Journal of Lie Theory
Volume **11** (2001) 415–426

Characterization of the Kantor–Koecher–Tits Algebra by a Generalized Ahlfors Operator

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Communicated by M. Cowling

Dedicated to Jacques Faraut on his 60th birthday

Abstract. In the context of certain generalized conformal structures we define a first order differential operator S generalizing the classical Ahlfors operator. We prove its invariance under the corresponding conformal group and show that, under certain conditions, the Lie algebra of this group (which is also known as the "Kantor–Koecher–Tits algebra") is precisely the space of solutions of the differential equation $SX = 0$.

0. Introduction

In his investigations on quasiconformal deformations, L. Ahlfors introduced the operator

$$SX = \frac{1}{2}(\mathrm{D}X + (\mathrm{D}X)^t) - \frac{1}{n}\,\mathrm{tr}(\mathrm{D}X) \tag{0.1}$$

(see [Ah74, Ah76]); it is a first order differential operator assigning to a vector field ("deformation") X on $\mathbb{R}^n$ a field of trace-free symmetric endomorphisms. He proved that this operator is invariant under the group of conformal transformations (which is the group generated by translations, similarities and the inversion) and that the solutions of the differential equation $SX = 0$ ("trivial deformations") are precisely the vector fields of the form

$$X(x) = a + Bx + 2\langle c, x\rangle x - \langle x, x\rangle c \tag{0.2}$$

with $a, c \in \mathbb{R}^n$ and B a sum of a skew-symmetric matrix and a multiple of the identity. This space of quadratic vector fields is nothing but the Lie algebra of the conformal group.

The Ahlfors operator S has a natural generalization in Riemannian geometry (see, for instance, [Bra97, PØ96]). Let (M, g) be a Riemannian manifold and

$$\nabla : \Gamma^\infty(TM) \to \Gamma^\infty(\mathrm{End}(TM)), \quad X \mapsto \nabla X \tag{0.3}$$

ISSN 0949–5932 / \$2.50

be its associated Levi-Civita connection; it assigns to a vector field X the field of endomorphisms ∇X defined by $(\nabla X)Y = \nabla_Y X$. Composing ∇ with a projection onto some subbundle of $\mathrm{End}(TM)$, we obtain what is nowadays often called a *generalized gradient* (usually in the more general context of connections on arbitrary bundles). In the case of a Riemannian manifold, the bundle $\mathrm{End}(TM)$ canonically decomposes into the sum of three subbundles, which are given in each tangent space T_xM respectively by skew-symmetric operators, multiples of the identity and symmetric trace-free operators (with respect to the metric g_x). Let $p_3 : \mathrm{End}(TM) \to \mathrm{End}(TM)$ denote the projection onto the last subbundle. Then the *Ahlfors operator* is the generalized gradient $S = p_3 \circ \nabla$.

In this paper we introduce a generalized gradient S on certain manifolds M having a "generalized conformal structure". As in the classical case, the corresponding conformal group is much "bigger" than the automorphism group of the connection ∇ we use (Theorem 2.1). The operator S shares an essential feature with the classical Ahlfors operator. Under certain conditions, the equation $SX = 0$ characterizes the Lie algebra of the conformal group (Theorem 2.5). This Lie algebra is known as the *Kantor–Koecher–Tits algebra* in the context of *Jordan algebras* and, more generally, of *Jordan triple systems*. Our approach is placed in this context. The basic ingredients of this approach are reviewed in Section 1; let us just mention here that the manifolds M we are interested in are *symmetric spaces* and ∇ is the canonical connection of such a space (see [KN69] or [Lo69] for the general theory). Jordan theory comes in by defining the subbundle of $\mathrm{End}(TM)$ onto which we project. This subbundle is an *additional* structure on the symmetric space which cannot be recovered from the usual theory of symmetric spaces; it is an aspect of what one may call a "generalized conformal structure on M" —in [Be01] and [Be00] the more neutral but not entirely satisfactory term "twist" is used; following a suggestion of the referee, we will say that M is a *Jordan symmetric space*. Most surprisingly, all classical and many of the exceptional symmetric spaces do admit such an additional structure and, as classification shows, it is in most cases unique. Thus there are many examples of the context in which we define our generalized Ahlfors operators; we mention some of them in Section 1.

The investigation of conformally invariant operators is a vast topic, see, for example, [CSS98] and [KR00]. In contrast to these approaches, our proof of conformal invariance is purely geometric and does not need any assumptions of irreducibility and the corresponding weight theory, thus giving a partial answer to a problem mentioned in the introduction of [KR00]. The fact that our operator is defined in geometric terms and not via representation theory also allows to give an algebraic formula for our operator (formula (2.7)) which is very similar to the classical formula (0.1).

Acknowledgement: The second author would like express his gratitude to the Institut Elie Cartan in Nancy, where part of this work was done, for its hospitality.

1. Preliminaries

1.1. Symmetric spaces. A *symmetric space* is a manifold M together with a complete torsion free affine connection ∇ such that the curvature R is covariantly constant: $\nabla R = 0$. If M is connected, then M can be written as a *homogeneous symmetric space*: $M = G/H$ is homogeneous under the action of a Lie group G such that H is an open subgroup of the group G^σ of fixed points of some nontrivial involution σ of G (see [Lo69] or [KN69]). Let $\mathfrak{g} = \mathfrak{h}_p \oplus \mathfrak{q}_p$ be the decomposition of the Lie algebra $\mathfrak{g}$ of G (assumed to act effectively on M) under the differential of the involution σ_p corresponding to the point $p \in M$. In this group theoretic set-up, the affine connection can then be recovered by the formula

$$(\nabla_X Y)_p = [L(X_p), Y]_p \qquad (p \in M) \tag{1.1}$$

where for a tangent vector $v \in T_pM$, $L(v)$ denotes the unique vector field $Z \in \mathfrak{q}_p$ such that $Z_p = v$ (see [Be01, Section I.2]).

1.2. The Jordan tensor. We assume that (M, ∇, T) is a *Jordan symmetric space*: (M, ∇) is a symmetric space and T is a *Jordan extension of the curvature* R, that is, a covariantly constant tensor field of the same type as R (that is, of type $(3,1)$) such that for all $u, v, w, x, y \in T_pM$, $p \in M$, the following identities hold

(JT1) (symmetry) $T_p(u, v, w) = T_p(w, v, u)$,
(JT2) $[T_p(u, v), T_p(x, y)] = T_p(T_p(u, v)x, y) - T_p(x, T_p(v, u)y)$,
where $T_p(a, b)c := T_p(a, b, c)$,

and such that T is related to R by

$$R(X, Y) = -(T(X, Y) - T(Y, X)). \tag{1.2}$$

The identities (JT1) and (JT2) mean that, for each $p \in M$, T_p is a *Jordan triple system* on the tangent space T_pM. It follows from the identity (JT2) that the subspace of $\mathrm{End}(T_pM)$ spanned by the operators $T_p(a, b)$, $a, b \in T_pM$, is a Lie algebra, called the (inner) structure algebra of T_p and denoted by $\mathfrak{str}(T_p)$. In most parts of the following text, the Jordan triple system T_p will be assumed to be *nondegenerate* in the sense that the *trace form*

$$\beta_p(u, v) := \operatorname{tr} T_p(u, v), \tag{1.3}$$

is nondegenerate. Then (see [Be01, Lemma V.2.4]) β_p is symmetric—and thus β is a pseudometric tensor field on M—and it is *associative* in the sense that the transpose of $T_p(u, v)$ with respect to β_p is given by

$$T_p(u, v)^\top = T_p(v, u). \tag{1.4}$$

1.3. Construction of Jordan symmetric spaces. The nondegenerate Jordan symmetric spaces are all obtained by the following construction (which is described in detail in [Be01]). One starts with a nondegenerate Jordan triple system T_0 on a finite-dimensional vector space V. With T_0 one associates a

Kantor–Koecher–Tits algebra $\mathfrak{co}(T) = \mathfrak{co}(T_0)$, that is the Lie algebra of quadratic vector fields of the form

$$\xi(x) = v + Hx + P(x)w, \tag{1.5}$$

for $v, w \in V$ and $H \in \mathfrak{str}(T_0)$, where

$$P(x)w = \frac{1}{2}T_0(x, w, x).$$

We will use the notation $\mathbf{v}$ for the constant vector field having value v on V and $A(w)$ for the homogeneous quadratic vector field given by

$$(A(w))(x) = P(x)w \tag{1.6}$$

so that $\xi = \mathbf{v} + H + A(w)$. Any local diffeomorphism φ of V preserving this Lie algebra (in the sense that the push-forward $\varphi^*\xi$ of every element $\xi \in \mathfrak{co}(T)$ coincides on its domain of definition with an element of $\mathfrak{co}(T)$) is in fact given by a birational formula, and hence extends to a birational map of V. The birational maps so obtained form a group $\mathrm{Co}(T) = \mathrm{Co}(T_0)$, called the *conformal group* or *Kantor–Koecher–Tits group.* It is isomorphic to the adjoint group of $\mathfrak{co}(T)$, hence is a Lie group. Moreover, the adjoint representation is faithful, and therefore $\mathfrak{co}(T)$ is the Lie algebra of $\mathrm{Co}(T)$. The group of translations τ_v, $v \in V$, is a subgroup of $\mathrm{Co}(T)$, as well as the *structure group*, which is the linear group $\mathrm{Str}(T) := \mathrm{Str}(T_0)$ defined by

$$\mathrm{Str}(T_0) := \{g \in \mathrm{GL}(V) | \, (\forall u, v, w \in V) \, gT_0(u, v, w) = T_0(gu, (g^\top)^{-1}v, gw)\}.$$

Let Q be the subgroup of $\mathrm{Co}(T)$ of all elements $\varphi \in \mathrm{Co}(T)$ which are defined at 0 and satisfy $\varphi(0) = 0$. It is known that $\mathrm{Co}(T)$ is semisimple and that Q is a maximal parabolic subgroup of $\mathrm{Co}(T)$ (see [Lo71]). The map

$$V \to V^c := \mathrm{Co}(T)/Q, \quad v \mapsto \tau_v Q \tag{1.7}$$

is an imbedding with open dense image, called the *conformal compactification* of V.

The Kantor–Koecher–Tits algebra carries an involution Θ interchanging constant and homogeneous quadratic vector fields. It is given by the formulas

$$\Theta(\mathbf{v}) = -A(v), \quad \Theta(T_0(u, v)) = -T_0(v, u), \quad \Theta(H) = -H^\top \ (H \in \mathfrak{str}(T_0)). \tag{1.8}$$

This involution induces an involution Θ on the adjoint group of $\mathfrak{co}(T)$ and hence on the conformal group $\mathrm{Co}(T)$. Let $G = \mathrm{Co}(T)_o^\Theta$ be the identity component of the fixed point group of this involution. Then

$$M = G.0 \subset V^c \tag{1.9}$$

is an open orbit carrying the structure of a homogeneous symmetric space G/H. Here H is the automorphism group of T_0, and therefore T_0 extends to a G-invariant tensor field T on M which is indeed a Jordan extension of the curvature tensor. The space M has a natural chart on a neighborhood of the base point 0, namely the "vectorialization" $V \cap M$; we call the associated coordinates *Jordan coordinates*. In Jordan coordinates, the geodesic symmetry with respect to the base point is simply given by $-\mathrm{id}_V$, and the extension $L(v)$ at the origin is given by

$$L(v) = \mathbf{v} - A(v), \quad v \in V = T_0 M, \tag{1.10}$$

see [Be01, Theorem VII.2.4].

1.4. Jordan algebras and spaces of the first kind. To a unital Jordan algebra $(V, \cdot)$ one associates the Jordan triple system T_0 given by

$$T_0(u, v, w) = 2(u(vw) - v(uw) + (uv)w). \tag{1.11}$$

If $e \in V$ is the unit element, then this Jordan triple system has the special property that $P(e) = \mathrm{id}_V$. More generally, we say that a Jordan triple system T_0 is *of the first kind* if it has *invertible elements*, that is, elements $x \in V$ such that $P(x)$ is invertible. All such Jordan triple systems are of the form $T_0^{(\alpha)}$, where T_0 is associated with the Jordan algebra as above, $\alpha \in \mathrm{GL}(V)$ satisfies

$$\alpha T_0(u, \alpha v, w) = T(\alpha u, v, \alpha w) \tag{1.12}$$

for all $u, v, w \in V$, and

$$T_0^{(\alpha)}(u, v, w) = T_0(u, \alpha v, w) \tag{1.13}$$

is the "α-modification of T_0" (see [Be01, Section XI.1]).

1.5. Examples. Classification shows that all classical and many exceptional symmetric spaces admit a (generically unique) Jordan extension of the curvature. The corresponding list is therefore very long (see [Be01]). Here we just mention some particularly interesting examples (all of them, with the exception of $\mathrm{GL}(n, \mathbb{R})$, are Riemannian); for the geometric realization of these data we refer to [Be01].

(1) Classical conformal space: $M = S^n = \mathrm{SO}(n+1)/\mathrm{SO}(n)$, conformal group: $\mathrm{SO}(n+1, 1)$, structure group: $\mathrm{SO}(n) \times \mathbb{R}^*$, Jordan triple system: $\mathbb{R}^n$ with $T_0(u, v, w) = \langle u, v\rangle w + \langle w, v\rangle u - \langle u, w\rangle v$ (thus the conformal Lie algebra is indeed given by (0.2)).

(2) Hyperbolic spaces: $M = H^n = \mathrm{SO}(n, 1)/\mathrm{SO}(n)$: conformal group and structure group as in (1), Jordan triple system: same space as in (1) with the negative of the product given there.

(3) Grassmannians: $M = \mathrm{O}(p+q)/(\mathrm{O}(p) \times \mathrm{O}(q))$, conformal group: $\mathrm{PGL}(p+q, \mathbb{R})$, structure group $\mathrm{P}(\mathrm{GL}(p, \mathbb{R}) \times \mathrm{GL}(q, \mathbb{R}))$, Jordan triple system $V = M(p, q; \mathbb{R})$ (real $p \times q$ matrices) with $T_0(X, Y, Z) = -(XY^tZ + ZY^tX)$. Here the P means that one considers the respective projective group where the multiples of the identity are factored out. The cases $p = 1$ (projective space) and $p = q$ (see (4)) play a somewhat special role. For $\mathbb{R}$ replaced by $\mathbb{C}$ or $\mathbb{H}$ there are similar formulae.

(4) Group case $M = \mathrm{GL}(n,\mathbb{R})$, conformal group: $\mathrm{P}(\mathrm{GL}(2n,\mathbb{R}))$, structure group: $\mathrm{P}(\mathrm{GL}(n,\mathbb{R}) \times \mathrm{GL}(n,\mathbb{R}))$, Jordan triple system $V = M(n,n,\mathbb{R})$ with

$$T_0(X,Y,Z) = XYZ + ZYX.$$

(5) Group case $M = \mathrm{U}(n)$, conformal group $\mathrm{SU}(n,n)$, structure group $\mathrm{GL}(n,\mathbb{C})/S^1$, Jordan triple system $V = \mathrm{Herm}(n,\mathbb{C})$ (Hermitian matrices) with

$$T_0(X,Y,Z) = -(XYZ + ZYX).$$

(6) Lagrangian Grassmannian $M = \mathrm{U}(n)/\mathrm{O}(n)$, conformal group $\mathrm{PO}(n,n)$, structure group $\mathrm{GL}(n,\mathbb{R})$, Jordan triple system $\mathrm{Sym}(n,\mathbb{R})$ (symmetric matrices) with $T_0(X,Y,Z) = -(XYZ + ZYX)$.

(7) Symmetric cones (see [FK94]): M isomorphic to the symmetric cone $\Omega = G/K$ associated with a Euclidean Jordan algebra V, G an open subgroup of the structure group, conformal group isomorphic to the automorphism group of the tube over Ω, Jordan triple system associated with V via formula (1.11) (notice that we use here the bounded realization $M = (-e+\Omega) \cap (e-\Omega)$ of Ω, see [Be01]).

(8) Bounded symmetric domains (see [Lo77]): $M = G/K$ is a Hermitian symmetric space in its Harish-Chandra realization in $\mathfrak{p}_\mathbb{C} \cong \mathbb{C}^n$, conformal group $G_\mathbb{C}$, structure group $K_\mathbb{C}$, Jordan triple system on $\mathfrak{p}_\mathbb{C}$ as defined in [Lo77].

The spaces of the first kind among the above examples are precisely (1), (2), (3) $p = q$, (4), (5), (6), (7), and the spaces of tube type from (8).

2. The generalized Ahlfors operator

Keeping the notation of the preceding section, we now decompose the bundle $\mathrm{End}(TV^c) = (TV^c)^* \otimes (TV^c)$ into three $\mathrm{Co}(T)$-invariant subbundles. Observe first that the space $\mathrm{End}(V)$ decomposes as

$$\mathrm{End}(V) = \{X \in \mathfrak{str}(T) \mid \mathrm{tr}(X) = 0\} \oplus \mathbb{R}\,\mathrm{id}_V \oplus \mathfrak{str}(T)^\perp, \tag{2.1}$$

where the orthocomplement is taken with respect to the usual trace form. Clearly this decomposition is invariant under $\mathrm{Str}(T)$, and therefore the three pieces can be transported to all tangent spaces of V^c in a well-defined way, giving rise to a $\mathrm{Co}(T)$-invariant decomposition of $\mathrm{End}(TV^c)$. We denote by $p_i : \mathrm{End}(TV^c) \to \mathrm{End}(TV^c)$, $i = 1,2,3$, the corresponding projections. (Here, p_2 is essentially the trace.)

The *generalized Ahlfors operator* is the map S assigning to a vector field X the section of $\mathrm{End}(TV^c)$ given by

$$SX := p_3(\nabla X). \tag{2.2}$$

Theorem 2.1. *The Ahlfors operator $S : \Gamma^\infty(TM) \to \Gamma^\infty(\mathrm{End}(TM))$ extends to a conformally invariant differential operator $S : \Gamma^\infty(TV^c) \to \Gamma^\infty(\mathrm{End}(TV^c))$.*

Proof. Let $g \in \mathrm{Co}(T)$. We denote by g_* the push-forward of tensor fields and differential operators and by $g^* = (g^{-1})_*$ the canonical pull-back. We have to show that g_*S and S coincide on the open dense set $g(M) \cap M \subset V^c$. Clearly, since the projections p_i are $\mathrm{Co}(T)$-invariant, we have on this set

$$g_*S - S = g_*(p_3 \circ \nabla) - p_3 \circ \nabla = p_3 \circ (g_*\nabla - \nabla).$$

The condition $p_3 \circ (g_*\nabla - \nabla) = 0$ means that, for all $p \in M \cap g(M)$ and vector fields X, $(g_*\nabla - \nabla)X_p \in \mathfrak{str}(T_p)$. To prove this, note first that the difference of two connections is a tensor field of type $(2,1)$, i.e. an algebra structure on each tangent space. If both connections are torsion free this algebra structure is commutative. The important fact, to be proved next, is that for all connections involved in our context the differences are *fields of Jordan algebras* which are derived from our Jordan triple system T_0 by fixing the middle argument. To make this precise, we introduce the following terminology.

Definition 2.2.

(i) For a Jordan triple system T_0 on a vector space V, let

$$W_0 := \{T_0(\cdot, v, \cdot) \,|\, v \in V\} \subset \mathrm{Hom}(V \otimes V, V)$$

be the space of algebra structures obtained by fixing the middle argument of T_0. (If T_0 is nondegenerate so that β_0 induces an isomorphism $V \to V^*, v \mapsto v^*$, it follows immediately from the definitions that the map $V^* \to W_0$, $v^* \mapsto T_0(\cdot, v, \cdot)$ is an isomorphism of vector spaces with inverse $W_0 \to V^*$, $c \mapsto (x \mapsto \mathrm{tr}\, c(x, \cdot).)$ The space W_0 is $\mathrm{Str}(T)$-invariant and hence gives rise to a $\mathrm{Co}(T)$-invariant subbundle W of $\mathrm{Hom}(TV^c \otimes TV^c, TV^c)$, called the *structure bundle*.

(ii) We say that two affine connections ∇^1 and ∇^2 on the open set $U \subset V^c$ are *conformally equivalent* if their difference $C := \nabla^2 - \nabla^1$ is a section of the structure bundle W over U. ■

Note that, if $c = T_0(\cdot, v, \cdot) \in W_0$, then $c(w, \cdot) = T_0(w, v) \in \mathfrak{str}(T_0)$; therefore, if ∇^1 and ∇^2 are conformally equivalent, then $((\nabla^1 - \nabla^2)X)_p \in \mathfrak{str}(T_p)$ for all vector fields X and points p where things are defined. Thus the theorem will be proved if we can show that $g_*\nabla$ and ∇ are conformally equivalent. More generally, we will prove

Theorem 2.3. *Let ∇^0 be the canonical flat connection on the vector part $V \subset V^c$. Then*

(i) *∇^0 and ∇ are conformally equivalent. More precisely, for $C := \nabla - \nabla^0$ we have the formula*

$$C_x(u, v) = T_0(u, (\mathrm{id} - P(x))^{-1}x, v), \quad u, v \in V, x \in M \cap V.$$

(ii) *For any $g \in \mathrm{Co}(T)$, the connections $\nabla, g_*\nabla$ and $g_*\nabla^0$ are all conformally equivalent.*

Proof. (i) We show first that $C_0 = 0$: in fact, for $u \in V = T_0M$, taking account of Equation (1.10), we get from Equation (1.1) $(\nabla_{\mathbf{u}}\mathbf{v})_0 = [L(u), \mathbf{v}]_0 = [\mathbf{u} - A(u), \mathbf{v}]_0 = 0$.

Suppose that $\varphi: M \to M$ is an affine diffeomorphism with respect to ∇ such that $x := \varphi(0) \in V$. We show that

$$C_x(u, v) = ((\mathrm{D}^2\varphi)(0))(\mathrm{D}\varphi(0)^{-1}u, \mathrm{D}\varphi(0)^{-1}v). \tag{2.3}$$

In fact, since $\varphi^*\nabla = \nabla$, we have

$$\varphi^*C - C = (\varphi^*\nabla - \nabla) - (\varphi^*\nabla^0 - \nabla^0) = (\nabla^0 - \varphi^*\nabla^0).$$

An elementary computation shows that the effect of φ on the canonical flat connection ∇^0 of a vector space is given by

$$(\varphi^*\nabla^0 - \nabla^0)_x(u, v) = (\mathrm{D}\varphi(x))^{-1} \cdot ((\mathrm{D}^2\varphi)(x))(u \otimes v)$$

(see [Be01, Appendix I.B]). Therefore

$$(\varphi^*C - C)_0(u, v) = -(\mathrm{D}\varphi(0))^{-1} \cdot ((\mathrm{D}^2\varphi)(0))(u \otimes v).$$

On the other hand, since $C_0 = 0$, the definition of φ^* gives

$$(\varphi^*C - C)_0(u, v) = (\mathrm{D}\varphi(0))^{-1}C_{\varphi(0)}(\mathrm{D}\varphi(0)u, \mathrm{D}\varphi(0)v).$$

Comparison of the last two equations yields (2.3). Next we have to calculate the second differential $\mathrm{D}^2\varphi(0)$. We claim that

$$((\mathrm{D}\varphi)(0))^{-1} \cdot \mathrm{D}^2(0)(\varphi)(u \otimes v) = T_0(u, \varphi^{-1}(0), v). \tag{2.4}$$

In fact, we may decompose $\varphi = vgn$ with $g = \mathrm{D}\varphi(0)$, $v = \tau_{\varphi(0)}$ and $n = \Theta(\tau_{\varphi^{-1}(0)}) = \exp(-A(\varphi^{-1}(0)))$ (see [Be01, Theorem VIII.2.3(5)]). Then $\mathrm{D}^2\varphi(0) = g \circ \mathrm{D}^2n(0)$ and

$$\mathrm{D}^2n(0) = -\mathrm{D}^2(A(-\varphi^{-1}(0)))(0) = T_0(\cdot, \varphi^{-1}(0), \cdot)$$

(for the last equality see [Be01, Prop. VIII.2.2 and its proof]). After multiplication by g^{-1} this gives (2.4). Putting (2.3) and (2.4) together, we have

$$\begin{aligned} C_{\varphi(0)}(u, v) &= \mathrm{D}\varphi(0)T_0((\mathrm{D}\varphi(0))^{-1}u, \varphi^{-1}(0), (\mathrm{D}\varphi(0))^{-1}v) \\ &= T_0(u, \Theta(\mathrm{D}\varphi(0))\varphi^{-1}(0), v), \end{aligned} \tag{2.5}$$

where the last equality is due to the fact that $\mathrm{D}\varphi(0)$ belongs to the structure group of T. Finally, we claim that, with $x = \varphi(0)$ in some neighborhood of the origin, the equality

$$\Theta(\mathrm{D}\varphi(0))\varphi^{-1}(0) = (\mathrm{id} - P(x))^{-1}x \tag{2.6}$$

holds. In fact, we may choose $\varphi = \exp(\mathbf{v} - A(v))$; then $x = \mathrm{Exp}(v)$ is given by $x = \tanh(v)$ in the sense of [Be01, Section X.4]. Here Exp denotes the exponential map of the symmetric space. On the other hand, $\Theta(\mathrm{D}\varphi(0)) = (\mathrm{D}\varphi^{-1}(0))^{-1}$ (see [Be01, VIII.2.3 (5)]. Note that $\varphi^{-1} = (-\mathrm{id}) \circ \varphi \circ (-\mathrm{id})$, and thus $\mathrm{D}\varphi^{-1} = \mathrm{D}\varphi$. Now [Be01, Prop. X.4.2] states that

$$\Theta(\mathrm{D}\varphi(0))\varphi^{-1}(0) = -(\mathrm{D}\varphi^{-1}(0))^{-1}\varphi(0) = \cosh^2(v)\tanh(v).$$

Define the power series $\tanh^2(v)$ to be $\sum a_k P(v)^k$, where $\sum a_k z^{2k}$ is the usual power series of $\tanh^2(z)$. By power associativity, the operator $\tanh^2(v)$ coincides with $P(\tanh v)$ on the Jordan span of v. Then the identity $\cosh^2 z = \left(1 - \tanh^2 z\right)^{-1}$ for complex z yields (again by power associativity) the identity

$$\cosh^2(v)w = (\mathrm{id} - \tanh^2(v))^{-1}w$$

for all w in the Jordan span of v, and in particular for $w = \tanh(v) = x$. Thus we get

$$\cosh^2(v)\tanh(v) = (\mathrm{id} - P(\tanh v))^{-1}\tanh(v).$$

Together with the preceding equation, this yields (2.6). Finally, the claim follows by combining (2.5) and (2.6).

(ii) Note first that conformal equivalence is indeed an equivalence relation (assuming that all objects are defined on Zariski-dense subsets). Now, it is proved in [Be01, Theorem VIII.1.11] that $g_*\nabla^0$ and ∇^0 are conformally equivalent. By (i), ∇^0 and ∇ are conformally equivalent. These two facts together imply that

$$g_*\nabla - \nabla = g_*(\nabla^0 + C) - (\nabla^0 + C) = g_*\nabla^0 - \nabla^0 + g_*C - C$$

is again a section of the structure bundle, whence the claim. ■

Corollary 2.4. *In Jordan coordinates, the Ahlfors operator is the constant operator* $(p_3)_0 \circ \mathrm{D}$ *obtained by composing* $(p_3)_0 : \mathrm{End}(V) \to \mathfrak{str}(T_0)^\perp$ *with the ordinary first differential, i.e.*

$$(SX)(p) = (p_3)_0(\mathrm{D}X)(p).$$

Proof. Since p_3 is invariant under the conformal group, and thus in particular under translations, it is represented in the chart V by the constant map $(p_3)_0$. Next, $SX = p_3\nabla(X) = p_3(\nabla^0 + C)(X) = p_3\nabla^0 X$ since $p_3 C(X) = 0$ by Theorem 2.3 (i). Now the claim follows since ∇^0 is nothing but the ordinary first differential: $(\nabla^0_Y X)(p) = \mathrm{D}X(p) \cdot Y(p)$. ■

More generally, the preceding argument shows that the Ahlfors operators which one can associate to any two conformally equivalent connections are the same.

Theorem 2.5. *Let M be the Jordan symmetric space associated with a Jordan triple system T of the first kind obtained from a simple Jordan algebra which is not isomorphic to $\mathbb{R}$ or $\mathbb{C}$ (or a direct sum of such Jordan triple systems), and let $S = p_3 \circ \nabla$ be the corresponding Ahlfors operator. Then a (possibly only locally defined) vector field X is conformal (i.e., it is the restriction of an element of the conformal Lie algebra $\mathfrak{co}(T)$) if and only if $SX = 0$.*

Proof. Let X be defined on the open set $U \subset V^c$. By possibly translating and shrinking U, we may assume that $U \subset V$. By definition of S, the condition $(SX)_p = 0$ for all $p \in U$ is equivalent to

$$(\nabla X)_p \in \mathfrak{str}(T_p)$$

for all $p \in U$. By the preceding corollary, this amounts to

$$\mathrm{D}X(p) \in \mathfrak{str}(T_0)$$

for all $p \in U$. This is the condition of $\mathfrak{str}(T_0)$-conformality from [Be96], and according to [Be96, Theorem 1.3.2] (see also [Be00] and [Be01, Chapter IX]) it describes precisely the conformal Lie algebra. (Note that $\mathfrak{str}(T) = \mathfrak{str}(T^{(\alpha)})$ for all α-modifications of T, allowing us to assume without loss of generality that T is already the Jordan triple system associated with a Jordan algebra.) ■

Remark 2.6. The proof of Theorem 2.3 shows that the condition $SX = 0$ is equivalent to $\mathfrak{str}(T_0)$-conformality in the sense of [Be96]. Thus the conclusion of the theorem holds whenever the condition $\mathrm{Hom}_s(V, \mathfrak{str}(T_0)) = 0$ from [Be96] is verified. In the literature one can find the remark (see, for example, [GK98, Go87]) that this condition holds not only in the Jordan *algebra* case but more generally for a simple Jordan triple system which is not isomorphic to a modification of the projective Jordan triple system $M(1, n; \mathbb{R})$. For the latter case it certainly does not hold since then $\mathfrak{str}(T_0)$ is the full general Lie algebra, and hence our Ahlfors operator is zero. ■

An explicit formula for the Ahlfors operator. Finally, by means of the Jordan tensor T and by using some results of K. Meyberg on traces in Jordan triple systems, we will give an "explicit formula" for our Ahlfors operator in the spirit of formula (0.1). To this end, given a nondegenerate Jordan triple system T_0 on V, let

$$\widetilde{T}_0 : V \otimes V^* \to \mathrm{End}(V), \quad v \otimes w^* \mapsto T_0(v, w)$$

(with w^* defined using the trace form). Using the isomorphism $V \otimes V^* \to \mathrm{End}(V)$, $v \otimes \varphi \mapsto (x \mapsto \varphi(x)v)$, we may and will consider $\widetilde{T}_0$ as a linear map $\widetilde{T}_0 \in \mathrm{End}(\mathrm{End}(V))$. The important point is that this map is $\mathrm{Str}(T)$-equivariant. Using this, Schur's lemma and some trace calculations, one gets the following result (see [Mey84,Mey93]).

Proposition 2.7. *Let $\mathrm{End}(V) = E_1 \oplus E_2 \oplus E_3$ be the decomposition given by Equation (2.1). Then $\widetilde{T}_0$ preserves the subspaces E_i, $i = 1, 2, 3$. Moreover,*

(i) *the restriction of $\widetilde{T}_0$ to E_3 is zero,*

(ii) $\widetilde{T}_0(\mathrm{id}_V) = \mathrm{id}_V$,

(iii) $\mathrm{tr} \circ \widetilde{T}_0 = \mathrm{tr}$,

(iv) *if the derived algebra* $\mathfrak{str}(T_0)'$ *is simple, then the restriction of* $\widetilde{T}_0$ *to* E_1 *acts by the scalar* $(\dim \mathfrak{str}(T_0) - 1)/(\dim V - 1)$. ■

Since we have always $(p_1)_0 A = A - n^{-1}\,\mathrm{tr}(A)\,\mathrm{id}_V$, we deduce that under the assumption of (iv), $(p_3)_0$ is given by

$$(p_3)_0(A) = A - \frac{s-1}{n-1}\widetilde{T}_0(A) + \frac{s-n}{n(n-1)}\,\mathrm{tr}(A)\,\mathrm{id}_V,$$

where $n = \dim V$, $s = \dim \mathfrak{str}(T_0)$. An explicit formula for the Ahlfors operator in Jordan coordinates is now deduced from Corollary 2.4 by letting $A = \mathrm{D}X$

$$SX = \mathrm{D}X - \frac{s-1}{n-1}\widetilde{T}_0(\mathrm{D}X) + \frac{s-n}{n(n-1)}\,\mathrm{tr}(\mathrm{D}X)\,\mathrm{id}_V\,. \tag{2.7}$$

We conclude by remarking that the assumption of (iv) is satisfied for all simple Jordan triple systems T except for the case $V = M(p,q;\mathbb{F})$ $(p,q > 1)$, for which $\mathfrak{str}(T)'$ is a direct product $\mathfrak{sl}(p,\mathbb{F}) \times \mathfrak{sl}(\mathfrak{q},\mathbb{F})$ $(\mathbb{F} = \mathbb{R},\mathbb{C},\mathbb{H})$. The conclusion of (iv) still holds if $p = q$ but is no longer true if $p \neq q$.

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Received April 25, 2000
and in final form December 4, 2000