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Reduction in contact geometry

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Abstract. The notion of a moment map for (exact) contact manifolds is introduced. We prove a reduction theorem analogous to the Marsden-Weinstein theorem in the (exact) symplectic case. Furthermore the existence of a local normal form for the moment map in a neighborhood of a point in the zero moment level is proved. This implies a local slice theorem for contact transformation groups.

1. Introduction and statement of results

Consider a smooth manifold C. A distribution $H \subseteq TC$ of hyperplanes in the tangent bundle $\pi: TC \to C$ induces a bundle morphism $\omega: H \times H \to L$, where $H \times H$ denotes the bundle product $\{(\xi, \eta) \mid \pi(\xi) = \pi(\eta)\}$ and L is the quotient bundle TC/H, given by the Lie bracket on the vector fields on C. In fact, when two vectors ξ and η in the fibre H_c over $c \in C$ are extended to vector fields X and Y respectively on C which are sections of H, then the class of $[X,Y]_c$ in $TC_c/H_c = L_c$ does not depend on the choice of the extensions; thus $\omega_c(\xi, \eta) = [X,Y]_c \bmod H_c \in L_c$ is well-defined. We say that the distribution $H \subseteq TC$ is of *constant rank*, if the rank of the skew-symmetric bilinear map ω_c does not depend on c. In case that ω is of maximal rank (i.e., ω is non-degenerate) H is called a contact structure on C and (C, H) is a contact manifold.

A special situation occurs when the distribution H is given as the kernel of a globally defined differential 1-form α on C, $H_c = \ker(\alpha_c)$. In this case we refer to the pair (C, α) as an *exact contact manifold.* If φ is a nowhere vanishing function on C, then the form $\varphi\alpha$ clearly induces the same contact structure on C. However, the exact contact manifold $(C, \varphi\alpha)$ is different from (C, α).

Let G be a Lie group. If (C, H) is a contact manifold and G acts on C by diffeomorphisms, $G \times C \to C$, $(g, c) \mapsto g.c$, we say that G acts by *contactomorphisms*, if any transformation $g: C \to C$, $c \mapsto g.c$ respects the contact structure, i.e., $g_*H = g^*H$. (We denote by g^*H the pullback bundle of H, i.e., $(g^*H)_c = H_{g.c}$ and by $g_*: TC \to g^*TC$ the differential of $g: C \to C$.) Similarly, if (C, α) is an exact contact manifold, then G is said to act by *exact*

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contactomorphisms, if any $g: C \to C$ respects α, i.e., $g^\sharp\alpha = \alpha$. (We denote by $g^*\alpha: g^*TC \to \mathbf{R}$ the pullback of α as a section and by $g^\sharp\alpha = g^*\alpha \circ g_*: TC \to \mathbf{R}$ the pullback of α as a 1-form on C.) In analogy to the exact symplectic case (see e.g. [8]) we define the corresponding *moment map* as the smooth map from C into the dual of the Lie algebra $\mathbf{g}$ of G, $\Phi: C \to \mathbf{g}^*$, given by

$$\Phi_a(c) := \langle \Phi(c), a\rangle := \langle \alpha_c, a_C(c)\rangle,$$

for any $a \in \mathbf{g}$. Here $\langle\cdot,\cdot\rangle$ denotes the natural pairing between a vector space and its dual and a_C denotes the vector field on C corresponding to the infinitesimal action of $\mathbf{g}$, $a_C(c) = \frac{d}{dt}_{|t=0}(\exp(ta).c)$. Φ is easily seen to be equivariant with respect to the given action on C and the coadjoint action on $\mathbf{g}^*$. Moreover Φ satisfies the so-called moment condition

$$d\Phi_a = -i_{a_C} d\alpha.$$

In the case of a contact manifold (C, H) we consider the natural bundle homomorphism $\alpha: TC \to L = TC/H$. We consider α as an L-valued differential 1-form and define the corresponding *moment map* as a section in the bundle $\mathbf{g}^* \otimes L$ over C. By abuse of notation we denote by $\mathbf{g}^*$ here the trivial vector bundle $C \times \mathbf{g}^*$ over C. Notice that the action $g.(c, \lambda) := (g.c, \mathrm{Ad}^*_g(\lambda))$ gives $\mathbf{g}^*$ the structure of a G-bundle over C. Since $H \subseteq TC$ is G-invariant, L carries the structure of a G-bundle as well. So $Y := \mathbf{g}^* \otimes L$ is a G-bundle. The moment map $\Phi: C \to Y = \mathrm{Hom}(\mathbf{g}, L)$ given by

$$\Phi_a(c) = \langle \Phi(c), a\rangle = \langle \alpha_c, a_C(c)\rangle \in L_c$$

turns out to be equivariant and moreover satisfies a certain moment condition as well (see proposition 2).

The first aim of this note is to describe the natural reduction procedure for contact structures and exact contact structures in the case where the Lie group acts freely and properly. In analogy to the symplectic and exact symplectic case we call this the *Marsden-Weinstein reduction*.

Theorem 1 (Reduction for contact manifolds). *Let (C, H) be a contact manifold and let G be a Lie group acting by contactomorphims freely and properly on C. Let $L = TC/H$, $\Phi: C \to Y := \mathbf{g}^* \otimes L$ the induced moment map and $\mathbf{o} \subseteq Y$ be the image of the zero section of $Y \to C$. Then the following holds:*

(a) *$M := \Phi^{-1}(\mathbf{o})$ is a smooth invariant submanifold of C;*

(b) *$i^\sharp H := i_*^{-1}(i^*H) \subseteq TM$ defines a distribution of constant rank on M, where $i: M \hookrightarrow C$ denotes the inclusion map;*

(c) *there exists a unique distribution $H_0 \subseteq TC_0$ of hyperplanes with $\pi^\sharp H_0 = i^\sharp H$, where $\pi: M \to C_0 := M/G$ denotes the natural G-fibration. Moreover H_0 is non-degenerate and defines therefore a contact structure on C_0.*

Observe that in this contact case a natural moment map always exists. This is in contrast to the symplectic case where a moment map corresponding to a symplectic action may not exist (cf. [3] for the obstructions which could occur).

Theorem 2 (Reduction for exact contact manifolds). *Let (C,α) be an exact contact manifold and let G be a Lie group acting by exact contactomorphisms freely and properly on C. Let $\Phi\colon C\to \mathbf{g}^*$ be the natural moment map. Then the following holds:*

(a) *$M:=\Phi^{-1}(0)$ is a smooth and invariant submanifold of C;*

(b) *there exists a unique 1-form α_0 on C_0 with $\pi^\sharp\alpha_0=i^\sharp\alpha$, where $i\colon M\hookrightarrow C$ denotes the inclusion and $\pi\colon M\to C_0:=M/G$ denotes the natural G-fibration. Moreover α_0 defines an exact contact structure on C_0.*

The corresponding statement in the exact symplectic case is also true. More precisely: If (M,α) is an exact symplectic manifold (i.e. $\omega:=-d\alpha$ is non-degenerate) where a Lie group G acts freely and properly by exact symplectomorphisms, then there exists a natural moment map and the symplectic reduction M_0 carries a canonical *exact* symplectic structure α_0 (see [4]).

As is well known (see [1], [2] e.g.), one associates to any contact manifold (C,H) its so-called (exact) symplectification (P,β) which is the $\mathbf{R}^*$-principal bundle $\pi\colon P\to C$ corresponding to the line bundle $L^*\to C$ together with a canonical 1-form β. Here L^* denotes the dual of the line bundle L. It turns out that an action of G by contactomorphisms on C lifts to an action on P by exact symplectomorphisms such that π is equivariant. Moreover the action of G stays free and proper. Finally the induced moment map $\Phi_P\colon P\to\mathbf{g}^*$, $\langle\Phi_P,a\rangle=\langle\beta,a_P\rangle$ for $p\in P$, $a\in\mathbf{g}$, is compatible with the moment map $\Phi_C\colon C\to\mathbf{g}^*\otimes L$ in the sense that

$$\langle\Phi_P(p),a\rangle=p(\langle\Phi_C\circ\pi(p),a\rangle),$$

for all $a\in\mathbf{g}$ and $p\in P$. Thus one may reduce (P,β) by the exact symplectic reduction procedure and obtains an exact symplectic manifold (P_0,β_0). On the other hand one can reduce (C,H) by contact reduction to (C_0,H_0) and it is worthwile to mention that (P_0,β_0) is the symplectification of (C_0,H_0). Thus symplectification and reduction commutes. (However, this is not the way how theorem 1 is proved. In fact, it is technically and conceptually simpler to stay in the contact category itself.)

The second aim of the paper is to investigate the case where G acts properly but not necessarily freely. So for a point $c\in C$ the isotropy group $K=G_c=\{g\in G\mid g.c=c\}$ may be non-trivial. (However, it is compact.) In this situation one would like to prove a so-called *slice theorem*, i.e., one looks for a certain model space D together with an embedding $j\colon G/K\hookrightarrow D$ such that G-invariant neighborhoods U of $j(G/K)\subseteq D$ and V of $i(G/K)\subseteq C$, $i\colon G/K\hookrightarrow C$, $gK\mapsto g.c$, are G-equivariantly contactomorphic.

As in the symplectic case (cf. [7] e.g.) such a normal form will be given for points in the zero moment level $M=\Phi^{-1}(\mathbf{o})\subseteq C$, where Φ is the natural moment map. Suppose that $c\in M$. Then the G-orbit of c is necessarily tangential, i.e., $\mathbf{g}c:=\{a_C(c)\mid a\in\mathbf{g}\}$ is contained in $H_c\subseteq TC_c$. In fact, by the definition of α and Φ one sees immediately that $0=\Phi_a(c)=\langle\alpha,a_C\rangle(c)$, i.e., $a_C(c)\in\ker\alpha_c=H_c$. Moreover $\mathbf{g}c\subseteq H_c$ is isotropic with respect to the I-valued symplectic form $\omega_c\colon H_c\times H_c\to I$. Here I denotes the quotient space TC_c/H_c which is a K-module in a natural way. Of course, ω_c is K-equivariant. One considers now the perpendicular $\mathbf{g}c^{\perp}$ of $\mathbf{g}c \subseteq H_c$ with respect to ω_c and defines

$$E := \mathbf{g}c^{\perp}/\mathbf{g}c.$$

Then the triple (I, E, τ), where $\tau: E \times E \to I$ is the non-degenerate skew-symmetric and K-equivariant bilinear form induced from ω_c, is called a *contact slice* for G at c. Associated to (I, E, τ) we construct a canonical contact structure on the vector bundle

$$D := G \times_K (\mathbf{k}^0 \otimes I \times E \times I).$$

Here $\mathbf{k}^0 \subseteq \mathbf{g}^*$ denotes the annihilator of $\mathbf{k} \subseteq \mathbf{g}$ and D is the G-bundle over G/K associated to the K-module $\mathbf{k}^0 \otimes I \times E \times I$ via the diagonal K-action with respect to the principal K-bundle $G \to G/K$. The induced moment map turns out to be the map $\Phi: D \to \mathbf{g}^* \otimes I \cong (\mathbf{k}^0 \oplus \mathbf{k}^*) \otimes I$,

$$\Phi([g, (\lambda, v, w)]) = \mathrm{Ad}_g^*(\lambda + \Phi_E(v)).$$

Here G acts on $\mathbf{g}^* \otimes I = \mathrm{Hom}(\mathbf{g}, I)$ by the coadjoint action in the first factor and $\Phi_E: E \to I$ is the natural moment map according to the I-valued symplectic structure on E, i.e., the homogeneous quadratic polynomial

$$\langle \Phi_E(v), b \rangle = \frac{1}{2}\tau(b.v, v)$$

for $b \in \mathbf{k}$.

Theorem 3 (Local contact slice theorem). *Let (C, H) be a contact manifold and let G act properly on C by contactomorphisms. Let Φ be the induced moment map and $c \in C$ in the zero moment level of Φ. Denote by (I, E, τ) the contact slice of G in c. Then there exists G-invariant neighborhoods U of the zero section $j: G/K \hookrightarrow D$ and V of $i: G/K \hookrightarrow C$, $gK \mapsto g.c$, and a G-equivariant contactomorphism $f: U \to V$ with $f \circ j = i$.*

In the special case where I is the trivial K-module (e.g., if K is connected this must be the case since K is compact and I is 1-dimensional) the contact slice (E, τ) gives also rise to the space $P = G \times_K (\mathbf{k}^0 \times E)$, which serves as a model space for the symplectic case. In fact, P carries even the structure of an *exact* symplectic manifold and G acts by *exact* symplectomorphisms. Now for any exact symplectic manifold (P, β) it is well-known (see e.g. [1], [2]) that there exists a natural (exact) contactification of P, namely $C = P \times \mathbf{R}$ with the contact form $\alpha = dz - \pi_1^*\beta$ (where π_1 is the projection onto the first factor and z is a coordinate on $\mathbf{R}$). As it turns out the model space D for the contact case is just the contactification of the model space P for the symplectic case.

2. Reduction of contact manifolds and exact contact manifolds

Let (C, α) be an exact contact manifold and let a Lie group G act by exact contactomorphisms, $g^\sharp\alpha = \alpha$ for all $g \in G$. The natural moment map is given by the map $\Phi: C \to \mathbf{g}^*$, $\Phi_a = \langle \Phi, a \rangle$ for $a \in \mathbf{g}$ and

$$\Phi_a = \langle \alpha, a_C \rangle,$$

where a_C denotes the vector field on C corresponding to $a \in \mathbf{g}$. As in the exact symplectic case it is now seen that Φ is equivariant with respect to the given action on C and the coadjoint action on $\mathbf{g}^*$. In fact, since the exponential map $\exp\colon \mathbf{g} \to G$ is equivariant with respect to the adjoint action Ad on $\mathbf{g}$ and the conjugation action on G, one finds that

$$(\mathrm{Ad}_g(a))_C(g.c) = \frac{d}{dt}_{|t=0}(g\exp(ta)g^{-1}.g.c) = g_*(a_C)(c)$$

and therefore

$$\langle \mathrm{Ad}^*_{g^{-1}}(\Phi(g.c)), a\rangle = \langle \Phi(g.c), \mathrm{Ad}_g(a)\rangle = \langle \alpha_{g.c}, (\mathrm{Ad}_g(a))_C(g.c)\rangle$$
$$= \langle \alpha_{g.c}, g_*(a_C)(c)\rangle = \langle g^\sharp\alpha_c, a_C(c)\rangle = \langle \alpha, a_C\rangle(c) = \Phi_a(c),$$

thus $\Phi(g.c) = \mathrm{Ad}^*_g(\Phi(c))$.

Similarly the moment condition for Φ transfers to the exact contact case. Since $g^\sharp\alpha = \alpha$ for all $g \in G$, the Lie derivative of α in the orbit direction vanishes, $\mathcal{L}_{a_C}\alpha = 0$ for all $a \in \mathbf{g}$. So

$$0 = \mathcal{L}_{a_C}\alpha = i_{a_C}d\alpha + di_{a_C}\alpha.$$

Denoting $\omega := -d\alpha$ this implies

$$d\Phi_a = i_{a_C}\omega$$

as in the symplectic case.

Proposition 1. *Let (C, α) be an exact contact manifold as above and $\Phi\colon C \to \mathbf{g}^*$ its associated moment map. For any $c \in \Phi^{-1}(0)$ the following is true:*

(a) *(i) If $S_c := \ker(D\Phi_c)$, then S_c is transversal to $H_c = \ker(\alpha_c)$, $S_c + H_c = TC_c$;*

(ii) if $K_c := S_c \cap H_c$, then the infinitesimal G-orbit $\mathbf{g}c$ is contained in K_c and is the kernel of the restriction of $\omega_c := -d\alpha_c$ to $K_c \times K_c$, $K_c^\perp = \mathbf{g}c$;

(b)

$$\mathrm{ran}(D\Phi_c) = \mathbf{k}^0,$$

where $\mathbf{k}^0 \subseteq \mathbf{g}^*$ is the annihilator of the infinitesimal isotropy $\mathbf{k} = \{a \in \mathbf{g} \mid a_C(c) = 0\}$ of c and $\mathrm{ran}(D\Phi_c)$ denotes the image of $D\Phi_c$.

Proof. **(a)** Recall that by definition of the contact structure the restriction of $d\alpha_c$ to $H_c \times H_c$ is non-degenerate, since

$$d\alpha(\xi, \eta) = -\alpha([X, Y]),$$

if X, Y are extensions of ξ, η with values in H. Therefore the kernel of $d\alpha\colon TC_c \times TC_c \to \mathbf{R}$, $R_c := \ker(d\alpha_c)$, is 1-dimensional and transversal to H_c, $H_c \oplus R_c = TC_c$. By the moment condition R_c is obviously contained in $\ker D\Phi_c = S_c$; thus S_c is transversal to H_c.

Now let $\xi \in K_c = S_c \cap H_c$. Then $0 = \langle D\Phi_c, \xi\rangle$, which means $0 = d\alpha(a_C(c), \xi)$ for all $a \in \mathbf{g}$. Thus $\xi \in K_c$ if and only if $\xi \in \mathbf{g}c^\perp \subseteq H_c$. On the other hand the orbit Gc is in $M = \Phi^{-1}(0)$ by the equivariance of Φ, thus $\mathbf{g}c \subseteq S_c$; and $\mathbf{g}c \subseteq H_c$ by the very definition of Φ, which shows that $\mathbf{g}c \subseteq \mathbf{g}c^\perp = K_c$. The kernel of the restriction of $d\alpha_c$ to $K_c \times K_c$ is therefore $\mathbf{g}c^{\perp\perp} = \mathbf{g}c$.

(b) It is clear that $\mathrm{ran}(D\Phi_c) \subseteq \mathbf{k}^0$ since for any $a \in \mathbf{k}$ by definition $a_C(c) = 0$ and therefore

$$\langle D\Phi_c(\xi), a\rangle = \langle d\Phi_a(c), \xi\rangle = -d\alpha(a_C(c), \xi) = 0$$

for all $\xi \in TC_c$. On the other hand $\dim\ker(D\Phi_c) = (2n - \dim(\mathbf{g}/\mathbf{k}) + 1$ by part (a) and therefore $\dim \mathrm{ran} D\Phi_c = \dim \mathbf{g}/\mathbf{k} = \dim \mathbf{k}^0$ which shows that $\mathrm{ran}(D\Phi_c) = \mathbf{k}^0$. ∎

Corollary. **(a)** *If G acts freely, then $0 \in \mathbf{g}^*$ is a regular value of Φ and $M = \Phi^{-1}(0)$ is a $\dim G$-codimensional and invariant submanifold of C;*

(b) *If G acts freely and properly, then $C_0 := M/G$ is a manifold and $\pi: M \to C_0$ is a G-principal bundle.*

Proof. **(a)** The equivariance of Φ shows the invariance of $M \subseteq C$ and, by part (b) of the proposition, Φ is submersive. So the result follows from the implicit function theorem.

(b) If G acts freely and properly on C, it acts freely and properly on M as well. Thus $M \to M/G$ is a G-principal bundle (by the standard slice theorem for proper actions). ∎

Consider now a contact manifold (C, H) and a Lie group G acting on C by contact diffeomorphisms, i.e., $g^\sharp H = H$. Thus $H \subseteq TC$ is a G-subbundle and the quotient bundle $L = TC/H$ inherits the structure of a G-bundle as well. The natural L-valued 1-form, i.e., the bundle homomorphism $\alpha: TC \to L$ given by the canonical projection, is then G-equivariant. Now the associated moment map, i.e., the section $\Phi: C \to \mathbf{g}^* \otimes L$ given by $\Phi_a = \langle \Phi, a\rangle$,

$$\Phi_a = \langle \alpha, a_C\rangle,$$

is equivariant with respect to the given action on C and the diagonal action on the tensor bundle $Y = \mathbf{g}^* \otimes L$. In particular the inverse image of the zero section $\mathbf{o} \subseteq Y$,

$$M = \Phi^{-1}(0),$$

is G-invariant since $\mathbf{o} \subseteq Y$ is G-invariant.

Fix now $c_0 \in M$ and suppose that G acts properly on M. Then the local slice theorem (see [6]) guarantees that there exists an invariant neighborhood U of c_0 where the line bundle L is G-equivariantly trivial, i.e., there exists a G-equivariant and nowhere vanishing section $s: U \to L|U$. This implies that for every $a \in \mathbf{g}$ there exists a unique function $\Psi_a: U \to \mathbf{R}$ such that

$$\Phi_a(c) = \Psi_a(c)s(c)$$

for all $c \in U$. Similarly there exists a unique and G-invariant 1-form β on U, $\beta \in \mathcal{E}^1(U)$, so that

$$\alpha|U = \beta \cdot s.$$

Now one sees that $\Psi_a = \langle \beta, a_C \rangle$ and, as in the exact case, a computation shows that

$$d\Psi_a = -i_{a_C} d\beta.$$

To put this formula into a global form recall that for any L-valued 1-form $\alpha : TC \to L$ one can define the exterior differential $d\alpha$ as an L-valued skew-symmetric bilinear form on $H = \ker(\alpha)$, i.e., $d\alpha : H \times H \to L$ by the following (cf. [5]). If $U \subseteq C$ is an open set where there exists a nowhere vanishing section s of L, then $\alpha = \beta s$ for a unique 1-form $\beta \in \mathcal{E}^1(U)$. Another such section $\tilde{s}$ is of the form $\tilde{s} = \varphi^{-1} s$ for $\varphi \in \mathcal{E}^*(U)$ and then $\alpha = \tilde{\beta}\tilde{s}$ with $\tilde{\beta} = \varphi\beta$. But $d\tilde{\beta} = d\varphi \wedge \beta + \varphi\, d\beta$; so $d\tilde{\beta}|H = \varphi\, d\beta|H$ since $H = \ker(\beta)$, and thus

$$d\tilde{\beta} \cdot \tilde{s} = d\beta \cdot s,$$

showing that $d\alpha$ $(= d\beta \cdot s$ on $U)$ is well defined. A contact structure on a manifold C can then be seen as a contact structure (H, ω) on the tangent bundle. i.e., $H \subseteq TC$ is a hyperplane bundle and

$$\omega : H \times H \to L := TC/H$$

is a non-degenerate skew-symmetric bilinear form, satisfying the *closeness condition*

$$-d\alpha = \omega,$$

where $\alpha : TC \to L$ is the canonical projection.

But now observe that in our case, where $\alpha : TC \to L$ is a G-equivariant bundle homomorphism not only the contraction of $\omega = -d\alpha$ by an element X with values in H is well defined as a bundle morphism on H, i.e., $i_X\omega : H \to L$, but moreover, if X comes from the group action on C, i.e., $X = a_C$ with $a \in \mathbf{g}$, the contraction $i_{a_C}(-d\alpha)$ makes sense even on the whole tangent bundle TC along M, i.e., $ia_C(-d\alpha)_c : TC_c \to L_c$. In fact, suppose that $U \subseteq C$ is G-invariant and $L \to C$ trivializes G-equivariantly over U as above via an equivariant section s. Then $\alpha = \beta s$ for an invariant 1-form β on U and one sets

$$i_{a_C}(-d\alpha) := i_{a_C}(-d\beta) \cdot s.$$

Namely, if $s = \varphi\tilde{s}$ with another equivariant nowhere vanishing section $\tilde{s}$, then necessarily φ is G-invariant. Infinitesimally this means $\langle d\varphi, a_C \rangle = 0$ for all $a \in \mathbf{g}$. Moreover, if $c \in M = \Phi^{-1}(\mathbf{o})$, then $\langle \beta, a_C \rangle(c) = 0$ by the definition of the moment map and therefore for $\alpha = \tilde{\beta}\tilde{s}$ with $\tilde{\beta} = \varphi\beta$ we see that

$$\begin{aligned} i_{a_C} d\tilde{\beta} &= i_{a_C}(d\varphi \wedge \beta + \varphi\, d\beta) \\ &= \langle d\varphi, a_C \rangle \beta - \langle \beta, a_C \rangle d\varphi + \varphi\, i_{a_C} d\beta = \varphi\, i_{a_C} d\beta, \end{aligned}$$

showing that the definition of $i_{a_C}(-d\alpha)_c : TC_c \to L_c$ is well-behaved for $c \in M$.

Finally, as for any vector bundle, the tangent space of L in an element $o_c \in L_c$ of the zero section splits canonically into

$$TL_{o_c} = TC_c \oplus L_c,$$

and we let $\pi_2 \colon TL_{o_c} \to L_c$ the projection onto the second factor. Now, using that the moment map $\Phi \colon C \to Y$ writes $\Phi | U = \Psi s$, where Ψ is the moment map for the exact contact manifold (U, β), one sees that the moment condition in the case of a contact transformation group reads as follows.

Proposition 2. *If G acts on a contact manifold (C, H) by contact diffeomorphisms and if Φ is the associated moment map, then for any $c \in M = \Phi^{-1}(\mathbf{o})$ we have the* moment condition

$$\pi_2 \circ (d\Phi_a)_c = -(i_{a_C} d\alpha)_c.$$

The reader will have no difficulty to prove the statements analogous to those given in proposition 1 for the non-exact contact case.

Proposition 3. *Let (C, H) be a contact manifold and $\Phi \colon C \to \mathbf{g}^* \otimes L$ (with $L = TC/H$) the associated moment map. Let $\mathbf{o} \subseteq Y = \mathbf{g}^* \otimes L$ be the zero section and $c \in M = \Phi^{-1}(\mathbf{o})$. Then the following is true:*

(a) *(i) If $S_c = D\Phi_c^{-1}(T\mathbf{o}_{\Phi(c)}) \subseteq TC_c$, then S_c is transversal to H_c, $H_c + S_c = TC_c$;*

(ii) if $K_c := S_c \cap H_c$, then the infinitesimal G-orbit $\mathbf{g}c$ in contained in K_c and in fact the kernel of the restriction of $\omega_c = -d\alpha_c \colon H_c \times H_c \to L_c$ to $K_c \times K_c \to L_c \cong S_c/K_c$.

(b) *If $\mathbf{k} \subseteq \mathbf{g}$ is the isotropy algebra of c, then*

$$\operatorname{ran}(\pi_2 \circ D\Phi_c) = \mathbf{k}^0,$$

the annihilator of $\mathbf{k}$ in $\mathbf{g}^*$.

Corollary. **(a)** *If G acts freely on C, then $\Phi \colon C \to Y = \mathbf{g}^* \otimes L$ is transversal to the zero section $\mathbf{o} \subseteq Y$ and $M = \Phi^{-1}(0)$ is a $\dim G$-codimensional and invariant submanifold of C.*

(b) *If G acts freely and properly, then $C_0 = M/G$ is a manifold and $\pi \colon M \to C_0$ is a G-principal bundle.*

At this point we want to recall what we mean by a contact vector space (cf. [5]). It is given by a triple (V, H, ω), where V is a (real) vector space, $H \subseteq V$ is a hyperplane and $\omega \colon H \times H \to V/H$ is a non-degenerate skew-symmetric bilinear map. The situation we arrived at above is the following. If $W \subseteq V$ is a transversal subspace, i.e., $W + H = V$, then (W, K, τ) with $K := W \cap H$ and $\tau = \omega | K \times K \to V/H \cong W/K$ is what we call a *precontact space*, i.e., K is a hyperplane and τ is skew-symmetric (not necessarily non-degenerate). One may reduce a precontact space (W, K, τ) in a natural way by the following. Denote by $N \subseteq K$ the kernel of τ and further $W_0 := W/N$, $K_0 := K/N$, and $\tau_0 \colon K_0 \times K_0 \to W_0/K_0 \cong W/K$ the induced structure from τ. It is clear that (W_0, K_0, τ_0) is a contact vector space.

We are now able to prove theorem 1 and theorem 2 in the introduction.

Proof. Let (C, H) be a (possibly non-exact) contact manifold and consider the L-valued 1-form $\alpha: TC \to L := TC/H$. Suppose that the Lie group G acts by contact transformations freely and properly and let $\Phi: C \to \mathbf{g}^* \otimes L$, $\Phi_a = \langle \alpha, a_C \rangle$, be the induced moment map. We have already seen that the preimage of the zero section $\mathbf{o} \subseteq \mathbf{g}^* \otimes L$, i.e., $M = \Phi^{-1}(0)$, is G-invariant and in fact a G-principal bundle over the quotient $C_0 = M/G$ by the natural projection $\pi: M \to C_0$.

Now fix $c_0 \in C_0$ and choose an element $c \in M$ in the fibre of π over c_0. The contact structure on TC_c, which is denoted by (H_c, ω_c) as usual, induces a precontact structure on $TM_c = S_c$ as was shown above, $K_c = H_c \cap S_c$, $\tau_c = \omega_c | K_c \times K_c$. Moreover, by proposition 3, the kernel of τ_c is exactly the infinitesimal G-orbit $\mathbf{g}c \subseteq K_c$. Therefore, by the preceding remark, $TM_c/\mathbf{g}c$ carries the structure (K_0, τ_0) of a contact vector space in a natural way. On the other hand the differential $D\pi_c: TM_c \to (TC_0)_{c_0}$ induces an isomorphism

$$f_c: TM/\mathbf{g}c \to (TC_0)_{c_0}.$$

Thus we define a contact structure $((H_0)_{c_0}, (\omega_0)_c)$ on $(TC_0)_{c_0}$ such that f_c is in fact a contact isomorphism.

Next we observe that this construction of $(H_0, \omega)_{c_0}$ is independent of the choice of the preimage $c \in \pi^{-1}(c_0)$. Indeed, if $\tilde{c}$ is another preimage, then $c = g.\tilde{c}$ for a group element $g \in G$ and by the invariance of the original contact structure H on C one finds that the construction of $(H_0, \omega_0)_{c_0}$ on $(TC_0)_{c_0}$ is in fact well defined.

Finally we have to check that (H_0, ω_0) does in fact define a contact structure on C_0 (not only on the tangent bundle TC_0). So let $\alpha_0: TC_0 \to L_0 := TC_0/H_0$ be the induced L_0-valued 1-form. We have to check that $-d\alpha_0: H_0 \times H_0 \to L_0$ satisfies $-d\alpha_0 = \omega_0$, which is the closeness condition for contact manifolds. But $\pi^\sharp(\omega_0) = i^\sharp(\omega)$ by construction of the structure ω_0, where $i: M \hookrightarrow C$ is the inclusion, and $\pi^\sharp\alpha_0 = i^\sharp\alpha$ by definition of $H_0 \subseteq TC_0$. Therefore

$$\pi^\sharp(-d\alpha_0) = d(-\pi^\sharp\alpha_0) = -d(i^\sharp\alpha) = -i^\sharp(d\alpha) = i^\sharp\omega = \pi^\sharp\omega_0,$$

and by the fact that $\pi: M \to C_0$ is a submersion we conclude that $-d\alpha_0 = \omega_0$. We have proved theorem 1. ∎

Proof. Let (C, α) be an exact contact manifold, G a Lie group acting by exact contactomorphisms freely and properly and $\Phi: C \to \mathbf{g}^*$, $\Phi_a = \langle \alpha, a_C \rangle$, be the induced moment map. The zero level $M = \Phi^{-1}(0)$ is a G-invariant submanifold and a G-principal bundle over the quotient $C_0 = M/G$ by proposition 1, $\pi: M \to C_0$. Moreover, the pullback $i^\sharp\alpha$ on M, $i: M \hookrightarrow C$ denoting the inclusion, defines an *exact precontact manifold* in the sense that $K = \ker(i^\sharp\alpha) \subseteq TM$ is a hyperplane bundle but $\tau := -d(i^\sharp\alpha) : K \times K \to \mathbf{R}$ is not necessarily non-degenerate. However, in our case it is of constant rank and its kernel is given exactly by the direction of the G-orbits, i.e.,

$$\ker(d(i^\sharp\alpha)_c) = \mathbf{g}c,$$

again by proposition 1. For any $c_0 \in C_0$ and $c \in \pi^{-1}(c_0)$ we denote by $D\pi_c$: $TM_c \to (TC_0)_{c_0}$ the differential of π in c as before. Thus there exists a unique homomorphism $(\alpha_0)_{c_0}: (TC_0)_{c_0} \to \mathbf{R}$ with $(\alpha_0)_{c_0} \circ D\pi_c = \alpha_c$. Moreover, there exists a unique skew-symmetric bilinear map $(\omega_0)_{c_0}: (TC_0)_{c_0} \times (TC_0)_{c_0} \to \mathbf{R}$ with $(\omega_0)_{c_0} \circ (D\pi_c \times D\pi_c) = \tau_c$ by the reduction procedure for precontact spaces.

We have to check that the definition of $(\alpha_0)_{c_0}$ and $(\omega_0)_{c_0}$ on $(TC_0)_{c_0}$ is independent of the chosen preimage $c \in \pi^{-1}(c_0)$. This follows from the invariance of α and ω and is left to the reader. We have defined now the global 1-form α_0 and the global 2-form ω_0 on C_0. Finally we have to check that $\omega_0 = -d\alpha_0$. But again $\pi^\sharp\alpha_0 = i^\sharp\alpha$ and $\pi^\sharp(\omega_0) = \tau = i^\sharp(\omega) = i^\sharp(-d\alpha)$ by construction and therefore

$$\pi^\sharp(-d\alpha_0) = -d(\pi^\sharp\alpha_0) = -d(i^\sharp\alpha) = i^\sharp(-d\alpha) = \pi^\sharp(\omega_0),$$

showing that $-d\alpha_0 = \omega_0$ and proving theorem 2. ■

Example. If M is a manifold (of dimension $n+1$), $m \in M$, then any hyperplane $c \subseteq TM_m$ is called a *contact element* on M. The space of all contact elements on M is therefore the projectivized ($\mathbf{P}^n$-) bundle $C = \mathbf{P}(T^*M)$. It carries a canonical (non-exact) contact structure given by: $\xi \in H_c \subseteq TC_c$ if $\pi_*\xi \in c$, where $\pi: C \to M$ is the bundle projection. Let G be a Lie group acting on M by smooth diffeomorphisms. Then G acts on $\mathbf{P}(T^*M))$ by bundle homomorphisms as well. Denote by $E_m = \mathbf{g}m \subseteq TM_m$ an infinitesimal orbit. It is seen directly from the definition of the moment map on C that $c \in \pi^{-1}(m)$ is in the zero moment level if and only if $c \supseteq E_m$. Thus $\Phi^{-1}(\mathbf{o}) = \mathbf{P}(E^0)$, where $E^0 \subseteq T^*M$ denotes the annihilator of E. It is clear that $E_m^0 \cong (TM_m/E_m)^*$ and therefore isomorphic to $(TM_0)^*_{m_0}$, when G is acting freely and properly on M and M_0 is its orbit space M/G ($m_0 = Gm$). This shows that the contact reduction of the manifold of all contact elements on M is nothing else than the manifold of all contact elements on the reduction M_0,

$$(\mathbf{P}(T^*M))_0 = \mathbf{P}(T^*M_0).$$

Example. A partial differential equation of the first order on an open set $\Omega \subseteq \mathbf{R}^n$ is given by an equation $F(x,u,p) = 0$ on $\Omega\times\mathbf{R}\times\mathbf{R}^n$ and a solution is a smooth function $u: \Omega \to \mathbf{R}$ so that $F(x, u(x), Du(x)) = 0$ for all $x \in \Omega$. A generalization of this concept to a manifold M of dimension n is given by a hypersurface Γ in the space J^1M of 1-jets of functions on M. A 1-jet of a smooth function f on M at $p \in M$ is by definition the element $j^1f_p := f \bmod \mathbf{m}_p^2 \in \mathcal{E}_p/\mathbf{m}_p^2$, where $\mathcal{E}_p$ denotes the germs of smooth functions at p and $\mathbf{m}_p \subseteq \mathcal{E}_p$ its maximal ideal. J^1M comes along with a natural vector bundle structure $\pi: J^1M \to M$ of rank $n+1$, $j^1f_p \mapsto p$. Now $C := J^1M$ carries a natural exact contact structure given by the 1-form α, $\alpha_c(\xi) = \pi_*\xi(c) - \mathrm{pr}_*(\xi)$, where $\mathrm{pr}: J^1M \to \mathbf{R}$ is given by $\mathrm{pr}(j^1f_p) = f(p)$. ($\pi_*\xi \in TM_p$ acts as a derivation on the 1-jet $c = j^1f_p \in \pi^{-1}(p)$.) For every function f on M one has the associated section $j^1f: M \to J^1M =: C$, $p \mapsto j^1f_p$, which defines a Legendre submanifold of C, i.e., an integral submanifold of the contact distribution of C of maximal dimension n. Given a hypersurface $\Gamma \subseteq C$ we say that f is a solution if the image of its section j^1f is contained in Γ.

Now if a Lie group G acts by (exact) contact transformations (freely and properly) on C and if Γ is a G-invariant hypersurface, it is clear how the reduction of C reduces the problem of finding a solution of the partial differential equation given by Γ to an appropriate Γ_0 on the reduced space C_0. In particular, if the action is induced from a G-action on M itself, it is not hard to see that C_0 coincides with the 1-jet bundle of $M_0 = M/G$.

3. A local slice theorem for contact transformation groups

We consider now a contact manifold (C,H), on which a Lie group G acts properly by contactomorphisms but not necessarily freely. Thus the isotropy groups $G_c = \{g \in G \mid g.c = c\} \subseteq G$ are not necessarily trivial. The aim of this section is to find a normal form of the G-action in the neighborhood of an orbit lying in the zero moment level $M = \Phi^{-1}(\mathbf{o}) \subseteq C$.

The basic ingredient is a tubular neighborhood theorem for contact geometry as it is proved in [5]. For convenience we repeat here its equivariant version. Suppose that G acts on (C,H) properly by contactomorphisms as above and let B be a G-manifold and $i: B \hookrightarrow C$ an equivariant *tangential embedding*, meaning that $i_*TB \subseteq i^*H$. One observes then that i_*TB is an isotropic subbundle of i^*H with respect to the i^*L-valued structure $i^*\omega: i^*H \times i^*H \to i^*L$ where $\omega = -d\alpha$, $\alpha: TC \to L = TC/H$. In fact, this is just the fact that the Lie bracket of two vector fields along i with values in i_*TB (i.e., vector fields on B) is again a vector field along i with values in i_*TB. Therefore one can build the so-called *symplectic normal bundle* $E := i_*TB^\perp / i_*TB$ over B. In fact, it is a G-vector bundle together with an I-valued and equivariant symplectic structure $\tau: E \times E \to I := i^*L$ inherited from $i^*\omega$. The tangential embedding theorem asserts that the triple (I,E,τ) characterizes a neighborhood of $i(B) \subseteq C$ up to contactomorphisms. More precisely, we say that two triples (I_1,E_1,τ_1) and (I_2,E_2,τ_2) are isomorphic, if there exist equivariant bundle isomorphisms $\theta: I_1 \to I_2$ and $\eta: E_1 \to E_2$ such that $\tau_2 \circ (\eta \times \eta) = \theta \circ \tau_1$.

Theorem (Equivariant tangential embedding). *Suppose that $i_1: B \hookrightarrow (C_1,H_1)$ and $i_2: B \hookrightarrow (C_2,H_2)$ are two equivariant tangential embeddings with isomorphic symplectic normal bundles (I_1,E_1,τ_1) and (I_2,E_2,τ_2). Then there exist invariant neighborhoods $U_1 \subseteq C_1$ of $i_1(B)$ and $U_2 \subseteq C_2$ of $i_2(B)$ and an equivariant contactomorphism $f: U_1 \to U_2$ satisfying $f \circ i_1 = i_2$.*

For the proof one has to check that the various steps in the proof of the tangential embedding theorem carry over to the equivariant case which could be done.

So what we have to do in order to determine the neighborhood geometry of a G-orbit in $\Phi^{-1}(\mathbf{o}) \subseteq C$ is to identify its symplectic normal bundle. In fact, by the very definition of $\Phi: C \to \mathbf{g}^* \otimes L$, a G-orbit in $M = \Phi^{-1}(\mathbf{o})$ is tangential. Moreover, $Gc \cong G/K$ (Gc denoting the orbit $\{g.c \mid g \in G\}$ and K the isotropy in c, $K = G_c$), and the symplectic normal bundle $(\mathbf{I}, \mathbf{E}, \bar{\tau})$ over Gc is a K-bundle. As such it is of the form

$$\mathbf{I} = G \times_K I, \qquad \mathbf{E} = G \times_K E$$

for the K-vector spaces I and E sitting as the fibres over c. Thus, all we have to do is to look for the triple (I,E,τ), where $I = TC_c/H_c$, $E = \mathbf{g}c^\perp/\mathbf{g}c$ and

$\tau: E \times E \to I$ is the symplectic structure induced from $i^*\omega_c: H_c \times H_c \to L_c$. We have called (I, E, τ) the *contact slice G at c*.

The rest of this section is therefore devoted to the construction of a certain model space D associated to the datum (I, E, τ), which will be a contact manifold where G acts by contactomorphisms and an equivariant tangential embedding $j: G/K \hookrightarrow D$ such that the contact slice at $K \in G/K$ is exactly (I, E, τ).

So let I be a 1-dimensional K-module, let E be a $2r$-dimensional K-module and let $\tau: E \times E \to I$ be a non-degenerate skew-symmetric and equivariant bilinear form. As a first step we consider the K-line bundle $\mathbf{I}$ over G associated to I, i.e., $\mathbf{I} := G \times I$ together with the action

$$k.(g, w) = (gk^{-1}, k.w).$$

On the other hand right multiplication on G induces a K-structure on the tangent bundle $T^*G \to G$ and thus we can form the tensor product $T^*G \otimes \mathbf{I}$ as a K-bundle via the diagonal action. The fibre over an element $g \in G$ is identified in a natural way with $\mathrm{Hom}(TG_g, I)$,

$$(T^*G \otimes \mathbf{I})_g = \mathrm{Hom}(TG_g, I);$$

thus $T^*G \otimes \mathbf{I}$ may be seen as the I-valued covectors on G. If one identifies T^*G via left multiplication on G with $G \times \mathbf{g}^*$, i.e.,

$$(g, \lambda) \mapsto (DL_g)_1^*(\lambda)$$

(where $L_g: G \to G$ denotes left multiplication with g on G), then the action of K on $T^*G \otimes \mathbf{I} \cong G \times \mathbf{g}^* \otimes I = G \otimes \mathrm{Hom}(\mathbf{g}, I)$ reads

$$k.(g, \lambda) = (gk^{-1}, k.\lambda),$$

where $k.\lambda(a) = k.\lambda(\mathrm{Ad}_{k^{-1}}(a))$ for $a \in \mathbf{g}$.

The next step is to see that $T^*G \otimes \mathbf{I}$ carries a canonical I-valued 1-form β. In terms of the trivialization $G \times \mathrm{Hom}(\mathbf{g}, I)$ the tangent space in a point (g, λ) may be identified with $TG_g \times \mathrm{Hom}(\mathbf{g}, I) \cong \mathbf{g} \times \mathrm{Hom}(\mathbf{g}, I)$ (again identified by left multiplication). The form β is then given simply by

$$\beta_{g,\lambda}(a, \mu) = \langle \lambda, a \rangle.$$

It is worthwile to note that just as a vector bundle over G (forgetting the K-action for a moment) $T^*G \otimes \mathbf{I}$ is isomorphic to T^*G since $I = \mathbf{R}$ and also the 1-form β is the canonical one. However, the K-bundle $T^*G \otimes \mathbf{I}$ is different from T^*G (if the action of K on I is not the trivial one). The essential point is then that the canonical I-valued 1-form is K-equivariant with respect to the action $T^*G \otimes \mathbf{I}$ and the action on I.

Now consider the whole datum (I, E, τ). We choose a basis $(\xi_1, \ldots, \xi_r, \eta_1, \ldots, \eta_r)$ of E and (ζ) of I with corresponding coordinates $(x^1, \ldots, x^r, y^1, \ldots, y^r) = (x, y)$ and z such that the form $\tau: E \times E \to I$ reads

$$\tau = dx \wedge dy \cdot \zeta.$$

Then one sees immediately that the I-valued 1-form

$$\beta := \frac{1}{2}(y\,dx - x\,dy)\cdot\zeta$$

is not only a potential of τ, i.e., $-d\beta = \tau$, but moreover, β is K-equivariant as well. Furthermore β is independent of the chosen Darboux-coordinates.

For the next step consider the product $P = (T^*G \otimes \mathbf{I}) \times E$ with its projectors π_1, π_2 onto the two factors. Denote by β_1 the canonical I-valued 1-form on $T^*G \otimes \mathbf{I}$ and β_2 the I-valued 1-form on E just constructed. The sum $\beta := \pi_1^\sharp\beta_1 + \pi_2^\sharp\beta_2$ gives an I-valued 1-form on P. The product

$$C = P \times I$$

with its projection pr_1 onto the first factor gets now the structure of a contact manifold via the I-valued 1-form

$$\alpha = dz\cdot\zeta - \mathrm{pr}_1^\sharp\beta.$$

By construction α is K-equivariant and defines a moment map $\Phi_R\colon C \to \mathbf{k}^*\otimes I = \mathrm{Hom}(\mathbf{k}, I)$. Via the trivialization $C = G \times \mathrm{Hom}(\mathbf{g}, I) \times E \times I$ it reads

$$\Phi_R(g, \lambda, v, w) = -\lambda|\mathbf{k} + \Phi_E(v).$$

Here $\Phi_E\colon E \to \mathbf{k}^*\otimes I = \mathrm{Hom}(\mathbf{k}, I)$ is the natural moment map of E with respect to β, i.e., for $b \in \mathbf{k}$ and $\Phi_b = \langle \Phi_R, b\rangle$ it is

$$\Phi_b(v) = \langle \beta_v, b.v\rangle = \frac{1}{2}\tau(b.v, v),$$

which is homogeneous quadratic in v. A choice of a K-invariant scalar product on $\mathbf{g}$ (recall that K is compact since the action is proper) allows us to embed $\mathbf{k}^*$ as a K-submodule of $\mathbf{g}^*$ via the orthogonal complement of $\mathbf{k}^0 \subseteq \mathbf{g}^*$, $\mathbf{g}^* = \mathbf{k}^0 \oplus \mathbf{k}^*$. Therefore we may identify $\Phi_R^{-1}(0)$ with $G \times (\mathbf{k}^0 \otimes I \times E \times I)$ via

$$(g, \lambda, v, w) \mapsto (g, \lambda + \Phi_E(v), v, w).$$

The contact quotient of C by K, i.e.,

$$D := G \times_K (\mathbf{k}^0 \otimes I \times E \times I),$$

inherits a canonical I-valued 1-form α_0. Observe that the bundle $G \times_K I$, i.e., the cocontact bundle of the contact structure $H_0 = \ker(\alpha_0)$ is now not trivial any longer (if I is not trivial, of course). This is the construction of the local model together with its embedding $j\colon G/K \hookrightarrow D$, which is the zero section of the vector bundle D.

Finally one has to look for the group action of G coming from left multiplication on G. Since it commutes with right multiplication, the whole group action goes through the construction. The G-moment map $\Phi\colon D \to \mathbf{g}^*\otimes I = (\mathbf{k}^0 \oplus \mathbf{k}^*)\otimes I$ reads

$$\Phi([g, \lambda, v, w]) = g.(\lambda + \Phi_E(v)).$$

That is the local normal form of the moment map. Up to the coadjoint action on $\mathrm{Hom}(\mathbf{g}, I)$ it is linear in the first summand and quadratic in the second ("linear" and "angular" momentum).

In fact, in order to prove theorem 3 from section 1 it is now only necessary to convince oneself that the contact slice of $j(K) \in D$, $j: G/K \hookrightarrow D$, is exactly (I, E, τ), which follws from the construction. An application of the equivariant tangential embedding theorem yields the result.

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