Journal of Lie Theory
Volume **11** (2001) 441–458

Invariant Theory for the Orthogonal Group via Star Products

Didier Arnal, Oumarou Boukary Baoua, Chal Benson, and Gail Ratcliff*

Communicated by J. Faraut

Abstract. In this paper we apply star products to the invariant theory for multiplicity free actions. The space of invariants for a compact linear multiplicity free action has two canonical bases which are orthogonal with respect to two different inner products. One of these arises in connection with the star product. We use this fact to determine the elements in the canonical bases for the invariants under the action of $SO(n,\mathbb{R})\times\mathbb{T}$ on $\mathbb{C}^n$. The formulae obtained improve prior results due to the last two authors and Jenkins.

1. Introduction

Let K be a compact Lie group acting unitarily on a finite dimensional Hermitian vector space V. One says that the action of K on V is *multiplicity free* when the space of polynomials $\mathbb{C}[V]$ decomposes into pair-wise inequivalent irreducible K-modules. Although this condition is a very restrictive, there is a rich family of examples which have, moreover, been completely classified [13, 5, 14]. The concept of multiplicity free action plays an organizational role in Classical Invariant Theory. We refer the reader to [11] for motivation and further references.

Suppose now that the action of K on V is multiplicity free and let

$$\mathbb{C}[V]=\sum_{\lambda\in\Lambda}P_\lambda$$

denote the decomposition of $\mathbb{C}[V]$ into K-irreducible subspaces. Here Λ is a countably infinite set of indices. Since the trivial representation can occur only once in $\mathbb{C}[V]$, there are no non-constant K-invariant polynomials in $\mathbb{C}[V]$. It is, however, of interest to consider K-invariants in $\mathbb{C}[V_\mathbb{R}]$, the ring of polynomial functions on the underlying real space. In this context, there are two canonical vector space bases for the algebra $\mathbb{C}[V_\mathbb{R}]^K$ of K-invariants, each indexed by Λ:

- a basis $\{p_\lambda\ :\ \lambda\in\Lambda\}$ consisting of homogeneous polynomials p_λ, and

ISSN 0949–5932 / \$2.50

- a basis $\{q_\lambda : \lambda \in \Lambda\}$ consisting of polynomials orthogonal with respect to the Fock inner product $\langle \cdot, \cdot \rangle_{\mathcal{F}}$ on $\mathbb{C}[V_\mathbb{R}]$.

The polynomials $\{q_\lambda\}$ are obtained from the p_λ's via Gram-Schmidt orthogonalization using the Fock inner product. Precise definitions will be given below in Section 3.

We will show that the p_λ's are themselves orthogonal with respect to the "star" inner product on $\mathbb{C}[V_\mathbb{R}]$, denoted $\langle \cdot, \cdot \rangle_*$, which is obtained from the Fock inner product on $\mathbb{C}[V]$ using $\mathbb{C}[V_\mathbb{R}] \simeq \mathbb{C}[V] \otimes \mathbb{C}[\overline{V}]$. This is a classical process which "doubles the variables". On the other hand, there is a more abstract point of view on the star inner product, which we adopt in Section 2 as the definition. In fact, $\langle f, g \rangle_* = tr(AB^*)$ where A, B are the operators on Fock space with Berezin (or Wick) symbols $\widehat{A}(z) = f(z)e^{-|z|^2/2}$ and $\widehat{B}(z) = g(z)e^{-|z|^2/2}$. This shows that the star inner product arises naturally in the context of the (Berezin) star product.

The current work extends results that are contained in the second author's Ph.D. dissertation [2]. Our goal is to show how the theory of star products can be used as a tool to study invariant theory for multiplicity free actions. The orthogonality of the polynomials $\{p_\lambda\}$ with respect to the star inner product follows from the observation that the projection operator onto P_λ has symbol $\dim(P_\lambda)p_\lambda(z)e^{-|z|^2/2}$. As a consequence, we obtain an orthogonalization procedure, Proposition 3.5, that determines the p_λ's from the *fundamental invariants*, a finite subset of $\{p_\lambda : \lambda \in \Lambda\}$ which generates $\mathbb{C}[V_\mathbb{R}]^K$ as an algebra.

The explicit determination of the p_λ's and q_λ's presents significant combinatorial difficulties. In Section 4 we consider a specific multiplicity free action, that of $K = SO(n, \mathbb{R}) \times \mathbb{T}$ on $V = \mathbb{C}^n$ $(n \geq 3)$. This example is the subject of [4], which contains recurrence relations for the p_λ's. In Proposition 4.1, we derive these relations using star product techniques. Using this result and orthogonality of the p_λ's with respect to the star inner product, we are able to obtain *explicit* formulae for the p_λ's in this example. (See Theorem 4.5.) The *generalized binomial coefficients*, which express the polynomials $\{q_\lambda\}$ in term of the p_λ's, are also given in [6] via recurrence relations. Here, in Theorem 4.8, we give an *explicit* formula for these numbers. Our proof again uses properties of the star inner product. The same techniques are used to prove Proposition 4.9, which gives product formulas for the p_λ's.

We see that the use of star products leads to new and more explicit formulae for the canonical invariants for the action of $SO(n, \mathbb{R}) \times \mathbb{T}$. We are optimistic that these methods can be put to good use in connection with further examples of multiplicity free actions.

We conclude this section by outlining a connection between the problems addressed here and analysis on the Heisenberg group. This provides motivation for the current work and was our original source of interest. If one extends V to form the Heisenberg group $H_V = V \times \mathbb{R}$ with suitable product, then K acts via automorphisms on H_V which fix the center $\mathbb{R}$. One says that the action of K on H_V yields a *Gelfand pair* when the K-invariant integrable functions $L^1_K(H_V)$ form a commutative algebra under convolution. This is the case if and only if the action of K on V is multiplicity free. There is, moreover, a well developed theory of spherical functions associated to such Gelfand pairs. The polynomials $\{q_\lambda : \lambda \in \Lambda\}$ determine a dense set of full measure in the space of bounded spherical functions. We refer the reader to [3] for details on this connection.

2. Preliminaries on the star product

This section contains background material concerning Berezin symbols and star products. Our viewpoint here is concrete and combinatorial. We will discuss only the most classical version of these ideas, involving Fock space and its reproducing kernel. (See Section 2.7 in [10] and the original source [8] for further details). For generalizations to the setting of a Hilbert space with reproducing kernel, we refer the reader to [9] and [15]. For connections to the orbit method in representation theory, see [1], [2] and [17].

Throughout this paper, V denotes a complex vector space with finite dimension n, (positive definite) Hermitian inner product $\langle\cdot,\cdot\rangle$, and norm $|\cdot|$. Often it will be convenient to identify $(V,\langle\cdot,\cdot\rangle)$ with $\mathbb{C}^n$ equipped with the standard Hermitian structure $\langle z,w\rangle = z\cdot\overline{w}$. This can always be done by choosing any orthonormal basis for V.

Fock space: The space $\mathbb{C}[V]$ of holomorphic polynomials on V can be equipped with the Fock (or Fischer) inner product

$$\langle f,g\rangle_{\mathcal{F}} = \left(\frac{1}{2\pi}\right)^n \int_V f(z)\overline{g(z)}e^{-|z|^2/2}dz. \tag{2.1}$$

Here "dz" denotes Lebesgue measure on the underlying real vector space $V_{\mathbb{R}} \simeq \mathbb{R}^{2n}$, normalized by the inner product on V. Fock space $\mathcal{F}=\mathcal{F}_n$ is the Hilbert space completion of $\mathbb{C}[V]$ with respect to $\langle\cdot,\cdot\rangle_{\mathcal{F}}$. This is the set of holomorphic functions on V which are square integrable with respect to the Gaussian measure $e^{-|z|^2/2}dz$. Equation (2.1) extends from $\mathbb{C}[V]$ to $\mathcal{F}$. See [10] for the completeness property of this space. We remark that the Bargmann transform yields an isometry from $L^2(\mathbb{R}^n)$ to $\mathcal{F}$. This fact, however, will play no role in the current work.

Identifying V with $\mathbb{C}^n$ yields an isomorphism $\mathbb{C}[V]\simeq\mathbb{C}[z_1,\ldots,z_n]$. The monomials $z^\alpha = z_1^{\alpha_1}\cdots z_n^{\alpha_n}$ form a complete system in $\mathcal{F}$ with

$$\langle z^\alpha, z^\beta\rangle_{\mathcal{F}} = \begin{cases} 2^{|\alpha|}\alpha! & \text{for } \beta=\alpha \\ 0 & \text{for } \beta\neq\alpha \end{cases}. \tag{2.2}$$

Here we have used the standard multi-index notation: $|\alpha| = \alpha_1+\cdots+\alpha_n = \deg(z^\alpha)$, $\alpha! = \alpha_1!\cdots\alpha_n!$.

The function

$$E_z(u) = e^{\frac{1}{2}\langle u,z\rangle} \tag{2.3}$$

is the *reproducing kernel* for Fock space. That is,

$$\langle f, E_z\rangle_{\mathcal{F}} = f(z) \tag{2.4}$$

for all $f\in\mathcal{F}$. To establish this, one can identify V with $\mathbb{C}^n$ and form the Taylor series

$$E_z(u) = \exp\left(\frac{u\cdot\overline{z}}{2}\right) = \sum_{j=1}^{\infty}\frac{1}{j!2^j}(u\cdot\overline{z})^j = \sum_{j=1}^{\infty}\frac{1}{j!2^j}\sum_{|\beta|=j}\frac{j!}{\beta!}u^\beta\overline{z}^\beta \tag{2.5}$$
$$= \sum_\beta \frac{1}{2^{|\beta|}\beta!}u^\beta\overline{z}^\beta.$$

Using Equation (2.2) we now see that $\langle z^\alpha, E_z\rangle_{\mathcal{F}} = z^\alpha$. This proves (2.4), as the monomials form a complete orthogonal system in $\mathcal{F}$.

Formula (2.1) also yields a Hermitian inner product on $\mathbb{C}[V_{\mathbb{R}}]$, the complex valued polynomial functions on $V_{\mathbb{R}}$. To avoid confusion between the spaces $\mathcal{F}$ and $\mathbb{C}[V_{\mathbb{R}}]$, we will refer to this as the Fock inner product *on* $\mathbb{C}[V_{\mathbb{R}}]$. For $f, g \in \mathbb{C}[V_{\mathbb{R}}]$, observe that $(2\pi)^n\langle f, g\rangle_{\mathcal{F}}$ coincides with the L^2-inner product of $f(z)e^{-|z|^2/4}$ with $g(z)e^{-|z|^2/4}$. Such functions arise naturally as spherical functions on the Heisenberg group modulo its center. This fact motivates our interest in the Fock inner product on $\mathbb{C}[V_{\mathbb{R}}]$.

Berezin symbols and the star product: The *Berezin symbol* for an operator A on $\mathcal{F}$ is the real analytic function $\widehat{A}$ on $V_{\mathbb{R}}$ defined by

$$\widehat{A}(z) = \frac{\langle AE_z, E_z\rangle_{\mathcal{F}}}{\langle E_z, E_z\rangle_{\mathcal{F}}} = \frac{(AE_z)(z)}{E_z(z)} = (AE_z)(z)e^{-\frac{1}{2}|z|^2}.$$

Here A may be unbounded on $\mathcal{F}$ but we require that the "coherent states" $\{E_z \ : \ z \in V\}$ belong to the domains of both A and A^*. When each of the operators A, B and AB satisfy these conditions we define the *star product* $\widehat{A} \star \widehat{B}$ of the two symbols $\widehat{A}$ and $\widehat{B}$ via

$$\widehat{A} \star \widehat{B} = (AB)\widehat{\ }.$$

The operators that play a role in this paper are polynomial coefficient differential operators and finite rank projections. We will need to use the following facts concerning Berezin symbols and star products:

(2.6) For $f \in \mathbb{C}[V]$, define the multiplication operator m_f on $\mathcal{F}$ by $m_f(g) = fg$. This operator has symbol $\widehat{m}_f = f$.

(2.7) Given $f \in \mathbb{C}[V] \simeq \mathbb{C}[z_1, \ldots, z_n]$, we let $f(\partial_z)$ be the operator obtained by replacing each variable z_j with $\partial/\partial z_j$. This is an unbounded operator on $\mathcal{F}$ with

$$f(\partial_z)^* = m_{\overline{f}(z/2)}, \quad m_f^* = \overline{f}(2\partial_z).$$

Here $\overline{f} \in \mathbb{C}[V]$ is the holomorphic polynomial obtained by conjugating the coefficients of f. We have $(f(\partial_z))\widehat{\ } = f(\overline{z}/2)$.

(2.8) Combining (2.6) and (2.7) we see that $f \in \mathbb{C}[V_{\mathbb{R}}] \simeq \mathbb{C}[z_1, \ldots, z_n, \overline{z_1}, \ldots, \overline{z_n}]$ is the symbol of the Wick ordered operator $f(z, 2\partial_z)$.

(2.9) The symbol $\widehat{\pi}$ for the orthogonal projection operator π onto a finite dimensional subspace of $\mathbb{C}[V]$ with orthonormal basis $\{v_j\}_{j=1}^m$ is

$$\widehat{\pi}(z) = \left(\sum_{j=1}^m v_j(z)\overline{v_j(z)}\right) e^{-|z|^2/2}.$$

(2.10) If A is a trace class operator, then $\int_V \widehat{A}(z)dz = (2\pi)^n tr(A)$.

(2.11) If $\widehat{A}$ is holomorphic or $\widehat{B}$ is antiholomorphic, then $(AB)\widehat{\ } = \widehat{A} \star \widehat{B} = \widehat{A}\widehat{B}$.

Note that, if we extend $\{v_j\}_{j=1}^m$ in (2.9) to an orthonormal basis $\{v_j\}_{j=1}^\infty$ in $\mathbb{C}[V]$, then we can write the decomposition (2.5) of E_z in the basis $\{v_j\}$ as $E_z(u) = \sum_{j=1}^\infty \overline{v_j(z)} v_j(u)$. Following [10], this series converges uniformly on any compact subset of V. Thus we obtain $\pi(E_z) = \sum_{j=1}^m \overline{v_j(z)} v_j$, which immediately yields (2.9).

For completeness, we recall one further result which will not be used in the sequel. One has

$$(Af)(z) = \left(\frac{1}{2\pi}\right)^n \int_V f(w)\widetilde{A}(w,z)e^{\langle z-w,w\rangle/2}dw \tag{2.12}$$

where $\widetilde{A}(w,z) = \langle AE_w, E_z\rangle_{\mathcal{F}} / \langle E_w, E_z\rangle_{\mathcal{F}}$. The multiplier $\widetilde{A}(w,z)$ is the unique function on $V \times V$ which is holomorphic in w and anti-holomorphic in z, with $\widetilde{A}(z,z) = \widehat{A}(z)$. Thus (2.12) shows that an operator A is completely determined by its symbol $\widehat{A}$.

The integral formula for the star product is then given by

$$(\widehat{A} \star \widehat{B})(z) = \left(\frac{1}{2\pi}\right)^n \int_V \widetilde{A}(w,z)\widetilde{B}(z,w)e^{-|z-w|^2/2}dw.$$

The star inner product: For $f, g \in \mathbb{C}[V_{\mathbb{R}}]$ we define

$$\langle f, g\rangle_* = \left(\frac{1}{2\pi}\right)^n \int_V \left(f(z)e^{-|z|^2/2}\right) \star \left(\overline{g(z)}e^{-|z|^2/2}\right) dz. \tag{2.13}$$

This gives another Hermitian inner product on $\mathbb{C}[V_{\mathbb{R}}]$, although it is not a priori clear that it is positive definite or even well-defined. To show this, we first make the identifications $V \simeq \mathbb{C}^n$, $\mathbb{C}[V_{\mathbb{R}}] \simeq \mathbb{C}[z_1, \ldots, z_n, \overline{z}_1, \ldots, \overline{z}_n]$ and apply formula (2.13) to a pair of monomials $z^\alpha \overline{z}^\beta$, $z^{\alpha'}\overline{z}^{\beta'}$. We will write m_j for the operator m_{z_j} and m_α for m_{z^α}.

The projection operator $\pi_0(f) = f(0)$ from $\mathcal{F}$ onto the constant polynomials in $\mathbb{C}[V]$ has symbol $\widehat{\pi}_0 = E_z(0)e^{-|z|^2/2} = e^{-|z|^2/2}$. Hence also $(m_\alpha \pi_0)\widehat{\ } = z^\alpha e^{-|z|^2/2}$ by (2.6) and (2.11). Now (2.7) gives $(2^{|\beta|}\partial_z^\beta)\widehat{\ } = \overline{z}^\beta$, and by (2.11) we obtain

$$z^\alpha \overline{z}^\beta e^{-|z|^2/2} = 2^{|\beta|}(m_\alpha\pi_0)\widehat{\ }(\partial_z^\beta)\widehat{\ } = 2^{|\beta|}\left(m_\alpha \pi_0 \partial_z^\beta\right)\widehat{\ }.$$

Thus we can write

$$\begin{aligned}\langle z^\alpha \overline{z}^\beta, z^{\alpha'}\overline{z}^{\beta'}\rangle_* &= \left(\frac{1}{2\pi}\right)^n \int_V 2^{|\beta|}2^{|\alpha'|}\left(m_\alpha \pi_0 \partial_z^\beta m_{\beta'}\pi_0\partial_z^{\alpha'}\right)\widehat{\ } dz \\ &= 2^{|\beta|}2^{|\alpha'|} tr\left(m_\alpha \pi_0 \partial_z^\beta m_{\beta'}\pi_0\partial_z^{\alpha'}\right)\end{aligned}$$

using (2.10). The operator $m_\alpha \pi_0 \partial_z^\beta m_{\beta'}\pi_0\partial_z^{\alpha'}$ is zero unless $\beta = \beta'$, in which case it sends $z^{\alpha'}$ to $\beta!\alpha'!z^\alpha$ and annihilates all other monomials in z. This gives, finally,

$$\langle z^\alpha \overline{z}^\beta, z^{\alpha'}\overline{z}^{\beta'}\rangle_* = \delta_{\alpha,\alpha'}\delta_{\beta,\beta'}\left(2^{|\alpha|}\alpha!\right)\left(2^{|\beta|}\beta!\right). \tag{2.14}$$

One can use (2.14) to compute the star inner product $\langle f, g\rangle_*$ for any pair of polynomials $f, g \in \mathbb{C}[V_{\mathbb{R}}]$. It is now clear that the star inner product is a well defined Hermitian inner product on $\mathbb{C}[V_{\mathbb{R}}]$. Moreover, (2.14) provides a concrete combinatorial viewpoint on the star inner product. In view of Equation (2.2), we can recast this as follows. The star inner product on $\mathbb{C}[V_{\mathbb{R}}]$ arises from $\langle\cdot,\cdot\rangle_{\mathcal{F}}$ on $\mathbb{C}[V]$ by *doubling the variables.* More precisely, we have:

Lemma 2.1. *The star inner product $\langle \cdot, \cdot \rangle_*$ on $\mathbb{C}[V_\mathbb{R}]$ coincides with $\langle \cdot, \cdot \rangle_\mathcal{F} \otimes \langle \cdot, \cdot \rangle_{\mathcal{F}^*}$ under the identification $\mathbb{C}[V_\mathbb{R}] = \mathbb{C}[V \oplus \overline{V}] \simeq \mathbb{C}[V] \otimes \mathbb{C}[\overline{V}] \subset \mathcal{F} \otimes \mathcal{F}^*$.*

One may choose to regard $\mathcal{F} \otimes \mathcal{F}^*$ as the space of Hilbert-Schmidt operators on $\mathcal{F}$. The subspace $\mathbb{C}[V_\mathbb{R}]$ consists of finite rank operators with image contained in $\mathbb{C}[V]$. Lemma 2.1 shows that the star norm on $\mathbb{C}[V_\mathbb{R}]$ agrees with the restriction of the Hilbert-Schmidt norm $|||T|||^2 = tr(T^*T)$ on $\mathcal{F} \otimes \mathcal{F}^*$. The star inner product on $\mathbb{C}[V_\mathbb{R}]$ is thus a classical object of study.

3. Polynomial invariants of a multiplicity free action

Now let K be a closed Lie subgroup of the unitary group $U(V)$ and consider the natural representations of K on $\mathbb{C}[V]$ and $\mathbb{C}[V_\mathbb{R}]$ defined by the rule

$$(k \cdot f)(z) = f(k^{-1}z).$$

We remark that the Bargmann transform $L^2(\mathbb{R}^n) \to \mathcal{F}$ intertwines the representation of K on $\mathbb{C}[V]$ with the restriction of the metaplectic representation to $K \subset U(n) \subset Sp(2n, \mathbb{R})$. The space $\mathbb{C}[V]$ decomposes as a countably infinite direct sum of finite dimensional K-irreducible subspaces:

$$\mathbb{C}[V] = \sum_{\lambda \in \Lambda} P_\lambda. \tag{3.1}$$

One says that the action of K on V is *multiplicity free* when the irreducible K-modules $\{P_\lambda : \lambda \in \Lambda\}$ are pair-wise inequivalent. In this case the decomposition given in (3.1) is canonical.

Proposition 3.1 below relates multiplicity free actions to the star product. First note that for $k \in U(V)$ one has $E_{k^{-1}z}(u) = e^{\langle u, k^{-1}z \rangle/2} = e^{\langle ku, z \rangle/2} = E_z(ku)$. So for operators A on $\mathcal{F}$ we have

$$\begin{aligned}(k \cdot \widehat{A})(z) &= \widehat{A}(k^{-1}z) = (AE_{k^{-1}z})(k^{-1}z)e^{-|k^{-1}z|^2/2} \\ &= \left(AU_k^{-1}E_z\right)(k^{-1}z)e^{-|z|^2/2} = \left(U_kAU_k^{-1}E_z\right)(z)e^{-|z|^2/2} = \left(U_kAU_k^{-1}\right)\widehat{\ }(z),\end{aligned}$$

where U_k denotes the unitary operator on $\mathcal{F}$ given by $(U_kf)(z) = f(k^{-1}z)$.

An operator A on $\mathcal{F}$ is said to be *K-invariant* when $U_kA = AU_k$ for all $k \in K$. The above calculation shows that A is K-invariant if and only if $k \cdot \widehat{A} = \widehat{A}$ for all $k \in K$. Since composites of K-invariant operators are K-invariant, it follows that the star product $\widehat{A} \star \widehat{B}$ of two K-invariant symbols is K-invariant. As noted in (2.8), $f \in \mathbb{C}[V_\mathbb{R}]$ is the symbol for the Wick ordered operator $f(z, 2\partial_z)$. So the symbol map gives an isomorphism between the vector space $\mathcal{PD}(V)^K$ of K-invariant polynomial coefficient differential operators on V and the the space $\mathbb{C}[V_\mathbb{R}]^K$ of K-invariant polynomials on $V_\mathbb{R}$. Moreover, this correspondence becomes an algebra isomorphism if we equip $\mathbb{C}[V_\mathbb{R}]$ with the star product. It is shown in [12] that the algebra $\mathcal{PD}(V)^K$ is commutative if and only if the action of K on V is multiplicity free. These remarks prove the following:

Proposition 3.1. *The space $\mathbb{C}[V_\mathbb{R}]^K$ of K-invariant polynomials on $V_\mathbb{R}$ forms an algebra under the star product. This algebra is abelian if and only if the action of K on V is multiplicity free.*

We now suppose that the action of K on V is multiplicity free. Since the spaces $\mathcal{P}_m(V)$ of homogeneous polynomials of degree m are $U(V)$-invariant, each P_λ is a subspace of some $\mathcal{P}_m(V)$. We let $|\lambda|$ denote the degree of homogeneity of the polynomials in P_λ, so that $P_\lambda \subset \mathcal{P}_{|\lambda|}(V)$, and write

$$d_\lambda = \dim(P_\lambda).$$

Homogeneous invariants: For each $\lambda \in \Lambda$, let $p_\lambda \in \mathbb{C}[V_\mathbb{R}]$ be defined by

$$p_\lambda(z) = \frac{1}{d_\lambda}\sum_{j=1}^{d_\lambda} v_j(z)\overline{v_j(z)}, \tag{3.2}$$

where $\{v_j\}_{j=1}^{d_\lambda}$ is any orthonormal basis for P_λ. In [3] it is shown that this definition does not depend on the choice of orthonormal basis for P_λ and that $\{p_\lambda \mid \lambda \in \Lambda\}$ is a vector space basis for the space $\mathbb{C}[V_\mathbb{R}]^K$. Note that p_λ is homogeneous of degree $2|\lambda|$. In view of (2.9), we obtain:

Lemma 3.2. *The symbol for the projection operator π_λ onto the subspace P_λ is given by*

$$\widehat{\pi}_\lambda(z) = d_\lambda p_\lambda(z) e^{-|z|^2/2}.$$

Lemma 3.3. *The polynomials $\{p_\lambda \mid \lambda \in \Lambda\}$ are pair-wise orthogonal with respect to the star inner product, and $\langle p_\lambda, p_\lambda\rangle_* = 1/d_\lambda$.*

Proof. Using Lemma 3.2, we compute:

$$\begin{aligned}\langle p_\lambda, p_\mu\rangle_* &= \left(\frac{1}{2\pi}\right)^n \frac{1}{d_\lambda d_\mu}\int_V \widehat{\pi}_\lambda \star \widehat{\pi}_\mu(z)dz = \left(\frac{1}{2\pi}\right)^n \frac{1}{d_\lambda d_\mu}\int_V (\pi_\lambda\pi_\mu)\widehat{}\,(z)dz \\ &= tr(\pi_\lambda\pi_\mu)/d_\lambda d_\mu = \delta_{\lambda,\mu} d_\lambda/d_\lambda d_\mu = \delta_{\lambda,\mu}/d_\lambda.\end{aligned}$$

Here we have also used (2.10) and the fact that $\overline{p_\mu(z)} = p_\mu(z)$. ■

Fundamental invariants: The index set Λ in (3.1) can be concretely realized as follows: Choose a maximal torus in K and a system of positive roots. These choices produce a simple ordering $\prec$ on the weights of the representations of K. We let Λ be the set of highest weights for the irreducible representations of K which occur in $\mathbb{C}[V]$, so that the irreducible component P_λ has highest weight $\lambda \in \Lambda$.

Following [12] we call the primitive elements of Λ *fundamental highest weights*. These are finite in number and freely generate Λ. Letting $\lambda_1, \lambda_2, \ldots, \lambda_r$ denote the fundamental highest weights listed in increasing order using $\prec$, we have

$$\Lambda = \{a_1\lambda_1 + \cdots + a_r\lambda_r \mid a_1, \ldots, a_r \in \mathbb{Z}_+\},$$

where $\mathbb{Z}_+$ denotes the non-negative integers. The *fundamental invariants* $\gamma_1, \ldots, \gamma_r$ are defined as

$$\gamma_j = p_{\lambda_j}.$$

One has $\mathbb{C}[V_{\mathbb{R}}]^K = \mathbb{C}[\gamma_1, \ldots, \gamma_r]$. For $\lambda = a_1\lambda_1 + \cdots + a_r\lambda_r \in \Lambda$ we define

$$\gamma^\lambda = \gamma_1^{a_1}\gamma_2^{a_2}\cdots\gamma_r^{a_r}.$$

In [7] it is shown that $p_\lambda = c_\lambda\gamma^\lambda + s_\lambda$ where c_λ is a positive constant and $s_\lambda \in Span\{p_\mu \ : \mu \in \Lambda, |\mu| = |\lambda|, \mu \prec \lambda\}$. (See Equation (2.5) in [7].) Thus we have:

Lemma 3.4. *For $\lambda \in \Lambda$,*

$$Span\{\gamma^\mu \ : \mu \in \Lambda, |\mu| = |\lambda|, \mu \preceq \lambda\} = Span\{p_\mu \ : \mu \in \Lambda, |\mu| = |\lambda|, \mu \preceq \lambda\}.$$

Lemmas 3.3 and 3.4 yield an orthogonalization procedure which determines the p_λ's.

Proposition 3.5. *For each $m \geq 0$ the polynomials $\{p_\lambda \ : \ |\lambda| = m\}$ are obtained from $\{\gamma^\lambda \ : \ |\lambda| = m\}$ via Gram-Schmidt orthogonalization using the star inner product, where*

- *the indices $\Lambda_m = \{\lambda \ : |\lambda| = m\}$ are ordered using the weight ordering $\prec$, and*
- *the p_λ's are normalized so that $\langle p_\lambda, p_\lambda\rangle_* = 1/d_\lambda$.*

One can also replace the ordinary products γ^λ in Proposition 3.5 by star products. Specifically, for $\lambda = a_1\lambda_1 + \cdots + a_r\lambda_r$ we let

$$\gamma^{\star\lambda} = \gamma_1^{\star a_1} \star \gamma_2^{\star a_2} \star \cdots \star \gamma_r^{\star a_r}$$

where $\gamma_j^{\star a_j}$ denotes the a_j-fold star product $\gamma_j \star \cdots \star \gamma_j$. Since $\mathbb{C}[V_{\mathbb{R}}]^K$ is commutative under the star product, 3.1) the definition of $\gamma^{\star\lambda}$ is independent of the ordering used for the factors γ_j. By (2.8) we see that $\gamma^{\star\lambda}$ is the symbol of $\gamma_1(z, 2\partial_z)^{a_1}\cdots\gamma_r(z, 2\partial_z)^{a_r}$, whereas γ^λ is the symbol of $(\gamma_1^{a_1}\cdots\gamma_r^{a_r})(z, 2\partial_z)$. In particular,

$$\gamma^{\star\lambda} = \gamma^\lambda + r_\lambda$$

where $r_\lambda \in \{\gamma^\mu \ : \ |\mu| < |\lambda| \text{ or } (|\mu| = |\lambda| \text{ and } \mu \prec \lambda)\}$. The following result is now a corollary of Proposition 3.5.

Proposition 3.6. *The polynomials $\{p_\lambda \ : \ \lambda \in \Lambda\}$ are obtained from $\{\gamma^{\star\lambda} \ : \ \lambda \in \Lambda\}$ via Gram-Schmidt orthogonalization using the star inner product, where*

- *Λ is ordered according to the rule*

$$\lambda < \lambda' \iff (|\lambda| < |\lambda'|) \text{ or } (|\lambda| = |\lambda'| \text{ and } \lambda \prec \lambda').$$

- *the p_λ's are normalized so that $\langle p_\lambda, p_\lambda\rangle_* = 1/d_\lambda$.*

Inhomogeneous invariants: The space $\mathbb{C}[V_{\mathbb{R}}]^K$ has a second canonical basis, denoted $\{q_\lambda \mid \lambda \in \Lambda\}$. These are the polynomials obtained from $\{p_\lambda \mid \lambda \in \Lambda\}$ via Gram-Schmidt orthogonalization using the Fock inner product on $\mathbb{C}[V_{\mathbb{R}}]$, where

- the index set Λ is ordered so that λ precedes λ' whenever $|\lambda| < |\lambda'|$, and
- the q_λ's are normalized so that $q_\lambda(0) = 1$.

It is shown in [3] that the polynomials q_λ are well defined, and independent of the ordering used within each $\Lambda_m = \{\lambda \in \Lambda \ : \ |\lambda| = m\}$. If we give Λ the ordering from Proposition 3.6 then Lemma 3.4 implies that we can use the polynomials $\{\gamma^\lambda\}$ in place of the p_λ's in the orthogonalization procedure. This is the content of Theorem 2.1 in [7].

The polynomial q_λ is inhomogeneous of total degree $2|\lambda|$. The homogeneous component of degree $2|\lambda|$ in q_λ is $(-1)^{|\lambda|}p_\lambda$. We express q_λ in the basis $\{p_\mu \mid \mu \in \Lambda\}$ by

$$q_\lambda = \sum_{\mu\in\Lambda} (-1)^{|\mu|} \begin{bmatrix}\lambda\\ \mu\end{bmatrix} p_\mu.$$

The values $\begin{bmatrix}\lambda\\ \mu\end{bmatrix}$ are called *generalized binomial coefficients.* We have $\begin{bmatrix}\lambda\\ \lambda\end{bmatrix} = 1$, and $\begin{bmatrix}\lambda\\ \mu\end{bmatrix} = 0$ for $|\mu| > |\lambda|$ or $|\mu| = |\lambda|$ with $\mu \neq \lambda$. The generalized binomial coefficients are non-negative rational numbers. (See [6, 7].)

There is a remarkable "Pieri formula," due to Yan, which provides another link between the bases $\{p_\lambda \ : \ \lambda \in \Lambda\}$ and $\{q_\lambda \ : \ \lambda \in \Lambda\}$:

$$\frac{1}{m!}\left(\frac{|z|^2}{2}\right)^m d_\mu p_\mu = \sum_{|\lambda|=|\mu|+m} \begin{bmatrix}\lambda\\ \mu\end{bmatrix} d_\lambda p_\lambda. \tag{3.3}$$

We refer the reader to [16] and [6] for proofs of this result. Using Lemma 3.3, we can rephrase this Pieri formula as

$$\begin{bmatrix}\lambda\\ \mu\end{bmatrix} = \frac{d_\mu}{m!}\left\langle \left(\frac{|z|^2}{2}\right)^m p_\mu, p_\lambda \right\rangle_\star \tag{3.4}$$

where $m = |\lambda| - |\mu|$.

4. The action of $K = SO(n,\mathbb{R}) \times \mathbb{T}$

The rest of this paper concerns the standard action of $K = SO(n,\mathbb{R}) \times \mathbb{T}$ on $V = \mathbb{C}^n$ for $n \geq 3$, namely

$$(A, c) \cdot z = cAz$$

for $A \in SO(n,\mathbb{R})$, $c \in \mathbb{T}$, $z \in V$.

The decomposition of $\mathbb{C}[V]$ under the action of K is determined by the classical theory of spherical harmonics. The polynomial

$$\varepsilon(z) = z_1^2 + \cdots + z_n^2$$

is invariant under the action of $SO(n,\mathbb{R})$ on $\mathbb{C}[V]$. We let

$$\Delta = \varepsilon(\partial_z) = \left(\frac{\partial}{\partial z_1}\right)^2 + \cdots + \left(\frac{\partial}{\partial z_n}\right)^2$$

and define the space of *harmonic* polynomials $\mathcal{H} = \ker(\Delta)$. Thus

$$\mathcal{H}_m = \mathcal{P}_m(V) \cap \mathcal{H}$$

is the space of harmonic polynomial which are homogeneous of degree m. For non-negative integers k, ℓ,

$$P_{k,\ell} = m_\varepsilon^\ell \mathcal{H}_k = \varepsilon^\ell \mathcal{H}_k$$

is an irreducible K-invariant subspace of $\mathcal{P}_{k+2\ell}(V)$. The space $P_{k,\ell}$ has dimension $d_{k,\ell} = h_k = \dim(\mathcal{H}_k)$, independent of ℓ. Explicitly, $h_0 = 1$, $h_1 = n$ and

$$h_k = \dim(\mathcal{P}_k(V)) - \dim(\mathcal{P}_{k-2}(V)) = \binom{k+n-1}{k} - \binom{k+n-3}{k-2}$$

for $k \geq 2$.

The action of K is multiplicity free because the $P_{k,\ell}$'s are pair-wise inequivalent K-modules. Indeed, $P_{k,\ell}$ and $P_{k',\ell'}$ have the same dimension only when $k = k'$, and the circle $\mathbb{T}$ acts in these spaces by the characters $c \mapsto c^{-(2k+\ell)}$ and $c \mapsto c^{-(2k'+\ell')}$ respectively. We write the decomposition (3.1) for this example as

$$\mathbb{C}[V] = \sum_{(k,\ell)\in\Lambda} P_{k,\ell}$$

where Λ denotes the set of all pairs of non-negative integers.

Homogeneous invariants: Here we will derive formulae for the polynomials $\{p_{k,\ell} \ : \ (k,\ell) \in \Lambda\}$. Recall that these form a canonical homogeneous basis for $\mathbb{C}[V_\mathbb{R}]^K$. Let $\gamma_1, \gamma_2 \in \mathbb{C}[V_\mathbb{R}]^K$ be the polynomials defined as

$$\gamma_1(z) = np_{1,0}(z) = |z|^2/2, \quad \gamma_2(z) = np_{0,1}(z) = |\varepsilon(z)|^2/4.$$

These are (up to multiples) the fundamental invariants in this example. We will express each $p_{k,\ell}$ as a polynomial in γ_1, γ_2. In fact, Theorem 5.12 in [4] solves this problem by providing recurrence relations for the $p_{k,\ell}$'s. We repeat this result below.

Proposition 4.1. *Writing p_k for $p_{k,0}$ one has*

(a) $p_{k,\ell} = p_k \gamma_2^\ell / c_{k,\ell}$ *where*

$$c_{k,\ell} = 4^\ell \ell! \prod_{j=1}^{\ell} (k + n/2 + \ell - j) = 4^\ell (\ell!)^2 \binom{k + n/2 + \ell - 1}{\ell}.$$

(b) p_k *is determined by the recurrence relation*

$$h_k p_k = \frac{\gamma_1^k}{k!} - \sum_{j=1}^{\lfloor k/2 \rfloor} h_{k-2j} p_{k-2j,j} = \frac{\gamma_1^k}{k!} - \sum_{j=1}^{\lfloor k/2 \rfloor} \frac{h_{k-2j} p_{k-2j} \gamma_2^j}{c_{k-2j,j}} \quad \textit{for } k \geq 2,$$

with initial conditions $p_0 = 1$, $p_1 = \gamma_1/n$.

We remark that the polynomials "$p_{k\,\ell}$" in [4] are not normalized by dividing by $h_k = \dim(P_{k,\ell})$. This accounts for the dimension factors in Proposition 4.1 that are absent in [4]. Here we will show how Proposition 4.1 can be derived using the machinery of star products. First we establish the following lemma.

Lemma 4.2. *The orthogonal projection operator $\pi_{k,\ell}$ of $\mathcal{F}$ onto $P_{k,\ell}$ satisfies*

$$\pi_{k,\ell} = \frac{1}{c_{k,\ell}} m_\varepsilon^\ell \pi_{k,0} \Delta^\ell.$$

Proof. Let $T_{k\ell}$ be the operator defined on $\mathcal{F}$ by

$$T_{k\ell} = \frac{1}{c_{k,\ell}} m_\varepsilon^\ell \pi_{k,0} \Delta^\ell.$$

First note that an easy calculation yields

$$\Delta(g\varepsilon^\ell) = 4\ell(k + n/2 + \ell - 1) g\varepsilon^{\ell-1}$$

for $g \in \mathcal{H}_k$. By induction we obtain $\Delta^\ell(g\varepsilon^\ell) = c_{k,\ell}g$. (This is Equation (5.14) in [4].) Thus we have $T_{k,\ell}(g\varepsilon^\ell) = g\varepsilon^\ell$ for $g \in \mathcal{H}_k$, and hence $Im(T_{k,\ell}) = P_{k,\ell}$. In addition, $\Delta^\ell m_\varepsilon^\ell \pi_{k,0} = c_{k,\ell}\pi_{k,0}$ shows us that $T_{k,\ell}^2 = T_{k,\ell}$. Next we note that $T_{k,\ell}$ is formally self-adjoint, since (2.7) shows $\Delta^* = m_\varepsilon/4$ and $m_\varepsilon^* = 4\Delta$. ∎

Proof. We turn now to the Proof of Proposition 4.1. Lemma 4.2 shows that

$$\widehat{\pi}_{k,\ell} = \frac{1}{c_{k,\ell}} \widehat{m}_\varepsilon^{\star\ell} \star \widehat{\pi}_{k,0} \star \widehat{\Delta}^{\star\ell}$$

where $\widehat{m}_\varepsilon^{\star\ell}$ and $\widehat{\Delta}^{\star\ell}$ denote ℓ-fold star products. From (2.6) and (2.7) we see that

$$\widehat{m}_\varepsilon(z) = \varepsilon(z), \quad \widehat{\Delta}(z) = \varepsilon(\partial_z)\widehat{\ }(z) = \varepsilon(\overline{z}/2) = \frac{\overline{\varepsilon(z)}}{4}.$$

As $\widehat{m}_\varepsilon$ is holomorphic and $\widehat{\Delta}$ is antiholomorphic, (2.11) shows that the star product above is an ordinary product:

$$\widehat{\pi}_{k,\ell} = \frac{1}{c_{k,\ell}} \widehat{m}_\varepsilon^\ell \widehat{\pi}_{k,0} \widehat{\Delta}^\ell = \frac{1}{c_{k,\ell}} \widehat{\pi}_{k,0} \varepsilon^\ell \left(\frac{\overline{\varepsilon}}{4}\right)^\ell = \frac{1}{c_{k,\ell}} \widehat{\pi}_{k,0} \gamma_2^\ell. \tag{4.1}$$

From Lemma 3.2 we have $\widehat{\pi}_{k,\ell} = h_k p_{k,\ell} e^{-|z|^2/2}$, and hence $p_{k,\ell} = p_k \gamma_2^\ell / c_{k,\ell}$.

To establish the second assertion in Proposition 4.1 we use the fact that

$$\sum_{\ell+2j=k} h_\ell p_{\ell,j} = \frac{\gamma_1^k}{k!}. \tag{4.2}$$

This follows from the decomposition $\pi_k = \sum_{\ell+2j=k} \pi_{\ell,j}$ of the projection operator π_k onto $\mathcal{P}_k(V)$ by passing to symbols. We use the orthonormal basis $\{z^\alpha/(2^k\alpha!)^{1/2} \ : \ |\alpha| = k\}$ for $\mathcal{P}_k(V)$ to obtain

$$\widehat{\pi}_k(z) e^{|z|^2/2} = \frac{1}{2^k} \sum_{|\alpha|=k} \frac{z^\alpha \overline{z}^\alpha}{\alpha!} = \frac{1}{2^k k!} \sum_{|\alpha|=k} \frac{k!}{\alpha!} z^\alpha \overline{z}^\alpha = \frac{1}{k!} \left(\frac{z \cdot \overline{z}}{2}\right)^k = \frac{\gamma_1(z)}{k!}^k.$$

∎

Corollary 4.3. *For all (k,ℓ) we have:*

$$\int_V \gamma_1^k \gamma_2^\ell e^{-|z|^2/2} dz = k!(2\pi)^n \sum_{j \le k/2} \frac{c_{k-2j,\ell+j}}{c_{k-2j,j}} \left[\binom{k-2j+n-1}{k-2j} - \binom{k-2j+n-3}{k-2j-2} \right].$$

Proof. By (4.1), (4.2), we see that

$$e^{-|z|^2/2}\gamma_1^k/k! = \sum_{j\le k/2} \widehat{\pi}_{k-2j,j} = \sum_{j \le k/2} \frac{1}{c_{k-2j,j}} \widehat{\pi}_{k-2j,0}\gamma_2^j,$$

and thus

$$e^{-|z|^2/2}\gamma_1^k\gamma_2^\ell = k! \sum_{j\le k/2} \frac{1}{c_{k-2j,j}} \widehat{\pi}_{k-2j,0}\gamma_2^{j+\ell} = k! \sum_{j \le k/2} \frac{c_{k-2j,j+\ell}}{c_{k-2j,j}} \widehat{\pi}_{k-2j,j+\ell}.$$

Integrating over V, we then obtain

$$\int_V \gamma_1^k\gamma_2^\ell e^{-|z|^2/2} dz = k!(2\pi)^n \sum_{j\le k/2} \frac{c_{k-2j,\ell+j}}{c_{k-2j,j}} tr(\pi_{k-2j,j+\ell}),$$

and the result follows. ∎

We remark that, in [4], this formula is given by

$$\int_V \gamma_1^k\gamma_2^\ell e^{-|z|^2/2} = k!(2\pi)^n 4^\ell (\ell!)^2 \binom{k+2l+n-1}{k}\binom{\ell+n/2-1}{\ell}.$$

We can find no independent proof of the combinatorial identity given by the right hand sides of these equations.

Lemma 3.4 shows that for fixed k, ℓ one has

$$Span\{\gamma_1^{k-2j}\gamma_2^{\ell+j} \ : \ j=0,\dots,\lfloor k/2 \rfloor\} = Span\{p_{k-2j,\ell+j} \ : \ j=0,\dots,\lfloor k/2\rfloor\}. \tag{4.3}$$

This can also be derived using Proposition 4.1 and induction on $m = k+2\ell$. (See Lemma 5.17 in [4]].) Propositions 3.5 and 3.6 specialize to the current example as:

Proposition 4.4. *For each $m \ge 0$ the polynomials $\{p_{k,\ell} \ : \ k+2\ell = m\}$ are obtained from $\{\gamma_1^k\gamma_2^\ell \ : \ k+2\ell=m\}$ via Gram-Schmidt orthogonalization using the star inner product, where*

- *the indices $\Lambda_m = \{(k,\ell) \ : \ k+2\ell = m\}$ are ordered so that (k,ℓ) precedes (k',ℓ') when $k<k'$, and*
- *the $p_{k,\ell}$'s are normalized so that $\langle p_{k,\ell}, p_{k,\ell}\rangle_* = 1/h_k$.*

Alternatively, one can obtain the polynomials $\{p_{k,\ell} \ : \ (k,\ell)\in\Lambda\}$ from $\{\gamma_1^{\star k} \star \gamma_2^{\star \ell} \ : \ (k,\ell)\in\Lambda\}$ via Gram-Schmidt orthogonalization using the star inner product, where

- *Λ is given the ordering*

$$(k,\ell) < (k',\ell') \iff (k+2\ell < k'+2\ell') \text{ or } (k+2\ell = k'+2\ell' \text{ and } k<k'),$$

and

- *the $p_{k,\ell}$'s are normalized so that $\langle p_{k,\ell}, p_{k,\ell}\rangle_* = 1/h_k$.*

The polynomials $\gamma_1^{\star k} \star \gamma_2^{\star \ell}$ in Proposition 4.4 can be computed by repeated application of the following identities:

$$\gamma_1 \star (\gamma_1^k\gamma_2^\ell) = \gamma_1^{k+1}\gamma_2^\ell + (k+2\ell)\gamma_1^k\gamma_2^\ell, \tag{4.4}$$

$$\gamma_2 \star (\gamma_1^k\gamma_2^\ell) = \gamma_1^k\gamma_2^{\ell+1} + 4\ell\gamma_1^{k+1}\gamma_2^\ell + 2k\gamma_1^{k-1}\gamma_2^{\ell+1} + k(k-1)\gamma_1^{k-2}\gamma_2^{\ell+1} + 4\ell(k+\ell+n/2-1)\gamma_1^k\gamma_2^\ell. \tag{4.5}$$

To establish these formulae, first note that by (2.8)

$$\gamma_1 = \sum_j z_j \frac{\overline{z_j}}{2} = \left[\sum_j m_j\partial_j\right]^\frown, \quad \gamma_2 = \varepsilon(z)\varepsilon\left(\frac{\overline{z}}{2}\right) = [m_\varepsilon\Delta]^\frown, \text{ and}$$

$$\begin{aligned}\gamma_1^k\gamma_2^\ell &= \left(\sum_j z_j\frac{\overline{z_j}}{2}\right)^k \varepsilon(z)^\ell\varepsilon\left(\frac{\overline{z}}{2}\right)^\ell = \sum_{|\alpha|=k}\frac{k!}{\alpha!}\varepsilon(z)^\ell z^\alpha\left(\frac{\overline{z}}{2}\right)^\alpha\varepsilon\left(\frac{\overline{z}}{2}\right)^\ell \\ &= \left[\sum_{|\alpha|=k}\frac{k!}{\alpha!}m_\varepsilon^\ell m_\alpha\partial_z^\alpha\Delta^\ell\right]^\frown.\end{aligned}$$

Thus we have:

$$\gamma_1\star(\gamma_1^k\gamma_2^\ell) = \left[\sum_{j,|\alpha|=k}\frac{k!}{\alpha!}m_j\partial_j m_\varepsilon^\ell m_\alpha\partial_z^\alpha\Delta^\ell\right]^\frown.$$

Since the commutator of ∂_j and $m_\varepsilon^\ell m_\alpha$ is multiplication by

$$\partial_j(\varepsilon^\ell z^\alpha) = 2\ell z_j\varepsilon^{\ell-1}z^\alpha + \alpha_j\varepsilon^\ell z^{\alpha-e_j},$$

we obtain

$$\begin{aligned}\gamma_1\star(\gamma_1^k\gamma_2^\ell) &= \left[\sum_{j,|\alpha|=k}\frac{k!}{\alpha!}(2\ell m_j^2 m_\varepsilon^{\ell-1}m_\alpha\partial_z^\alpha\Delta^\ell + \alpha_j m_\varepsilon^\ell m_\alpha\partial_z^\alpha\Delta^\ell + m_\varepsilon^\ell m_{\alpha+e_j}\partial_z^{\alpha+e_j}\Delta^\ell)\right]^\frown \\ &= 2\ell\gamma_1^k\gamma_2^\ell + k\gamma_1^k\gamma_2^\ell + \left[\sum_{|\beta|=k+1}\sum_{j:\beta_j\ge 1}\frac{k!}{(\beta-e_j)!}m_\varepsilon^\ell m_\beta\partial_z^\beta\Delta^\ell\right]^\frown \\ &= (k+2\ell)\gamma_1^k\gamma_2^\ell + \left[\sum_{|\beta|=k+1}\frac{k!}{\beta!}\left(\sum_j\beta_j\right)m_\varepsilon^\ell m_\beta\partial_z^\beta\Delta^\ell\right]^\frown \\ &= (k+2\ell)\gamma_1^k\gamma_2^\ell + \left[\sum_{|\beta|=k+1}\frac{(k+1)!}{\beta!}m_\varepsilon^\ell m_\beta\partial_z^\beta\Delta^\ell\right]^\frown \\ &= (k+2\ell)\gamma_1^k\gamma_2^\ell + \gamma_1^{k+1}\gamma_2^\ell.\end{aligned}$$

This proves (4.4). The argument for (4.5) is more complicated but uses the same techniques.

Next we will show that the orthogonalization procedure in Proposition 4.4 can be carried out to obtain *explicit* formulae for the $p_{k,\ell}$'s in terms of γ_1 and γ_2. We will prove:

Theorem 4.5.

$$h_k p_{k,\ell} = \frac{1}{c_{k,\ell}} \sum_{j=0}^{\lfloor k/2 \rfloor} \frac{(-1)^j}{c_{k-j-1,j}} \frac{\gamma_1^{k-2j}\gamma_2^{\ell+j}}{(k-2j)!}.$$

The values $c_{k,\ell}$ are defined for $k, \ell \geq 0$ as in Proposition 4.1. In particular, we have $c_{k,0} = 1$ for all $k \geq 0$. In addition, we define $c_{-1,0} = 1$, so that the formula reduces to $p_{0,\ell} = \gamma_2^\ell / c_{0,\ell}$ when $k = 0$. This is consistent with Proposition 4.1(a). Our proof of Theorem 4.5 uses two further lemmas.

Lemma 4.6. *For non-negative integers k, ℓ, k', ℓ' one has*

(a) $\langle \gamma_1^k \gamma_2^\ell, p_{k',\ell'} \rangle_* = 0$ *if* $k' + 2\ell' \neq k + 2\ell$,

(b) $\langle \gamma_1^k \gamma_2^\ell, p_{k',\ell'} \rangle_* = 0$ *if* $k' + 2\ell' = k + 2\ell$ *and* $\ell > \ell'$

(c) $\langle \gamma_1^k \gamma_2^\ell, p_{k',\ell'} \rangle_* = k! c_{k',\ell'} / c_{k',\ell'-\ell}$ *if* $k' + 2\ell' = k + 2\ell$ *and* $\ell \leq \ell'$

Proof. Since $\gamma_1^k\gamma_2^\ell \in \mathcal{P}_{k+2\ell}(V) \otimes \overline{\mathcal{P}_{k+2\ell}(V)}$ and $p_{k',\ell'} \in \mathcal{P}_{k'+2\ell'}(V) \otimes \overline{\mathcal{P}_{k'+2\ell'}(V)} \subset \mathbb{C}[V_\mathbb{R}]$, (a) follows immediately from Equation (2.14).

Equation (4.3) shows that $\gamma_1^k\gamma_2^\ell$ can be written as a linear combination of $\{p_{k-2j,\ell+j} \,:\, j = 0, \ldots, \lfloor k/2 \rfloor\}$. When $\ell > \ell'$ we have $\langle p_{k-2j,\ell+j}, p_{k',\ell'} \rangle_* = 0$ for all $j = 0, \ldots, \lfloor k/2 \rfloor$ by Lemma 3.3. This proves (b).

To show (c), we first use Proposition 4.1 to write

$$\frac{\gamma_1^k\gamma_2^\ell}{k!} = \sum_{j=0}^{\lfloor k/2 \rfloor} \frac{h_{k-2j} p_{k-2j} \gamma_2^{j+\ell}}{c_{k-2j,j}} = \sum_{j=0}^{\lfloor k/2 \rfloor} h_{k-2j} \frac{c_{k-2j,j+\ell}}{c_{k-2j,j}} p_{k-2j,j+\ell}.$$

By Lemma 3.3, $\langle p_{k-2j,j+\ell}, p_{k',\ell'} \rangle_* = \delta_{k-2j,k'} \delta_{j+\ell,\ell'} / h_{k-2j}$, and thus

$$\left\langle \frac{\gamma_1^k\gamma_2^\ell}{k!}, p_{k',\ell'} \right\rangle_* = \sum_{j=0}^{\lfloor k/2 \rfloor} h_{k-2j} \frac{c_{k-2j,j+\ell}}{c_{k-2j,j}} \langle p_{k-2j,j+\ell}, p_{k',\ell'} \rangle_* = \frac{c_{k',\ell'}}{c_{k',\ell'-\ell}}.$$

■

Lemma 4.7. *The polynomial $Q_m(x)$ defined for $m \geq 2$ as*

$$Q_m(x) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} (x-j-2) \cdots (x-j-m)$$

is identically zero.

Proof. Write $(x-j-2)\cdots(x-j-m)=\sum_{k=0}^{m-1}A_k(x)j^k$ where $A_k(x)$ is a polynomial in x independent of j. We then have

$$Q_m(x)=\sum_{k=0}^{m-1}\left(\sum_{j=0}^{m}(-1)^j\binom{m}{j}j^k\right)A_k(x)=0$$

since $\sum_{j=0}^{m}(-1)^j\binom{m}{j}j^k=0$ for $k=0,\dots,m-1$. The latter fact is a standard combinatorial identity which can be proved by noting that

$$\sum_{j=0}^{m}(-1)^j\binom{m}{j}j^kx^j=\left(x\frac{d}{dx}\right)^k[(1-x)^m]$$

vanishes at $x=1$ for $k=0,\dots,m-1$. ■

Proof (of Theorem 4.5).
The discussion following the statement of Theorem 4.5 shows that we can assume $k>0$. Let $\widetilde{p}_{k,\ell}=\frac{1}{c_{k,\ell}}\sum_{j=0}^{\lfloor k/2\rfloor}\frac{(-1)^j}{c_{k-j-1,j}}\frac{\gamma_1^{k-2j}\gamma_2^{\ell+j}}{(k-2j)!}$. In view of Lemma 3.3, it suffices to show that

$$\langle\widetilde{p}_{k,\ell},p_{k',\ell'}\rangle_*=\delta_{k,k'}\delta_{\ell,\ell'}$$

for all k', ℓ'.

Parts (a) and (b) in Lemma 4.6 show that $\langle\widetilde{p}_{k,\ell},p_{k',\ell'}\rangle_*=0$ when $k'+2\ell'\neq k+2\ell$ and when $k'+2\ell'=k+2\ell$ but $\ell>\ell'$. Indeed, $\langle\gamma_1^{k-2j}\gamma_2^{\ell+j},p_{k',\ell'}\rangle_*=0$ in all such cases for $j=0,\dots,\lfloor k/2\rfloor$.

Now suppose that $k'+2\ell'=k+2\ell$ and $\ell\le\ell'$. Let $m=\ell'-\ell$ so that $k'=k-2m$ and $\ell'=\ell+m$. Parts (b) and (c) in Lemma 4.6 yield

$$\begin{aligned}\langle\widetilde{p}_{k,\ell},p_{k',\ell'}\rangle_* &= \frac{1}{c_{k,\ell}}\sum_{j=0}^{\lfloor k/2\rfloor}\frac{(-1)^j}{c_{k-j-1,j}(k-2j)!}\langle\gamma_1^{k-2j}\gamma_2^{\ell+j},p_{k',\ell'}\rangle_* \qquad (4.6)\\ &= \frac{1}{c_{k,\ell}}\sum_{j=0}^{m}\frac{(-1)^j}{c_{k-j-1,j}(k-2j)!}\frac{(k-2j)!c_{k',\ell'}}{c_{k',\ell'-(\ell+j)}}\\ &= \frac{c_{k-2m,\ell+m}}{c_{k,\ell}}\sum_{j=0}^{m}\frac{(-1)^j}{c_{k-j-1,j}c_{k-2m,m-j}}\end{aligned}$$

When $m\ge 2$, we substitute $c_{k-j-1,j}=4^jj!(k+n/2-2)!/(k+n/2-2-j)!$ and $c_{k-2m,m-j}=4^{m-j}(m-j)!(k+n/2-m-j-1)!/(k+n/2-2m-1)!$ in (4.6) and simplify to obtain

$$\langle\widetilde{p}_{k,\ell},p_{k',\ell'}\rangle_*=\frac{c_{k-2m,\ell+m}}{c_{k,\ell}}\frac{(k+n/2-2m-1)!}{4^mm!(k+n/2-2)!}Q_m(k+n/2)=0$$

by Lemma 4.7.

When $m=1$ Equation (4.6) reads

$$\langle\widetilde{p}_{k,\ell},p_{k',\ell'}\rangle_*=\frac{c_{k-2,\ell+1}}{c_{k,\ell}}\left[\frac{1}{c_{k-1,0}c_{k-2,1}}-\frac{1}{c_{k-2,1}c_{k-2,0}}\right]=0.$$

Finally, when $m=0$ Equation (4.6) reduces to

$$\langle\widetilde{p}_{k,\ell},p_{k',\ell'}\rangle_*=\langle\widetilde{p}_{k,\ell},p_{k,\ell}\rangle_*=\frac{c_{k,\ell}}{c_{k,\ell}}\frac{1}{c_{k-1,0}c_{k,0}}=1$$

as desired. ■

Inhomogeneous invariants: Recall that the space of $SO(n,\mathbb{R}) \times \mathbb{T}$-invariant polynomials on $\mathbb{C}^n$ has a canonical basis $\{q_{k,\ell} \;:\; (k,\ell) \in \Lambda\}$ that is orthogonal with respect to the Fock inner product on $\mathbb{C}[V_{\mathbb{R}}]$. The polynomials $q_{k,\ell}$ are obtained via Gram-Schmidt orthogonalization from the $p_{k,\ell}$'s using a suitable ordering and normalization.

The polynomials $q_{k,\ell}$ are expressed in terms of the $p_{k,\ell}$'s in the form

$$q_{K,L} = \sum_{k+2\ell \le K+2L} (-1)^{k+2\ell} \begin{bmatrix} K, L \\ k, \ell \end{bmatrix} p_{k,\ell}.$$

Our goal here is to obtain explicit formulae for the $q_{k,\ell}$'s. Since Theorem 4.5 gives the $p_{k,\ell}$'s explicitly in terms of γ_1 and γ_2, we need only provide a formula for the generalized binomial coefficient $\left[\begin{smallmatrix} K,L \\ k,\ell \end{smallmatrix}\right]$. Recurrence relations for these coefficients were given in [6]. These are quite complicated and will not be repeated here, as we will prove:

Theorem 4.8. *For $(k,\ell), (K,L) \in \Lambda$ one has*

$$\begin{bmatrix} K, L \\ k, \ell \end{bmatrix} = \frac{c_{K,L}}{c_{k,\ell}} \sum_{j=0}^{\min(\lfloor k/2 \rfloor, L-\ell)} \binom{K+2L-2(\ell+j)}{k-2j} \frac{(-1)^j}{c_{k-j-1,j}\, c_{K,L-\ell-j}}$$

when $k+2\ell \le K+2L$ and $\ell \le L$. Otherwise $\left[\begin{smallmatrix} K,L \\ k,\ell \end{smallmatrix}\right] = 0$.

Proof. For $(k,\ell),(K,L) \in \Lambda$ with $|(k,\ell)| = k+2\ell \le K+2L = |(K,L)|$, Equation (3.4) yields

$$\begin{bmatrix} K, L \\ k, \ell \end{bmatrix} = \left\langle \frac{\gamma_1^m}{m!} h_k p_{k,\ell}, p_{K,L} \right\rangle_\star$$

where $m = (K+2L) - (k+2\ell)$. Using Theorem 4.5, this becomes

$$\begin{bmatrix} K, L \\ k, \ell \end{bmatrix} = \frac{1}{c_{k,\ell}} \sum_{j=0}^{\lfloor k/2 \rfloor} \frac{(-1)^j}{m!(k-2j)!\, c_{k-j-1,j}} \langle \gamma_1^{m+k-2j} \gamma_2^{\ell+j}, p_{K,L} \rangle_*.$$

Parts (b) and (c) in Lemma 4.6 show that

$$\langle \gamma_1^{m+k-2j} \gamma_2^{\ell+j}, p_{K,L} \rangle_* = \begin{cases} 0 & \text{if } j > L-\ell \\ \frac{(m+k-2j)!\, c_{K,L}}{c_{K,L-\ell-j}} & \text{if } j \le L-\ell \end{cases}.$$

Thus we have $\left[\begin{smallmatrix} K,L \\ k,\ell \end{smallmatrix}\right] = 0$ when $\ell > L$ and

$$\begin{aligned} \begin{bmatrix} K, L \\ k, \ell \end{bmatrix} &= \frac{1}{c_{k,\ell}} \sum_{j=0}^{\min(\lfloor k/2 \rfloor, L-\ell)} (-1)^j \frac{(m+k-2j)!}{m!(k-2j)!} \frac{c_{K,L}}{c_{k-j-1,j}\, c_{K,L-\ell-j}} \\ &= \frac{c_{K,L}}{c_{k,\ell}} \sum_{j=0}^{\min(\lfloor k/2 \rfloor, L-\ell)} \binom{K+2L-2(\ell+j)}{k-2j} \frac{(-1)^j}{c_{k-j-1,j}\, c_{K,L-\ell-j}} \end{aligned}$$

when $\ell \le L$. ∎

Product formulae: The method used to prove Theorem 4.8 can also be applied to products of the form $p_{k,\ell}p_{k',\ell'}$. We first consider a product $p_k p_{k'} = p_{k,0}p_{k',0}$. Since $p_k p_{k'}$ is $SO(n,\mathbb{R})\times\mathbb{T}$-invariant and homogeneous of degree $2(k+k')$, we have

$$p_k p_{k'} = \sum_{K+2L=k+k'} A_{k,k'}^{K,L} p_{K,L}$$

for some numbers $A_{k,k'}^{K,L}$. In fact

$$A_{k,k'}^{K,L} = \frac{c_{K,L}h_K}{h_k h_{k'}} \sum_{\substack{0\le j\le\lfloor k/2\rfloor \\ 0\le j'\le\lfloor k'/2\rfloor \\ j+j'\le L}} \binom{k-2j+k'-2j'}{k-2j} \frac{(-1)^{j+j'}}{c_{k-j-1,j}c_{k'-j'-1,j'}c_{K,L-j-j'}}. \tag{4.7}$$

Indeed, using Lemma 3.3, Theorem 4.5 and Lemma 4.6 we compute

$$\begin{aligned} A_{k,k'}^{K,L} &= h_K\langle p_k p_{k'}, p_{K,L}\rangle_* \\ &= \frac{h_K}{h_k h_{k'}} \sum_{j=0}^{\lfloor k/2\rfloor}\sum_{j'=0}^{\lfloor k'/2\rfloor} \frac{(-1)^{j+j'}\langle \gamma_1^{k-2j+k'-2j'}\gamma_2^{j+j'}, p_{K,L}\rangle_*}{(k-2j)!(k'-2j')!c_{k-j-1,j}c_{k'-j'-1,j'}} \\ &= \frac{h_K}{h_k h_{k'}} \sum_{\substack{0\le j\le\lfloor k/2\rfloor \\ 0\le j'\le\lfloor k'/2\rfloor \\ j+j'\le L}} \frac{(-1)^{j+j'}(k-2j+k'-2j')!c_{K,L}}{(k-2j)!(k'-2j')!c_{k-j-1,j}c_{k'-j'-1,j'}c_{K,L-j-j'}}. \end{aligned}$$

The formula for $p_{k,\ell}p_{k',\ell'}$ follows from the above by using Proposition 4.1(a). One obtains:

Proposition 4.9.

$$p_{k,\ell}p_{k',\ell'} = \frac{1}{c_{k,\ell}c_{k'\ell'}} \sum_{K+2L=k+k'} \frac{c_{K,\ell+\ell'+L}}{c_{K,L}} A_{k,k'}^{K,L} p_{K,\ell+\ell'+L}$$

where $A_{k,k'}^{K,L}$ is given by (4.7).

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D. Arnal and O. B. Baoua
Département de Mathématiques
Université de Metz
Ile du Saulcy 57045
Metz cedex 01
France
arnal@poncelet.univ-metz.fr
boukary@poncelet.univ-metz.fr

C. Benson and G. Ratcliff
Dept. of Math. and Comp. Science
Univ. of Missouri-St. Louis
St. Louis, MO 63121
U.S.A.
benson@arch.umsl.edu
ratcliff@arch.umsl.edu

Received Mai 24, 2000
and in final form January 23, 2001