the standard one:

$$\langle\Phi(h_1),\Phi(h_2)\rangle = c_{(l_1,l_2),m}\langle h_1,h_2\rangle_{(l_1,l_2),m} \quad h_1,h_2\in\mathcal{H}_{(l_1,l_2),m}.$$

Then, $c_{(l_1,l_2),m}\in\mathbb{R}$. Furthermore, if $\langle\cdot,\cdot\rangle$ is positive definite, $c_{(l_1,l_2),m}>0$.

For example, since we shifted the parameter of the parabolic induction to preserve unitarity, the whole $S^c(\mathbf{X}^+)$ is unitarizable for $c'\in i\mathbb{R}$. We call this the "unitary axis". It corresponds to $c_{(l_1,l_2),m}=1$ for all the $K_1$-types.

Conversely, since the $K_1$-types are mutually orthogonal (this follows from the fact that the decomposition into $K_1$-types is multiplicity free), the constants $\{c_{(l_1,l_2),m}\}$ completely determine the inner product.

The condition for the positive definite $K_1$-invariant inner product $\langle\cdot,\cdot\rangle$ to be $G^+$-invariant is rephrased as

$$\langle Xf_1,f_2\rangle+\langle f_1,Xf_2\rangle=0 \quad \forall X\in\mathfrak{p},\ \forall f_1,f_2\in S^c(\mathbf{X}^+). \tag{$\sharp$}$$

**Lemma 8.2.** *($\sharp$) is equivalent to:*

$$\begin{aligned}
(c-l_1)c_{(l_1-1,l_2),m}+(\overline{c}+l_1+p-1)c_{(l_1,l_2),m}&=0,\\
(c-l_1)c_{(l_1+1,l_2),m+1}+(\overline{c}+l_1+p-1)c_{(l_1,l_2),m}&=0,\\
(c-l_2+1)c_{(l_1,l_2-1),m-1}+(\overline{c}+l_2+p-2)c_{(l_1,l_2),m}&=0,\\
(c-l_2+1)c_{(l_1,l_2+1),m}+(\overline{c}+l_2+p-2)c_{(l_1,l_2),m}&=0,
\end{aligned}$$

*whenever $c_{(\cdot,\cdot),\cdot}$ is well-defined on the irreducible constituent of $S^c(\mathbf{X}^+)$.*

**Proof.** Writing $f_1=\Phi(h_1)$ $(h_1\in\mathcal{H}_{(l_{1,1},l_{1,2}),m_1})$ and $f_2=\Phi(h_2)$ $(h_2\in\mathcal{H}_{(l_{2,1},l_{2,2}),m_2})$, we use Prop. 5.1 for $X\Phi(h_1)$ and $X\Phi(h_2)$.

$$\begin{aligned}
&\langle X(\Phi(h_1)),\Phi(h_2)\rangle+\langle\Phi(h_1),X(\Phi(h_2))\rangle\\
=&\Big\langle (c-l_{1,1})\left(\Phi(T^{(1,0,0)}_{(l_{1,1},l_{1,2}),m_1}(X\otimes h_1))+\Phi(T^{(1,0,1)}_{(l_{1,1},l_{1,2}),m_1}(X\otimes h_1))\right)\\
&+(c-l_{1,2}+1)\left(\Phi(T^{(0,1,-1)}_{(l_{1,1},l_{1,2}),m_1}(X\otimes h_1))+\Phi(T^{(0,1,0)}_{(l_{1,1},l_{1,2}),m_1}(X\otimes h_1))\right)\\
&+(c+l_{1,1}+p-2)\left(\Phi(T^{(-1,0,-1)}_{(l_{1,1},l_{1,2}),m_1}(X\otimes h_1))+\Phi(T^{(-1,0,0)}_{(l_{1,1},l_{1,2}),m_1}(X\otimes h_1))\right)\\
&+(c+l_{1,2}+p-3)\left(\Phi(T^{(0,-1,0)}_{(l_{1,1},l_{1,2}),m_1}(X\otimes h_1))+\Phi(T^{(0,-1,1)}_{(l_{1,1},l_{1,2}),m_1}(X\otimes h_1))\right),\Phi(h_2)\Big\rangle\\
&+\Big\langle\Phi(h_1),(c-l_{2,1})\left(\Phi(T^{(1,0,0)}_{(l_{2,1},l_{2,2}),m_2}(X\otimes h_2))+\Phi(T^{(1,0,1)}_{(l_{2,1},l_{2,2}),m_2}(X\otimes h_2))\right)\\
&+(c-l_{2,2}+1)\left(\Phi(T^{(0,1,-1)}_{(l_{2,1},l_{2,2}),m_2}(X\otimes h_2))+\Phi(T^{(0,1,0)}_{(l_{2,1},l_{2,2}),m_2}(X\otimes h_2))\right)
\end{aligned}$$

$$+(c+l_{2,1}+p-2)\left(\Phi(T^{(-1,0,-1)}_{(l_{2,1},l_{2,2}),m_2}(X\otimes h_2))+\Phi(T^{(-1,0,0)}_{(l_{2,1},l_{2,2}),m_2}(X\otimes h_2))\right)$$
$$+(c+l_{2,2}+p-3)\left(\Phi(T^{(0,-1,0)}_{(l_{2,1},l_{2,2}),m_2}(X\otimes h_2))+\Phi(T^{(0,-1,1)}_{(l_{2,1},l_{2,2}),m_2}(X\otimes h_2))\right)\Big\rangle.$$

By the orthogonality property of the $K_1$-types,

$$\begin{aligned}
&\langle X(\Phi(h_1)),\Phi(h_2)\rangle+\langle\Phi(h_1),X(\Phi(h_2))\rangle\\
=&\delta_{l_{1,1}+1,l_{2,1}}\delta_{l_{1,2},l_{2,2}}\delta_{m_1,m_2}(c-l_{1,1})\Big\langle\Phi(T^{(1,0,0)}_{(l_{1,1},l_{1,2}),m_1}(X\otimes h_1)),\Phi(h_2)\Big\rangle\\
&+\delta_{l_{1,1}+1,l_{2,1}}\delta_{l_{1,2},l_{2,2}}\delta_{m_1+1,m_2}(c-l_{1,1})\Big\langle\Phi(T^{(1,0,1)}_{(l_{1,1},l_{1,2}),m_1}(X\otimes h_1)),\Phi(h_2)\Big\rangle\\
&+\delta_{l_{1,1},l_{2,1}}\delta_{l_{1,2}+1,l_{2,2}}\delta_{m_1-1,m_2}(c-l_{1,2}+1)\Big\langle\Phi(T^{(0,1,-1)}_{(l_{1,1},l_{1,2}),m_1}(X\otimes h_1)),\Phi(h_2)\Big\rangle\\
&+\delta_{l_{1,1},l_{2,1}}\delta_{l_{1,2}+1,l_{2,2}}\delta_{m_1,m_2}(c-l_{1,2}+1)\Big\langle\Phi(T^{(0,1,0)}_{(l_{1,1},l_{1,2}),m_1}(X\otimes h_1)),\Phi(h_2)\Big\rangle\\
&+\delta_{l_{1,1}-1,l_{2,1}}\delta_{l_{1,2},l_{2,2}}\delta_{m_1-1,m_2}(c+l_{1,1}+p-2)\Big\langle\Phi(T^{(-1,0,-1)}_{(l_{1,1},l_{1,2}),m_1}(X\otimes h_1)),\Phi(h_2)\Big\rangle\\
&+\delta_{l_{1,1}-1,l_{2,1}}\delta_{l_{1,2},l_{2,2}}\delta_{m_1,m_2}(c+l_{1,1}+p-2)\Big\langle\Phi(T^{(-1,0,0)}_{(l_{1,1},l_{1,2}),m_1}(X\otimes h_1)),\Phi(h_2)\Big\rangle\\
&+\delta_{l_{1,1},l_{2,1}}\delta_{l_{1,2}-1,l_{2,2}}\delta_{m_1,m_2}(c+l_{1,2}+p-3)\Big\langle\Phi(T^{(0,-1,0)}_{(l_{1,1},l_{1,2}),m_1}(X\otimes h_1)),\Phi(h_2)\Big\rangle\\
&+\delta_{l_{1,1},l_{2,1}}\delta_{l_{1,2}-1,l_{2,2}}\delta_{m_1+1,m_2}(c+l_{1,2}+p-3)\Big\langle\Phi(T^{(0,-1,1)}_{(l_{1,1},l_{1,2}),m_1}(X\otimes h_1)),\Phi(h_2)\Big\rangle\\
&+\delta_{l_{1,1},l_{2,1}+1}\delta_{l_{1,2},l_{2,2}}\delta_{m_1,m_2}(\overline{c-l_{2,1}})\Big\langle\Phi(h_1),\Phi(T^{(1,0,0)}_{(l_{2,1},l_{2,2}),m_2}(X\otimes h_2))\Big\rangle\\
&+\delta_{l_{1,1},l_{2,1}+1}\delta_{l_{1,2},l_{2,2}}\delta_{m_1,m_2+1}(\overline{c-l_{2,1}})\Big\langle\Phi(h_1),\Phi(T^{(1,0,1)}_{(l_{2,1},l_{2,2}),m_2}(X\otimes h_2))\Big\rangle\\
&+\delta_{l_{1,1},l_{2,1}}\delta_{l_{1,2},l_{2,2}+1}\delta_{m_1,m_2-1}(\overline{c-l_{2,2}+1})\Big\langle\Phi(h_1),\Phi(T^{(0,1,-1)}_{(l_{2,1},l_{2,2}),m_2}(X\otimes h_2))\Big\rangle\\
&+\delta_{l_{1,1},l_{2,1}}\delta_{l_{1,2},l_{2,2}+1}\delta_{m_1,m_2}(\overline{c-l_{2,2}+1})\Big\langle\Phi(h_1),\Phi(T^{(0,1,0)}_{(l_{2,1},l_{2,2}),m_2}(X\otimes h_2))\Big\rangle\\
&+\delta_{l_{1,1},l_{2,1}-1}\delta_{l_{1,2},l_{2,2}}\delta_{m_1,m_2-1}(\overline{c+l_{2,1}+p-2})\Big\langle\Phi(h_1),\Phi(T^{(-1,0,-1)}_{(l_{2,1},l_{2,2}),m_2}(X\otimes h_2))\Big\rangle\\
&+\delta_{l_{1,1},l_{2,1}-1}\delta_{l_{1,2},l_{2,2}}\delta_{m_1,m_2}(\overline{c+l_{2,1}+p-2})\Big\langle\Phi(h_1),\Phi(T^{(-1,0,0)}_{(l_{2,1},l_{2,2}),m_2}(X\otimes h_2))\Big\rangle\\
&+\delta_{l_{1,1},l_{2,1}}\delta_{l_{1,2},l_{2,2}-1}\delta_{m_1,m_2}(\overline{c+l_{2,2}+p-3})\Big\langle\Phi(h_1),\Phi(T^{(0,-1,0)}_{(l_{2,1},l_{2,2}),m_2}(X\otimes h_2))\Big\rangle\\
&+\delta_{l_{1,1},l_{2,1}}\delta_{l_{1,2},l_{2,2}-1}\delta_{m_1,m_2+1}(\overline{c+l_{2,2}+p-3})\Big\langle\Phi(h_1),\Phi(T^{(0,-1,1)}_{(l_{2,1},l_{2,2}),m_2}(X\otimes h_2))\Big\rangle.
\end{aligned}$$

Here, $\delta_{i,j}$ is 1 if $i=j$, and 0 otherwise.
Because of these $\delta$'s, most terms vanish. There are some remaining terms only when $(l_{1,1},l_{1,2}),m_1$ and $(l_{2,1},l_{2,2}),m_2$ are "adjacent". ($\sharp$) becomes:

$$\begin{gathered}
(c-l_1)c_{(l_1-1,l_2),m}\langle T^{(1,0,0)}_{(l_1,l_2),m}(X\otimes h_1),h_2\rangle_{(l_1-1,l_2),m}\\
+(\overline{c}+l_1+p-1)c_{(l_1,l_2),m}\langle h_1,T^{(-1,0,0)}_{(l_1+1,l_2),m}(X\otimes h_2)\rangle_{(l_1,l_2),m}=0\\
(\forall h_1\in\mathcal{H}_{(l_1,l_2),m},\quad\forall h_2\in\mathcal{H}_{(l_1+1,l_2),m}),
\end{gathered}$$

$$\begin{gathered}
(c-l_1)c_{(l_1+1,l_2),m+1}\langle T^{(1,0,1)}_{(l_1,l_2),m}(X\otimes h_1),h_2\rangle_{(l_1+1,l_2),m+1}\\
+(\overline{c}+l_1+p-1)c_{(l_1,l_2),m}\langle h_1,T^{(-1,0,-1)}_{(l_1+1,l_2),m+1}(X\otimes h_2)\rangle_{(l_1,l_2),m}=0\\
(\forall h_1\in\mathcal{H}_{(l_1,l_2),m},\quad\forall h_2\in\mathcal{H}_{(l_1+1,l_2),m+1}),
\end{gathered}$$

$$(c-l_2+1)c_{(l_1,l_2+1),m-1}\langle T^{(0,1,-1)}_{(l_1,l_2),m}(X\otimes h_1),h_2\rangle_{(l_1,l_2+1),m-1}$$
$$+(\overline{c}+l_2+p-2)c_{(l_1,l_2),m}\langle h1,T^{(0,-1,1)}_{(l_1,l_2+1),m-1}(X\otimes h_2)\rangle_{(l_1,l_2),m}=0$$
$$(\forall h_1\in\mathcal{H}_{(l_1,l_2),m},\quad \forall h_2\in\mathcal{H}_{(l_1,l_2+1),m-1}),$$

$$(c-l_2+1)c_{(l_1,l_2+1),m}\langle T^{(0,1,0)}_{(l_1,l_2),m}(X\otimes h_1),h_2\rangle_{(l_1,l_2+1),m}$$
$$+(\overline{c}+l_2+p-2)c_{(l_1,l_2),m}\langle h_1,T^{(0,-1,0)}_{(l_1,l_2+1),m}(X\otimes h_2)\rangle_{(l_1,l_2),m}=0$$
$$(\forall h_1\in\mathcal{H}_{(l_1,l_2),m},\quad \forall h_2\in\mathcal{H}_{(l_1,l_2+1),m}).$$

By the case of unitary axis, the above relations hold for $\forall c'=-2c-p+1\in i\mathbb{R}$ with $c_{(l_1,l_2),m}=1$ for all pairs $(l_1,l_2,m)$. Substituting, we get

$$\langle T^{(1,0,0)}_{(l_1,l_2),m}(X\otimes h_1),h_2\rangle_{(l_1+1,l_2),m}=\langle h_1,T^{(-1,0,0)}_{(l_1+1,l_2),m}(X\otimes h_2)\rangle_{(l_1,l_2),m}$$
$$(\forall h_1\in\mathcal{H}_{(l_1,l_2),m},\quad \forall h_2\in\mathcal{H}_{(l_1+1,l_2),m}),$$

$$\langle T^{(1,0,1)}_{(l_1,l_2),m}(X\otimes h_1),h_2\rangle_{(l_1+1,l_2),m+1})=\langle h_1,T^{(-1,0,-1)}_{(l_1+1,l_2),m+1}(X\otimes h_2)\rangle_{(l_1,l_2),m}$$
$$(\forall h_1\in\mathcal{H}_{(l_1,l_2),m},\quad \forall h_2\in\mathcal{H}_{(l_1+1,l_2),m+1}),$$

$$\langle T^{(0,1,-1)}_{(l_1,l_2),m}(X\otimes h_1),h_2\rangle_{(l_1,l_2+1),m-1})=\langle h1,T^{(0,-1,1)}_{(l_1,l_2+1),m-1}(X\otimes h_2)\rangle_{(l_1,l_2),m}$$
$$(\forall h_1\in\mathcal{H}_{(l_1,l_2),m},\quad \forall h_2\in\mathcal{H}_{(l_1,l_2+1),m-1}),$$

$$\langle T^{(0,1,0)}_{(l_1,l_2),m}(X\otimes h_1),h_2\rangle_{(l_1,l_2+1),m})=\langle h_1,T^{(0,-1,0)}_{(l_1,l_2+1),m}(X\otimes h_2)\rangle_{(l_1,l_2),m}$$
$$(\forall h_1\in\mathcal{H}_{(l_1,l_2),m},\quad \forall h_2\in\mathcal{H}_{(l_1,l_2+1),m}).$$

Thus, we get the assertions. ■

From this Lemma, we get $c_{(l_1,l_2),m}=c_{(l_1,l_2),m\pm1}$. So, we can again ignore the parameter $m$, and this settles the difference between $G^+$ and $G$. From now on, we write $c_{(l_1,l_2)}$ instead of $c_{(l_1,l_2),m}$.

The recursive relations become

$$A^{+0}(l_1,l_2)c_{(l_1+1,l_2)}+\overline{A^{-0}(l_1+1,l_2)c_{(l_1,l_2)}}=0,$$
$$A^{-+}(l_1,l_2)c_{(l_1,l_2+1)}+\overline{A^{+-}(l_1,l_2+1)c_{(l_1,l_2)}}=0.$$

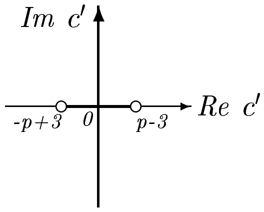
The condition $c_{(l_1,l_2)}\in\mathbb{R}$ is equivalent to $c'\in\mathbb{R}$ or $c'\in i\mathbb{R}$. The latter case is the unitary axis, the module is irreducible and unitarizable.

In case $c'\in\mathbb{R}$, we get

$$c_{(l_1+1,l_2)}=\frac{A^{-0}(l_1+1,l_2)}{A^{+0}(l_1,l_2)}c_{(l_1,l_2)},$$
$$c_{(l_1,l_2+1)}=\frac{A^{+-}(l_1,l_2+1)}{A^{-+}(l_1,l_2)}c_{(l_1,l_2)}.$$

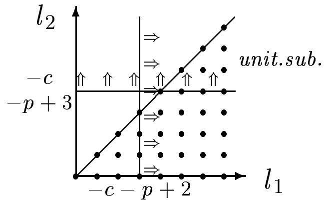
Once we fix a $c_{(l_1,l_2)}$ in an irreducible constituent, the other $c_{(\cdot,\cdot)}$'s are automatically determined.
To make all the $c_{(\cdot,\cdot)}$'s positive, we have to look at the signature of $A^{\cdot\cdot}(l_1,l_2)$. A careful examination of these formulae and Thm.7.2 yields the following. The numbers correspond to those of Thm.7.2, except that we have to single out the case of trivial representation in case (1) and (5).

**Theorem 8.3.** *The Harish-Chandra module of* $S^c(\mathbf{X}^+)$ *is irreducible and unitary if and only if* $c' = -2c-p+1 \in i\mathbb{R}$ *(unitary axis) or* $(0),(3)$ $-p+2 < c < -1$.

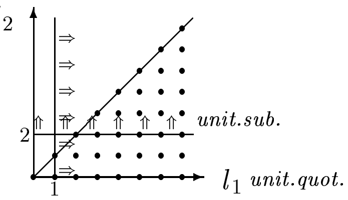



*When it is reducible (i.e.* $c \in \mathbb{Z}$*),*

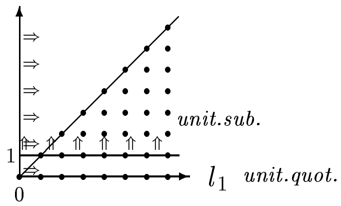
(1) $c \leq -p$ : *1 non-unitary finite-dimensional quotient, 1 non-unitary subquotient, 1 unitary submodule.*



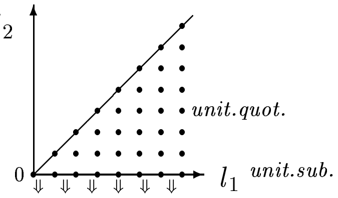

(1bis) $c = -p+1$ : *1 unitary 1-dimensional quotient, 1 non-unitary subquotient, 1 unitary submodule.*



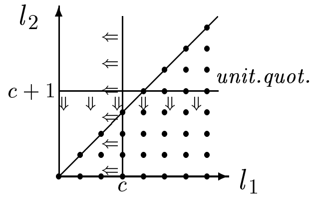

(2) $c = -p+2$ : *1 unitary quotient, 1 unitary submodule.*




(4) $c = -1$ : *1 unitary submodule, 1 unitary quotient.*

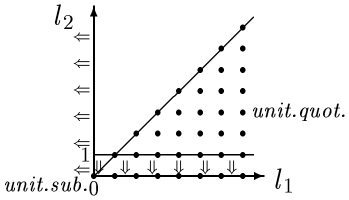



(5) $c \geq 1$ : *1 non-unitary finite-dimensional submodule, 1 non-unitary subquotient, 1 unitary quotient.*




(5bis) $c = 0$ : *1 unitary 1-dimensional submodule, 1 non-unitary subquo-*

*tient, 1 unitary quotient.*

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