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A manifold structure for analytic isotropy Lie pseudogroups of infinite type

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Abstract. We give the solution of Lie's third fundamental problem for the class of infinite dimensional Lie algebras corresponding to the isotropy subpseudogroups of the flat transitive analytic Lie pseudogroups of infinite type. The associated Lie groups are regular Gâteaux-analytic infinite-dimensional Lie groups whose compatible manifold structure is modelled on locally convex topological vector spaces (countable inductive limits of Banach spaces) of vector fields by charts involving countable products exponential mappings. This structure theorem is applied to the local automorphisms pseudogroups of Poisson, symplectic, contact and unimodular structures. In particular the local analytic Lie-Poisson algebra associated to any finite dimensional real Lie algebra is shown to be integrable into a unique connected and simply connected regular infinite-dimensional Gâteaux-analytic Lie group.

1. Introduction

An analytic Lie pseudogroup Γ^ω of transformations of a manifold M is a subpseudogroup of the pseudogroup of analytic local diffeomorphisms of M which forms the general solution of an involutive system $\mathcal{S}$ of analytic partial differential equations. If the system $\mathcal{S}$ is completely integrable (in the sense of the Frobenius theorem), then the elements of Γ^ω depend on a finite number of parameters and the Lie pseudogroup Γ^ω is said to be of *finite type.* For instance, the Lie pseudogroup acting on the real line and defined by the differential equation

$$\frac{d^3\bar{x}}{dx^3} - \frac{3}{2}(\frac{d^2\bar{x}}{dx^2})^2/(\frac{d\bar{x}}{dx}) = 0,$$

is the Lie pseudogroup of finite type corresponding to the homographic transformations of the real line. If $\mathcal{S}$ is involutive, but not completely integrable, then the Lie pseudogroup Γ^ω is said of *infinite type* since its elements are then parametrized by arbitrary functions. For example, the locally conformal transformations of the complex plane $\mathbb{C}$ form the Lie pseudogroup of infinite type associated with the Cauchy-Riemann equations.

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It is well-known that the Lie pseudogroups of finite type have led to the classical theory of Lie groups. Indeed, Élie Cartan proved that given a Lie pseudogroup of finite type, there exists a unique connected and simply connected global Lie group whose Lie algebra is isomorphic to the Lie algebra of infinitesimal transformations of the pseudogroup. Any pseudogroup Γ^ω of finite type can therefore be regarded as the local action of an abstract global Lie group. For instance, the Lie pseudogroup acting on the real line and defined by the differential equation of order 3 given above can be regarded as a local action of $PSL(2,\mathbb{R})$. The first and crucial step in the proof of Cartan's theorem is to show that there exists a local Lie group whose Lie algebra is isomorphic to the given Lie algebra of infinitesimal transformations. This result, which was already known to Lie, is fundamentally analytical in nature since it is based on the Frobenius integration theorem [8]. The second and final step in the proof is to embed a neighbourhood of the identity in the local group in a neighbourhood of the identity in a global Lie group, which is essentially a topological problem. These results of Lie and Cartan are known as the local and global versions of Lie's Third Fundamental Theorem.

Starting with Sophus Lie himself, many mathematicians have risen to the challenge of extending this abstract viewpoint, even in the local case, to pseudogroups of infinite type. We should particularly mention the important papers of Lie [22, 23], Cartan [5], Kuranishi [18], Guillemin, Sternberg [13], Leslie [19, 20, 21], Ebin, Marsden [11], Milnor [24], Omori, de la Harpe [27, 28, 29], Souriau [35], Coste, Dazord and Weinstein [7, 10, 37]. One has first to understand the local problem in infinite dimensions. Since the Frobenius theorem and the standard existence and uniqueness theorems for ordinary differential equations do not hold in most of the relevant infinite-dimensional spaces, a different approach needs to be developed. This difficulty was already recognized by Cartan [6], who bypassed the use of infinitesimal transformations altogether in his study of the classification problem for infinite Lie pseudogroups of transformations. Indeed, what he did was to develop for infinite Lie pseudogroups a structure theory based on a geometric generalization of the Maurer-Cartan equations and on his theory of exterior differential systems in involution. The modern incarnation of Cartan's approach appears in the theory of G-structures of infinite type.

Following Cartan's ideas, the problem of assigning an abstract structure to pseudogroups of infinite type was reconsidered by Kuranishi in [18]. Almost simultaneously Guillemin and Sternberg [36, 13] developed a more algebraic approach based on the natural filtration of the Lie algebra of infinitesimal transformations associated to any Lie pseudogroup of infinite type. The pathological behaviour of the exponential mapping, even for Lie algebras of infinitesimal transformations defined by analytic vector fields, led Kuranishi and Sternberg to work with *formal* local Lie groups. An analytic framework suitable for the convergent situation was thus still missing at that time. An important breakthrough came with the work of Leslie [20], who showed that the group of analytic diffeomorphisms of an analytic compact manifold can be given the structure of a *smooth* infinite dimensional Lie group in Milnor's sense [24]. A powerful body of ideas also comes from the seminal work of Écalle [12], who in the context of his iterative theory, carried out an elaborate analysis of various transformation groups on the real line and obtained some important new results in both the formal and convergent cases. A basic ingredient in Écalle's theory is the notion of a *regular sequence*, calibrating power series into the analytic, quasi-analytic and Gevrey classes. This latter point of view is helpful for defining the appropriate locally convex topology for analytic Lie pseudogroups.

We succeed in this paper in proving an integration theorem for the class of isotropy Lie algebras arising from transitive and flat analytic Lie pseudogroups of infinite type. Our result is thus an infinite-dimensional counterpart of Cartan's global theorem for finite-dimensional Lie algebras. We prove precisely that for any regular sequence π in the sense of Écalle and any flat infinite Lie pseudogroup of transformations of class $\mathcal{C}(\pi)$, the corresponding isotropy Lie algebra is the Lie algebra of a unique connected and simply connected regular Gâteaux analytic Lie group G_0^π (Theorem 6.9). The natural differentiable structure associated to any flat isotropy Lie pseudogroup is thus obtained by applying a combination of Leslie's approach with Écalle's results.

Our paper is organized as follows. In Section 3., we briefly review the notion of a regular infinite-dimensional Lie group in the sense of Milnor. In Section 4., we show that to any flat Lie pseudogroup and any regular sequence in the sense of Écalle, one can associate a locally convex topological Lie algebra of infinitesimal transformations. In Section 5., we prove our main result for the pseudogroup of analytic local diffeomorphisms of $\mathbb{R}^n$, namely that for any regular sequence, the topological Lie algebra corresponding to the isotropy sub-pseudogroup is the Lie algebra of a Gâteaux-analytic group which is regular in the sense of Milnor. Our proof is based on a new regularity theorem that we establish for the exponential mapping. In Section 6., we use the above results to prove our generalization of Cartan's global version of Lie's Third Fundamental Theorem for the class of topological Lie algebras associated to the isotropy sub-pseudogroups of flat transitive Lie pseudogroups. Finally, Section 7. contains some applications of our structure theorems to the fundamental pseudogroups of local automorphisms of Poisson, symplectic, contact and unimodular structures on a manifold. Our main results had been conjectured in [16]. Their proof was annouced in [15].

2. Differential Calculus

Our goal in this section is to define the category of topological vector spaces that will serve as model spaces for the infinite-dimensional manifolds that we will consider in this paper. We begin by recalling some basic results on the Gâteaux calculus. We refer the reader to [24] and the references therein for details.

Gâteaux differentiability: Let E, F be two Hausdorff locally convex topological vector spaces. Let U be an open subset of E. A map $f : U \to F$ is said to be of class C^0 if it is continuous.

Given any point x in U and any vector v in E, the directional derivative of f at x in the direction v is defined by

$$Df(x;v) = Df_x(v) = \lim_{t\to 0} \frac{f(x+tv) - f(x)}{t}$$

if the limit exists.

Definition 2.1. The map f is of class C^1 in U if it is of class C^0 in U and if $Df(x;v)$ is well defined in $U \times E$ and continuous as a function of two variables.

Higher order derivatives are defined as follows.

Definition 2.2. A map $f : U \subseteq\rightarrow F$ is said to be of class C^2 in U if it is of class C^1 in U and if, for each fixed vector v_1 in E, the map defined in $U \subseteq E$ with values in F by $x \mapsto Df(x; v_1)$ is of class C^1, its directional derivative

$$D^2 f(x; v_1, v_2) = \lim_{t\to 0} \frac{Df(x + tv_2; v_1) - Df(x; v_1)}{t}$$

being defined for all v_2 and being continuous as a function of three variables.

We call $D^2 f(x; v_1, v_2)$ the second directional derivative of f at x in the directions v_1 and v_2. The third directional derivative and C^3 functions are similarly defined and so on.

A function of class C^n for any integer n is called a C^∞ map or briefly a smooth map.

Using standard techniques, namely the Hahn-Banach theorem and the differential calculus in finite dimensions, one proves that for any function of class C^n in U and any point x in U, the map $D^n f_x$ defined on $E \times \cdots \times E$ (n times) by

$$D^n f_x(v_1, \dots , v_n) = D^n f(x; v_1, \dots , v_n)$$

is a continuous symmetric n-multilinear map. One also shows that this notion of differentiability satisfies the classical theorems such as the chain rule, Taylor's theorem with Lagrangian remainder (when F is sequentially complete), etc.

This notion of differentiability coincides with the usual one in finite dimensions. If E and F are infinite-dimensional Banach spaces, this notion is a little weaker than the Fréchet differentiation. However the C^{n+1} differentiability in Gâteaux sense implies the C^n differentiability in Fréchet sense [29]. Hence every infinitely Gâteaux differentiable mapping is, in the latter context, infinitely Fréchet differentiable and conversely. For more details see [1, 24].

Complex and real analytic mappings:

Definition 2.3. [Complex analyticity] Let E, F be two locally convex complex topological vector spaces. A map $f : U \subseteq E \to F$ from an open subset U of E to F is said to be *complex-analytic* or *holomorphic* if it is of class C^1, and its derivative $Df_x : E \to F$ a complex linear map at each point x of U .

Cauchy's formula holds in this infinite-dimensional context.

Definition 2.4. [Real analyticity] A map $f : U \subseteq E \to F$ between two real locally convex topological vector spaces is said to be *real analytic* in the open subset U if it can be extended to a complex analytic map, defined on some open subset $U_{\mathbb{C}}$ of $E_{\mathbb{C}} = E \bigoplus iE$ containing U with values in $F_{\mathbb{C}}$.

We conclude this section by reviewing the notion of Gâteaux-analitycity. A homogeneous polynomial of degree n defined on a vector space E with values in a vector space F is a mapping $f_n : E \to F$ which is determined by an n-linear mapping

$$\bar{f}_n : \overbrace{E \times \cdots \times E}^{n \text{ copies}} \to F$$

by $f_n(h) = \bar{f}_n(h, \ldots, h)$ for all h in E. Let $\mathcal{P}^k(E,F)$ denote the space of *continuous* homogeneous polynomials of degree k from E to F.

A formal sum $\sum_{k=0}^{\infty} f_k$ where, for all $k \in \mathbb{N}$, f_k is a homogeneous polynomial of degree k, is said to be a *formal series.* The space of formal series with continuous coefficients (i.e. in $\mathcal{P}^k(E,F)$) is denoted by $\mathcal{S}(E,F)$.

A formal series is said to converge in an open set U of E if for each continuous semi-norm q of F and all $x \in U$, the series

$$\sum_{n=0}^{\infty} q(f_n(x))$$

converges.

Definition 2.5. [Gâteaux analyticity] Let E and F be real locally convex topological vector spaces and $f : U \subseteq E \to F$ be a continuous function defined in the open set U of E. The function f is said to be *(real) Gâteaux analytic* in U if for each $x \in U$ there exists a series $\sum_{n=0}^{\infty} f_{n,x} \in S(E,F)$ such that $f(x+h) = \sum_{n=0}^{\infty} f_{n,x}(h)$ for all h belonging to a neighbourhood of zero in E.

More details can be found in [9], [2], [3].

Model spaces: It is known from the fundamental work of Leslie [19] and Omori [28] that the topology of the group of global smooth diffeomorphisms of a compact manifold is given by a *projective limit* of Banach spaces. We will see in contrast that in the case of Lie pseudogroups of analytic transformations, the natural topology is that of an *inductive limit* of Banach spaces. This structural difference is strong enough to prevent us from being able to use Leslie and Omori's existence theorems for the exponential map in the context of pseudogroups of analytic transformations.

Bornological spaces: Our purpose in this section is to recall some of the basic properties of bornological spaces that will be needed in the remainder of this paper. We refer the reader to Bourbaki [4], Köthe [17] and Hogbé-Nlend [14] for details.

Let E be a locally convex real topological vector space.

A subset S in E is said to *absorb* a subset D if there is a $\rho > 0$ such that $D \subseteq \rho S$. We then say that a subset B is *bounded* if and only if each neighbourhood U of the origin absorbs B.

A map $f : E \to F$ between two locally convex spaces is said to be *bounded* if for any bounded set B in E, $f(B)$ is a bounded set in F.

By definition, a *disk* is a subset D satisfying the following property: if $x, y \in D$ and λ, μ are such that $\mid \lambda \mid + \mid \mu \mid \leq 1$ then $\lambda x + \mu y \in D$.

The *disked cover* of a subset $S = \bigcup_\alpha S_\alpha$ of E is denoted by $\bigwedge(S)$ or $\bigwedge_\alpha S_\alpha$ and consists of all elements of the form $\sum_{i=1}^{n} \rho_i x_i$, $\rho_i \in \mathbb{R}$ and $x_i \in S$ with $\sum_{i=1}^{n} \mid \rho_i \mid \leq 1$.

A linear mapping between two normed spaces is said to be *compact* if the image of the closed unit ball is relatively compact.

Following Bourbaki [4], we will say that a locally convex topological vector space is *bornological* if every disk which absorbs all the bounded sets is a neighbourhood of the origin.

Silva spaces and properties: The model spaces adapted to the study of analytic Lie pseudogroups are special bornological spaces known as Silva spaces. They are defined as follows, [17], [14]:

Definition 2.6. A *Silva space* is a countable topological inductive limit of Banach spaces $\lim_{\substack{\rightarrow \\ n\in\mathbb{N}}} E_n$, where the injection $E_n \hookrightarrow E_{n+1}$ is compact for all n.

If for each n, $\{U_p^n\}$ denotes a base of neighbourhoods of the origin in E_n, then the disked covers $\bigwedge_n U_p^n$ form a base of neighbourhoods of the origin in the Silva space. Silva spaces are thus indeed bornological. The sequence $(E_n)_{n\in\mathbb{N}}$ of Banach spaces is called a defining sequence for the Silva space.

Let E be a Silva space with defining sequence $(E_n)_{n\in\mathbb{N}}$. We let $\|\cdot\|_n$ denote a norm associated to the Banach space E_n. Without loss of generality these norms may be chosen in such a way that $\|u\|_{n+1} \leq \|u\|_n$ for all $u \in E_n$ for all n. Let then B_n be the closed unit ball in E_n, i.e.

$$B_n = \{x \in E_n, \|x\|_n \leq 1\}.$$

This gives rise to the infinite nested sequence of disks

$$B_1 \subset \cdots \subset B_n \subset B_{n+1} \subset \cdots \subset E$$

in E. Notice also that, since by assumption the injection $E_n \hookrightarrow E_{n+1}$ is compact and E_n is complete for any n, the disk B_n is a compact subset of E_{n+1} for any integer n.

The following three propositions are easily established:

Proposition 2.7. *A Silva space is a complete Hausdorff locally convex vector space.*

Proposition 2.8. *The topological product of two Silva spaces* $\lim_{\substack{\rightarrow \\ n\in\mathbb{N}}} E_n$ *and* $\lim_{\substack{\rightarrow \\ n\in\mathbb{N}}} F_n$ *is a Silva space with defining sequence* $(E_n \times F_n)_{n\in\mathbb{N}}$.

Proposition 2.9. *A Silva space* $\lim_{\substack{\rightarrow \\ n\in\mathbb{N}}} E_n$ *is topologically isomorphic to the inductive limit* $\lim_{\substack{\rightarrow \\ a_n, n\in\mathbb{N}}} E_{a_n}$ *for any strictly increasing sequence* $(a_n)_{n\in\mathbb{N}}$ *of integers.*

We will also need the following useful characerization of bounded sets in a Silva space.

Proposition 2.10. *A subset* B *of a Silva space* $E = \lim_{\substack{\rightarrow \\ n\in\mathbb{N}}} E_n$ *is bounded if and only if there is an integer* n_0 *such that* $B \subset E_{n_0}$ *and* B *is bounded in the Banach space* E_{n_0}.

Proof. Since the disked covers $\bigwedge_n U_p^n$ form a base of neighbourhoods of the origin in the Silva space (see definition 2.6) it is clear that the condition is sufficient.

Conversely if B is bounded in the Silva space E, we show that there exists an integer n_0 such that B is contained in some E_{n_0} with $B \subset n_0 B_0$ (B_n is the closed unit ball of E_n). Suppose that this were not true. Then, for every integer n, there exists an $x_n \in B$ such that $x_n \notin nB_n$. We are going to construct a disked neighbourhood of the origin in the Silva space E that doesn't absorb B, thereby reaching a contradiction.

Let $y_n = \frac{x_n}{n}$. Since $y_1 \notin B_1$ and B_1 is closed in E_2, there exists $0 < \lambda_2 \leq 1$ such that $y_1 \notin B_1 + \lambda_2 B_2$. In particular $y_1, y_2 \notin (B_1 + \lambda_2 B_2) \bigcap B_2$. Since $(B_1 + \lambda_2 B_2) \bigcap B_2$ is compact in E_3, it is closed and so there exists $0 < \lambda_3 \leq 1$ with $y_1, y_2 \notin (B_1 + \lambda_2 B_2) \bigcap B_2 + \lambda_3 B_3$. In particular $y_1, y_2, y_3 \notin ((B_1 + \lambda_2 B_2) \bigcap B_2 + \lambda_3 B_3) \bigcap B_3$. This construction gives rise to an increasing sequence of disks D_n defined by $D_1 = B_1$ and $D_n = (D_{n-1} + \lambda_n B_n) \bigcap B_n$ for $n > 1$. Setting $\lambda_1 = 1$ we have $\lambda_n B_b \subset D_n$ for all n and $y_1, \ldots, y_n \notin D_n$. Let

$$V = \bigcup_{n=1}^{+\infty} D_n.$$

It is a disked neighbourhood of the origin in the Silva space E. Moreover V does not absorb B since $x_n \notin nV$ for any integer n. This completes the proof. ∎

Proposition 2.11. *A subset U in the Silva space $\lim_{\substack{\rightarrow \\ n\in\mathbb{N}}} E_n$ is an open subset if and only if for any element $x \in U$ and any integer n there is $\rho_n > 0$ with $x + \rho_n B_n \subset U$ where B_n is the closed unit ball in the Banach space E_n.*

Proof. It suffices to show that for any $x \in U$ there exists a disked bornivorous set D_x containing x and contained in U. The construction of D_x can be done inductively as in the proof of proposition 2.10. ∎

The following theorem provides us with a simple characterization of continuity for mappings of bornological spaces in terms of boundedness. This result will be very useful in our study of model spaces for the Lie algebras of vector fields associated to analytic Lie pseudogroups.

Theorem 2.12. *A linear mapping between bornological spaces is continuous if and only if it is bounded.*

Proof. Let E, F be bornological locally convex spaces and let

$$u : E \to F$$

be a linear map. We first prove that if u is continuous, then it is bounded. Let B be a bounded subset of E and V be a neighbourhood of the origin in F. Since u is continuous, $u^{-1}(V)$ is a neighbourhood of the origin in E and hence absorbs B. So there exists $\rho > 0$ with $B \subset \rho u^{-1}(V) = u^{-1}(\rho V)$ and $u(B) \subset \rho V$. Therefore V absorbs $u(B)$ which proves that u is bounded.

Conversely let $u : E \to F$ be a bounded linear map. For every disked neighbourhood V of the origin in F, $u^{-1}(V)$ is a disk that absorbs every bounded set in E, hence a neighborhood of the origin since E is a bornological locally convex vector space. ∎

Remark 2.13. That theorem is a fortiori valid for multilinear mappings from products of bornological spaces into bornological spaces.

We conclude this section with a characterization of continuous mappings between Silva spaces.

Theorem 2.14. *Let* $E = \lim_{\substack{\to \\ n\in\mathbb{N}}} E_n$ *and* F *be Silva spaces. Then* $f : E \to F$ *is continuous if and only if the image* $(f(x_n))_{n\in\mathbb{N}}$ *of any* bounded sequence $(x_n)_{n\in\mathbb{N}}$ *converging to* x_0 *is converging to* $f(x_0)$ *in* F.

Proof. The condition is obviously necessary. Conversely let x belong to $f^{-1}(V)$ where V is an arbitrary neighbourhood in F. Let B_n be the closed unit disk of E_n. The image by f of any sequence of elements in $x + B_n$ that converges to x will converge to $y = f(x)$. It follows that there is a strictly positive real number ρ_n such that $x + \rho_n B_n \subset f^{-1}(V)$. Hence $\bigcup_{n\in\mathbb{N}}(x + \rho_n B_n)$ is contained in $f^{-1}(V)$. Since this holds for any $x \in f^{-1}(V)$, it follows that $f^{-1}(V)$ is open by proposition 2.11. ∎

3. Infinite-dimensional Lie groups

By a differentiable manifold, we mean a sequentially complete Hausdorff topological space modelled via an atlas on a locally convex Hausdorff and complete topological vector space. We use the Gâteaux differential calculus. Following Milnor, [24], we define an infinite-dimensional Lie group to be a group endowed with a smooth manifold structure defined as above, compatible with the group operations.

If G is a Lie group, the model space of its underlying manifold structure is identified with its Lie algebra $\mathcal{L}(G)$. The existence for all v in $\mathcal{L}(G)$ of a one parameter subgroup $\gamma_v : \mathbb{R} \to G$ with v as tangent vector at the identity is of course not always guaranteed.

Let $\mathcal{I} = [0, 1]$. The space $C^\infty(\mathcal{I}, \mathcal{L}(G))$ of smooth paths of the Lie algebra $\mathcal{L}(G)$ is endowed with the C^∞ uniform convergence topology. The group G is said to be *regular* if the ordinary differential equation

$$g^{-1}\dot{g}(t) = v(t) \text{ with } g(0) = e, \tag{1}$$

where the dot denotes the derivative with respect to t, admits a smooth solution γ_v for all smooth paths v in $\mathcal{L}(G)$ and if the correspondence $v \mapsto \gamma_v(1)$, from $C^\infty(\mathcal{I}, \mathcal{L}(G))$ with values in G, is smooth. The mapping $\mathrm{Exp} : \mathcal{L}(G) \to G$ that associates to the constant path v the group element $\gamma_v(1)$ is called the *exponential mapping* of the group.

For instance the group $\mathrm{Diff}^\infty(M)$ of smooth diffeomorphisms of a compact manifold M is a regular Lie group modelled on the Fréchet space $\chi^\infty(V)$ of smooth vector fields on M [19].

If G admits the exponential mapping as a manifold chart near the identity we say that G is a Lie group of the *first kind*. The class of Lie groups of the first kind is denoted by $\mathcal{EXP}$. If in addition G is analytic then it is said of Campbell-Baker-Hausdorff (CBH for short) type. The class of CBH Lie groups is denoted by $\mathcal{CBH}$. Finite dimensional Lie groups, gauge groups and Banach Lie groups are examples of CBH Lie groups. In contrast, the Lie group of analytic global diffeomorphisms of a compact analytic manifold is not of CBH type.

We will say that G is a Lie group of the *second kind* if its Lie algebra $\mathcal{L}(G)$ decomposes into a direct sum $\mathcal{L}(G) = \bigoplus_{i=1}^{m} \mathcal{G}_i$ wit the property that the mapping $\prod_{i=1}^{m} \mathrm{Exp}_i$, which associates to $(X_1, \ldots, X_m) \in \mathcal{G}_1 \times \cdots \times \mathcal{G}_m$ the product $\mathrm{Exp}(X_1) \circ \cdots \circ \mathrm{Exp}(X_m) \in G$, defines a manifold chart near the identity. The integer m denotes the multiplicity of the decomposition. In this case we will say that G is of the second kind and of order m and belongs to the class $\mathcal{E}XP^m$. We generalize this class to the class $\mathcal{E}XP^\infty$ of Lie groups of the second kind and of countable order. We have the obvious inclusions $\mathcal{C}BH \subset \mathcal{E}XP \subset \mathcal{E}XP^m \subset \mathcal{E}XP^{m'} \subset \mathcal{E}XP^\infty$ for $m < m'$.

4. Lie algebras of infinitesimal transformations

In this section it is shown that for any analytic Lie pseudogroup and any regular sequence in the sense of Écalle the corresponding Lie algebra of calibrated infinitesimal transformations is a Silva topological Lie algebra.

Notations: Let M be a connected analytic manifold and let z_0 be a point of M. We choose a coordinate chart $z = (z^1, \ldots, z^n)$ such that $z_0 = (0, \ldots, 0)$. For any n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ of non-negative integers we let

$$\mid \alpha \mid = \alpha_1 + \cdots + \alpha_n$$

(its length) $\mid \alpha \mid' = \mid \alpha \mid -1$ (its reduced length) and

$$z^\alpha = (z^1)^{\alpha_1} \cdots (z^n)^{\alpha_n}.$$

Lie algebras of infinitesimal transformations: Let

$$\chi(n) = \chi_{-1}(n)$$

denote the Lie algebra of formal vector fields based at the origin O of $\mathbb{R}^n$. For every non-negative integer q, let $\chi_q(n)$ denote the Lie subalgebra of formal vector fields tangent to order q to the zero vector field. This defines the decreasing filtration naturally associated to $\chi(n)$. In component form, any formal vector field V of $\chi_q(n)$ can be written as $V = \sum_{i=1}^{n} V^i \partial_i$ with $V^i = \sum_{|\alpha|>q} v_\alpha^i z^\alpha$. A formal vector field V will be said *positive* (resp. *negative*) if all its coefficients v_α^j are positive (resp. negative). To each formal vector field V we associate a positive formal vector field $\mid V \mid$ defined by $\mid V^j \mid = \sum_{|\alpha|=0}^{+\infty} \mid v_\alpha^j \mid z^\alpha$.

We endow $\chi(n)$ with the natural Tychonov topology.

Let now Γ be a transitive Lie pseudogroup of analytic local diffeomorphisms of the connected analytic manifold M. To Γ one associates a Lie algebra sheaf of germs of vector fields whose local flows generate Γ, [34]. Since Γ is transitive, the stalks of this Lie algebra sheaf at any two points are isomorphic. We will thus consider the stalk at the point z_0 and we will denote it by $\hat{\mathcal{L}}(\Gamma)$. If Γ is a transitive Lie pseudogroup, then the corresponding Lie algebra $\hat{\mathcal{L}}(\Gamma)$ determines a closed Lie subalgebra $\mathcal{L}(\Gamma)$ of $\chi(n)$ consisting of the formal vector fields obtained by formally expanding the elements of $\hat{\mathcal{L}}(\Gamma)$ in Taylor series about the origin.

Let $\mathcal{L}_q(\Gamma) = \mathcal{L}(\Gamma) \cap \chi_q(n)$. We have $\mathcal{L}(\Gamma)/\mathcal{L}_0(\Gamma) \simeq \mathbb{R}^n$ by transitivity. We associate to $\mathcal{L}(\Gamma)$ the Lie algebra $L(\Gamma)$ defined by $L(\Gamma) = \oplus_{q=-1}^{\infty} \mathcal{L}_q(\Gamma)/\mathcal{L}_{q+1}(\Gamma)$. A transitive Lie pseudogroup Γ^ω is said to be *flat* if its formal Lie algebra $\mathcal{L}(\Gamma)$ and the associated Lie algebra $L(\Gamma)$ are isomorphic. The pseudogroup of local transformations that preserve a volume form and the Hamiltonian pseudogroup of local symplectomorphisms are important examples of flat Lie pseudogroups.

Analytic, quasi-analytic and Gevrey's settings: Let

$$M = \mathbb{R}^n$$

and fix a norm on M. This defines for the space $M \otimes S^p(M^\star)$ of homogeneous vector fields of degree p a norm $\| \,.\, \|_p$. Any non decreasing sequence $\pi = \{\pi_n\}$ $(n \geq 1)$ of positive real numbers such that $\limsup \pi_{n+1}/\pi_n < \infty$ will be called a sequence of type $\mathcal{R}$, or of regular type [12]. We will say that a formal vector field

$$V = \sum_{k=0}^{\infty} V_k$$

(where, for all k, $V_k \in M \otimes S^k(M^\star)$) belongs to the class $\mathcal{C}(\pi)$ if its coefficients satisfy $\limsup \| V_n \|_n^{\frac{1}{n}} / \pi_n < \infty$.

Lemma 4.1. *The class $\mathcal{C}(\pi)$ does not depend on the choice of norm. A formal series*

$$\Sigma = \sum_{i=1}^{n} (\sum_{|\alpha| \geq 0} \sigma_\alpha^i z^\alpha) \partial_i$$

is of class $\mathcal{C}(\pi)$ if and only if there exists a positive constant ρ such that for all i and α we have,

$$| \sigma_\alpha^i | \leq (\pi_{|\alpha|})^{|\alpha|} \rho^{|\alpha|}.$$

If $\pi \equiv \{1\}$ the corresponding class $\mathcal{C}(\pi)$ is the analytic class. If $\pi_n = \log n$ or more generally $\pi_n = \prod_{r=1}^{N} \log_{(r)} n$ where $\log_{(r)} n$ stands for

$$(\log(\log \cdots (\log n)) \cdots)$$

r times and where N is fixed, the corresponding class is the quasi-analytic class of Denjoy. For $\pi_n = n^\theta$ $(\theta > 0)$, we obtain Gevrey's class of index θ. We denote $\{n^\theta\}_{n \geq 1}$ by Θ. If Γ is any Lie pseudogroup, the subspace $\mathcal{L}^\pi(\Gamma) \subset \mathcal{L}(\Gamma)$ of formal vector fields of class $\mathcal{C}(\pi)$ is also a Lie subalgebra. All these spaces admit a natural locally convex topology.

Let ρ be a positive real number and let $\mathcal{L}_\rho^\pi(\Gamma)$ denote the subspace of $\mathcal{L}^\pi(\Gamma)$ of V's such that $\limsup \| V_n \|_n / (\pi_n)^n \rho^n) < +\infty$. We have $\mathcal{L}^\pi(\Gamma) = \bigcup_{\rho > 0} \mathcal{L}_\rho^\pi(\Gamma)$. Each $\mathcal{L}_\rho^\pi(\Gamma)$ is naturally endowed with a Banach space structure with the norm

$$\| V \|_\rho = \sup_n \frac{\| V_n \|_n}{(\pi_n)^n \rho^n}.$$

For $\rho < \rho'$ the injection $\mathcal{L}_\rho^\pi(\Gamma) \hookrightarrow \mathcal{L}_{\rho'}^\pi(\Gamma)$ is continuous and compact. Hence $\mathcal{L}^\pi(\Gamma)$ is a complete Hausdorff locally convex topological vector space. Its associated topology is the locally convex strict inductive limit topology

$$\mathcal{L}^\pi(\Gamma) = \varinjlim_{n \in \mathbb{N}} \mathcal{L}_n^\pi(\Gamma).$$

This endows $\mathcal{L}^\pi(\Gamma)$ with a Silva topological Lie algebra structure.

5. The general pseudogroups

In this section we establish a regular Gâteaux-analytic infinite-dimensional Lie group structure for the isotropy subgroup of the origin in the general Lie pseudogroup of local diffeomorphisms of class $\mathcal{C}(\pi)$ defined in a neighbourhood of the origin of $\mathbb{R}^n$. This will enable us to establish a general existence and regularity theorem for the exponential mapping, which we will use in Section 8 to solve the integration problem for isotropy subalgebras of analytic Lie pseudogroups of infinite type.

Lie pseudogroups of class $\mathcal{C}(\pi)$: We represent formal transformations defined in a neighborhood of the origin in $\mathbb{R}^n$ by formal series

$$\Phi = \sum_{i=1}^{n} (\sum_{|\alpha|\geq 0} \phi^i_\alpha z^\alpha)\partial_i,$$

where the matrix (ϕ^i_j) defined by the linear terms in Φ is invertible. We denote by $G^\pi(n) = G^\pi_{-1}(n)$ the set of formal transformations of class $\mathcal{C}(\pi)$ in n variables. For any given positive integer q, we let $G^\pi_q(n)$ be the subset of transformations tangent at order q to the identity transformation I.

Thus $G^\pi_0(n)$ is the set of formal series

$$\Phi = \sum_{i=1}^{n} (\sum_{|\alpha|\geq 1} \phi^i_\alpha z^\alpha)\partial_i$$

of class $\mathcal{C}(\pi)$ with $\det(\phi^i_j) \neq 0$ $(i, j = 1, \ldots, n)$. The set $G^\pi_1(n)$ is the subset of $G^\pi_0(n)$ defined by $(\phi^i_j) = I_n$ where I_n is the n-by-n identity matrix. Finally for any integer $q \geq 2$, $G^\pi_q(n)$ is the set of formal transformations of class $\mathcal{C}(\pi)$ given by

$$\Phi = \sum_{i=1}^{n} (z^i + \sum_{|\alpha|\geq q+1} \phi^i_\alpha z^\alpha)\partial_i.$$

We will write these for short as

$$\Phi = I + \sum_{i=1}^{n} (\sum_{|\alpha|\geq q+1} \phi^i_\alpha z^\alpha)\partial_i.$$

The set of formal transformations (without any restriction concerning the growth of coefficients) will be denoted by $G_q(n)$.

We are first going to show that for any non-negative integer q and any regular sequence π, the set $G^\pi_q(n)$ together with the composition of transformations. In a second step we will show how such a group can be regarded as a Gâteaux analytic regular Lie group the sense of Milnor.

Let Σ_k $k = 1, 2$ be formal series with coefficients $\sigma^i_{k\alpha}$, i.e.

$$\Sigma_k = \sum_{i=1}^{n} (\sum_{|\alpha|\geq 0} \sigma^i_{k\alpha} z^\alpha)\partial_i.$$

We introduce on the set of formal transformations a partial order, denoted by $\ll$ and defined as follows: $\Sigma_1 \ll \Sigma_2$ if $\mid \sigma^i_{1\alpha} \mid \leq \mid \sigma^i_{2\alpha} \mid$ for all indices i and α.

A formal transformation Φ will be said to be *positive* whenever its coefficients ϕ^i_α are all positive real numbers.

We associate to each regular sequence π a scale transformation denoted by ${}^\pi$ which acts on formal series by mapping

$$\Phi = \sum_{i=1}^{n} (\sum_{|\alpha| \geq 0} \phi^i_\alpha z^\alpha) \partial_i.$$

to the formal series ${}^\pi\Phi$ defined by

$${}^\pi\Phi = \sum_{i=1}^{n} (\sum_{|\alpha| \geq 0} \pi_{|\alpha|}^{|\alpha|'} \phi^i_\alpha z^\alpha) \partial_i.$$

Note that $\pi_{|\alpha|}$ is raised to the power $\mid \alpha \mid' = \mid \alpha \mid -1$ the reduced length of the multi-index α.

The scale transformation ${}^\pi$ defines a bijection between the analytic class and the class $\mathcal{C}(\pi)$.

The expression of the group law induced by composition on the set $G_0(n)$ of formal transformations preserving the origin in $\mathbb{R}^n$ is determined as follows. If Φ and $\Psi \in G_0(n)$ are given by

$$\Phi = \sum_{i=1}^{n} (\sum_{|\alpha| \geq 1} \phi^i_\alpha z^\alpha) \partial_i$$

and

$$\Psi = \sum_{i=1}^{n} (\sum_{|\alpha| \geq 1} \psi^i_\alpha z^\alpha) \partial_i = \sum_{i=1}^{n} \psi^i(z) \partial_i,$$

then

$$\Phi \circ \Psi = \sum_{i=1}^{n} (\sum_{|\alpha| \geq 1} \phi^i_\alpha (\psi^1(z))^{\alpha_1} \cdots (\psi^n(z))^{\alpha_n}) \partial_i.$$

Replacing $\psi^i(z)$ by $\sum_{|\alpha| \geq 1} \psi^i_\alpha z^\alpha$ we get

$$\Phi \circ \Psi = \sum_{i=1}^{n} (\sum_{|\alpha| \geq 1} \phi^i_\alpha (\sum_{|\beta_1| \geq 1} \psi^1_{\beta_1} z^{\beta_1})^{\alpha_1} \cdots (\sum_{|\beta_n| \geq 1} \psi^n_{\beta_n} z^{\beta_n})^{\alpha_n}) \partial_i,$$

where $\beta_1, \ldots, \beta_n$ are arbitrary multi-indices. The coefficients of $\Phi \circ \Psi$ are then easily obtained. The linear terms compose as matrices and each coefficient of the resulting series $\Phi \circ \Psi$ is a polynomial in the coefficients of Φ and Ψ. Moreover the coefficients of these polynomials are positive. This proves that the composition of two positive transformations in $G_0(n)$ is also a positive transformation of $G_0(n)$. One proceeds likewise to show that $G_0(n)$ is closed under inversion.

Regarding the scale transformation ${}^\pi$, we have:

Lemma 5.1. *For any regular sequence π, the composition of transformations in $G_0(n)$ satisfies*

$${}^\pi\Phi \circ {}^\pi\Psi \ll {}^\pi(\Phi \circ \Psi).$$

Proof. Let $\Phi = \sum \phi^i_\alpha z^\alpha \partial_i$ and $\Psi = \sum \psi^i_\alpha z^\alpha \partial_i$ be formal transformations in $G_0(n)$. We have $\Phi \circ \Psi = \sum (\phi \circ \psi)^i_\alpha z^\alpha \partial_i$, with

$$(\phi \circ \psi)^i_\gamma = (\phi^i \circ \psi)_\gamma = \sum_{\substack{n^{j_1}_{\beta_1}+\cdots+n^{j_p}_{\beta_p}=|\alpha| \\ |\alpha|'+n^{j_1}_{\beta_1}|\beta_1|'+\cdots+n^{j_p}_{\beta_p}|\beta_p|'=|\gamma|'}}^{j_k\in\{1,\dots,n\},p\in\mathbb{N}} C^{i\,j_1\quad j_p}_{\alpha\beta_1\cdots\beta_p \atop 1n^{j_1}_{\beta_1}\cdots n^{j_p}_{\beta_p}} \phi^i_\alpha (\psi^{j_1}_{\beta_1})^{n^{j_1}_{\beta_1}} \cdots (\psi^{j_p}_{\beta_p})^{n^{j_p}_{\beta_p}}$$

where the constants C are non-negative integers. Hence

$$({}^\pi\phi \circ {}^\pi\psi)^i_\gamma = \sum C^{i\,j_1\quad j_p}_{\alpha\beta_1\cdots\beta_p \atop 1n^{j_1}_{\beta_1}\cdots n^{j_p}_{\beta_p}} \pi^{|\alpha|'}_{|\alpha|} \pi^{n^{j_1}_{\beta_1}|\beta_1|'}_{|\beta_1|} \cdots \pi^{n^{j_p}_{\beta_p}|\beta_p|'}_{|\beta_p|} \phi^i_\alpha (\psi^{j_1}_{\beta_1})^{n^{j_1}_{\beta_1}} \cdots (\psi^{j_p}_{\beta_p})^{n^{j_p}_{\beta_p}}.$$

The sequence π being increasing, each of the factors $\pi_{|\alpha|}, \pi_{|\beta_1|}, \dots, \pi_{|\beta_p|}$ is less than $\pi_{|\gamma|}$.Therefore we have $\pi^{|\alpha|'}_{|\alpha|} \pi^{n^{j_1}_{\beta_1}|\beta_1|'}_{|\beta_1|} \cdots \pi^{n^{j_p}_{\beta_p}|\beta_p|'}_{|\beta_p|} \leq \pi^{|\gamma|'}_{|\gamma|}$ since

$$|\ \alpha\ |' + n^{j_1}_{\beta_1}\ |\ \beta_1\ |' + \cdots + n^{j_p}_{\beta_p}\ |\ \beta_p\ |' = |\ \gamma\ |'.$$

We conclude that ${}^\pi\Phi \circ {}^\pi\Psi \ll {}^\pi(\Phi \circ \Psi)$. ■

If Φ is a formal transformation of $G_1(n)$ we will say that Φ is a *negative* transformation whenever its coefficients ϕ^i_α are negative real numbers. In this case Φ takes the form

$$\Phi = I - \sum_{i=1}^{n} (\sum_{|\alpha|\geq 2} |\ \phi^i_\alpha\ |\ z^\alpha)\partial_i,$$

with I being the identity transformation. The class of negative transformations has a remarkable behaviour under the group inversion.

Lemma 5.2. *The group inversion restricted to $G_1(n)$ sends negative transformations into positive transformations. Moreover*

$$({}^\pi\Phi)^{-1} \ll {}^\pi(\Phi^{-1}).$$

Proof. Let

$$\Phi = \sum_{i=1}^{n} (z^i - \sum_{|\alpha|\geq 2} |\ \phi^i_\alpha\ |(z^1)^{\alpha_1} \cdots (z^n)^{\alpha_n})\partial_i$$

be a negative formal transformation of $G_1(n)$ and put

$$\Psi = \sum_{j=1}^{n} (z^j + \sum_{|\beta|\geq 2} |\ \psi^j_\beta\ |(z)^\beta)\partial_j = \sum_{j=1}^{n} \psi^j \partial_j.$$

We want to solve the equation $\Phi \circ \Psi = I$ for Ψ, give Φ. To do so, we substitute in Φ each z^i by ψ^i and then set for $i = 1, \dots, n$ the coefficients of z^β to 0 (for $|\ \beta\ | > 1$). It follows that

$$\psi^i_\beta = P_{\beta,i}(|\ \phi^l_\alpha\ |, \psi^m_\gamma)$$

where $P_{\beta,i}$ is a polynomial in $|\ \phi^l_\alpha\ |$ and ψ^m_γ for l,m in $\{1,\dots,n\}$, $|\ \alpha\ |\leq|\ \beta\ |$ and $|\ \gamma\ |<|\ \beta\ |$. The coefficients of $P_{\beta,i}$ are positive integers. We conclude that ψ^i_β takes the form

$$\psi^i_\beta = |\ \phi^i_\beta\ | + \sum_{\substack{p\in\mathbb{N}, |\gamma_l|<|\beta| l=1,\dots,p \\ n^{j_1}_{\gamma_1}|\gamma_1|'+\dots+n^{j_p}_{\gamma_p}|\gamma_p|'=|\beta|'}} J^{j_1\ \dots\ j_p}_{\gamma_1\dots\gamma_p \atop n^{j_1}_{\gamma_1}\dots n^{j_p}_{\gamma_p}} \ |\ \phi^{j_1}_{\gamma_1}\ |^{n^{j_1}_{\gamma_1}} \cdots |\ \phi^{j_p}_{\gamma_p}\ |^{n^{j_p}_{\gamma_p}}$$

where the constants J are positive integers.

Hence the group inversion restricted to $G_1(n)$ sends negative transformations to positive transformations. Following the proof of lemma 5.1, we conclude that $({}^\pi\Phi)^{-1} \ll {}^\pi(\Phi^{-1})$. ■

Theorem 5.3. *For any non-negative integer q and any regular sequence π the set $G^\pi_q(n)$ of isotropy transformations is a Gâteaux analytic Lie group.*

Proof. Let us first show that for any non-negative integer q and any regular sequence π, the subset $G^\pi_q(n)$ of isotropic local transformations of class $\mathcal{C}(\pi)$ is a group. This is a well known result for analytic transformations. Let ϕ and ψ be elements of $G^\pi_q(n)$. Denote by ${}^{\pi^{-1}}$ the inverse of the scale transformation ${}^\pi$. The transformations ${}^{\pi^{-1}}\phi$ and ${}^{\pi^{-1}}\psi$ being analytic, ${}^{\pi^{-1}}\phi\circ{}^{\pi^{-1}}\psi$ is analytic and ${}^\pi({}^{\pi^{-1}}\phi\circ{}^{\pi^{-1}}\psi)$ is a tranformation of class $\mathcal{C}(\pi)$. But by lemma 5.1, we have

$$\phi\circ\psi ={}^\pi ({}^{\pi^{-1}}\phi)\circ{}^\pi ({}^{\pi^{-1}}\psi) \ll{}^\pi ({}^{\pi^{-1}}\phi\circ{}^{\pi^{-1}}\psi).$$

We conclude that the composition is well-defined inside the class $\mathcal{C}(\pi)$.

For the inversion, let us first remark that any transformation ϕ in $G_0(n)$ can be decomposed as $\phi = L\circ\psi$, where L is linear and ψ belongs to $G_1(n)$. Moreover if ϕ is of class $\mathcal{C}(\pi)$, then so is $\psi = L^{-1}\circ\phi$. Since the inverse of ϕ takes the form $\phi^{-1} = \psi^{-1}\circ L^{-1}$, all we need to do in order prove that the inversion respects the class $\mathcal{C}(\pi)$ is to prove that property for transformations in $G_1(n)$. As before consider ${}^{\pi^{-1}}\psi$ that is analytic whenever ψ is of class $\mathcal{C}(\pi)$. Since $({}^{\pi^{-1}}\psi)^{-1}$ is analytic ${}^\pi[({}^{\pi^{-1}}\psi)^{-1}]$ is of class $\mathcal{C}(\pi)$. By lemma 5.2

$$\psi^{-1} = ({}^\pi({}^{\pi^{-1}}\psi))^{-1} \ll{}^\pi [({}^{\pi^{-1}}\psi)^{-1}]$$

which proves that ψ^{-1} is also of class $\mathcal{C}(\pi)$.

Therefore, for any non-negative integer q and any regular sequence π, the set $G^\pi_q(n)$ is a group.

We now define an analytic manifold structure on $G^\pi_q(n)$. Consider first $G^\pi_0(n)$ the full isotropy group of transformations of class $\mathcal{C}(\pi)$, which is the set of series

$$\Phi = \sum_{i=1}^n \Big(\sum_{|\alpha|\geq 1}\phi^i_\alpha z^\alpha\Big)\partial_i$$

of class $\mathcal{C}(\pi)$ for which $\det(\phi^i_j)\neq 0$. As such, it is an open subset of the space of series of class $\mathcal{C}(\pi)$, which we identify with the Lie algebra $\mathcal{L}^\pi_0(n)$ of infinitesimal transformations of class $\mathcal{C}(\pi)$. This provides in a natural way an analytic manifold structure to $G^\pi_0(n)$.

When $q > 0$, $G_q^\pi(n)$ is not an open subset of $\mathcal{L}_q^\pi(n)$ and the above method is therefore no longer applicable. We must proceed differently. Let ∇ be the canonical flat Riemannian connection in $\mathbb{R}^n$. The corresponding exponential mapping $(Exp_\nabla)_x$ associates to each analytic vector field X the local analytic mapping

$$Exp_\nabla(X) : x \mapsto (Exp_\nabla)_x(X(x)).$$

With our choice of connection, we have $Exp_\nabla(X) = I + X$. We now define Exp_∇ for formal vector fields in $\mathcal{L}_0^\pi(n)$ by the same formula. The mapping Exp_∇ is a bijection between $\mathcal{L}_q^\pi(n)$ and $G_q^\pi(n)$ for all regular sequences π and all integers $q > 0$. It thus defines the analytic manifold structure of $G_q^\pi(n)$. In particular, the topology of $G_q^\pi(n)$ for $q > 0$ is nothing but the topology induced by $G_0^\pi(n)$.

Next, we establish the Gâteaux analyticity of the multiplication in $G_0^\pi(n)$. It is convenient for this purpose to associate to each series

$$\Xi = \sum \xi_\alpha^i z^\alpha \partial_i$$

of class $\mathcal{C}(\pi)$ the bounded subset $\mathcal{B}_{|\Xi|}$ of $\mathcal{L}_0^\pi(n)$ of series $\Lambda = \sum \lambda_\alpha^i z^\alpha \partial_i$ satisfying $| \lambda_\alpha^i | \leq | \xi_\alpha^i |$ for all indices i, α. The collection of bounded subsets $\mathcal{B}_{|\Xi|}$ forms a basis for the bornology of $\mathcal{L}_0^\pi(n)$ in the sense that each bounded subset is contained in some $\mathcal{B}_{|\Xi|}$.

Let $\Phi = \sum \phi_\alpha^i z^\alpha \partial_i$ and $\Psi = \sum \psi_\alpha^i z^\alpha \partial_i$ be elements of $G_0^\pi(n)$. We have

$$\Phi \circ \Psi = \sum (\phi \circ \psi)_\alpha^i z^\alpha \partial_i,$$

where

$$(\phi \circ \psi)_\gamma^i = (\phi^i \circ \psi)_\gamma = \sum_{\substack{n_{\beta_1}^{j_1} + \cdots + n_{\beta_p}^{j_p} = |\alpha| \\ |\alpha|' + n_{\beta_1}^{j_1} |\beta_1|' + \cdots + n_{\beta_p}^{j_p} |\beta_p|' = |\gamma|'}}^{j_k \in \{1, \ldots, n\}, p \in \mathbb{N}} C^{i\, j_1 \;\; j_p}_{\alpha \beta_1 \cdots \beta_p \atop 1 n_{\beta_1}^{j_1} \cdots n_{\beta_p}^{j_p}} \phi_\alpha^i (\psi_{\beta_1}^{j_1})^{n_{\beta_1}^{j_1}} \cdots (\psi_{\beta_p}^{j_p})^{n_{\beta_p}^{j_p}}.$$

and the constants C are non-negative integers. It follows that

$$| \Xi | = \sum | \xi_\alpha^i | z^\alpha \partial_i$$

is the maximal element in $\mathcal{B}_{|\Xi|}$ with respect to the composition. Indeed for any pair Φ_1, Φ_2 of series belonging respectively to $\mathcal{B}_{|\Xi_1|}, \mathcal{B}_{|\Xi_2|}$, we have

$$\Phi_1 \circ \Phi_2 \ll | \Xi_1 | \circ | \Xi_2 | .$$

In other words, we have

$$\mathcal{B}_{|\Xi_1|} \circ \mathcal{B}_{|\Xi_2|} \subseteq \mathcal{B}_{|\Xi_1| \circ |\Xi_2|},$$

and the composition is a bounded mapping.

Now given two transformations Φ_0 and Ψ_0 of $G_0^\pi(n)$, their product can be uniquely decomposed as a series $\sum_{n=0}^\infty f_{n,x_0}$ of homogeneous polynomials of degree n in a neighbourhood of $x_0 = (\Phi_0, \Psi_0)$ in $\mathcal{L}_0^\pi(n)$. The positiveness of the coefficients implies that

$$f_{n,x_0}(\mathcal{B}_{|\Xi|}) \subset \mathcal{B}_{|\Phi_0| + |\Xi|} \circ \mathcal{B}_{|\Psi_0| + |\Xi|} \subset \mathcal{B}_{|\Omega|}$$

where $\mid \Omega \mid = (\mid \Phi_0 \mid + \mid \Xi \mid) \circ (\mid \Psi_0 \mid + \mid \Xi \mid)$. Hence each homogeneous polynomial appearing the decomposition is continuous by virtue of Theorem 2.12. Using Definition 2.5, we conclude that the multiplication is Gâteaux analytic. Indeed, observe that for any $0 \leq t \leq 1$, we have

$$(\Phi_0 + \mathcal{B}_{t|\Theta|}) \circ (\Psi_0 + \mathcal{B}_{t|\Gamma|}) \subset \Phi_0 \circ \Psi_0 + t\mathcal{B}_{|\Upsilon|}$$

where $\mid \Upsilon \mid = (\mid \Phi_0 \mid + \mid \Theta \mid) \circ (\mid \Psi_0 \mid + \mid \Gamma \mid) - \mid \Phi_0 \mid \circ \mid \Psi_0 \mid$. It follows that the image of any bounded sequence in $G_0^\pi(n) \times G_0^\pi(n)$ converging to (Φ_0, Ψ_0) by the multiplication map converges to $\Phi_0 \circ \Psi_0$ in $G_0^\pi(n)$. Hence the multiplication is Gâteaux analytic in the group $G_0^\pi(n)$ by Theorem 2.14. The Gâteaux analyticity of the multiplication in $G_q^\pi(n)$ for any $q > 0$ is established similarly.

We now establish the analyticity of the inversion. We first consider the groups $G_q^\pi(n)$ with $q > 0$. To each transformation

$$Exp_\nabla(\Xi) = I + \sum_{|\alpha| \leq 2} \xi_\alpha^i z^\alpha \partial_i$$

we associate as before a bounded subset $\mathcal{B}_{|\Xi|}$ of $\mathcal{L}_q^\pi(n)$, consisting of those series

$$\Upsilon = \sum_{|\alpha| \geq 2} \upsilon_\alpha^i z^\alpha \partial_i$$

for which $\mid \upsilon_\alpha^i \mid \leq \mid \xi_\alpha^i \mid$ for all i, α. This time the maximal element in $\mathcal{B}_{|\Xi|}$ for the inversion is $- \mid \Xi \mid$. Indeed, it follows from the positiveness of the coefficients J introduced in the proof of Lemma 5.2 shows that

$$Exp_\nabla^{-1} \circ [Exp_\nabla(\mathcal{B}_{|\Xi|})]^{-1} \subset \mathcal{B}_{Exp_\nabla^{-1} \circ [Exp_\nabla(-|\Xi|)]^{-1}}.$$

In particular the inversion is a bounded mapping.

Given a transformation $\Phi_0 = Exp_\nabla(\Xi_0)$ in $G_q^\pi(n)$ its inverse can be uniquely decomposed as a series

$$\sum_{n=0}^{\infty} g_{n,\Xi_0}$$

of homogeneous polynomials of degree n. We have

$$g_{n,\Xi_0}(\mathcal{B}_{|\Omega|}) \subset \mathcal{B}_{Exp_\nabla^{-1}[Exp_\nabla(-|\Xi_0|-|\Omega|)^{-1}]}$$

so that each of these homogeneous polynomials is continuous (Theorem 2.12). In addition, we have for any $0 \leq t \leq 1$

$$Exp_\nabla^{-1}[(Exp_\nabla(\Xi_0 + \mathcal{B}_{t|\Omega|})^{-1}] \subset Exp_\nabla^{-1}[Exp_\nabla(\Xi_0)^{-1}] + \mathcal{B}_{t|\Delta|},$$

where

$$\mid \Delta \mid = Exp_\nabla^{-1}[Exp_\nabla(- \mid \Xi_0 \mid - \mid \Omega \mid)^{-1}] - Exp_\nabla^{-1}(Exp_\nabla(- \mid \Xi_0 \mid)^{-1}).$$

It follows that the inversion in $G_q^\pi(n)$, $q \geq 1$ is continuous and thus Gâteaux analytic(theorem 2.14).

Finally, for the full isotropy group $G_0^\pi(n)$, we proceed as follows. The Lie algebra $\mathcal{L}_0^\pi(n)$ can be decomposed as a direct sum $\mathcal{L}_0^\pi(n) = L \bigoplus \mathcal{L}_1^\pi(n)$ where L is the Lie sub-algebra of vector fields with homogeneous linear coefficients. Consider the map

$$id_L \bigotimes Exp_\nabla : G_0^\pi(n) \subset \mathcal{L}_0^\pi(n) \to G_0^\pi(n)$$

given by

$$id_L \bigotimes Exp_\nabla(A + \Xi) = A \circ Exp_\nabla(\Xi)$$

for $A \in L$ and $\Xi \in \mathcal{L}_1^\pi(n)$. We can check as before that this map is an analytic isomorphism. But

$$(A \circ Exp_\nabla(\Xi))^{-1} = (Exp_\nabla \Xi)^{-1} \circ A^{-1}.$$

Since the inversion is analytic in $G_1^\pi(n)$ and the composition is analytic in $G_0^\pi(n)$ we conclude that the inversion is Gâteaux analytic in the full group $G_0^\pi(n)$. ∎

Exponential mapping and regularity: If $\mathcal{V}$ is a local analytic vector field in $\mathbb{R}^n$ and f an analytic function, $\mathcal{V}f$ denotes as usual $df(\mathcal{V})$.

Proposition 5.4. *The exponential map sends, whenever defined, positive vector fields into positive transformations and preserves the class $\mathcal{C}(\pi)$. It is, for each non-negative integer q, an entire Gâteaux analytic mapping from $\mathcal{L}_q^\pi(n)$ into $G_q^\pi(n)$ given by $(Exp\mathcal{V})(z^1, \dots, z^n) = (w^1, \dots, w^n)$ with $w^i = (I + \sum_{k=1}^\infty \frac{1}{k!}\mathcal{V}^k)z^i$ for all $1 \le i \le n$. Symbolically, we have*

$$Exp\mathcal{V} = I + \sum_{n=1}^\infty \frac{1}{n!}\mathcal{V}^n.$$

Proof. Let ϕ_t denote the exponential $\mathrm{Exp}t\mathcal{V}$ of $t\mathcal{V}$. By definition $\dot{\phi_t} = \mathcal{V} \circ \phi_t$ where the dot denotes the derivative with respect to t. Hence $\phi_t^{(k)} = \mathcal{V}^k \circ \phi_t$ for the k-th derivative. Taylor's formula applied to ϕ_t at $t = 0$ gives the formal expansion $\mathrm{Exp}\mathcal{V}(z) = (I + \sum_{n=1}^\infty \frac{1}{n!}\mathcal{V}^n)(z)$. If we put $\mathrm{Exp}\mathcal{V} = \sum(\mathrm{Exp}\mathcal{V})_\alpha^i z^\alpha \partial_i$ we obtain

$$(\mathrm{Exp}\mathcal{V})_\alpha^i = \sum_{n_{\beta_1}^{j_1}|\beta_1|' + \cdots + n_{\beta_p}^{j_p}|\beta_p|' = |\alpha|'} E_{n_{\beta_1}^{j_1} \cdots n_{\beta_p}^{j_p}}^{\beta_1 \cdots \beta_p \atop j_1 \quad j_p} (\mathcal{V}_{\beta_1}^{j_1})^{n_{\beta_1}^{j_1}} \cdots (\mathcal{V}_{\beta_p}^{j_p})^{n_{\beta_p}^{j_p}},$$

where teh coefficients E are positive integers. Using the scale transformation $^\pi$, the convergence of the exponential mapping for analytic pseudogroups and the method introduced in the proof of Theorem 5.3 we obtain that the exponential mapping from $\mathcal{L}_q^\pi(n)$ into $G_q^\pi(n)$ is an entire Gâteaux analytic mapping. ∎

This Proposition shall in fact be extended as follows:

Theorem 5.5. *For any given continuous path (resp. of class C^k) v of the Lie algebra $\mathcal{L}_q^\pi(n)$ $(q \ge 0)$, the the left logarithmic derivative ordinary differential equation $g^{-1}\dot{g} = v$, with initial condition given by $g(0) = e$, admits a unique solution. This solution γ_v is of class C^1 (resp. C^{k+1}). Moreover the correspondence $v \mapsto \gamma_v$ is continuous and Gâteaux analytic (entire). It is given by the expansion*

$$\gamma_v(t) = I + \int_0^t v(\tau_1)d\tau_1 + \cdots + \int_0^t \!\!\cdots\! \int_0^{\tau_n} v(\tau_{n+1}) \cdots v(\tau_1)d\tau_{n+1} \cdots d\tau_1 + \cdots$$

Proof. The uniqueness is a well-known fact [24]. Following Picard's method, we set $g_0 = I$ and define g_n recursively as $\dot{g}_{n+1} = v g_n$. We thus obtain

$$g_{n+1}(t) = I + \int_0^t v(\tau_1)d\tau_1 + \cdots + \int_0^t \cdots \int_0^{\tau_n} v(\tau_{n+1}) \cdots v(\tau_1)d\tau_{n+1} \cdots d\tau_1.$$

We claim that the solution γ_v is nothing but the limit as n tends to infinity of g_n in the locally convex topology of $\mathcal{L}_q^\pi(n)$. If $\mathcal{V}$ and $\mathcal{W}$ are two vector fields of $\mathcal{L}^\pi(n)$, let $\mathcal{V}W$ denote, by abuse of notation, the vector field $\sum_{i=1}^n (\mathcal{V}\mathcal{W}^i)\partial_i$. The non-associative algebra structure defined on $\mathcal{L}^\pi(n)$ by $(\mathcal{V}, \mathcal{W}) \mapsto \mathcal{V}W$ admits the same filtration as the Lie algebra structure. That is to say $\mathcal{L}_p(n) \cdot \mathcal{L}_q(n) \subseteq \mathcal{L}_{p+q}(n)$. Hence the convergence is clear in the formal Lie groups $G_q(n)$ for $q > 0$. Since moreover,

$$\mathcal{L}_0(n)/\mathcal{L}_1(n) \cdot \mathcal{L}_p(n)/\mathcal{L}_{p+1}(n) \subseteq \mathcal{L}_p(n)/\mathcal{L}_{p+1}(n),$$

the convergence comes from the finite dimensional theory for $q = 0$. Now remark that the range by v of the compact segment $\mathcal{I}$ is bounded in the Silva space $\mathcal{L}_q^\pi(n)$. Hence there exists a positive formal vector field v_{max} belonging to $\mathcal{C}(\pi)$ such that $\mid v(t) \mid \leq v_{max}$ for all t in $\mathcal{I}$. We conclude that $\mid \gamma_v(t) \mid \leq \mathrm{Exp} t v_{max}$ for all t in $\mathcal{I}$ which proves that the class $\mathcal{C}(\pi)$ is also preserved. ∎

General isotropy group structure: We now summarize results obtained in Section 7.

Theorem 5.6. *For every regular sequence π and every non-negative integer q, the general isotropy group $G_q^\pi(n)$ is a regular Gâteaux analytic Lie group modelled on the Silva space $\mathcal{L}_q^\pi(n)$.*

6. Lie's third fundamental theorem

Recall the following result concerning the formal structure of isotropy groups, [33].

Theorem 6.1. *The isotropy subgroup $G_0(n)$ of $G(n)$ is, in the one-dimensional case a non-CBH exponential Lie group and, for $n > 1$ an analytic Lie group of second kind and second order. In all cases $G_1(n)$ is a CBH Lie subgroup and $G(n)$ is a Lie pseudogroup of the second kind and second order. That is $G_0(1) \notin \mathcal{C}BH$ but $G_0(1) \in \mathcal{E}XP$, and $G(n > 1) \in \mathcal{C}BH^2 \subset \mathcal{E}XP^2$.*

By making use of Écalle's results on the iterative theory [12], one can show that for most regular sequences π, $G^\pi(\Gamma)$ is of second order, its Lie subgroup $G_1^\pi(\Gamma)$ being then a CBH Lie group. For a regular sequence $\pi = \{\pi_n\}$, let us define $d\pi$ as $d\pi = \{n\pi_n\}$. The following is a re-formulation of Theorem 3 in Chapter II of [12].

Theorem 6.2. *For any given integer $q > 0$, the logarithm is well defined in $G_q(\Gamma)$ and sends $G_q^\pi(\Gamma)$ into $\mathcal{L}_q^{d\pi}(\Gamma)$. For all Gevrey sequences Θ of index $\theta \geq 1$ the Lie subalgebra $\mathcal{L}_1^\Theta(\Gamma)$ is of CBH type. Hence $G^\Theta(\Gamma)$ is a Lie pseudogroup of second kind and order 2.*

Lie's second fundamental theorem: Let us define a *compatible chart* of a Lie group G with Lie algebra $\mathcal{L}$ to be a pair (U, ϕ) consisting of an open neighbourhood U of the origin in $\mathcal{L}$ and of a pointed local diffeomorphism $\phi : (0, U) \to (e, G)$ compatible with the underlying manifold structure of G. We will need the following (see [30],[31][32]).

Theorem 6.3. Lie II *Let G be a Lie group with Lie algebra $\mathcal{L}$ having (U, ϕ) as compatible chart. Let $\mathcal{H} \hookrightarrow \mathcal{L}$ be a closed Lie subalgebra. Suppose that $\phi^{-1} \circ \phi \otimes \phi$ restricted to $(V \cap \mathcal{H}) \times (V \cap \mathcal{H})$ takes its values in $\mathcal{H}$ for some open set $V \subset U$. Then $\mathcal{H}$ is the Lie algebra of a unique connected Lie subgroup H of G.*

Charts of the second kind and countable order: Let Γ be a flat Lie pseudogroup and let Exp_q denote the restriction to the vector subspace

$$\mathcal{L}_{q-1}(\Gamma)/\mathcal{L}_q(\Gamma)$$

of the exponential mapping. Consider the left product exponential mapping PExp_λ defined in $\mathcal{L}_0(n)$ with values in $G_0(n)$, that associates to $X = \sum_{k=1}^{\infty} X_k$ the formal local diffeomorphism

$$\mathrm{PExp}_\lambda(X) = \lim_N \mathrm{Exp}_N(X_N) \otimes \cdots \otimes \mathrm{Exp}_1(X_1).$$

It is clear that $(\mathcal{L}_0(n), \mathrm{PExp}_\lambda)$ is a compatible chart for the formal isotropy group $G_0(n)$.

Proposition 6.4. *Let π be any regular sequence. The left product exponential mapping $PExp_\lambda$ is well defined and realizes a local Gâteaux analytic isomorphism between $\chi_0^\pi(n)$ and $G_0^\pi(n)$.*

Remark 6.5. It is sufficient to restrict our attention to the subalgebra $\chi_1^\pi(n)$ and its associated group $G_1^\pi(n)$.

The proof of Proposition 6.4 rests on the following lemmas.

Lemma 6.6. *The inverse formal mapping $LogP_\lambda$ sends any transformation of the form $I - X$ where X is positive into a negative formal vector field.*

Lemma 6.7. *Let ϕ be any local diffeomorphism of the form $I - X$, X being a positive vector field. Then $\Delta = \phi \circ ExpX = I - \sum_{k\geq 1} \frac{k}{(k+1)!} X^{k+1}$.*

Proof. We have $\Delta = \mathrm{Exp}X - X \circ \mathrm{Exp}X$. But X is $\mathrm{Ad}(\mathrm{Exp}X)$ invariant. Hence $d\mathrm{Exp}X(X) = X \circ \mathrm{Exp}X$ with $d\mathrm{Exp}X(X) = X + \sum_{k\geq 1} \frac{1}{k!} X^{k+1}$ from Proposition 5.4. This completes the proof. ∎

The proof of Lemma 6.6 follows from a recurrence based on Lemma 6.7.

Lemma 6.8. *The formal exponential mapping $PExp_\rho : \chi(n) \to G(n)$ that associates to $X = \sum_{k=-1}^{+\infty} X_k$ the infinite product $ExpX_{-1} \circ ExpX_0 \circ \cdots \circ ExpX_p \circ \cdots$ sends positive formal vector fields into positive formal transformations. Moreover if $g = PExp_\rho X$ with $X = (X_1, \ldots, X_n)$ and $X_j = \sum_{|\alpha|=2}^{\infty} a_\alpha^j x^\alpha$ positive, then $g_\alpha^j \geq a_\alpha^j$.*

Proof. This is a simple consequence of Lemma 5.1 and Proposition 5.4. ∎

Proof. (of Proposition 6.4). Let $g = I + Z$ be an arbitrary formal transformation of class π tangent at order 1 to the identity. From lemma 6.6 it is clear that $\mathrm{LogP}_\lambda(g)$ belongs to $\chi_1^\pi(n)$ if $\mathrm{LogP}_\lambda(\mid g \mid)$ does, where $\mid g \mid = I - \mid Z \mid$. Since $\mid g \mid \in G_1^\pi(n)$ and $G_1^\pi(n)$ is a group $\mid g \mid^{-1} \in G_1^\pi(n)$. But

$$\mid g \mid^{-1} = \mathrm{PExp}_\rho(-\mathrm{LogP}_\lambda \mid g \mid)$$

and $-\mathrm{LogP}_\lambda \mid g \mid$ is positive according to Lemma 6.6. Lemma 6.8 leads to the conclusion that $-\mathrm{LogP}_\lambda \mid g \mid$ belongs to $\chi_1^\pi(n)$ completing the proof. ∎

The Lie-Cartan extension for isotropy algebras: The resulst of Section 6 may be summarized as follows:

Theorem 6.9. Lie III *Let π be any regular sequence and Γ a flat transitive Lie pseudogroup of* infinite type. *For any given non-negative integer q, the isotropy Lie subalgebra $\mathcal{L}_q^\pi(\Gamma)$ of the formal Lie algebra $\mathcal{L}(\Gamma)$ is the Lie algebra of a unique connected and simply connected regular Gâteaux-analytic Lie group $\tilde{G}_q^\pi(\Gamma)$ of the second kind and of countable order. A canonical chart for the analytic Lie pseudogroup $G_0^\pi(\Gamma)$ is given by $ExpX_0 \circ \cdots \circ ExpX_p \cdots$.*

7. Examples

We will say for short that a Lie pseudogroup is of the second kind and countable order if its isotropy Lie subpseudogroup is a Lie group of the second kind and countable order.

Example 7.1. The isotropy Lie group of local holomorphic transformations of $\mathbb{C}^n$ leaving the origin invariant is a Gâteaux holomorphic Lie group of the second kind and countable order.

In what follows M is a differentiable manifold M of dimension n.

Example 7.2. Let ω be a smooth volume form defined on M. Then the isotropy Lie group of analytic volume preserving local transformations $\mathcal{P}_{loc}(\omega) = \{\phi \in \mathrm{Diff}_{loc}(M)/\phi^\star\omega = \omega\}$ is a Lie group of the second kind and countable order. The same holds for the one-dimensional extension

$$\tilde{\mathcal{P}}_{loc}(\omega) = \{\phi \in \mathrm{Diff}_{loc}(M)/\phi^\star\omega = \lambda\omega, \lambda \in \mathbb{R}\}.$$

Example 7.3. Let Ω be a closed C^∞ two-form defined on M and satisfying $\Omega^r \neq 0$, $\Omega^{r+1} = 0$, $r < n$ being constant. According to Darboux's theorem, there exist local coordinates $w^1, \ldots, w^n$ such that

$$\Omega = dw^1 \wedge dw^2 + \cdots + dw^{2r-1} \wedge dw^{2r}.$$

Hence the Lie pseudogroup of local analytic transformations

$$\mathcal{P}_{loc}^\omega(\Omega) = \{\phi \in \mathrm{Diff}_{loc}^\omega(M)/\phi^\star\Omega = \Omega\}$$

is a Lie pseudogroup of second kind and countable order.

Example 7.4. Hamiltonian Lie pseudogroups of analytic local transformations are Lie pseudogroups of the second kind and countable order.

Example 7.5. Let α be a smooth Pfaffian form on M of constant rank r. By Pfaff's theorem there exists a local coordinate system $w^1, \ldots, w^n$ such that

$$\alpha = dw^1 + w^2 dw^3 + \cdots + w^{2r} dw^{2r+1}.$$

The local Lie pseudogroup $\mathcal{P}_{loc}(\alpha) = \{\phi \in \mathrm{Diff}_{loc}(M) / \phi^\star \alpha = \alpha\}$ is not a flat transitive Lie pseudogroup. But it decomposes into an extension involving $\mathcal{P}_{loc}(d\alpha)$. Hence it is a Lie pseudogroup of second kind and countable order.

Example 7.6. The Poisson Lie algebra of local real analytic functions of a symplectic manifold is a Lie algebra integrable into a Lie pseudogroup of second kind and countable order.

Let us now consider a Lie-Poisson structure associated with a r dimensional Lie algebra $\mathcal{G}$. The Lie-Poisson bracket defined on the dual $\mathcal{G}^\star$ is defined by $\{f,h\}(x) =< x, [df(x), dh(x)] >$ for any $x \in \mathcal{G}^\star$ [25].

Theorem 7.7. *The local analytic Lie-Poisson algebra $\mathcal{P}$ associated with any finite dimensional Lie algebra $\mathcal{G}$ is integrable into a unique connected and simply connected Gâteaux analytic regular Lie group of the second kind and countable order.*

Proof. Let α be the mapping that associates to each function $f \in \mathcal{P}$ its associated Hamiltonian vector field X_f. It is well known that X_f is an infinitesimal automorphism of the Lie-Poisson structure. If c_{ij}^k, $i,j,k = 1, \ldots, r$ are the structure constants of $\mathcal{G}$ relative to a basis, the Lie-Poisson bracket can take the coordinate form

$$\{f,h\} = \sum_{i,j,k=1}^{r} c_{ij}^k x^k \partial_i f \partial_j h.$$

As a consequence the local Lie pseudogroup $Aut(\mathcal{P})$ of automorphisms of the Lie-Poisson structure is a flat isotropy group. This is equally true for the image of $\mathcal{P}$ by the mapping α. Hence this latter is integrable into a unique connected and simply connected Gâteaux analytic regular Lie group I of the second kind and countable order. The Poisson algebra $\mathcal{P}$ splits naturally into the direct sum $Ker(\alpha) \bigoplus \alpha(\mathcal{P})$. Since $Ker(\alpha)$ is nothing but the centre of $\mathcal{P}$, we conclude that $\mathcal{P}$ is integrable into the direct product of Lie groups $Ker(\alpha) \times I$. The result follows from the regularity of I. ∎

Remark 7.8. The natural topology for the Lie algebra of local smooth vector fields defined near the origin in $\mathbb{R}^n$ is a (non-strict) inductive limit topology of Fréchet spaces [33]. In this topology the Lie algebra of analytic local vector fields is a dense Lie subalgebra. Our results can be regarded as density theorems in the smooth case.

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