Journal of Lie Theory
Volume **11** (2001) 559–604

On Asymptotic Behavior and Rectangular Band Structures in SL(2,$\mathbb{R}$)

Brigitte E. Breckner* **and Wolfgang A.F. Ruppert**

Communicated by K. H. Hofmann

Abstract. We associate with every subsemigroup of $\mathrm{Sl}(2,\mathbb{R})$, not contained in a single Borel group, an 'asymptotic object,' a rectangular band which is defined on a closed subset of a torus surface. Using this concept we show that the *horizon set* (in the sense of LAWSON [10]) of a connected open subsemigroup of $\mathrm{Sl}(2,\mathbb{R})$ is always convex, in fact the interior of a three dimensional Lie semialgebra. Other applications include the classification of all exponential subsemigroups of $\mathrm{Sl}(2,\mathbb{R})$ and the asymptotics of semigroups of integer matrices in $\mathrm{Sl}(2,\mathbb{R})$.

1991 *Mathematics Subject Classification:* 22E15, 22E67, 22E46, 22A15, 22A25

Key words and phrases: asymptotic objects, asymptotic property, subsemigroups of $\mathrm{Sl}(2,\mathbb{R})$, Lie semigroups, Lie semialgebras and their classification, compression semigroups, diamond product, rectangular domain, umbrella sets, Control Theory in Lie groups, asymptotics of integer matrix semigroups, semialgebraic sets.

1. Introduction

During the last two decades *Lie semigroups* and *Lie wedges* have been the central objects of interest in the structure theory of subsemigroups of Lie groups. With the Lie theory of groups as a model and some control theoretic ideas as additional input, a useful technical apparatus has been developed, allowing the passage from infinitesimal to local and global objects, and, conversely, from local or global Lie semigroups to their infinitesimal objects, which are Lie wedges. This machinery provided us with fairly detailed information about interesting types of Lie semigroups and about the interplay between infinitesimal, local and global properties of semigroups. A substantial part of the theory has been published in books and monographs, we only mention [5], [6], [14] and [8].

For some questions, however, these well established tools have to be complemented by definitely non-local concepts and methods. Most prominent among such problems is the determination and structural description of maximal

* This author was supported by the Deutsche Forschungsgemeinschaft.

ISSN 0949–5932 / $2.50

subsemigroups in a connected Lie group. Let S be a maximal subsemigroup with nonempty interior of a connected Lie group G. It has been shown by J.D. LAWSON [9] that if G is solvable then S must be a half space subsemigroup bounded by a closed subgroup of codimension one, so in this case S is a Lie semigroup. (A similar result holds if the radical of G is cocompact, cf. [6].) But if G is semisimple then S no longer needs to be a Lie semigroup. In fact, by the work of L.A.B. SAN MARTIN and P.A. TONELLI (cf., e.g., [18] and [19]), we know how to construct, if G is noncompact semisimple and has finite center, maximal closed subsemigroups S with nonempty interior which do not contain the identity. These semigroups appear as compression semigroups of certain subsets of a flag manifold G/P. In this theory an important question is whether a subsemigroup S of G meets a certain open subset of G, the control set of the action. This is a definitely non-local problem, which, as D. MITTENHUBER ([12],[13]) has pointed out, is closely related with the *horizon* concept of J.D. LAWSON in [10].

The *horizon* of a subsemigroup S with nonempty interior of a Lie group G is defined as the set of all vectors in the Lie algebra for which the corresponding one parameter subsemigroup eventually stays in the interior of S. Obviously, the horizon of S is closed under multiplication with positive scalars, and if two elements in the horizon commute then their sum also lies in the horizon. So far further information about the structure of the horizon seems to be scarce, except in a few special cases.

If the subsemigroup S of G neither does cluster at the identity nor has interior points (in particular, if S is discrete) then there is usually little chance that the infinitesimal or local theory of subsemigroups of Lie groups will yield any structural information about S. But, even for S discrete we are faced with the natural analytic question asking for the behavior of S 'at large.'

All these situations call for the study of asymptotic properties and asymptotic objects, alongside and in conjunction with infinitesimal properties and tangent objects. The present paper is intended to produce evidence for the feasibility of such a theory, offering a fairly comprehensive treatment of asymptotic properties in the special case of subsemigroups of $\mathrm{Sl}(2,\mathbb{R})$. We shall show that, at least in this special case,

- workable definitions of asymptotic objects, associated with not necessarily connected subsemigroups, can be given,
- these asymptotic objects bear a 'rectangular band' multiplicative structure, which is obtained easily from the multiplication of S by a limiting process,
- the so defined asymptotic objects can be explicitly computed from the Lie wedge $L(S)$ if S is a Lie semigroup (thus infinitesimal and asymptotic structures supplement each other nicely),
- the asymptotic objects are very helpful for the determination and structural description of the horizon sets of LAWSON and similar objects ('umbrella sets', defined below), they can be used also for the determination of the conjugacy classes of exponential subsemigroups of $\mathrm{Sl}(2,\mathbb{R})$.

Perhaps the most surprising among the results of this paper is the fact that for a connected open subsemigroup $S \neq \mathrm{Sl}(2,\mathbb{R})$ the umbrella set (which in this case coincides with LAWSON's horizon set)

$$\mathrm{Umb}(S) = \{X \in \mathfrak{sl}(2,\mathbb{R}) \mid \exp \mathbb{R}^+ X \cap S \neq \emptyset\}$$

is an open convex cone whose closure is the Lie wedge of an exponential subsemigroup of $\mathrm{Sl}(2,\mathbb{R})$. This feature certainly does not hold in general for connected open subsemigroups of Lie groups.

In a subsequent paper ([17]) we shall exploit the present concepts and the ensuing theory for the explicit construction of Bohr compactifications of subsemigroups of $\mathrm{Sl}(2,\mathbb{R})$.

We temporarily postpone, however, the further extension of the theory to higher dimensions. Many important aspects of the structure of $\mathrm{Sl}(2,\mathbb{R})$ are no longer present in a more general context, and we cannot hope to develop a comparably satisfying theory by straightforward generalization. For instance, as mentioned above, in $\mathrm{Sl}(2,\mathbb{R})$ a central role is played by the exponential subsemigroups with nonvoid interior, whereas for noncompact simple Lie groups of dimension > 3 no exponential subsemigroups with interior points exist. The group $\mathrm{Sl}(2,\mathbb{R})$ has a unique standing among simple Lie groups (similar to the unique role of $\mathbb{R}$ among abelian Lie groups) and it is also the aim of this paper to further elucidate its special structure.

Acknowledgements. We want to thank especially KARL HEINRICH HOFMANN, who evoked the interest of both authors in questions which eventually led to investigations some of which are presented here, and who always has taken an active interest in these studies. Special thanks go also to the Deutsche Forschungsgemeinschaft (DFG) for giving financial support to the first author during her stay at Darmstadt and at Vienna while working on this project.

2. Setting the stage

2.1. Notation. Throughout these notes we use the symbol $\mathbb{R}^+$ for the set of strictly positive reals and $\mathbb{R}_0^+$ for the set of nonnegative reals, we write $\mathbb{N}$ for the positive integers and $\mathbb{N}_0$ for $\mathbb{N} \cup \{0\}$. The interior of a subset A of a topological space is denoted by $\mathrm{Int}\, A$, the closure of A by $\overline{A}$.

We denote the Killing form of $\mathfrak{sl}(2,\mathbb{R})$ by Kill and write

$$\mathsf{Kill}^+ \stackrel{\text{def}}{=} \{X \in \mathfrak{sl}(2,\mathbb{R}) \mid -\det(X) = \frac{1}{8}\mathsf{Kill}(X,X) > 0\}.$$

For any $X \in \mathsf{Kill}^+$ we let

$$\Delta(X) \stackrel{\text{def}}{=} \sqrt{-\det(M)}.$$

Similarly, Kill^0 $[\mathsf{Kill}^-]$ will denote the set of all $X \in \mathfrak{sl}(2,\mathbb{R})$ with $\mathsf{Kill}(X,X)=0$ $[\mathsf{Kill}(X,X)<0]$. The set Kill^0 will be also called the *light cone.*

Furthermore, in accordance with the notation of [5]:

$$H=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad P=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \quad Q=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}.$$

Note that the Borel algebras in $\mathfrak{sl}(2,\mathbb{R})$ are exactly the conjugates of $\mathbb{R}H+\mathbb{R}P$.

2.2. The image of the exponential function.

(i) $\overline{\exp(\mathsf{Kill}^+)} = \exp(\overline{\mathsf{Kill}^+})$ *and the restriction of* $\exp$ *to* $\overline{\mathsf{Kill}^+}$ *is a diffeomorphism.*

(ii) $\mathrm{Sl}(2,\mathbb{R}) = \exp(\mathfrak{sl}(2,\mathbb{R})) \cup (-1)\cdot\overline{\exp(\mathsf{Kill}^+)}$ *and* $\overline{\exp(\mathsf{Kill}^+)} \cap (-1)\cdot\overline{\exp(\mathsf{Kill}^+)} = \emptyset$.

Proof. These assertions are long known, they can be inferred, e.g., from the discussion in [5], p. 416ff. ∎

Notation. In view of 2.2(i) we can extend the usual log-function $\exp(B)\to B$, where B is a Campbell-Hausdorff neighborhood, to a map

$$\exp(B)\cup\exp(\overline{\mathsf{Kill}^+}) \to B\cup\overline{\mathsf{Kill}^+}.$$

This extension is unique, we also denote it by $\log$.

2.3. Characterizations involving the trace. *Let* $g=\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$ *be an element in* $\mathrm{Sl}(2,\mathbb{R})$. *Then the following assertions hold:*

(i) $g\in\exp(\mathfrak{sl}(2,\mathbb{R}))$ *if and only if either* $\mathrm{trace}(g)>-2$ *or* $g=-\mathbf{1}$.

(ii) $-2<\mathrm{trace}(g)$ *if and only if* g *lies in the interior of* $\exp(\mathfrak{sl}(2,\mathbb{R}))$.

(iii) $-2<\mathrm{trace}(g)<2$ *if and only if* g *is conjugate to a rotation* $x\in SO(2,\mathbb{R})$ *with* $x\neq\mathbf{1},-\mathbf{1}$.

(iv) $\mathrm{trace}(g)>2$ *if and only if* g *lies in* $\exp(\mathsf{Kill}^+)$.

(v) $\mathrm{trace}(g)=2$ *if and only if* g *is unipotent, i.e.,* $g\in\exp(\mathsf{Kill}^0)$.

(vi) $\mathrm{trace}(g)=2\cos(q\pi)\notin\{-2,2\}$ *for some nonzero rational* q *if and only if* g *has finite order* >2.

Proof. In view of 2.2 and by inspection these assertions follow readily if g is either an upper triangular matrix or a rotation matrix. Since every matrix in $\mathrm{Sl}(2,\mathbb{R})$ is conjugate to either an upper triangular or a rotation matrix, and since the trace is invariant under conjugation, this implies the assertions. ∎

2.4. The reduced logarithm. For technical convenience we introduce the *reduced logarithm* as the map

$$\mathrm{rlog}\colon \exp(\mathsf{Kill}^+) \to \Delta^{-1}(1),\ g \mapsto \tfrac{1}{\Delta(\log(g))} \cdot \log(g).$$

2.5. Explicit formulas for $\exp$ **and** rlog**.** *Let* $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

(i) *If* $\mathrm{trace}(g) > 2$ *then*

$$\mathrm{rlog}(g) = \begin{pmatrix} \frac{a-d}{u} & \frac{2b}{u} \\ \frac{2c}{u} & -\frac{a-d}{u} \end{pmatrix} \quad \textit{with} \quad u = \sqrt{(a+d)^2 - 4}.$$

Furthermore

$$\log(g) = \log(\frac{a+d+u}{2}) \cdot \mathrm{rlog}(g).$$

(ii) *Conversely, for any real numbers* α, β, γ *with* $\alpha^2 + \beta\gamma = 1$ *and any nonzero* $t \in \mathbb{R}$ *we have*

$$\exp t \cdot \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \exp(\mathsf{Kill}^+),$$

where

$$a = \cosh(t) + \alpha \sinh(t) = \frac{1+\alpha}{2} e^t + \frac{1-\alpha}{2} e^{-t}, \qquad b = \sinh(t)\beta,$$
$$c = \sinh(t)\gamma, \qquad d = \cosh(t) - \alpha \sinh(t) = \frac{1-\alpha}{2} e^t + \frac{1+\alpha}{2} e^{-t}.$$

Proof. The formulas in (i),(ii) can be checked quickly by straightforward computation; they are just variants of the formulas in [5], p.417. ∎

2.6. Remark. If $X \in \mathfrak{sl}(2,\mathbb{R})$ is nilpotent then $X^2 = 0$ and $\exp tX = \mathbf{1} + tX$. Conversely, if $g \in \mathrm{Sl}(2,\mathbb{R})$ is unipotent then $\log(g) = g - \mathbf{1}$.

2.7. Lie wedges and Lie semigroups. (cf. [6], p.19ff) Recall that if S is a closed subsemigroup with $\mathbf{1} \in S$ of a Lie group G with Lie algebra $\mathfrak{g}$ then the *Lie wedge* $L(S)$ *of* S is the set of all $X \in \mathfrak{g}$ with $\exp(\mathbb{R}_0^+ X) \subseteq S$. The Lie wedge $L(S)$ is indeed a wedge, i.e., a closed convex set which is closed under addition. Furthermore, for every $X \in L(S) \cap -L(S)$ we have $e^{\mathrm{ad}X} L(S) = L(S)$.

If S is the smallest closed subsemigroup generated by $\exp(L(S))$ then S is called a *Lie semigroup* and $W = L(S)$ is said to be a *global Lie wedge with respect to* G.

2.8. Lie semialgebras. (cf. [5], p.86.) Let W be a wedge in a Lie algebra $\mathfrak{g}$. Then W is called a *Lie semialgebra* if it is a local semigroup with respect to the Campbell-Hausdorff multiplication $*$, i.e., there exists a Campbell-Hausdorff neighborhood B such that $(B \cap W) * (B \cap W) \subseteq W$.

In the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ every Lie semialgebra with nonvoid interior is the intersection of half spaces bounded by a Borel algebra (cf. [5], II.3.7, p.109).

3. Basic properties of subsemigroups of $\mathrm{Sl}(2, \mathbb{R})$

3.1. This section is intended to provide a handy summary of those basic properties of subsemigroups of $\mathrm{Sl}(2, \mathbb{R})$ which are used (or supposed to be known) in the sequel. Most of the results are from [5] and/or belong to the general folklore of the subject.

3.2. The interior of a subsemigroup in a topological group. *Let S be a subsemigroup of a topological group G with nonvoid interior. Then the following assertions hold:*

(i) *The set* $\operatorname{Int} S$ *of interior points of S is an ideal in S.*

(ii) *If* $\mathbf{1} \in \overline{\operatorname{Int} S}$ *then* $\operatorname{Int} S$ *is dense in S and* $\operatorname{Int}(\overline{S}) = \operatorname{Int} S$.

(iii) *Suppose that G is locally connected and that S is connected. If* $\mathbf{1} \in \overline{S}$ *then* $\operatorname{Int} S$ *is connected.*

Proof. Assertions (i), (ii) form Proposition V.0.15 of [5], p.370.

(iii) Pick $a, b \in \operatorname{Int} S$ and let U, V be connected open sets with $a \in U \subseteq \operatorname{Int} S$, $b \in V \subseteq \operatorname{Int} S$. Since $\mathbf{1} \in \overline{S}$ the intersections $U \cap aS$ and $V \cap Sb$ are nonvoid, so $U \cup aS$ and $V \cup Sb$ are connected subsets of $\operatorname{Int} S$. Since $aS \cap Sb$ contains the nonvoid subset aSb we therefore conclude that the union $U \cup aS \cup V \cup Sb$ is a connected subset of $\operatorname{Int} S$ containing both a and b. This implies (iii). ∎

3.3. Connected components of a topological semigroup. (cf.[16] Proposition 3.1) Let S be a topological semigroup. For $x \in S$ we denote with $C(x)$ the connected component of S which contains x.

(i) For every pair $x, y \in S$ we have $C(x)C(y) \subseteq C(xy)$ (note that $C(x)C(y)$ is connected and contains xy). In other words: the relation $x \sim_C y \iff C(x) = C(y)$ is a congruence on S.

(ii) $x^2 \in C(x)$ if and only if $C(x)$ is a semigroup.

(iii) For every $x \in S$ the union $S_c(x) = \bigcup_{k \in \mathbb{N}} C(x^k)$ is a subsemigroup of S.

3.4. Proposition. *Let S be a subsemigroup of* $\mathrm{Sl}(2,\mathbb{R})$ *with nonvoid interior* $\operatorname{Int} S$ *in* $\mathrm{Sl}(2,\mathbb{R})$ *and suppose that* $S \neq \mathrm{Sl}(2,\mathbb{R})$. *Then the following assertions hold:*

(i) $\operatorname{Int} S \subseteq \exp(\mathsf{Kill}^+) \cup (-1)\cdot\exp(\mathsf{Kill}^+)$ *and* $\operatorname{Int} S \cap \exp(\mathsf{Kill}^+)$ *is nonempty.*

(ii) *The closure of S cannot contain a one dimensional torus group.*

(iii) *If S is connected then* $S \subseteq \overline{\exp(\mathsf{Kill}^+)}$ *and* $\operatorname{Int} S \subseteq \exp(\mathsf{Kill}^+)$.

(iv) *If* $\mathbf{1} \in \overline{\operatorname{Int} S}$ *then* $S_{\mathbf{1}} \overset{\mathrm{def}}{=} \overline{\operatorname{Int} S \cap \exp(\mathfrak{sl}(2,\mathbb{R}))}$ *and its interior* $\operatorname{Int}(S_{\mathbf{1}}) = \operatorname{Int} S \cap \exp(\mathsf{Kill}^+)$ *are connected semigroups.*

Proof. (i) Since $\operatorname{Int} S \neq \mathrm{Sl}(2,\mathbb{R})$ and since every neighborhood of a compact element contains an element of finite order we conclude that $\operatorname{Int} S$ cannot contain compact elements. Similarly there are compact elements, and hence elements of finite order, in every vicinity of a unipotent element, so $\operatorname{Int} S$ cannot contain unipotent elements. Thus by the discussion in 2.3 above $\operatorname{Int} S \cap \exp(\mathfrak{sl}(2,\mathbb{R})) \subseteq \exp(\mathsf{Kill}^+)$ and the first assertion follows. If $s = -g \in \operatorname{Int} S$ with $g \in \exp(\mathsf{Kill}^+)$ then $s^2 = g^2 \in \operatorname{Int} S \cap \exp(\mathsf{Kill}^+)$, so $\operatorname{Int} S \cap \exp(\mathsf{Kill}^+)$ cannot be empty.

(ii) Suppose that, contrary to our assertion, $\overline{S}$ contains a one dimensional torus subgroup T. Pick an inner point $x \in \operatorname{Int} S$. But $\operatorname{trace}(-x) = -\operatorname{trace}(x)$ and T is connected, so we conclude that there is a $t \in T$ with $\operatorname{trace}(tx) = 0$. For all $s \in S$ we have $sx \in \operatorname{Int} S$ and for s sufficiently near to t we also have $\operatorname{trace}(sx) \in]-1,1[$, which means that the interior of S contains a compact element (2.3(iii)), a contradiction.

(iii) Since S is connected and contains inner points, the set $\operatorname{trace}(S)$ is a nondegenerate interval. If there were a point $g \in S \setminus \overline{\exp(\mathsf{Kill}^+)}$ then by 2.3(iv) $\operatorname{trace}(g) < 2$. On the other hand, by (i), there is an element $h \in S \cap \exp(\mathsf{Kill}^+)$, and $\operatorname{trace}(h) > 2$, by 2.3(iv). But S is connected, so $\operatorname{trace}(S)$ would be a neighborhood of 2 and therefore we could find a $k \in S$ such that $\operatorname{trace}(k) = 2\cos(\alpha)$, with α rationally independent of π. Since the powers k^n of such an element are dense in a one dimensional torus group, we arrive at a contradiction to (ii). This implies $S \subseteq \overline{\exp(\mathsf{Kill}^+)}$. The remaining inclusion follows from 2.3(iv) and (v).

(iv) Suppose that $\mathbf{1} \in \overline{\operatorname{Int} S}$. Then $\operatorname{Int} S$ is dense in S. We assume, without losing generality, that S is open. It was shown in [16] that in $\mathrm{PSl}(2,\mathbb{R}) = \mathrm{Sl}(2,\mathbb{R})/\{\mathbf{1},-\mathbf{1}\}$ every open subsemigroup clustering at $\mathbf{1}$ is connected. Let $q\colon \mathrm{Sl}(2,\mathbb{R}) \to \mathrm{PSl}(2,\mathbb{R}) = \mathrm{Sl}(2,\mathbb{R})/\{\mathbf{1},-\mathbf{1}\}$ be the natural quotient morphism. Then $q(S)$ is open and clusters at $\mathbf{1}$, hence is connected. Its inverse image $q^{-1}q(S) = S \cup (-\mathbf{1})S$ is an open semigroup and has exactly two connected components, namely $(S \cup -S) \cap \exp(\mathsf{Kill}^+)$ and $(S \cup -S) \cap -\exp(\mathsf{Kill}^+)$. The connected component $(S \cup -S) \cap \exp(\mathsf{Kill}^+)$ is a semigroup (by 3.3(ii)) and contains $S \cap \exp(\mathsf{Kill}^+)$. Thus the semigroup generated by $S \cap \exp(\mathsf{Kill}^+)$ is contained in $\exp(\mathsf{Kill}^+)$, so $S \cap \exp(\mathsf{Kill}^+)$ is a semigroup. The restriction of q to $S \cap \exp(\mathsf{Kill}^+)$ is an isomorphic embedding into $\mathrm{PSl}(2,\mathbb{R})$. Since the open semigroup $q(S \cap \exp(\mathsf{Kill}^+))$ clusters at $\mathbf{1}$, it is connected and therefore $S \cap \exp(\mathsf{Kill}^+)$ is connected. The assertion follows. ∎

3.5. Proposition. *Let S be an open subsemigroup of* $\mathrm{Sl}(2,\mathbb{R})$ *with* $S \neq \mathrm{Sl}(2,\mathbb{R})$. *Then for every $x \in S$ the following assertions hold:*

(i) *The subsemigroup $S_c(x) = \bigcup_{k\in\mathbb{N}} C(x^k)$ is open.*

(ii) *Suppose that x belongs to the exponential image in* $\mathrm{Sl}(2,\mathbb{R})$. *Then $S_c(x)$ is contained in* $\exp(\mathsf{Kill}^+)$ *and the sequence $\langle C(x^n)\rangle$ is eventually constant: there exists a natural number n_0 such that $C(x^n) = C(x^{n_0})$ for $n \geq n_0$. Moreover, $C(x^{n_0})$ is an ideal in $S_c(x)$ and $S_c(x)/\sim_C$ is a finite nilpotent semigroup.*

(iii) *Suppose that x does not belong to the exponential image in* $\mathrm{Sl}(2,\mathbb{R})$. *Then the powers of $C(x)$ in the semigroup $S/\sim_C$ form a finite subsemigroup $N(x)$ and the minimal ideal of $N(x)$ is a cyclic group of order two.*

Proof. Assertion (i) follows from the fact that in a locally connected space the connected components of open subsets are open.

(ii) If $x \in \exp(\mathfrak{sl}(2,\mathbb{R}))$ then $x \in \exp(\mathsf{Kill}^+)$ (by 3.4(i)) and $x^k \in \exp(\mathsf{Kill}^+)$, for any $k \in \mathbb{N}$. Since $C(x^k)$ is connected and open we see from 3.4(i) that $C(x^k) \subseteq \exp(\mathsf{Kill}^+)$, for any $k \in \mathbb{N}$, so $S_c(x) \subseteq \exp(\mathsf{Kill}^+)$.

Let $x = \exp X$. Since $C(x)$ is open we find a number $n \in \mathbb{N}$ with $y \stackrel{\mathrm{def}}{=} \exp((1+\frac{1}{n})X) \in C(x)$. Then $x^{n+1} = y^n \in C(x)^n \subseteq C(x^n)$. By connectedness, $C(x)C(x^n) \cup C(x^n)C(x) \subseteq C(x^n)$. Also, we conclude by induction that $C(x)^nC(x^n) \subseteq C(x^n)$, so the connected set $C(x^n)C(x^n)$ meets the component $C(x^n)$ and therefore is contained in $C(x^n)$. This finishes the proof of (ii).

(iii) The powers of x with odd exponents belong to $-\exp(\mathsf{Kill}^+)$, those with even exponents to $\exp(\mathsf{Kill}^+)$. Now (ii) implies the assertion. ∎

3.6. Exponential subsemigroups of a Lie group. Recall that a closed subsemigroup S of a Lie group G is called *exponential* if it is the exponential image $\exp(W)$ of its Lie wedge $W = L(S)$.

A wedge in the Lie algebra $\mathfrak{g}$ of G is called *exponential with respect to G*, or *G-exponential*, or *exponential* for short, if it is the Lie wedge of an exponential subsemigroup of G. Note that if $G \to G_1$ is a covering homomorphism (so that $\mathfrak{g}$ can be considered as the Lie algebra of G_1 as well) and if W is an exponential wedge with respect to G then W is exponential also with respect to G_1— *provided that W is global with respect to G_1*.

3.7. The exponential subsemigroup $\mathrm{Sl}(2,\mathbb{R})^+$. (cf.[5], p. 419ff)

Following [5] we write $\mathrm{Sl}(2,\mathbb{R})^+$ for the semigroup of matrices with nonnegative entries in $\mathrm{Sl}(2,\mathbb{R})$. The corresponding Lie wedge

$$L(\mathrm{Sl}(2,\mathbb{R})^+) = \mathbb{R}H + \mathbb{R}_0^+P + \mathbb{R}_0^+Q$$

is denoted by $\mathfrak{sl}(2,\mathbb{R})^+$. Obviously, $\mathfrak{sl}(2,\mathbb{R})^+$ is a Lie semialgebra and $\mathrm{Sl}(2,\mathbb{R})^+$ is an exponential subsemigroup of $\mathrm{Sl}(2,\mathbb{R})$. It has been shown in [5] (p.421, V.4.30) that

(i) *if a subsemigroup S of* $\mathrm{Sl}(2,\mathbb{R})$ *contains* $\mathrm{Sl}(2,\mathbb{R})^+$ *then its $\mathbf{1}$-component coincides with* $\mathrm{Sl}(2,\mathbb{R})^+$ *and* $S \subseteq \mathrm{Sl}(2,\mathbb{R})^+ \cup (-\mathbf{1})\mathrm{Sl}(2,\mathbb{R})^+$.

In particular, $\mathrm{Sl}(2,\mathbb{R})^+$ is a maximal connected subsemigroup of $\mathrm{Sl}(2,\mathbb{R})$. If W is the Lie wedge of a closed subsemigroup with nonvoid interior $S \neq \mathrm{Sl}(2,\mathbb{R})$ then W cannot meet Kill^- (by 3.4(ii)), hence for each of the convex halves of Kill^- there exists a support plane separating it from W. These two support planes dissect $\mathfrak{sl}(2,\mathbb{R})$ into four open quadrants and W is contained in the closure W_0 of one of the two quadrants not meeting Kill^-. The edge $H(W_0) = W_0 \cap -W_0$ of W_0 is a Cartan subalgebra, hence conjugate to $\mathbb{R}H$. It follows that W_0 must be conjugate to $\mathfrak{sl}(2,\mathbb{R})^+$. Thus

(ii) *the Lie wedge of a Lie subsemigroup with nonvoid interior* $S \neq \mathrm{Sl}(2,\mathbb{R})$ *is always contained in a conjugate of* $\mathfrak{sl}(2,\mathbb{R})^+$,

(iii) *every Lie subsemigroup with nonvoid interior* $S \neq \mathrm{Sl}(2,\mathbb{R})$ *is contained in a conjugate of* $\mathrm{Sl}(2,\mathbb{R})^+$.

3.8. Lie wedges of three dimensional exponential subsemigroups. *Let* W *be a wedge in* $\mathfrak{sl}(2,\mathbb{R})$, *let* $S = \exp(W) \subseteq \mathrm{Sl}(2,\mathbb{R})$ *and assume that* W *does not lie in a subalgebra of dimension* < 3. *Then the following assertions are equivalent:*

(i) S *is a semigroup;*

(ii) S *is a closed semigroup;*

(iii) W *is the intersection of conjugates of* $\mathfrak{sl}(2,\mathbb{R})^+$.

(iv) W *is a Lie semialgebra contained in* $\overline{\mathsf{Kill}^+}$.

(v) W *is an exponential wedge.*

Proof. The implications (ii) $\Longrightarrow$ (i), (ii) $\Longleftrightarrow$ (v), and (iii) $\Longrightarrow$ (iv) are trivial.

(i) $\Longrightarrow$ (iv) We first note that $S \neq \mathrm{Sl}(2,\mathbb{R})$, since $\mathrm{Sl}(2,\mathbb{R})$ is not exponential, and that $\operatorname{Int} S$ is nonvoid, since W is not contained in a two dimensional subalgebra of $\mathfrak{sl}(2,\mathbb{R})$ ([5], Theorem V.1.10, p.377, and p.382). Also, S is connected, hence 3.4(iii) applies and shows that S is contained in $\overline{\exp(\mathsf{Kill}^+)}$, so $W = \log(S) \subseteq \overline{\mathsf{Kill}^+}$. Since the restriction of exp to $\overline{\mathsf{Kill}^+}$ is a diffeomorphic embedding it follows that for any Campbell-Hausdorff neighborhood B

$$\begin{aligned}(B\cap W)*(B\cap W) &= \log(\exp((B\cap W)*(B\cap W)))\\ &= \log(\exp(B\cap W)\exp(B\cap W))\\ &\subseteq \log(\exp(W)) = W,\end{aligned}$$

so W is a semialgebra.

(iv) $\Longrightarrow$ (iii) By Theorem II.3.7 of [5] (p. 109) we know that W is the intersection of a family $\mathcal{F}$ of half space semialgebras each of which is bounded by a Borel algebra. Since W is a Lie semialgebra, and not contained in a Borel subalgebra, it has nonvoid interior. Let K_1, K_2 be the two connected components of Kill^-. Then each half space semialgebra contains exactly one of either K_1 or K_2. By 3.7(ii) there exist at least one pair $W_1, W_2 \in \mathcal{F}$ with $K_1 \subseteq W_1$, $K_2 \subseteq W_2$, and for every such pair the intersection $W_1 \cap W_2$ is conjugate to $\mathfrak{sl}(2,\mathbb{R})^+$ and contains W. Clearly, W is the intersection of all such intersection wedges $W_1 \cap W_2$. Thus W is obtained as the intersection of conjugates of $\mathfrak{sl}(2,\mathbb{R})^+$.

(iii) $\Longrightarrow$ (ii) The restriction of exp to $\overline{\mathsf{Kill}^+}$ is a diffeomorphism and the exponential image of $\mathfrak{sl}(2,\mathbb{R})^+$ is the closed subsemigroup $\mathrm{Sl}(2,\mathbb{R})^+$ of all matrices in $\mathrm{Sl}(2,\mathbb{R})$ with nonnegative entries. Thus we see that $\exp(W)$, being the intersection of conjugates of $\mathrm{Sl}(2,\mathbb{R})^+$, must be a closed subsemigroup of $\mathrm{Sl}(2,\mathbb{R})$. ■

4. The diamond product

4.1. In the present notes we shall not discuss possible formal definitions for the terms 'asymptotic property' and 'asymptotic object.' Let us emphasize only that a natural requirement for an 'asymptotic property' seems to be: if S has this property then each of its two-sided ideals has this property. In the same vein, an 'asymptotic object' ought to be a functor assigning to certain semigroups S a structure $A(S)$, so that $A(S) = A(I)$ for every two-sided ideal I of S. Also, taking as a model the case of additive subsemigroups of $\mathbb{R}^n$, we would like to have the asymptotic objects carry good information about 'asymptotic directions,' i.e., limits of matrices $X \in \mathfrak{sl}(2,\mathbb{R})$ with $\exp \mathbb{R}^+ X \cap S \neq \emptyset$.

The guiding idea of our subsequent definitions is to identify directions with matrices in $\mathfrak{sl}(2,\mathbb{R})$ of length 1 and to define a special kind of multiplication of directions. Instead of introducing an arbitrary Euclidean distance we measure the length in terms of the Killing form, this is technically more convenient and better adapted to the structures we are interested in.

4.2. The fundamental hyperboloid. We henceforth call the hyperboloid of revolution

$$\mathsf{Hyp} \stackrel{\text{def}}{=} \Delta^{-1}(1) = \{\alpha H + \beta P + \gamma Q \mid \alpha^2 + \beta\gamma = 1\}$$

the *fundamental hyperboloid.* Note that the "light cone" $\mathsf{Kill}^0 = \{\alpha H + \beta P + \gamma Q \mid \alpha^2 + \beta\gamma = 0\}$ is the asymptotic cone of Hyp.

For our purposes the point of interest is that the points of the fundamental hyperboloid correspond to the one parameter subsemigroups leading into $\exp(\mathsf{Kill}^+)$. (These are exactly those one parameter subsemigroups which meet open subsemigroups $S \neq \mathrm{Sl}(2,\mathbb{R})$.) A drawback of working with Hyp is that it is not a compact surface, so we shall embed it later into a product of flag manifolds. The advantage of Hyp is that it allows easy calculations and that the multiplication introduced below has a convincing geometric interpretation.

4.3. Horizontal and vertical lines. For any $X \in \mathsf{Hyp}$ let $\mathfrak{p}_X$, $\mathfrak{q}_X$ be the (one-dimensional) eigenspaces of $\operatorname{ad} X$ with, respectively, eigenvalue 2,-2. Then we define the *horizontal line through* X,

$$\mathsf{hor}(X) \stackrel{\text{def}}{=} X + \mathfrak{p}_X,$$

and the *vertical line through* X

$$\mathsf{vert}(X) \stackrel{\mathrm{def}}{=} X + \mathfrak{q}_X .$$

Both of these sets are contained in the fundamental hyperboloid Hyp.

4.4. Elementary Properties of horizontal and vertical lines. (i) For every $X \in \mathsf{Hyp}$ the convex span of $\mathsf{hor}(X) \cup \mathsf{vert}(X)$ is the tangent plane of Hyp at X and $\mathsf{hor}(X) \cap \mathsf{vert}(X) = \{X\}$.

(ii) Because of $\mathfrak{p}_X = \mathfrak{q}_{-X}$ we see that $\mathsf{hor}(X) = -\mathsf{vert}(-X)$ and $\mathsf{vert}(X) = -\mathsf{hor}(-X)$, so the lines $\mathsf{hor}(X)$ and $\mathsf{vert}(-X)$ are parallel to each other.

(iii) For later use we also note that for every $Y \in \mathsf{hor}(-X) \cup \mathsf{vert}(-X)$ the sum $X+Y$ is nilpotent.

(iv) The linear span of $\mathsf{hor}(X)$ is the Borel algebra $\mathfrak{b}_1 = \mathbb{R}X + \mathfrak{p}_X = \mathbb{R} \cdot \mathsf{hor}(X) \cup \mathfrak{p}_X$, the linear span of $\mathsf{vert}(X)$ is the Borel algebra $\mathfrak{b}_2 = \mathbb{R}X + \mathfrak{q}_X = \mathbb{R} \cdot \mathsf{vert}(X) \cup \mathfrak{q}_X$. Note that $\mathfrak{b}_1 \cap \mathsf{Hyp} = \mathsf{hor}(X) \cup \mathsf{vert}(-X)$ and $\mathfrak{b}_2 \cap \mathsf{Hyp} = \mathsf{vert}(X) \cup \mathsf{hor}(-X)$.

(v) Distinct horizontal [vertical] lines do not intersect. (Indeed, $Y \in \mathsf{hor}(X)$ implies $\mathsf{hor}(Y) = \mathsf{hor}(X)$, and $Y \in \mathsf{vert}(X)$ implies $\mathsf{vert}(X) = \mathsf{vert}(Y)$.)

(vi) The horizontal line $\mathsf{hor}(X)$ intersects the vertical line $\mathsf{vert}(Y)$ if and only if $Y \notin -\mathsf{hor}(X)$ (or, equivalently, if $X \notin -\mathsf{vert}(Y)$).

(vii) Under any automorphism of $\mathfrak{sl}(2,\mathbb{R})$ horizontal lines go to horizontal lines and vertical lines go to vertical lines.

4.5. Explicit formulas for $\mathsf{hor}(X)$ and $\mathsf{vert}(X)$. For $X = \alpha H + \beta P + \gamma Q \in \mathsf{Hyp}$ we have $\mathsf{hor}(X) = X + \mathbb{R}A$, $\mathsf{vert}(X) = X + \mathbb{R}B$ with

$$A = \begin{cases} \beta(\alpha-1)H + \beta^2 P - (\alpha-1)^2 Q & \text{if } (\alpha,\beta) \neq (1,0); \\ -\gamma H + 2P - \frac{\gamma^2}{2} Q & \text{if } (\alpha,\beta) = (1,0); \end{cases}$$

$$B = \begin{cases} \beta(\alpha+1)H + \beta^2 P - (\alpha+1)^2 Q & \text{if } (\alpha,\beta) \neq (-1,0); \\ \gamma H + 2P - \frac{\gamma^2}{2} Q & \text{if } (\alpha,\beta) = (-1,0). \end{cases}$$

We leave the straightforward proof to the reader.

4.6. Remarks about the orbit structure of Hyp.

(i) The set Hyp is the orbit of any of its points X under the adjoint action of $\mathrm{Sl}(2,\mathbb{R})$ on $\mathfrak{sl}(2,\mathbb{R})$. Thus, fixing $X = H$, we identify Hyp with the homogeneous space $\mathrm{Sl}(2,\mathbb{R})/Z\exp(\mathbb{R}H)$, where $Z = \{I,-I\}$ is the center of $\mathrm{Sl}(2,\mathbb{R})$. (For $X = P+Q$ the isotropy group is just $SO(1,1)$.)

(ii) The space Hor of all horizontal lines in Hyp (endowed with the Vietoris topology) can be identified with the flag manifold (the simplest possible) $\mathrm{Sl}(2,\mathbb{R})/Z\exp(\mathbb{R}H + \mathbb{R}P)$. This homogeneous space is homeomorphic with $\mathrm{SO}(2)/Z$, which is also homeomorphic with the one dimensional torus $\mathbb{T}$. The injection $\mathbb{R}H \to \mathbb{R}H + \mathbb{R}P$ induces the natural map

$$\mathrm{Sl}(2,\mathbb{R})/Z\exp(\mathbb{R}H) \to \mathrm{Sl}(2,\mathbb{R})/Z\exp(\mathbb{R}H + \mathbb{R}P),$$

which to every $Z\exp(\mathbb{R}H)$-coset assigns the $Z\exp(\mathbb{R}H+\mathbb{R}P)$-coset containing it; this mapping corresponds to the map $\mathsf{hor}\colon \mathsf{Hyp} \to \mathsf{Hor}$, $X \mapsto \mathsf{hor}(X)$. Thus we see that hor is a continuous and open mapping.

(iii) Similarly, the space Vert of all vertical lines is identified with the homogeneous space $\mathrm{Sl}(2,\mathbb{R})/Z\exp(\mathbb{R}H+\mathbb{R}Q)$, which is homeomorphic with the one dimensional torus $\mathbb{T}$, and we note that the map $\mathsf{vert}\colon \mathsf{Hyp} \to \mathsf{Vert}$, $X \mapsto \mathsf{vert}(X)$, is a continuous and open mapping. It is convenient to write the quotient space $\mathrm{Sl}(2,\mathbb{R})/Z\exp(\mathbb{R}H+\mathbb{R}Q)$ as the set of left cosets $Z\exp(\mathbb{R}H+\mathbb{R}Q)\cdot g$, $g \in \mathrm{Sl}(2,\mathbb{R})$.

(iv) We conclude from 4.4(vi) that $\mathsf{vert}(\mathsf{hor}(X)) = \mathsf{Vert}\setminus\{\mathsf{vert}(-X)\}$ and that $\mathsf{hor}(\mathsf{vert}(X)) = \mathsf{Hor}\setminus\{\mathsf{hor}(-X)\}$.

(v) Further information about actions on Hyp and about its order structure can be found, e.g., in [6], p.57ff.

4.7. Proposition. *We retain the notation of the preceding paragraph. The following assertions hold:*

(i) *The map* $c\colon \mathsf{Hyp} \to \mathsf{Hor}\times\mathsf{Vert}$, $X \mapsto (\mathsf{hor}(X), \mathsf{vert}(X))$, *is an open topological embedding of* Hyp *into the space* $\mathsf{Hor}\times\mathsf{Vert} \cong \mathbb{T}\times\mathbb{T}$.

(ii) c *maps* Hyp *onto the set* $c(\mathsf{Hyp}) = \{(h,v) \in \mathsf{Hor}\times\mathsf{Vert} \mid h \neq -v\}$, *which is dense in* $\mathsf{Hor}\times\mathsf{Vert}$.

(iii) *The inverse of the corestriction* $\mathsf{Hyp} \to c(\mathsf{Hyp})$ *of* c *sends every pair* (h,v) *with* $h \neq -v$ *to the unique point* $X \in h\cap v$.

Remark. In terms of the identifications in 4.6 the map c is defined by

$$c(g\cdot Z\exp(\mathbb{R}H)) = (g\cdot Z\exp(\mathbb{R}H+\mathbb{R}P), Z\exp(\mathbb{R}H+\mathbb{R}Q)\cdot g^{-1}).$$

Its inverse is given (under the proviso that $h\cap v \neq \emptyset$) by $(h,v) \mapsto g\cdot Z\exp(\mathbb{R}H)$, for any $g \in h\cap v$.

Proof. (i) By 4.6(ii) and (iii) we know that $\mathsf{Hor}\times\mathsf{Vert}$ is homeomorphic with $\mathbb{T}\times\mathbb{T}$ and that the map c is continuous. By 4.4(i) c is injective, hence its restriction to any compact subset of Hyp is a topological embedding. It follows by the invariance of domains that c is an open map, hence an open topological embedding.

Assertions (ii) and (iii) follow from 4.4(vi). ∎

4.8. The compactification (Hyp^-, c)**.** The above proposition shows that the space $\mathsf{Hor}\times\mathsf{Vert}$, together with the map c, is a compactification of Hyp. We abbreviate $\mathsf{Hor}\times\mathsf{Vert}$ by Hyp^-.

4.9. Nilpotency Points in Hyp^-**.** We henceforth call an element of the form $(\mathsf{hor}(X), -\mathsf{hor}(X)) = (\mathsf{hor}(X), \mathsf{vert}(-X))$, $X \in \mathsf{Hyp}$, a *nilpotency point.*

(i) The set of all nilpotency points in Hyp^- is the remainder $\mathsf{Hyp}^-\setminus c(\mathsf{Hyp})$ of the compactification (Hyp^-, c).

(ii) The set of all nilpotency points in Hyp^- can be identified with the space of all one dimensional subspaces $\mathbb{R}\cdot N$, $N \in \mathsf{Kill}^0$. Under this identification the nilpotency point $(\mathsf{hor}(X), -\mathsf{hor}(X))$ corresponds to the line $\mathsf{hor}(X) - X$, for every $X \in \mathsf{Hyp}$.

(iii) For every $X \in \mathsf{Hyp}$ the closure of $c(\mathsf{hor}(X))$ $[c(\mathsf{vert}(X)]$ in Hyp^- contains exactly one nilpotency point, namely the point $(\mathsf{hor}(X), -\mathsf{hor}(X))$ $[(-\mathsf{vert}(X), \mathsf{vert}(X))]$.

(iv) Let $h \in \mathsf{Hor}$, $v \in \mathsf{Vert}$. If (h, v) is not a nilpotency point then $\mathbb{R}h \cap \mathbb{R}v = \mathbb{R}(h \cap v)$. If $h = \mathsf{hor}(X)$ and $v = -\mathsf{hor}(X)$ then $\mathbb{R}h = \mathbb{R}v$, moreover $\overline{\mathbb{R}h} = \mathbb{R}h \cup \mathfrak{p}_X$.

4.10. Proposition. *Let $\langle X_n \rangle$ be a sequence in* Hyp *such that for suitably chosen real numbers t_n the limit $N = \lim t_n X_n$ exists and is a nonzero nilpotent matrix with $N \in \mathsf{hor}(X) - X$ for some $X \in \mathsf{Hyp}$. Then $\lim c(X_n) = (\mathsf{hor}(X), -\mathsf{hor}(X))$.*

Proof. We assume, with no loss of generality, that $X = H, N = P$. Write $X_n = \alpha_n H + \beta_n P + \gamma_n Q$. Then

$$|t_n| = \Delta(t_n X_n) \to 0, \quad t_n\alpha_n \to 0, \quad t_n\beta_n \to 1, \quad t_n\gamma_n \to 0.$$

In particular, $|\beta_n| \to \infty$. Applying our formulas in 4.5 we see that for every $s \in \mathbb{R}$

$$X_n + s(\beta_n(\alpha_n - 1)H + \beta_n^2 P - (\alpha_n - 1)^2 Q) \in \mathsf{hor}(X_n),$$
$$X_n + s(\beta_n(\alpha_n + 1)H + \beta_n^2 P - (\alpha_n + 1)^2 Q) \in \mathsf{vert}(X_n).$$

(Note that $\beta_n \neq 0$ for sufficiently large n.) We now put $s = -1/\beta_n$ and find

$$\mathsf{hor}(X_n) = \mathsf{hor}\left(H + \frac{2(1-\alpha_n)}{\beta_n} Q\right), \qquad \mathsf{vert}(X_n) = \mathsf{vert}\left(-H + \frac{2(1+\alpha_n)}{\beta_n} Q\right).$$

Next we choose n so large that $t_n \neq 0$ and compute

$$\lim_{n\to\infty} \frac{2(1 \mp \alpha_n)}{\beta_n} = \lim_n \frac{2(t_n \mp t_n\alpha_n)}{t_n\beta_n} = 0$$

which implies that $\lim \mathsf{hor}(X_n) = \mathsf{hor}(H)$ and $\lim \mathsf{vert}(X_n) = \mathsf{vert}(-H)$. The assertion follows. ∎

4.11. Remark. Note that by Proposition 4.10 the space Hyp^- can be considered as the quotient space of the space of all one parameter subsemigroups contained in $\exp(\overline{\mathsf{Kill}^+})$, where for each $N \in \mathsf{Kill}^0$ the one parameter semigroups $\exp(\mathbb{R}_0^+ N)$ and $\exp(-\mathbb{R}_0^+ N)$ are identified. Occasionally this identification requires some care when nilpotency points are involved.

4.12. Rectangular bands. (i) Recall that every cartesian product $S = M \times N$ of nonvoid sets is an idempotent semigroup with respect to the multiplication $(m,n)(m',n') = (m,n')$. Such semigroups are called *rectangular bands* (cf., e.g., [3] I, p.25).

(ii) Note that with respect to this multiplication every subset $M_n = M \times \{n\}$, $n \in N$, is a left zero semigroup, and every subset $N_m = \{m\} \times N$, $m \in M$, is a right zero semigroup. Moreover, for fixed $(m,n) \in S$ the map $M_n \times N_m \to S$, $((x,n),(m,y)) \mapsto (x,n)(m,y) = (x,y)$, is an isomorphism.

(iii) Every rectangular band S is a *simple semigroup*, that is, if I is a two sided ideal in S then $I = S$.

4.13. Elementary properties of topological rectangular bands. *Let M and N be two nonvoid topological spaces and let $S = M \times N$ be the associated rectangular band. Then the following assertions hold:*

(i) *S is a topological semigroup, i.e., the multiplication is jointly continuous. Moreover, the multiplication of S is an open mapping.*

(ii) *A subset S' of S is a subsemigroup [open subsemigroup] [closed subsemigroup] [connected subsemigroup] if and only if there exist nonvoid [nonvoid open] [nonvoid closed] [nonvoid connected] subspaces $M' \subseteq M$ and $N' \subseteq N$ with $S = M' \times N'$.*

(iii) *The interior* $\operatorname{Int} S_1$ *of a subsemigroup S_1 of S is either void or a subsemigroup of S.*

Proof. Assertions (i) and (ii) follow immediately from the definition of the multiplication and the continuity and openness of the projections $p_1\colon M \times N \to M$ and $p_2\colon M \times N \to N$.

(iii) By (ii) we know that $S' = M' \times N'$, where $M' \subseteq M$, $N' \subseteq N$. Since $\operatorname{Int}(M' \times N') = \operatorname{Int}(M') \times \operatorname{Int}(N')$ our assertion follows from (ii). ∎

4.14. The diamond product. We now put $M = \mathsf{Hor}$, $N = \mathsf{Vert}$ and write Hyp^- for $\mathsf{Hor} \times \mathsf{Vert}$, endowed with the rectangular band structure of 4.12. Then we pull back the multiplication of Hyp^- to a partial multiplication, the *diamond product* $\diamond$, on Hyp. This amounts to:

If $\mathsf{hor}(X)$ intersects $\mathsf{vert}(Y)$ then the *diamond product $X \diamond Y$ of X and* Y is defined as the point of intersection, that is, $\{X \diamond Y\} = \mathsf{hor}(X) \cap \mathsf{vert}(Y)$.

(i) By the above discussion the diamond product $X \diamond Y$ is defined for all $X, Y \in \mathsf{Hyp}$ with $\mathsf{hor}(X) \neq -\mathsf{vert}(Y)$, it is jointly continuous and associative as long as all products involved are defined.

(ii) In particular, all vertical and all horizontal lines are semigroups with respect to the diamond product. The horizontal lines are right zero semigroups, and the vertical lines are left zero semigroups with respect to $\diamond$.

(iii) If $X \diamond Y$ is not defined then $Y \in -\mathsf{hor}(X)$. If both $X \diamond Y$ and $Y \diamond X$ are not defined then $X = -Y$.

4.15. Example. The diamond products $A \diamond B$ and $B \diamond A$ of $A = H + \beta P$ and $B = H + \gamma Q$ compute to

$$A \diamond B = H, \qquad B \diamond A = \frac{1}{4+\beta\gamma}\left((4-\beta\gamma)H + 4\beta P + 4\gamma Q\right).$$

Note that $B \diamond A$ exists if and only if $\beta\gamma \neq -4$. In the special case $\beta = \gamma = 2$ we get $A \diamond B = P + Q$ ($= T$ in the notation of [5]).

4.16. Remark. The space Hyp^- not only bears the natural rectangular band structure of above but also two natural actions of $\mathrm{Sl}(2,\mathbb{R})$:

$$\begin{aligned} \mathrm{Sl}(2,\mathbb{R}) \times \mathsf{Hyp}^- \to \mathsf{Hyp}^-, &\quad (g,(h,v)) \mapsto g\cdot(h,v) = (g\cdot h, v),\\ \mathsf{Hyp}^- \times \mathrm{Sl}(2,\mathbb{R}) \to \mathsf{Hyp}^-, &\quad ((h,v),g) \mapsto (h,v)\cdot g = (h, v\cdot g). \end{aligned}$$

These actions induce partial actions of $\mathrm{Sl}(2,\mathbb{R})$ on Hyp, which we also denote with a "$\cdot$." For instance, if $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \mathrm{Sl}(2,\mathbb{R})$ with $a \neq 0$ then $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\cdot H = H + \frac{2c}{a}Q$, and $H \cdot \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) = H + \frac{2b}{a}P$. We do not use this additional structure as yet, but shall exploit it in [17] for the construction of compactifications.

5. Rectangular domains

5.1. Connected $\diamond$-semigroups.

(i) *A $\diamond$-subsemigroup D of* Hyp *is connected if and only if for every two elements $X, Y \in D$ lying on the same horizontal or vertical line the line segment joining X and Y lies in D.*

(ii) *The intersection of an arbitrary family D_i of connected $\diamond$- subsemigroups of* Hyp *is connected.*

Proof. Assertion (i) follows from 4.13(ii), assertion (ii) is then an immediate consequence of (i). ∎

5.2. Rectangular domains. An open connected $\diamond$-semigroup $D \subset \mathsf{Hyp}$ is called a *rectangular domain.* Note that in a rectangular domain *all* diamond products $X \diamond Y$ are defined. (Thus Hyp itself is not a rectangular domain.) It can be deduced from what is shown below that every rectangular domain is an *open domain* (in Hyp) in the topological sense, i.e., it is the interior of its closure (Kuratowski, cf.[4], p.37).

5.3. Examples. (i) The intersection $D(+)$ of Hyp with the interior of $\mathfrak{sl}(2,\mathbb{R})^+$ is a rectangular domain, which is bounded by the four half lines $\pm H + \mathbb{R}_0^+ \cdot P$ and $\pm H + \mathbb{R}_0^+ \cdot Q$. More specifically, a straightforward computation verifies that the map

$$j\colon \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow D(+),\ (\lambda,\mu) \longmapsto \frac{1-\lambda\mu}{1+\lambda\mu}\cdot H + \frac{2\mu}{1+\lambda\mu}\cdot P + \frac{2\lambda}{1+\lambda\mu}\cdot Q$$

is bijective (with inverse $j^{-1}(\alpha H + \beta P + \gamma Q) = (\frac{1-\alpha}{\beta}, \frac{\beta}{1+\alpha})$); it is a homomorphism if $\mathbb{R}^+ \times \mathbb{R}^+$ is endowed with the rectangular band structure of 4.12. Note that the identities

$$\begin{aligned} j(\lambda,\mu) &= H + 2\mu P + \frac{2\lambda\mu}{1+\lambda\mu}\cdot(-H - \mu P + \frac{1}{\mu}Q) \\ &= H + 2\lambda Q + \frac{2\lambda\mu}{1+\lambda\mu}\cdot(-H + \frac{1}{\lambda}P - \lambda Q) \end{aligned}$$

show that j maps the sets $\{\lambda\} \times \mathbb{R}^+$ onto horizontal line segments, the sets $\mathbb{R}^+ \times \{\mu\}$ onto vertical line segments.

(ii) If the intersection of an arbitrary family of conjugates of $D(+)$ is nonvoid and open then it is a rectangular domain. This follows from 5.1(ii).

5.4. Basic properties of rectangular domains. *For a rectangular domain D the following assertions hold:*

(i) *Let $X \in D$. Then $-\mathsf{hor}(X) \cap D = -\mathsf{vert}(X) \cap D = \emptyset$.*
(ii) *The closure $\overline{c(D)}$ of $c(D)$ in Hyp^- is the product $A \times B$ of two proper closed arcs $A \subseteq \mathsf{Hor}$, $B \subseteq \mathsf{Vert}$.*
(iii) *The nilpotency points of $\overline{c(D)}$, if they exist, are corner points, i.e., contained in $\partial A \times \partial B$, and no two of them lie on the same bounding arc. Thus $\overline{c(D)}$ contains at most two nilpotency points.*
(iv) *The interior $\mathrm{Int}(\overline{D})$ of $\overline{D}$ in Hyp coincides with D.*
(v) *The closure of D is compact if and only if $\overline{c(D)}$ does not contain nilpotency points.*

Proof. Assertion (i) follows from the fact that $X \diamond Y$ is not defined if $Y \in -\mathsf{hor}(X)$ and $Y \diamond X$ is not defined if $Y \in -\mathsf{vert}(X)$.

(ii) Since D is connected and c is an open and isomorphic embedding we know that $c(D)$ is an open connected subsemigroup of Hyp^-. By (i) we can have neither $\overline{\mathsf{hor}(D)} = \mathsf{Hor}$ nor $\overline{\mathsf{vert}(D)} = \mathsf{Vert}$. Thus $\overline{c(D)} = A \times B$, where A is an arc in Hor with two distinct endpoints a_1, a_2, and B is an arc in Vert with two distinct endpoints b_1, b_2.

(iii) Since $c(D)$ is the interior of $\overline{c(D)}$, the nilpotency points of $\overline{c(D)}$ must lie on the boundary of $c(D)$. The boundary of D is the union of the four line segments

$$h_i \overset{\text{def}}{=} c^{-1}(\{a_i\} \times B), \qquad v_j \overset{\text{def}}{=} c^{-1}(A \times \{b_j\}), \qquad i = 1,2,\ j = 1,2.$$

Thus every nilpotency point in $\overline{c(D)}$ must be an endpoint of one of the arcs $\overline{c(h_i)}$, $i = 1,2$. Since $\overline{c(h_i)}$ contains at most one nilpotency point, this implies the assertion.

(iv) Since c is continuous and open we have $\operatorname{Int}(\overline{D}) = c^{-1}\big(\operatorname{Int}\,(\overline{c(D)})\big)$, hence the assertion follows from (ii) and (iii).

Assertion (v) is a consequence of 4.9(i). ∎

5.5. Rectangular domains and exponential subsemigroups of $\mathrm{Sl}(2,\mathbb{R})$**.**

(i) *If D is a rectangular domain then the set $W_0 = \mathbb{R}^+ D$ is open and convex, and its closure $W = \overline{W_0}$ is a Lie semialgebra. Moreover,* $\exp(W)$ *is a closed subsemigroup of* $\mathrm{Sl}(2,\mathbb{R})$.

(ii) *Conversely, if W is a Lie semialgebra with nonvoid interior in* $\mathfrak{sl}(2,\mathbb{R})$ *and* $\exp W$ *is a semigroup then the intersection $D = W_0 \cap \mathsf{Hyp}$ of its interior $W_0 = \operatorname{Int} W$ with Hyp is a rectangular domain.*

Proof. (i) We first notice that the map

$$f\colon \mathbb{R}^+ \times \mathsf{Hyp} \to \mathsf{Kill}^+,\ (t,X) \longmapsto tX,$$

is a homeomorphism. Since D is an open subset of Hyp and since f is an open map, the image $f(\mathbb{R}^+ \times D) = \mathbb{R}^+ D$ is an open subset of Kill^+. Now pick two elements $X, Y \in D$. We write $S(X,Y)$ for the connected $\diamond$-semigroup generated by the line segments $\operatorname{conv}\{X, X\diamond Y\}$ and $\operatorname{conv}\{X, Y\diamond X\}$. First we will show that $\mathbb{R}_0^+ S(X,Y)$ is a Lie semialgebra. If X and Y lie on the same horizontal or vertical line then $\mathbb{R}_0^+ S(X,Y)$ is a wedge in the Borel algebra generated by $S(X,Y)$, hence it is a Lie semialgebra. Suppose now that X and Y do not lie on the same horizontal or vertical line. Then the polyhedral wedge

$$C = \mathbb{R}_0^+ X + \mathbb{R}_0^+ X\diamond Y + \mathbb{R}_0^+ Y\diamond X + \mathbb{R}_0^+ Y$$

is the intersection of four closed half spaces, each bounded by one of the four Borel algebras generated by the two dimensional faces

$$\mathbb{R}_0^+ X + \mathbb{R}_0^+ X\diamond Y,\quad \mathbb{R}_0^+ X + \mathbb{R}_0^+ Y\diamond X,\quad \mathbb{R}_0^+ Y + \mathbb{R}_0^+ X\diamond Y,\quad \mathbb{R}_0^+ Y + \mathbb{R}_0^+ Y\diamond X.$$

Thus C is a Lie semialgebra. Moreover, since the boundary of C does not contain nonzero nilpotent elements, we conclude that $C\setminus\{0\} \subseteq \mathsf{Kill}^+$.

Since f is a homeomorphism, it maps the boundary $\partial(\mathbb{R}^+ \times S(X,Y)) = \mathbb{R}^+ \times \partial(S(X,Y))$ of $\mathbb{R}^+ \times S(X,Y)$ in $\mathbb{R}^+ \times \mathsf{Hyp}$ onto the boundary of $f(\mathbb{R}^+ \times S(X,Y))$ in Kill^+. Hence

$$\mathbb{R}^+ \partial(S(X,Y)) = \partial(\mathbb{R}^+ S(X,Y)).$$

On the other hand, the boundary of $C\setminus\{0\}$ in Kill^+ is exactly $\mathbb{R}^+\partial(S(X,Y))$. Thus $C\setminus\{0\}$ and $\mathbb{R}^+ S(X,Y)$ have the same boundary in Kill^+. Also, these two sets have non-empty intersection. Since both of these sets have dense and connected interior, they must coincide, i.e., $\mathbb{R}^+ S(X,Y) = C\setminus\{0\}$. Hence $\mathbb{R}_0^+ S(X,Y) = C$, which shows that $\mathbb{R}_0^+ S(X,Y)$ is a Lie semialgebra.

In particular, we see that $W_0 = \mathbb{R}^+ D$ is convex, so $W = \overline{W_0}$ is a wedge. Since every two elements of W_0 lie in a Lie semialgebra contained in W, Corollary II.2.16 of [5], p. 89, applies and shows that W is a Lie semialgebra. Also, $W \subseteq \overline{\mathsf{Kill}^+}$, hence $\exp(W)$ is a closed subsemigroup of $\mathrm{Sl}(2,\mathbb{R})$, by 3.8.

(ii) Suppose that W is a wedge with nonvoid interior in $\mathfrak{sl}(2,\mathbb{R})$ such that $\exp W$ is a semigroup. Then by 3.8 W is the intersection of certain conjugates of $\mathfrak{sl}(2,\mathbb{R})^+$. Note that for any two elements $X, Y \in \mathfrak{sl}(2,\mathbb{R})^+ \cap \mathsf{Hyp}$ the diamond product $X \diamond Y$, if defined, also lies in $\mathfrak{sl}(2,\mathbb{R})^+$. It follows that $W \cap \mathsf{Hyp}$ is closed under the $\diamond$-product, as far as the latter is defined. The diamond product is defined for all X, Y in the interior $W_0 \cap \mathsf{Hyp} = \mathrm{Int}(W \cap \mathsf{Hyp})$ (since $W_0 \cap \mathsf{Hyp}$ lies in a conjugate of $\mathrm{Int}(\mathfrak{sl}(2,\mathbb{R})^+) \cap \mathsf{Hyp}$). Recalling that the diamond product is an open map (4.13(i)) we see that $W_0 \cap \mathsf{Hyp}$ is a $\diamond$-semigroup. The inverse image $f^{-1}(W_0) = \mathbb{R}^+ \times (W_0 \cap \mathsf{Hyp})$ of W_0 under the map f in the proof of (i) is connected, so $W_0 \cap \mathsf{Hyp}$ is connected. Thus $W_0 \cap \mathsf{Hyp}$ is a rectangular domain.∎

5.6. Remark. Since the boundary of a rectangular domain consists of two horizontal and two vertical line segments the above assertion (ii) implies that the Lie wedge of an exponential subsemigroup of $\mathrm{Sl}(2,\mathbb{R})$ with inner points is the intersection of at most four half spaces bounded by Borel algebras. This assertion was formulated first in [5] (p.110; its proof was left to the reader as exercise E.II.1).

5.7. Nilpotency points and nilpotent elements. *Let D be a rectangular domain and write $W_0 = \mathbb{R}^+ D$, $W = \overline{W_0}$.*

(i) *If $\overline{c(D)}$ contains a nilpotency point $(a, -a)$ then the Borel algebra $\mathfrak{b} = \mathbb{R} \cdot a$ intersects W in a closed face which in its algebraic interior contains a nonzero nilpotent element N.*

(ii) *Conversely, if W contains a nonzero nilpotent element N then N lies in the algebraic interior of a closed two dimensional face of W and $\overline{c(D)}$ contains a nilpotency point $(a, -a)$, where a is the horizontal line contained in the Borel algebra generated by N.*

(iii) *Let k be the cardinality of the set of nilpotency points in $\overline{c(D)}$. Then $W_0 = \mathbb{R}^+ D$ is the intersection of exactly $4 - k$ distinct open half spaces in $\mathfrak{sl}(2,\mathbb{R})$, each of which is bounded by a Borel algebra. The closed wedge W contains exactly k linearly independent nilpotent elements.*

Proof. (i) Let $X \in a \cap W$, $Y \in -a \cap W$. Then X, Y are nonzero and lie in the boundary of W, also $N = X + Y$ is a nilpotent element (by 4.4(iii)). If $N \neq 0$ then the assertion follows. If $N = 0$ then $X = -Y \in H(W)$ and thus $W \cap \mathfrak{b}$ is one of the two closed half spaces of $\mathfrak{b}$ bounded by $\mathbb{R}X$, and these must contain a nonzero nilpotent element.

(ii) Suppose that N is a nonzero nilpotent element in W. Then $N \in \partial W$, since $\mathrm{Int}\, W \subset \mathsf{Kill}^+$. Let F be a closed two dimensional face of W with $N \in F$. We have seen in 5.6 that $F = \mathfrak{b} \cap W$, where $\mathfrak{b}$ is a Borel algebra. Since the Borel algebra $\mathfrak{b}$ passing through N is uniquely defined, we conclude that N must be contained in the algebraic interior of F. It follows that $\overline{D}$ meets both the horizontal line a and the vertical line $-a$ in $\mathfrak{b}$, and thus $\overline{c(D)}$ must contain the nilpotency point $(a,-a)$.

The proof of (iii) is left to the reader. ■

6. Examples of connected subsemigroups of $\mathrm{Sl}(2,\mathbb{R})$

6.1. In this section we discuss a collection of typical examples of connected subsemigroups of $\mathrm{Sl}(2,\mathbb{R})$. Some of these will be used later to illustrate the application of our main results.

The main focus of attention is given to three dimensional exponential subsemigroups and their Lie wedges. In fact, we shall see in the next section that our examples include a representative of each conjugacy class of three dimensional exponential subsemigroups.

It is instructive (and very convenient, too) to represent our examples as compression semigroups, some aspects of this approach, in particular the observation that all semigroups of our list are defined on 'semialgebraic' sets, seem to be of independent interest.

6.2. Compression semigroups. Let S be a semigroup which acts on some space X. Then for every subset M of X we define the *compression semigroup in S of M* as the set

$$\mathrm{compr}_S(M) \stackrel{\mathrm{def}}{=} \{s \in S \mid sM \subseteq M\}.$$

The original definition of compression semigroup (as used, e.g., in [6], p.203) supposes that S is a group, since the main application is to find maximal subsemigroups of a group. In our present setting S will usually be a subsemigroup of $G = \mathrm{Sl}(2,\mathbb{R})$ (acting on $\mathbb{R}^2$) and it will make a difference whether the compression semigroup is formed only within S or in the whole group G. In general, an inclusion $S \subseteq S'$ implies a corresponding inclusion $\mathrm{compr}_S(M) \subseteq \mathrm{compr}_{S'}(M)$, but this inclusion is proper in most cases, even if S has inner points in S' (cf. the example in 6.10 below).

6.3. Remark. The following observation is checked easily by straightforward calculation.

Let S be a subsemigroup of a group G which acts on a space X. Then for any subset $Y \subseteq X$ and $a \in G$ we have

(i) $\mathrm{compr}_G(aY) = a\,\mathrm{compr}_G(Y)a^{-1}$*;*

(ii) $\mathrm{compr}_S(aY) = S \cap a\,\mathrm{compr}_G(Y)a^{-1}$. ■

6.4. Example. The semigroup $\mathrm{Sl}(2,\mathbb{R})^+$, with Lie wedge $\mathfrak{sl}(2,\mathbb{R})^+$, is the compression semigroup in $\mathrm{Sl}(2,\mathbb{R})$ of the cone $\mathbb{R}_0^+ \times \mathbb{R}_0^+$. Note that $\mathrm{Sl}(2,\mathbb{R})^+ = \exp\left(\mathfrak{sl}(2,\mathbb{R})^+\right)$ is an exponential semigroup.

6.5. Example. It is well known (cf.[5], p.419) that the set

$$\mathrm{Sl}(2,\mathbb{R})^{++} \overset{\mathrm{def}}{=} \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sl}(2,\mathbb{R})^+ \mid 1 \le a \right\}$$

is a closed subsemigroup of $\mathrm{Sl}(2,\mathbb{R})^+$ with Lie wedge $W = \mathbb{R}_0^+ H + \mathbb{R}_0^+ P + \mathbb{R}_0^+ Q$. Also, the decomposition

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a^{-1}c & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$$

shows that $\mathrm{Sl}(2,\mathbb{R})^{++}$ is the Lie semigroup generated by W. It is checked easily that $\mathrm{Sl}(2,\mathbb{R})^{++}$ is the compression semigroup in $\mathrm{Sl}(2,\mathbb{R})$ of the set $M_x \overset{\mathrm{def}}{=} \{(x,y) \in \mathbb{R}^2 \mid x \ge 1, y \ge 0\}$.

Likewise the set

$$\mathrm{Sl}(2,\mathbb{R})^{+-} \overset{\mathrm{def}}{=} \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sl}(2,\mathbb{R})^+ \mid 1 \le d \right\}$$

is a Lie subsemigroup of $\mathrm{Sl}(2,\mathbb{R})^+$ with Lie wedge $W = -\mathbb{R}_0^+ H + \mathbb{R}_0^+ P + \mathbb{R}_0^+ Q$, and this semigroup is the compression semigroup in $\mathrm{Sl}(2,\mathbb{R})$ of the set $M_y \overset{\mathrm{def}}{=} \{(x,y) \in \mathbb{R}^2 \mid x \ge 0,\ y \ge 1\}$.

The semigroup $\mathrm{Sl}(2,\mathbb{R})^{+-}$ is the image of $\mathrm{Sl}(2,\mathbb{R})^{++}$ under the (outer) automorphism which, on the level of Lie algebras, takes H to $-H$ and interchanges P and Q. Both of these semigroups fail to be exponential.

6.6. Example. Now we form the intersection of the two semigroups in the preceding example:

$$\begin{aligned}\mathrm{Sl}(2,\mathbb{R})^{++,+-} &\overset{\mathrm{def}}{=} \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sl}(2,\mathbb{R})^+ \mid 1 \le a \text{ and } 1 \le d \right\} \\ &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sl}(2,\mathbb{R})^+ \mid 1 \le a \le 1 + bc \right\}\end{aligned}$$

is a closed subsemigroup of $\mathrm{Sl}(2,\mathbb{R})^+$ with Lie wedge $W = \mathbb{R}_0^+ P + \mathbb{R}_0^+ Q$. We claim that every element $s = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sl}(2,\mathbb{R})^{++,+-}$ decomposes in the form

$$s = \exp(qQ)\exp(pP)\exp(q'Q) \in \exp(W)^3, \qquad \text{with } p, q, q' \in \mathbb{R}_0^+,$$

so $\mathrm{Sl}(2,\mathbb{R})^{++,+-} = \exp(W)^3$. Indeed, this decomposition is trivial if $b = 0$ (and thus $a = d = 1$), whereas for $b > 0$ we have the decomposition

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{d-1}{b} & 1 \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{a-1}{b} & 1 \end{pmatrix}.$$

6.7. Example. The compression semigroup $\mathrm{compr}_G(M_x \cap M_y)$ of the intersection $M_x \cap M_y = \{(x,y) \in \mathbb{R}^2 \mid x \geq 1,\ y \geq 1\}$ in $G = \mathrm{Sl}(2,\mathbb{R})$ is the semigroup

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sl}(2,\mathbb{R})^+ \mid a+b \geq 1 \text{ and } c+d \geq 1 \right\},$$

which properly contains $\mathrm{Sl}(2,\mathbb{R})^{++,+-}$. (For example, the matrix $\left(\begin{smallmatrix} \frac{1}{2} & 1 \\ 1 & 4 \end{smallmatrix}\right)$ belongs to this compression semigroup, but not to $\mathrm{Sl}(2,\mathbb{R})^{++,+-}$.)

6.8. Example. For a fixed real $\lambda > 0$ we define the cone

$$C_\lambda = \{(x,y) \in \mathbb{R}^2 \mid x \geq \lambda y \geq 0\}$$

and write S_λ for the compression semigroup of C_λ in $\mathrm{Sl}(2,\mathbb{R})^+$. Then

$$(*) \qquad \begin{aligned} S_\lambda &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sl}(2,\mathbb{R})^+ \,\middle|\, a + \frac{1}{\lambda} b \geq \lambda c + d \right\} \\ &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sl}(2,\mathbb{R})^+ \,\middle|\, (\lambda a + b)(a - \lambda c) \geq \lambda \right\}. \end{aligned}$$

Indeed, for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sl}(2,\mathbb{R})^+$ the inequalities

$$a + b/\lambda \geq \lambda c + d \quad \text{and} \quad (\lambda a + b)(a - \lambda c) \geq \lambda$$

are equivalent, since multiplying the first inequality with $a > 0$ yields (note that $ad = 1 + bc$)

$$a(a + b/\lambda) \geq \lambda ac + ad = \lambda ac + 1 + bc = 1 + \lambda c(a + b/\lambda),$$

and this is equivalent to $(\lambda a + b)(a - \lambda c) \geq \lambda$. Thus in $(*)$ the two sets on the right are the same.

Looking at the special case $x = 1, y = 1/\lambda$ we see that for every matrix $s = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sl}(2,\mathbb{R})^+$ with $sC_\lambda \subseteq C_\lambda$ the inequality $a + b/\lambda \geq \lambda c + d$ holds. Conversely, if $s = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sl}(2,\mathbb{R})^+$ with $(\lambda a + b)(a - \lambda c) \geq \lambda$ then $a - \lambda c > 0$ and

$$s \begin{pmatrix} 1 \\ 1/\lambda \end{pmatrix} = \begin{pmatrix} a + b/\lambda \\ c + d/\lambda \end{pmatrix} \in C_\lambda \quad \text{as well as} \quad s \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} \in C_\lambda,$$

therefore $sC_\lambda = s(\mathbb{R}_0^+ \begin{pmatrix} 1 \\ \frac{1}{\lambda} \end{pmatrix} + \mathbb{R}_0^+ \begin{pmatrix} 1 \\ 0 \end{pmatrix}) \subseteq C_\lambda$.

Note that all semigroups S_λ are conjugate to S_1:

$$S_\lambda = \begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & 1/\sqrt{\lambda} \end{pmatrix} S_1 \begin{pmatrix} 1/\sqrt{\lambda} & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix},$$

and that $S_\lambda \subseteq S_\mu$ whenever $\lambda \geq \mu > 0$.

We now compute the Lie wedge of S_1 using the following one parameter subsemigroups of S_1:

$$S_A = \left\{ \begin{pmatrix} a & 0 \\ a-1/a & 1/a \end{pmatrix} \mid a \geq 1 \right\} \text{ with infinitesimal generator } A = 2Q + H,$$
$$S_H = \left\{ \begin{pmatrix} h & 0 \\ 0 & 1/h \end{pmatrix} \mid h \geq 1 \right\} \text{ with infinitesimal generator } H,$$
$$S_B = \left\{ \begin{pmatrix} 1/b & b-1/b \\ 0 & b \end{pmatrix} \mid b \geq 1 \right\} \text{ with infinitesimal generator } B = 2P - H.$$

For every element $s = \left(\begin{smallmatrix} u & v \\ w & x \end{smallmatrix}\right) \in S_1$ we have the decomposition

$$s = \begin{pmatrix} a & 0 \\ a-1/a & 1/a \end{pmatrix} \begin{pmatrix} h & 0 \\ 0 & 1/h \end{pmatrix} \begin{pmatrix} 1/b & b-1/b \\ 0 & b \end{pmatrix},$$

where

$$a = \sqrt{\frac{u}{u-w}}, \quad b = \sqrt{\frac{u+v}{u}}, \quad h = \sqrt{(u+v)(u-w)}$$

(recall that by $(*)$ the inequality $u+v \geq x+w$ is equivalent to $(u+v)(u-w) \geq 1$). Thus the Lie wedge W_1 of S_1 is spanned by the three vectors A, B, H:

$$W_1 = \mathbb{R}_0^+ H + \mathbb{R}_0^+ (H + 2Q) + \mathbb{R}_0^+ (-H + 2P),$$

and S_1 is the Lie semigroup generated by W_1,

$$S_1 = S_A S_H S_B = S_H S_A S_B = S_H S_B S_A.$$

Note also that $S_A S_B = S_B S_A$, $S_A S_H = S_H S_A$, but $S_H S_B \supsetneqq S_B S_H$. Since W_1 is a Lie semialgebra contained in $\overline{\mathsf{Kill}^+}$ we deduce from 3.8 that S_1 is exponential.

6.9. Remark. It is not difficult to deduce right from its definition that S_1 is divisible:

Since the dyadic fractions are dense in $\mathbb{R}$, and since the restriction of exp to $\overline{\mathsf{Kill}^+}$ is a diffeomorphic embedding, it suffices to show that if for some $s = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sl}(2,\mathbb{R})^+$ the square $s^2 \in S_1$ then $s \in S_1$. Applying the definition of S_1 to

$$s^2 = \begin{pmatrix} a^2 + bc & b(a+d) \\ c(a+d) & d^2 + bc \end{pmatrix}$$

we see that $s^2 \in S_1$ implies $a^2 + bc + b(a+d) \geq c(a+d) + d^2 + bc$, which is equivalent to $(a+d)(a-d+b-c) \geq 0$, or, since $a+d > 0$, to $a+b \geq c+d$.

6.10. Example. The compression semigroup of C_1 with respect to the whole group $G = \mathrm{Sl}(2,\mathbb{R})$ properly contains S_1. Since

$$C_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} (\mathbb{R}_0^+ \times \mathbb{R}_0^+)$$

Example 6.4 and Remark 6.3 show that $\mathrm{compr}_G(C_1)$ is conjugate to $\mathrm{Sl}(2,\mathbb{R})^+$, namely

$$\mathrm{compr}_G(C_1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mathrm{Sl}(2,\mathbb{R})^+ \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

This set contains, e.g., the element $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 3 & -1 \\ 1 & 0 \end{smallmatrix}\right) \notin S_1$. Furthermore,

$$S_1 = \mathrm{Sl}(2,\mathbb{R})^+ \cap \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mathrm{Sl}(2,\mathbb{R})^+ \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

A more specific (but straightforward) calculation shows that

$$\mathrm{compr}_G(C_1) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sl}(2,\mathbb{R}) \mid a+b \geq c+d \geq 0 \text{ and } a \geq c \geq 0 \right\}.$$

6.11. Example. The anti-isomorphism sending every matrix s to its transpose s^T maps S_1 onto the semigroup

$$\begin{aligned} S^1 &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sl}(2,\mathbb{R})^+ \mid a+c \geq b+d \right\} \\ &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sl}(2,\mathbb{R})^+ \mid (a-b)(a+c) \geq 1 \right\}. \end{aligned}$$

Its Lie wedge is $L(S^1) = \mathbb{R}_0^+ \cdot H + \mathbb{R}_0^+ \cdot A + \mathbb{R}_0^+ \cdot B$, where $A = 2P + H$, $B = 2Q - H$. This semigroup is the compression semigroup of the dual cone $C^1 = \{(x,y) \mid x \geq 0, y \geq -x\}$ of C_1.

More generally, the map $s \mapsto s^T$ maps each semigroup S_λ onto a semigroup S^λ. Each S^λ is the compression semigroup in $\mathrm{Sl}(2,\mathbb{R})^+$ of the cone $C^\lambda = \{(x,y) \mid x \geq 0, y \geq -\lambda x\}$ and all semigroups S^λ are conjugate to S^1.

6.12. Example. For $\lambda, \mu \in \mathbb{R}^+$ we let $S_\lambda^\mu = S_\lambda \cap S^\mu$. Then S_λ^μ is the compression semigroup in $\mathrm{Sl}(2,\mathbb{R})^+$ of the union $C_\lambda \cup C^\mu = \{(x,y) \in \mathbb{R}^2 \mid x \geq 0, -\mu x \leq y \leq \frac{1}{\lambda}x\}$. Note that every semigroup S_λ^μ is exponential. Also, S_λ^μ is conjugate to $S_{\lambda'}^{\mu'}$ if and only if $\lambda\mu = \lambda'\mu'$. The Lie wedge of S_λ^1 is spanned by the vectors H, $H + 2P$, $H + \frac{2}{\lambda}Q$ and $(\lambda - 1)H + 2\lambda P + 2Q$ (the Lie wedges for general λ, μ can be deduced readily). We leave the straightforward proofs to the reader.

6.13. Semialgebraic Semigroups. Recall that a subset of $\mathbb{R}^n$ is called a *semialgebraic set* if it is the solution set (in some $\mathbb{R}^n$) of finitely many polynomial equalities and inequalities, or a finite union of such sets (cf. [2], p.162). It is natural to say that a semigroup S is a *semialgebraic semigroup*, if it is defined on a semialgebraic set and the multiplication is defined by polynomial functions. In this sense all semigroups of this section are semialgebraic. More generally, if we have an algebraic action of a semialgebraic semigroup S then every compression semigroup in S of a semialgebraic set is a semialgebraic semigroup.

7. Conjugacy classes of exponential Lie wedges in $\mathfrak{sl}(2,\mathbb{R})$

7.1. In this section we give a complete list of the conjugacy classes of Lie semialgebras with nonvoid interior in $\overline{\mathsf{Kill}^+}$. Since by 5.5 these Lie semialgebras are in 1-1-correspondence with the rectangular domains in Hyp this amounts to giving a list of the conjugacy classes of rectangular domains in Hyp. A first natural classification of the rectangular domains in Hyp is based on counting the number of nilpotency points.

7.2. The three types of rectangular domains. A rectangular domain D is said to be of *infinity type* k, or a *type* k *rectangular domain* for short, if $\overline{c(D)}$ contains exactly k nilpotency points in Hyp^-. (Note that $0 \leq k \leq 2$.)

We say that a wedge W is a *semialgebra of infinity type* k, or a type k *semialgebra* for short, if $\operatorname{Int} W = \mathbb{R}^+ D$, where D is a type k rectangular domain.

Proposition 5.7 says that every rectangular domain belongs to exactly one of the above types. Also, a Lie semialgebra W is of type k if and only if it does not meet the interior of the light cone Kill^0 and is the intersection of exactly $4-k$ half space semialgebras (or equivalently: contains exactly k linearly independent nilpotent elements).

7.3. The rectangular domains of infinity type 2. *Every type 2 rectangular domain is conjugate to* $\operatorname{Int}(\mathfrak{sl}(2,\mathbb{R})^+) \cap \mathsf{Hyp}$.

Proof. This is an immediate consequence of the well known fact that all Lie semialgebras (indeed all Lie wedges) in $\mathfrak{sl}(2,\mathbb{R})$ with nontrivial edge and nonvoid interior are conjugate to $\mathfrak{sl}(2,\mathbb{R})^+$ (cf., e.g., [5], p.109f). ∎

7.4. The rectangular domains of infinity type 1. *Every type 1 rectangular domain is conjugate to the rectangular domain*

(i) D_1 *with the three corner points* H, $H+2P$, $H-2Q$
or to

(ii) *the transpose* D_1^T *of* D_1, *with the three corner points* H, $H-2P$, $H+2Q$.

Proof. Let D be a rectangular domain of type 1. Then by assumption, the boundary of D in Hyp has exactly three corner points, say A, B, C. We may assume that $A = B \diamond C$. Upon applying a suitable inner automorphism we enforce that $A = H$, so that $B = H + \lambda P$ and $C = H + \mu Q$ for some nonzero $\lambda, \mu \in \mathbb{R}$. Also, the diamond product $C \diamond B$ does not exist, hence, as we have seen in Example 4.15, we must have $\lambda\mu = -4$. Now the inner automorphism $e^{t\,\mathrm{ad}H}$ with $t = 1/2(\log 2 - \log|\lambda|)$ carries the triple (A, B, C) either to the triple $(H, H + 2P, H - 2Q)$ or to the triple $(H, H - 2P, H + 2Q)$. It follows that D is conjugate either to D_1 or to D_1^T. ■

(Note that D_1^T is mapped to $-D_1$ by conjugation with the rotation matrix $g = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.)

7.5. For the list of conjugacy classes of type 0 rectangular domains we need a device which to every such domain assigns a convenient reference point and an area.

(i) Consider the element $H \in \mathsf{Hyp}$. We noticed already in 4.6 that the inner automorphisms of $\mathfrak{sl}(2, \mathbb{R})$ act transitively on the space Hyp, so for every $X \in \mathsf{Hyp}$ there is an inner automorphism φ carrying X to H. For every such inner automorphism φ we also have $\varphi(\mathsf{hor}(X)) = \mathsf{hor}(H) = H + \mathbb{R}P$ and $\varphi(\mathsf{vert}(X)) = \mathsf{vert}(H) = H + \mathbb{R}Q$.

(ii) The inner automorphisms fixing H are exactly the maps $e^{t\,\mathrm{ad}H}$, $t \in \mathbb{R}$, and these act transitively on each connected component of $\mathsf{hor}(H) \setminus \{H\}$. Thus if $X, Y \in \mathsf{Hyp}$ with $Y \in \mathsf{hor}(X) \setminus \{X\}$ then there is exactly one inner automorphism φ with $\varphi(X) = H$ and either $\varphi(Y) = H + P$ or $\varphi(Y) = H - P$.

(iii) Similarly, if $X, Y \in \mathsf{Hyp}$ with $Y \in \mathsf{vert}(X) \setminus \{X\}$ then there is a unique inner automorphism ψ such that $\psi(X) = H$ and either $\psi(Y) = H + Q$ or $\psi(Y) = H - Q$.

7.6. Orientation. For any $X \in \mathsf{Hyp}$ we say that

(i) an element $Y \in \mathsf{Hyp}$ lies on the *right* of X, or X lies on the *left* of Y, in symbols: $X <_h Y$, if there exists an inner automorphism of $\mathfrak{sl}(2, \mathbb{R})$ sending X to H and Y to $H + P$;

(ii) an element $Y \in \mathsf{Hyp}$ lies *above* X, or X lies *below* Y, in symbols: $X <_v Y$, if there exists an inner automorphism of $\mathfrak{sl}(2, \mathbb{R})$ sending X to H and Y to $H + Q$.

7.7. Remark. (i) On each horizontal line the relation $<_h$ induces a total order. (Note that there is an inner automorphism α with $\alpha(H - P) = H$ and $\alpha(H) = H + P$.) Similarly, $<_v$ induces a total order on every vertical line.

(ii) The partial orders induced by $<_h$ and $<_v$ are preserved under every inner automorphism. (The outer automorphisms reverse them.)

7.8. Lower left corner and characteristic of a type 0 rectangular domain. Let D be a type 0 rectangular domain. Then the boundary of D in Hyp has four corners. The corner point A of ∂D will be called the *lower left corner* of D if $A <_h B$ for every $B \in \partial D \cap \mathsf{hor}(A)$ and $A <_v C$ for every $C \in \partial D \cap \mathsf{vert}(A)$. It is immediate that a lower left corner of D exists and that it is uniquely defined. Since by definition inner automorphisms respect the ordering on horizontal or vertical lines, the lower left corner of a rectangular domain D is carried to the lower left corner of $\varphi(D)$ by every inner automorphism φ of $\mathfrak{sl}(2,\mathbb{R})$.

We now assign to each type 0 rectangular domain D a *characteristic* $\delta(D)$. Let A be the lower left corner of D and let B, C be the corner points adjacent to A, so that $B \in \mathsf{hor}(A)$, $C \in \mathsf{vert}(A)$. Then the Lie bracket $[B-A, C-A]$ is a positive multiple $r \cdot A$ of A, we define $\delta(D) = \sqrt{r}$. Note that δ is invariant under inner automorphisms. For $A = H$ the number $\delta(D)^2$ is just the Euclidean area of the rectangle spanned by the vectors $B - A$ and $C - A$.

7.9. Remark. The characteristic $\delta(D)$ can be computed also in terms of opposite corners of D (without knowing the lower left corner of D), after conjugating one of them onto H. If H and $Y = \alpha H + \beta P + \gamma Q$ are opposite corners of D then $\delta(D) = \frac{2}{1+\alpha}\sqrt{|\beta\gamma|}$. (We shall see in 8.8 that $1 + \alpha > 0$.) Observing that $H \diamond Y = H + \frac{2\beta}{1+\alpha} P$ and $Y \diamond H = H + \frac{2\gamma}{1+\alpha} Q$ this formula is checked by straightforward calculation.

7.10. The conjugacy classes of type 0 rectangular domains. *Let D be a rectangular domain of type 0 with $\delta(D) = \lambda$. Then D is conjugate to the rectangular domain $D_0(\lambda)$ with the four corner points*

$$\begin{gathered} A = H, \quad B = H + \lambda P, \quad C = H + \lambda Q, \\ C \diamond B = \frac{1}{4+\lambda^2}\left((4-\lambda^2)H + 4\lambda P + 4\lambda Q\right). \end{gathered}$$

Proof. Applying a suitable inner automorphism we enforce that H is the lower left corner of D. Then, similar to the proof of 7.4, we apply an inner automorphism $e^{t\,\mathrm{ad}H}$ for a suitable real t which maps D onto $D_0(\lambda)$. ■

7.11. Conjugacy classes of exponential Lie semialgebras. *Let W be a Lie semialgebra with nonvoid interior which is contained in $\overline{\mathsf{Kill}^+}$. Then exactly one of the following assertions holds:*

(i) *W is conjugate to $\mathfrak{sl}(2,\mathbb{R})^+$;*

(ii) *W is conjugate either to*

$$W(D_1) = \mathbb{R}_0^+ H + \mathbb{R}_0^+ (H + 2P) + \mathbb{R}_0^+ (H - 2Q)$$

or to

$$W(D_1^T) = \mathbb{R}_0^+ H + \mathbb{R}_0^+ (H - 2P) + \mathbb{R}_0^+ (H + 2Q);$$

(iii) *there exists a unique positive real number* λ *such that* W *is conjugate to* $W(D_0(\lambda)) = \mathbb{R}_0^+ A + \mathbb{R}_0^+ B + \mathbb{R}_0^+ C + \mathbb{R}_0^+ C \diamond B$, *with*

$$A = H, \quad B = H + \lambda P, \quad C = H + \lambda Q,$$
$$C \diamond B = \frac{1}{4+\lambda^2}\left((4-\lambda^2)H + 4\lambda P + 4\lambda Q\right).$$

Proof. This is immediate from the above classification of rectangular domains in view of the 1-1 correspondence between rectangular domains and Lie semialgebras with nonvoid interior in $\overline{\mathsf{Kill}^+}$. ∎

7.12. Lie wedges of exponential compression semigroups. In section 4 we have seen that the compression semigroups S_λ, S^μ and $S_\lambda^\mu = S_\lambda \cap S^\mu$ are exponential. We now explicitly indicate the conjugacy classes of the corresponding Lie wedges. The following assertions can be checked by straightforward calculation:

(i) The Lie wedge W_λ of S_λ is conjugate to $W(D_1)$. In fact,

$$W_\lambda = u_\lambda W(D_1) u_\lambda^{-1} \quad \text{with } u_\lambda = \begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

(ii) The Lie wedge W^μ of S^μ is conjugate to $W(D_1^T)$, with conjugating element $(u_\mu^{-1})^T$.

(iii) The Lie wedge W_λ^μ of S_λ^μ is conjugate to $W(D_0(\frac{2}{\sqrt{\lambda\mu}}))$. In fact,

$$W_\lambda^\mu = vW(D_0(\frac{2}{\sqrt{\lambda\mu}}))v^{-1} \quad \text{with } v = \begin{pmatrix} \sqrt[4]{\frac{\lambda}{\mu}} & 0 \\ 0 & \frac{1}{\sqrt[4]{\frac{\mu}{\lambda}}} \end{pmatrix}.$$

7.13. The above list shows that every conjugacy class of exponential Lie semialgebras with nonvoid interior in $\mathfrak{sl}(2,\mathbb{R})$ contains a representative which is the Lie wedge of an exponential compression semigroup. The following proposition formulates this fact in terms of exponential semigroups.

7.14. Conjugacy classes of exponential subsemigroups. *Let* S *be a three dimensional exponential subsemigroup of* $\mathrm{Sl}(2,\mathbb{R})$. *Then* S *is conjugate to exactly one of the following semigroups:*

(i) $\mathrm{Sl}(2,\mathbb{R})^+$;

(ii) $S_1 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sl}(2,\mathbb{R})^+ \mid a+b \geq c+d \right\}$;

(iii) $S^1 = (S_1)^T = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sl}(2,\mathbb{R})^+ \mid a+c \geq b+d \right\}$;

(iv) $S_\lambda^1 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sl}(2,\mathbb{R})^+ \mid a+c \geq b+d \text{ and } a+\frac{1}{\lambda}b \geq \lambda c + d \right\}$, *for some* $\lambda \in \mathbb{R}^+$. ∎

7.15. Remark. The above proposition says that all three dimensional exponential subsemigroups of $\mathrm{Sl}(2,\mathbb{R})$ are semialgebraic semigroups in the sense of 6.13. (The exponential subsemigroups of dimension 0,1,2 are trivially semialgebraic.)

8. An asymptotic formula for the diamond product

8.1. Proposition. *For two points X, Y in Hyp the following assertions are equivalent:*

(i) *There exist a positive real s and a bound $B \in \mathbb{N}$ such that for all natural numbers $m > B$ and $n > B$ the product $\exp(msX)\exp(nsY)$ lies in $\exp(\mathsf{Kill}^+)$.*

(ii) *There exists a positive real s such that $\exp(sX)$ and $\exp(sY)$ (algebraically) generate a semigroup which is contained in $\exp(\mathsf{Kill}^+)$.*

(iii) *The semigroup generated by $\exp(\mathbb{R}^+X) \cup \exp(\mathbb{R}^+Y)$ is contained in $\exp(\mathsf{Kill}^+)$.*

(iv) *There exists a Lie semialgebra $W \subseteq \overline{\mathsf{Kill}^+}$ containing both X and Y in its interior.*

(v) *X and Y lie in the same connected component of the open set $\mathsf{Hyp} \setminus \tau$, where τ denotes the tangent plane of Hyp at $-X$.* (Note that $\tau \cap \mathsf{Hyp} = \mathsf{hor}(-X) \cup \mathsf{vert}(-X)$.)

(vi) *The set $\mathbb{R}^+X + \mathbb{R}^+Y$ (equivalently, the line segment $\operatorname{conv}\{X, Y\}$) does not meet the light cone Kill^0.*

Proof. The implications (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i) are obvious. To see the implication (iv) $\Longrightarrow$ (iii) we only have to recall that $\exp W$ is a closed semigroup if W is a Lie semialgebra with nonvoid interior contained in $\overline{\exp(\mathsf{Kill}^+)}$, and that the restriction of exp to $\overline{\mathsf{Kill}^+}$ is a homeomorphic embedding.

In order to establish the implications (i) $\Longrightarrow$ (v), (v) $\Longrightarrow$ (iv), and (v) $\Longleftrightarrow$ (vi) we simplify matters by applying a suitable inner automorphism so that $X = H$. We write $Y = \alpha H + \beta P + \gamma Q$. Then the tangent plane of Hyp at $-X$ is the plane $-X + \mathbb{R}\cdot P + \mathbb{R}\cdot Q$, so (v) is equivalent to $\alpha > -1$.

Pick $\lambda, \mu \in \mathbb{R}$ and let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \exp(\mu Y)$. Using the formulas for exp we compute the number $\tau(\lambda, \mu) = \operatorname{trace}(\exp \lambda X \exp \mu Y)$:

(∗)

$$\begin{aligned}\tau(\lambda,\mu) = ae^{\lambda} + de^{-\lambda} &= (\cosh(\mu) + \alpha\sinh(\mu))e^{\lambda} + (\cosh(\mu) - \alpha\sinh(\mu))e^{-\lambda} \\ &= (1+\alpha)\cosh(\lambda+\mu) + (1-\alpha)\cosh(\lambda-\mu).\end{aligned}$$

(i) $\Longrightarrow$ (v) By 2.3 we know that a matrix lies in $\exp(\mathsf{Kill}^+)$ if and only if its trace is > 2, so assertion (i) implies that $\tau(\lambda,\mu) > 2$ for any $\lambda = ms, \mu = ns$ with $m, n \in B + \mathbb{N}$. But if $\alpha < -1$ then $\lim_{\lambda\to\infty} \tau(\lambda,\lambda) = -\infty$, and if $\alpha = -1$ then $\tau(\lambda,\lambda) = 2$. Thus we see that $\alpha > -1$, and therefore (v) holds.

(v) $\Longrightarrow$ (iv) Case 1: $|\alpha| < 1$. Then we conclude from $\alpha^2 + \beta\gamma = 1$ that β and γ have the same sign. Applying an inner automorphism of the form $e^{h \operatorname{ad} H}$, $h \in \mathbb{R}$, we enforce that $\beta = \gamma$. Then both $X = H$ and $Y = \alpha H + \beta(P+Q)$ lie in the plane $\mathbb{R}H + \mathbb{R}(P+Q)$. Thus a suitable rotation $e^{\sigma \operatorname{ad}(P-Q)}$ will map $\mathfrak{sl}(2,\mathbb{R})^+$ onto a semialgebra W which contains both X and Y in its interior.

Case 2: $\alpha = 1$. Then either $\beta = 0$ or $\gamma = 0$, so $Y = H + \beta P$ or $Y = H + \gamma Q$. Then a rotation $e^{\sigma \operatorname{ad}(P-Q)}$ will map $\mathfrak{sl}(2,\mathbb{R})^+$ onto a semialgebra which contains both X and Y. (For instance, if $Y = H + 2P$ then $\sigma = \frac{\pi}{8}$ will do the job. This situation can be reached by applying suitable automorphisms and/or anti-automorphisms.)

Case 3: $\alpha > 1$. Then β and γ have opposite sign. Using a suitable inner automorphism $e^{h \operatorname{ad} H}$ we enforce that $\beta = -\gamma$. Then $\alpha^2 + \beta\gamma = 1$ implies that $|\beta| < \alpha$. A straightforward calculation shows that both $X = H$ and $Y = \alpha H + \beta(P-Q)$ lie in the interior of the semialgebra $\mathbb{R}(P+Q) + \mathbb{R}_0^+(H + P - Q) + \mathbb{R}_0^+(H - P + Q)$. In fact, $H = \frac{1}{2}(H+P-Q) + \frac{1}{2}(H-P+Q)$ and $\alpha H + \beta(P-Q) = \frac{\alpha+\beta}{2}(H+P-Q) + \frac{\alpha-\beta}{2}(H-P+Q)$.

(v)$\Longleftrightarrow$(vi) Let s and t be positive reals. Then $sX + tY = (s+\alpha t)H + \beta tP + \gamma tQ \in \mathsf{Kill}^0$ if and only if $(s-t)^2 + 2st(\alpha+1) = 0$, i.e., $\alpha + 1 = -\frac{(s-t)^2}{2st}$. This implies the equivalence of assertions (v) and (vi). ∎

8.2. Remarks. For later use we record the following observation of the above proof: If $X = H$ and $Y = \alpha H + \beta P + \gamma Q$ then assertion 8.1(v) just means that $\alpha > -1$.

8.3. Scholium. Suppose that $X, Y \in \mathsf{Hyp}$ do not belong to the same Borel algebra. Then the assertions listed in 8.1 are also equivalent to

(i′) There exist positive reals s, t and a bound $B \in \mathbb{N}$ such that for all natural numbers $m > B$ and $n > B$ the product $\exp(msX)\exp(ntY)$ lies in $\exp(\mathsf{Kill}^+)$.

Proof. The implication (i) $\Longrightarrow$ (i′) is obvious. To see that (i′) $\Longrightarrow$ (v), we suppose, without losing generality, that $X = H$ and $Y = \alpha H + \beta P + \gamma Q$. Then we choose two sequences $\langle m_k \rangle$ and $\langle n_k \rangle$ of natural numbers such that $\lim m_k = \lim n_k = \infty$ and $\lim(m_k s - n_k t) = 0$. Let $\lambda_k = m_k s$ and $\mu_k = n_k t$. If $\alpha < -1$ then relation $(*)$ of the proof of 8.1 implies that $\lim \tau(\lambda_k, \mu_k) = -\infty$. Since (i′) holds and since X and Y do not belong to the same Borel algebra, we conclude that $\alpha > -1$. ∎

8.4. Scholium. For two points $X, Y \in \mathsf{Hyp}$ the following assertions are equivalent:

(i) The line segment $\operatorname{conv}\{X, Y\}$ meets the interior Kill^- of the light cone.

(ii) For any choice of positive reals s, t there exist natural numbers $m, n \in \mathbb{N}$ such that the product $\exp(msX)\exp(ntY)$ does not belong to the image of the exponential function.

(iii) There exist positive reals s,t and natural numbers $m,n\in\mathbb{N}$ such that the product $\exp(msX)\exp(ntY)$ does not belong to the image of the exponential function.

Proof. The implication (ii) $\Longrightarrow$ (iii) is obvious.

(i) $\Longrightarrow$ (ii) Assertion (i) implies that X and Y cannot lie in the same Borel algebra. Choose m_k and n_k as in the proof of Scholium 8.3. Then for large k the trace $\tau(m_k s, n_k t)$ of $\exp(m_k sX)\exp(n_k tY)$ is <-2, so assertion (ii) follows from 2.3.

(iii) $\Longrightarrow$ (i) If (iii) holds then the line segment $\operatorname{conv}\{X,Y\}$ cannot be tangent to the light cone, otherwise X and Y would lie in the same Borel algebra $\mathfrak{b}$, which is impossible since $\exp\mathfrak{b}$ is a group. Thus the implication (iii) $\Longrightarrow$ (i) follows from 8.1. ■

8.5. Proposition. *Let X,Y be two elements in Hyp, not belonging to the same Borel algebra. Then the following assertions hold:*

(i) *There exists a bound $B>0$ such that $\exp(sX)\exp(tY)\notin\exp(\mathsf{Kill}^0)$, for all $s,t>B$.*

(ii) *Let $s,t\in\mathbb{R}^+$ such that the elements $\exp(sX)$ and $\exp(tY)$ generate a semigroup S with $S\subset\overline{\exp(\mathsf{Kill}^+)}$. Then X and Y satisfy the equivalent conditions of* 8.1.

Proof. (i) Pick $s,t\in\mathbb{R}^+$, and, for notational convenience, put $\exp(tY)=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$. By 2.3(v) we have to show that the trace of $\exp(sX)\exp(tY)$ is $\neq 2$ for all sufficiently large coefficients s,t.

We assume, without losing generality, that $X=H$ and write $Y=\alpha H+\beta P+\gamma Q$. Since X and Y do not belong to the same Borel algebra, we must have $\beta\gamma\neq 0$. Now the equation $\operatorname{trace}(\exp(sX)\exp(tY))=2$ boils down to $e^s a+e^{-s}d=2$, which means that e^s satisfies the quadratic equation $ax^2-2x+d=0$. If t is sufficiently large to ensure that $a\neq 0$ then the solutions of this quadratic equation are

$$(**)\qquad \frac{1\pm\sqrt{1-ad}}{a}=\frac{1\pm\sqrt{-bc}}{a}=\frac{1\pm\sinh(t)\sqrt{-\beta\gamma}}{\cosh(t)+\alpha\sinh(t)}$$

and these are real only if $\beta\gamma<0$. But the expressions on the right of $(**)$ remain bounded if $t\to\infty$, so if s and t are chooser sufficiently large then e^s cannot satisfy the above quadratic equation and the assertion follows.

Assertion (ii) is a consequence of (i) and 8.3. ■

8.6. Corollary. *Let $X,Y\in\overline{\mathsf{Kill}^+}$ and let S be the semigroup generated by $x=\exp(X)$ and $y=\exp(Y)$. Then the following assertions are equivalent:*

(i) *The semigroup S is contained in $\overline{\exp(\mathsf{Kill}^+)}$.*

(ii) *There exists a Lie semialgebra in $\overline{\mathsf{Kill}^+}$ which contains both X and Y. Equivalently, either X and Y are linearly dependent or S is contained in a conjugate of $\mathrm{Sl}(2,\mathbb{R})^+$.*

(iii) *The line segment between* X *and* Y *does not meet the interior* Kill^- *of the light cone* Kill^0.

Proof. Case 1: X and Y belong to the same Borel algebra. Then all three assertions (i)–(iii) are satisfied, so their equivalence is obvious. (Note that if X and Y are linearly independent then $\mathbb{R}_0^+ X + \mathbb{R}_0^+ Y$ is a two dimensional Lie semialgebra which can be conjugated onto a Lie subsemialgebra of either $\mathbb{R}H + \mathbb{R}_0^+ P$ or $\mathbb{R}H - \mathbb{R}_0^+ P$.)

Case 2: We assume that both X and Y belong to Kill^+ but do not belong to the same Borel algebra. Then, in particular, X and Y cannot be linearly dependent. Note that every exponential subsemigroup is conjugate to a subsemigroup of $\mathrm{Sl}(2,\mathbb{R})^+$, by 3.8. Put

$$X_0 = \frac{1}{\Delta(X)}X, \quad Y_0 = \frac{1}{\Delta(Y)}Y.$$

Note that X_0, Y_0 sit in Hyp but not in a single Borel algebra. By 8.5(ii) our assertion (i) implies that X_0, Y_0 satisfy the equivalent assertions (i)–(vi) of 8.1, conversely, if X_0, Y_0 satisfy 8.1(i)–(vi) then our (i) follows. In other words, $S \subseteq \overline{\exp(\mathsf{Kill}^+)}$ if and only if X_0, Y_0 satisfy the equivalent conditions of 8.1.

Also, our assertion (iii) holds if and only if X_0, Y_0 satisfy 8.1(vi), and if 8.1(iv) is true for X_0, Y_0 then our (ii) follows. Finally (ii) $\Longrightarrow$ (i), so the equivalence of (i)–(iii) follows.

Case 3: At least one of X, Y belongs to Kill^0, and X, Y do not belong to the same Borel algebra. With no loss of generality we assume that $X = P$ and write $Y = \alpha H + \beta P + \gamma Q$. Note that $\gamma \neq 0$. We first remark that in this case assertion (iii) is equivalent to $\gamma > 0$. Indeed, since by assumption $\alpha^2 + \beta\gamma \geq 0$, the number

$$\Delta(\lambda X + \mu Y) = \Delta(\mu\alpha H + (\lambda + \mu\beta)P + \mu\gamma Q) = \mu^2(\alpha^2 + \beta\gamma) + \lambda\mu\gamma$$

is nonnegative for all $\lambda, \mu \in \mathbb{R}^+$ if and only if $\gamma > 0$. Thus the set $\mathbb{R}^+ X + \mathbb{R}^+ Y$ does not meet the interior Kill^- of the light cone if and only if $\gamma > 0$.

Next we show that (i) implies that $\gamma > 0$. Write $y = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Then for $m \in \mathbb{N}$ we have

$$\operatorname{trace}(x^m y) = \operatorname{trace}(y x^m) = \operatorname{trace}(y) + mc.$$

If $\gamma < 0$ then $c < 0$ and, for m sufficiently large, $\operatorname{trace}(x^m y) < -2$, so S is not contained in the image of the exponential function, *a fortiori* not in $\overline{\exp(\mathsf{Kill}^+)}$. It follows that (i) implies $\gamma > 0$ and hence (i) $\Longrightarrow$ (iii).

Since (ii) trivially implies (i) it remains to show that (iii) $\Longrightarrow$ (ii). Suppose that (iii) holds. Then for every $n \in \mathbb{N}$ the elements $X_n = (1 - \frac{1}{n})X + \frac{1}{n}Y$, $Y_n = \frac{1}{n}X + (1 - \frac{1}{n})Y$ are in Kill^+ and do not lie in the same Borel algebra, hence by the discussion of case 2 the line segment between X_n and Y_n is contained in a Lie semialgebra which lies in $\overline{\mathsf{Kill}^+}$. Write W_n for the semialgebra W_n generated by X_n and Y_n. Then the W_n's are monotone increasing, hence the closure of the union $\bigcup W_n$ is a Lie semialgebra (cf. [5], Corollary II.2.16, p. 89) which contains both X and Y. This finishes the proof. ∎

8.7. Remark. Suppose that X and Y are linearly independent elements in Kill^0. Then $\exp(\mathbb{R}^+ X)\exp(\mathbb{R}^+ Y) \cap \exp(\mathsf{Kill}^0) = \emptyset$.

Proof. Let us assume, with no loss of generality, that $X = P$, and write $Y = \alpha H + \beta P + \gamma Q$, with $\alpha^2 + \beta\gamma = 0$. For any $s, t \in \mathbb{R}^+$ we have $\operatorname{trace}(\exp(sX)\exp(tY)) = 2 + st\gamma$, so $\operatorname{trace}(\exp(sX)\exp(tY)) = 2$ if and only if $\gamma = 0$, equivalently, Y is a multiple of $X = P$. This implies the assertion. ∎

We now come to the main result of this section, a useful formula which expresses the diamond product $\diamond$ in terms of products in $\mathrm{Sl}(2,\mathbb{R})$. For investigations "in the large" it plays the role the Trotter Product Formula plays for investigations "near the identity." This formula is based on the following observation:

8.8. Remark. Let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sl}(2,\mathbb{R})$ and suppose that the products

$$\ell(s) = \begin{pmatrix} e^s & 0 \\ 0 & e^{-s} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad r(s) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e^s & 0 \\ 0 & e^{-s} \end{pmatrix}$$

lie in $\overline{\exp(\mathsf{Kill}^+)}$ for certain positive reals s which can be chosen arbitrarily large. Then $a > 0$ (for otherwise $\operatorname{trace}(\ell(s)) < 2$ if s is large) and for *any* sufficiently large $s \in \mathbb{R}^+$ the elements $\ell(s)$ and $r(s)$ sit in $\exp(\mathsf{Kill}^+)$. Now our formulas for the function rlog yield

$$(\dagger) \qquad \operatorname{rlog}\ell(s) = \frac{e^s a - e^{-s}d}{u}\cdot H + \frac{2e^s b}{u}\cdot P + \frac{2e^{-s}c}{u}\cdot Q,$$

$$(\ddagger) \qquad \operatorname{rlog} r(s) = \frac{e^s a - e^{-s}d}{u}\cdot H + \frac{2e^{-s} b}{u}\cdot P + \frac{2e^{s}c}{u}\cdot Q,$$

where $u = \sqrt{(e^s a + e^{-s}d)^2 - 4}$. This implies

$$\lim_{s\to\infty} \operatorname{rlog}\ell(s) = H + \frac{2b}{a}P, \qquad \lim_{s\to\infty} \operatorname{rlog} r(s) = H + \frac{2c}{a}Q.$$

Note that in the terminology and notation of 4.16 we have $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\cdot H = H + \frac{2c}{a}Q$ and $H \cdot \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = H + \frac{2b}{a}P$. Thus the two formulas in the display above can be considered as formulas for the natural partial left and right action of $\mathrm{Sl}(2,\mathbb{R})$ on Hyp.

8.9. Theorem: An asymptotic formula for the diamond product. *Assume that* $X, Y \in \mathsf{Hyp}$ *satisfy one (hence all) of the conditions listed in* Proposition 8.1. *Then* $X \diamond Y$ *is defined and*

$$\begin{aligned} X \diamond Y &= \lim_{(s,t)\to(\infty,\infty)} \operatorname{rlog}\big(\exp(sX)\exp(tY)\big) \\ &= \lim_{s\to\infty}\lim_{t\to\infty} \operatorname{rlog}\big(\exp(sX)\exp(tY)\big) \\ &= \lim_{t\to\infty}\lim_{s\to\infty} \operatorname{rlog}\big(\exp(sX)\exp(tY)\big). \end{aligned}$$

Proof. We first note that if the pair (X, Y) satisfies the conditions in 8.1 then so does $(-Y, -X)$. Thus if we can show that under our assumptions the above three limits of $\operatorname{rlog}\big(\exp(sX)\exp(tY)\big)$ exist and lie in $\mathsf{hor}(X)$ then, by applying this result to the pair $(-Y, -X)$, we find that each of the corresponding limits of

$$\operatorname{rlog}\big(\exp(sX)\exp(tY)\big) = -\operatorname{rlog}\big(\exp(-tY)\exp(-sX)\big)$$

exists and lies in $-\mathsf{hor}(-Y) = \mathsf{vert}(Y)$, hence is contained in $\mathsf{hor}(X) \cap \mathsf{vert}(Y) = \{X \diamond Y\}$, and the assertion follows.

Applying a suitable inner automorphism we enforce that $X = H$. Pick two sufficiently large positive reals s and t. Write $Y = \alpha H + \beta P + \gamma Q$ and $\exp(tY) = \binom{a\ b}{c\ d}$. Then $a = \cosh(t) + \alpha\sinh(t)$, $d = \cosh(t) - \alpha\sinh(t)$, $b = \beta\sinh(t)$, $c = \gamma\sinh(t)$ and inserting this into formula (†):

$$\operatorname{rlog}(\exp sX \exp tY) = \frac{e^s a - e^{-s} d}{u} \cdot H + \frac{2e^s b}{u} \cdot P + \frac{2e^{-s} c}{u} \cdot Q,$$

with $\sqrt{(e^s a + e^{-s} d)^2 - 4}$, we see that

$$\begin{aligned}
\lim_{(s,t)\to(\infty,\infty)} & \operatorname{rlog}\big(\exp(sX)\exp(tY)\big) \\
&= \lim_{s\to\infty}\lim_{t\to\infty} \operatorname{rlog}\big(\exp(sX)\exp(tY)\big) \\
&= \lim_{t\to\infty}\lim_{s\to\infty} \operatorname{rlog}\big(\exp(sX)\exp(tY)\big) \\
&= H + \frac{2\beta}{1+\alpha} P \in \mathsf{hor}(H).
\end{aligned}$$

(note that by 8.2 we already know that $1 + \alpha > 0$) and the assertion follows. ■

8.10. Remark.

(i) The arguments of the above proof can be used almost verbatim to show that under the hypothesis of 8.9 the following slightly more general equality holds:

$$X \diamond Y = \lim_{\substack{(s,t)\to(\infty,\infty) \\ (X',Y')\to(X,Y)}} \operatorname{rlog}\big(\exp(sX')\exp(tY')\big).$$

Details are left to the reader.

(ii) For $X = H$, $Y = \alpha H + \beta P + \gamma Q \in \mathsf{Hyp}$ with $\alpha > -1$, the proof of Theorem 8.9 also yields, without taking recourse to 4.5, the special formula $H \diamond (\alpha H + \beta P + \gamma Q) = H + \frac{2\beta}{1+\alpha} P$.

8.11. Corollary. *Assume that $X, Y \in \mathsf{Hyp}$ satisfy one (hence all) of the conditions listed in Proposition 8.1. Let s, t be arbitrary positive reals and denote by S the semigroup generated by $\exp(sX)$ and $\exp(tY)$. Then $S \subseteq \exp(\mathsf{Kill}^+)$, moreover, the diamond products $X \diamond Y$ and $Y \diamond X$ exist and lie in the closure of the set $\operatorname{rlog}(S)$.*

Proof. This is an immediate consequence of Theorem 8.9. ■

8.12. Remark. The formulas of Theorem 8.9 cannot be extended to the closure of the set of all pairs (X, Y) satisfying the conditions of Proposition 8.1. The boundary of this 'admissible' set consists exactly of the pairs (X, Y) with $Y \in \mathsf{hor}(-X) \cup \mathsf{vert}(-X)$, which means that at least one of the two products $X \diamond Y$ and $Y \diamond X$ does not exist. For instance, if $Y \in \mathsf{hor}(-X)$ then there are arbitrarily large numbers $s, t \in \mathbb{R}^+$ such that $\exp(sX)\exp(tY)$ is unipotent, and thus $\operatorname{rlog}(\exp(sX)\exp(tY))$ is not defined.

As an example consider the case $X = H - 2Q$, $Y = H + 2P$. Here $X \diamond Y$ is not defined (cf. 4.15), whereas $Y \diamond X = H$. The elements X and Y span a Borel subalgebra, which does not contain H. Thus no formula involving only products of the form $\exp(sX)$ or $\exp(tY)$ can yield $Y \diamond X$.

9. Rectangular bands coming from subsemigroups of $\mathrm{Sl}(2, \mathbb{R})$

9.1. Asymptotic rectangular bands. For a subset S of $\mathrm{Sl}(2, \mathbb{R})$ we write

$$\begin{aligned} U_0(S) &\overset{\text{def}}{=} \{X \in \mathsf{Hyp} \mid \exp(\mathbb{R}^+ X) \cap S \neq \emptyset\} = \operatorname{rlog}(S \cap \exp(\mathsf{Kill}^+)), \\ U(S) &\overset{\text{def}}{=} \overline{U_0(S)}. \end{aligned}$$

If S is a subsemigroup and contained in $\overline{\exp(\mathsf{Kill}^+)}$ and if $U_0(S)$ is not contained in a single Borel algebra then

$$\mathsf{Asy}(S) \overset{\text{def}}{=} \overline{c(U(S))} = \overline{c(U_0(S))} \subset \mathsf{Hyp}^-$$

is called the *asymptotic rectangular band* of S.

This is an ambitious definition, the term 'rectangular band' and the epitheton ornans 'asymptotic' have yet to be justified. We need some preparation.

9.2. Remark. Note that for any subset S of $\mathrm{Sl}(2, \mathbb{R})$ we have $U_0(\overline{S}) = \operatorname{rlog}(\overline{S} \cap \exp(\mathsf{Kill}^+)) \subseteq \overline{U_0(S)} = U(S)$, and therefore $U(S) = U(\overline{S})$.

9.3. Proposition. *Suppose that $X, Y \in \mathsf{Hyp}$ with $Y \in -\mathsf{hor}(X) \setminus \{-X\}$. Consider two arbitrary positive reals s, t and denote by S the semigroup generated by $\exp(sX)$ and $\exp(tY)$. Then $\mathsf{Asy}(S)$ contains the nilpotency point $(\mathsf{hor}(X), -\mathsf{hor}(X))$.*

Proof. Without losing generality we assume that $X = H$ and $Y = -H + \beta P$, where $\beta \in \mathbb{R} \setminus \{0\}$. Let m, n be natural numbers so that $ms > nt$. Using the formulas for exp and rlog we compute

$$Z(m, n) := \operatorname{rlog}(\exp(msX)\exp(ntY)) = H + \sigma P \quad \text{with } \sigma = \frac{(e^{2nt} - 1)\beta}{1 - e^{-2ms+2nt}}.$$

Note that $\mathsf{hor}(Z(m,n)) = H + \mathbb{R}P = \mathsf{hor}(X)$. Choose now sequences $\langle m_k \rangle$ and $\langle n_k \rangle$ of natural numbers such that $\lim n_k = \lim(m_k s - n_k t) = \infty$. Then all elements of the sequence $\langle Z(m_k, n_k) \rangle$ lie on the horizontal line $\mathsf{hor}(X)$, but no subsequence of $\langle Z(m_k, n_k) \rangle$ converges in Hyp. Thus (cf. 4.9(iii)) we must have $\lim c(Z(m_k, n_k)) = (\mathsf{hor}(X), -\mathsf{hor}(X))$, which finishes the proof. ∎

Now here is the justification for introducing the terminology of 9.1.

9.4. Theorem. *Let $S \subseteq \overline{\exp(\mathsf{Kill}^+)}$ be a subsemigroup of $\mathrm{Sl}(2,\mathbb{R})$. Suppose that $U(S)$ is not contained in a single Borel subalgebra of $\mathfrak{sl}(2,\mathbb{R})$. Then the following assertions hold:*

(i) *The subset $U(S)$ is a full partial subsemigroup of Hyp, that is, if $X, Y \in U(S)$ and $X \diamond Y$ is defined then $X \diamond Y \in U(S)$.*

(ii) *$\mathsf{Asy}(S)$ is a subsemigroup of the rectangular band Hyp^-.*

(iii) *Let I be a two sided ideal of S. Then $U(S) = U(I)$ and $\mathsf{Asy}(S) = \mathsf{Asy}(I)$.*

Proof. Pick X, $Y \in U_0(S)$.

Case 1: X and Y do not belong to the same Borel algebra. Then by 8.5(ii) the equivalent conditions of 8.1 are satisfied and hence by Corollary 8.11 the diamond product $X \diamond Y$ exists and lies in $U(S)$. Moreover, if either X or Y belongs to $U_0(I)$ then $X \diamond Y \in U(I)$, in view of 8.9.

Case 2: X and Y belong to the same horizontal or vertical line. Then $X \diamond Y = X$ or $X \diamond Y = Y$ and hence $X \diamond Y \in U(S)$, trivially. Furthermore, the line segment joining X and Y does not intersect Kill^0 so the asymptotic formula of Theorem 8.9 shows that $X \diamond Y \in U(I)$ whenever X or Y lies in $U_0(I)$.

Case 3: X and Y belong to the same Borel algebra $\mathfrak{b}$ but $Y \in -\mathsf{hor}(X)$. Then the diamond product $X \diamond Y$ does not exist and the product $c(X)c(Y)$ is the nilpotency point $(\mathsf{hor}(X), -\mathsf{hor}(X))$. If $Y \neq -X$ then Proposition 9.3 shows that $(\mathsf{hor}(X), -\mathsf{hor}(X)) \in \mathsf{Asy}(S)$. Also, if X or Y belongs to $U_0(I)$ then 9.3 implies that $(\mathsf{hor}(X), -\mathsf{hor}(X)) \in \mathsf{Asy}(I)$.

Thus we may assume that $Y = -X$. This means that there are numbers $s, t \in \mathbb{R}^+$ such that $\exp(sX) \in S$ and $\exp(-tX) \in S$. If s, t are rationally dependent then we may assume $s = t$. If s, t are independent over the rationals then the set $\mathbb{N}s - \mathbb{N}t$ is dense in $\mathbb{R}$ and therefore $\exp(-sX) \in \overline{S}$.

For simplicity we suppose, without losing generality, that $X = H$. Since $U_0(S)$ is not contained in a single Borel algebra there exists an element $Z \in U_0(S) \setminus (\mathbb{R}H + \mathbb{R}Q)$, say $Z = \alpha H + \beta P + \gamma Q$ with $\beta \neq 0$. Pick $\zeta \in \mathbb{R}^+$ with $\exp(\zeta Z) \in S$. Then for every $n \in \mathbb{N}$ the element $\exp(nsH)\exp(\zeta Z)\exp(-nsH)$ lies in $\exp(\mathsf{Kill}^+) \cap \overline{S}$, hence

$$\begin{aligned} Z_n &\overset{\mathrm{def}}{=} \mathrm{rlog}(\exp(nsH)\exp(\zeta Z)\exp(-nsH)) = e^{ns\cdot\mathrm{ad}H} Z \\ &= \alpha H + e^{2ns}\beta P + e^{-2ns}\gamma Q \end{aligned}$$

belongs to $U(S)$. Now $\lim_{n\to\infty} \frac{1}{\beta e^{2ns}} Z_n = P$, so Proposition 4.10 shows that $\lim c(Z_n) = (\mathsf{hor}(H), -\mathsf{hor}(H)) \in \mathsf{Asy}(S)$. Moreover, if either $X \in U_0(I)$ or

$Y \in U_0(I)$ then $Z_n \in U(I)$, for all n, hence $\lim c(Z_n) = (\mathsf{hor}(H), -\mathsf{hor}(H)) \in \mathsf{Asy}(I)$.

Case 4: X and Y span a Borel algebra $\mathfrak{b}$ but $X \in -\mathsf{hor}(Y)$, with $Y \neq -X$. Then $X \diamond Y$ exists. Here we assume, with no loss of generality, that $Y = H$. Then X can be written in the form $X = -H + pP$ with $p \neq 0$. We claim that there exists an element $Z \in U_0(S)$ with $Z \notin \mathbb{R}H + \mathbb{R}P$ and $Z \notin -H + \mathbb{R}Q$. Indeed, since $U_0(S)$ is not contained in $\mathbb{R}H + \mathbb{R}P$ we can find a $Z \in U_0(S)$, say $Z = \alpha H + \beta P + \gamma Q$ with $\gamma \neq 0$. Let $s, t \in \mathbb{R}^+$ with $\exp(tZ) \in S$, $\exp(sH) \in S$. Recalling formula (‡) of Remark 8.8 we find

$$\lim_{s \to \infty} \operatorname{rlog}(\exp(tZ)\exp(sH)) = H + \frac{2\gamma \sinh(t)}{\cosh(t) + \alpha \sinh(t)} Q.$$

Replacing Z by $\operatorname{rlog}(\exp(tZ)\exp(sH))$ for suitably large s we now enforce that $Z \notin -H + \mathbb{R}Q$, since $\gamma \neq 0$ we also have $Z \notin \mathbb{R}H + \mathbb{R}P$. Thus, applying what we have learned in Case 1 and Case 2, we see that $Z \diamond Y \in U(S)$. Note that with $Z = \alpha H + \beta P + \gamma Q$ we have $Z \diamond Y = H + \frac{2\gamma}{1+\alpha} Q$. Let $\langle Y_n = \alpha_n H + \beta_n P + \gamma_n Q \rangle$ be a sequence in $U_0(S)$ with $\lim Y_n = Z \diamond Y$. Then $\alpha_n \to 1$, $\beta_n \to 0$, $\gamma_n \not\to 0$.

We claim that X and Y_n, for sufficiently large n, satisfy the condition (v) of 8.1, that is, $Y_n \notin -\mathsf{hor}(X)$ and $Y_n \notin -\mathsf{vert}(X)$. Since $-\mathsf{vert}(X) = \mathsf{hor}(-X) = H + \mathbb{R}P$ we see that $Y_n \notin -\mathsf{vert}(X)$, for large n. Also, the formula in 4.5 yields $-\mathsf{hor}(X) = H - pP + \mathbb{R}(-2pH + p^2 P - 4Q)$. Thus $Y_n \in -\mathsf{hor}(X)$ if and only if the equations

$$\alpha_n = 1 - 2px, \quad \beta_n = -p + p^2 x, \quad \gamma_n = -4x$$

have a solution $x \in \mathbb{R}$. But since $\alpha_n \to 1$ and $\gamma_n \not\to 0$ such a solution cannot exist for all large n.

It follows that we can apply 8.11 once more, and conclude that $X \diamond Y_n \in U(S)$, so $X \diamond Y = \lim_n X \diamond Y_n \in U(S)$.

Furthermore, if $X \in U_0(I)$ then, by the discussions in Case 1 and Case 2, the diamond products $X \diamond Y_n$ lie in $U(I)$, hence $X \diamond Y \in U(I)$. Similarly, if $Y \in U_0(I)$ then, invoking the results in Case 1 and Case 2, we see that the diamond product $Z \diamond Y$ lies in $U(I)$. Hence we can choose $Y_n \in U_0(I)$ and therefore $X \diamond Y = \lim X \diamond Y_n \in U(I)$.

Summarizing the results established case by case above we now note that for fixed $X, Y \in U_0(S)$ the diamond product $X \diamond Y$ belongs to $U(S)$ if it exists and that $c(X)c(Y) \in \mathsf{Asy}(S)$ otherwise. Passing to limits, this proves (i) and (ii). In the same vein we note that $X \diamond Y \in U(I)$ if $X \diamond Y$ exists and either X or Y lies in $U_0(I)$; if $X \diamond Y$ does not exist then $c(X)c(Y) \in \mathsf{Asy}(I)$. Passage to limits shows that $\mathsf{Asy}(I)$ is a two sided ideal of $\mathsf{Asy}(S)$. But the rectangular bands are simple semigroups (by 4.12), so we conclude $\mathsf{Asy}(S) = \mathsf{Asy}(I)$, and whence $U(S) = c^{-1}(\mathsf{Asy}(S)) = c^{-1}(\mathsf{Asy}(I)) = U(I)$. ∎

9.5. Remark. In the course of the above proof we also have seen that, under the assumptions of the theorem, the set

$$\{X \diamond Y \mid X, Y \in U_0(S) \text{ and } X,\ Y \text{ satisfy the conditions of 8.1}\}$$

is dense in $U(S)$.

9.6. Proposition. *Let $S \subseteq \overline{\exp(\mathsf{Kill}^+)}$ be a subsemigroup of $\mathrm{Sl}(2,\mathbb{R})$. Suppose that the set $U_0(S)$ contains at least two elements and that it satisfies one of the following conditions:*

(i) *$U_0(S)$ is contained in a horizontal line,*

(ii) *$U_0(S)$ is contained in a vertical line,*

(iii) *$U_0(S)$ is not contained in a Borel algebra.*

Then $U(S)$ is a perfect set, i.e., none of its points is isolated.

Proof. Consider an arbitrary element $X \in U_0(S)$. Each of the conditions (i), (ii), and (iii) implies the existence of an element $Y \in U_0(S) \setminus \{X\}$ such that X and Y satisfy one (hence all) of the conditions listed in 8.1. (In particular, the semigroup generated by $\exp(\mathbb{R}^+ X) \cup \exp(\mathbb{R}^+ Y)$ is a subset of $\exp(\mathsf{Kill}^+)$.) Without any loss of generality let us assume that $X = H$. Put $Y = \alpha H + \beta P + \gamma Q$. Since $X \neq Y$ we can also assume that $\beta \neq 0$. (The case $\gamma \neq 0$ is treated analogously.) Fix a positive t such that $\exp(tY) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in S$ and $a \neq 0$. Pick positive reals s and s' such that $\exp(sH)$ and $\exp(s'H)$ lie in S. Then $\mathrm{rlog}(\exp(sH)\exp(tY)\exp(s'H)) \in U_0(S)$. Furthermore, applying formula (†) of 8.8 we obtain that

$$\lim_{s'\to\infty} \lim_{s\to\infty} \mathrm{rlog}(\exp(sH)\exp(tY)\exp(s'H)) = \lim_{s'\to\infty} \left(H + \frac{2be^{-2s'}}{a} P \right) = H.$$

Since $b \neq 0$, the matrix H cannot be an isolated point of $U(S)$, which finishes the proof. ∎

9.7. Example. The semigroup

$$\mathrm{Sl}(2,\mathbb{N}_0) \stackrel{\text{def}}{=} \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{N} \cup \{0\}, ad - bc = 1 \right\}$$

is a discrete submonoid of $\mathrm{Sl}(2,\mathbb{R})^+$, it is the compression semigroup in $\mathrm{Sl}(2,\mathbb{R})$ of the lattice $\mathbb{N} \times \mathbb{N}$. We show that

(i) $\mathrm{Sl}(2,\mathbb{N}_0) \setminus \{\mathbf{1}\}$ *is a free semigroup, generated by the matrices* $x = \exp(P)$ *and* $y = \exp(Q)$*;*

(ii) $U(\mathrm{Sl}(2,\mathbb{N}_0)) = \mathfrak{sl}(2,\mathbb{R})^+ \cap \mathsf{Hyp}$.

Remark. It is well known that the *group* generated by x and y is $\mathrm{Sl}(2,\mathbb{Z})$, which is not a free group (it contains the elements $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -1 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ of order 4 and 3, respectively.)

Proof. (i) Write $F(\mathsf{x},\mathsf{y})$ for the free semigroup in two generators, and write $f\colon F(\mathsf{x},\mathsf{y}) \to \mathrm{Sl}(2,\mathbb{N}_0)$, $w(\mathsf{x},\mathsf{y}) \mapsto w(x,y)$, for the canonical evaluation map. Let $\mathbf{1} \neq s = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sl}(2,\mathbb{N}_0)$. We have to show that $s = f(w)$ for some uniquely defined word $w \in F(\mathsf{x},\mathsf{y})$. We use induction on the number $n = bc$. If $bc = 0$ then $a = d = 1$ and either $b = 0$ or $c = 0$, so either $s = x^b$ or $s = y^c$ and the assertion is obvious. Suppose now that the assertion is true for all matrices $s' = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) \in \mathrm{Sl}(2,\mathbb{N}_0)$ with $0 \le b'c' < bc$.

Case 1: $a \le b$. Then $ad = 1 + bc \le bd$ hence $1 \le b(d-c)$ and thus $c < d$. It follows that for $a \le b$ we have

$$sx^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b-a \\ c & d-c \end{pmatrix} \in \mathrm{Sl}(2,\mathbb{N}_0),$$

whereas

$$sy^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} a-b & b \\ c-d & d \end{pmatrix} \notin \mathrm{Sl}(2,\mathbb{N}_0).$$

By the induction hypothesis there is a unique word w_1 with $f(w_1) = sx^{-1}$. We conclude that $f(w) = s$ for $w = w_1\mathsf{x}$. This representation is unique since there is no word $w' \in F(\mathsf{x},\mathsf{y})$ with $f(w')y = s$.

Case 2: $a > b$. Then $ad = 1 + bc > bd$ and therefore $1 > b(d-c)$, so $0 \ge b(d-c)$. Since $bc \neq 0$ we conclude that $d \le c$, similar to case 1 this implies

$$sy^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} a-b & b \\ c-d & d \end{pmatrix} \in \mathrm{Sl}(2,\mathbb{N}_0).$$

whereas

$$sx^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b-a \\ c & d-c \end{pmatrix} \notin \mathrm{Sl}(2,\mathbb{N}_0).$$

As in case 1 it follows that there is a unique word w with $f(w) = s$.

(ii) Let ρ be a positive rational number. Then there are relatively prime positive integers p, q such that $\rho = \frac{p}{q}$. Remembering elementary number theory we find infinitely many pairs $m, n \in \mathbb{N}$ such that $pm - qn = 1$. The quotients $\frac{n}{m}$ of such pairs (m,n) converge, and $\lim \frac{n}{m} = \lim \frac{pm-1}{qm} = \frac{p}{q} = \rho$. Furthermore,

$$\operatorname{rlog}\begin{pmatrix} m & n \\ q & p \end{pmatrix} = \frac{m-p}{\sqrt{(p+m)^2-4}}H + \frac{2n}{\sqrt{(p+m)^2-4}}P + \frac{2q}{\sqrt{(p+m)^2-4}}Q$$

and for $m \to \infty$ this matrix converges to $H + 2\rho P$. Thus, by continuity, $H + \mathbb{R}_0^+ P \subset U(\mathrm{Sl}(2,\mathbb{N}_0))$. In the same way, or using the anti-automorphism $s \mapsto s^T$ we see that $H + \mathbb{R}_0^+ Q \subset U(\mathrm{Sl}(2,\mathbb{N}_0))$. Similarly, we find

$$\operatorname{rlog}\begin{pmatrix} p & n \\ q & m \end{pmatrix} = \frac{p-m}{\sqrt{(p+m)^2-4}}H + \frac{2n}{\sqrt{(p+m)^2-4}}P + \frac{2q}{\sqrt{(p+m)^2-4}}Q$$

and we see that $-H + \mathbb{R}_0^+ P$ and $-H + \mathbb{R}_0^+ Q$ lie in $U(\mathrm{Sl}(2,\mathbb{N}_0))$. Because of the rectangular structure of $U(\mathrm{Sl}(2,\mathbb{N}_0))$ this implies the assertion. ∎

9.8. Example. The semigroup $S = \mathrm{Sl}(2, \mathbb{N}_0) \cap S_1^1$, where S_1^1 is defined as in 6.12, is a discrete subsemigroup of $\mathrm{Sl}(2, \mathbb{N}_0)$. Moreover, it can be checked easily that for nonnegative integers a, b, c, d with $ad - bc = 1$ the following assertions (i)–(iii) are equivalent:

(i) $a + b \geq c + d$, and $a + c \geq b + d$;

(ii) $(a + b)(a - c) \geq 1$, and $(a + c)(a - b) \geq 1$;

(iii) $a > b$, and $a > c$.

Moreover, if $a = d$ then these inequalities imply $b = c$, hence $a^2 = b^2 + 1$, which means $a = 1$, $b = 0$. A straightforward calculation, along the lines of the preceding example, shows that for $\mathbf{1} \neq x \in \mathrm{Sl}(2, \mathbb{N}_0)$ we have $xS \cap S \neq \emptyset$ if and only if $x \in \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\mathrm{Sl}(2, \mathbb{N}_0)$, and similarly, $Sx \cap S \neq \emptyset$ if and only if $x \in \mathrm{Sl}(2, \mathbb{N}_0)\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$. Thus it follows that

$$S = \{\mathbf{1}\} \cup \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mathrm{Sl}(2, \mathbb{N}_0) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Also, by a well known criterion (cf., e.g., [3] II, Corollary 9.8, p.119), this implies that $S \setminus \{\mathbf{1}\}$ is a free semigroup. (It can be shown that S has infinitely many generators.) Recall from 6.12 that the Lie wedge W_1^1 of S_1^1 is spanned by the four vectors H, $H + 2P$, $H + 2Q$, $2(P + Q)$. By the preceding example we know that $U_0(\mathrm{Sl}(2, \mathbb{N}_0))$ is dense in $\mathfrak{sl}(2, \mathbb{R})^+ \cap \mathsf{Hyp}$, so we conclude that $U_0(S)$ is dense in $W_1^1 \cap \mathsf{Hyp}$, hence $U(S) = W_1^1 \cap \mathsf{Hyp}$.

10. Rectangular domains and umbrella sets

10.1. The umbrella set of a subset in a Lie group. Let S be a subset of a Lie group G with Lie algebra $\mathfrak{g}$. Then we define the *umbrella set* $\mathrm{Umb}(S)$ *of* S as:

$$\mathrm{Umb}(S) \stackrel{\text{def}}{=} \{X \in \mathfrak{g} \mid \exists t_0 > 0 : \ \exp(tX) \in S, \ \forall t > t_0\}.$$

In the present notes S will always be a subsemigroup of G.

The term 'umbrella set' is derived from our situation in $\mathrm{Sl}(2, \mathbb{R})$. If S is open and $S \neq \mathrm{Sl}(2, \mathbb{R})$ then $\mathrm{Umb}(S) = \mathbb{R}^+ U_0(S)$, so $\mathrm{Umb}(S)$ is formed by 'unfolding' the set $U_0(S)$ along the half lines passing through its points, like a 'knirps.' For general semigroups S the set $\mathrm{Umb}(S)$ is not fit to serve as an asymptotic object, but its closure $\overline{\mathrm{Umb}}(S)$ is.

10.2. Remark. The above definition provides a variant of LAWSON's 'horizon' concept ([10], p.21). If S is a subsemigroup with nonempty interior of a Lie group G then LAWSON defines the *horizon of* S as the set

$$\mathcal{L}_\infty(S) \stackrel{\text{def}}{=} \{X \in \mathfrak{g} \mid \exp(tX) \in \mathrm{Int}\, S \text{ for some } t > 0\}.$$

It is not difficult to see that $\mathcal{L}_\infty(S) = \mathrm{Umb}(\mathrm{Int}\, S)$ (see the discussion below). This observation as well as other elementary properties of $\mathrm{Umb}(S)$ in the case of open semigroups S are contained (at least implicitly) in [10]. The major reason for our slight deviation from LAWSON's concept is that we are interested also in one parameter subsemigroups meeting boundary points of S and in semigroups S with $\mathrm{Int}\, S = \emptyset$.

10.3. Properties of $\mathrm{Umb}(S)$.

(i) Obviously $\mathrm{Umb}(S)$ is closed under multiplication with positive scalars, but except for the case $[X,Y] = 0$ it is not clear from the definition that $X, Y \in \mathrm{Umb}(S)$ implies that $X + Y \in \mathrm{Umb}(S)$. In fact, examples show that in the general context $\mathrm{Umb}(S)$ need not be additively closed, even if S is open and G is nilpotent (cf. [11],[12]). Also, if S is closed then $\mathrm{Umb}(S)$ need not be a closed subset of $\mathfrak{g}$. To see this, consider, for example, the subsemigroup $T = \{(x,y) \in \mathbb{R}^2 \mid y \geq 1\}$ of $\mathbb{R}^2$. A simple computation shows that $\mathrm{Umb}(T) = \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$.

(ii) Suppose that $\exp(X) \in \mathrm{Int}\, S$. Then there exist a neighborhood U of X in $\mathfrak{g}$ and a positive number T such that $\exp(tY) \in \mathrm{Int}\, S$ for all $t \geq T$ and all $Y \in U$. In particular, $U \subseteq \mathrm{Umb}(\mathrm{Int}\, S)$. This also shows that $\mathrm{Umb}(S)$ is open whenever S is open.

Proof of (ii): Since exp is continuous there is an $\varepsilon > 0$ and a neighborhood U of X in $\mathfrak{g}$ such that $\exp([1, 1+\varepsilon] \cdot U) \subseteq \mathrm{Int}\, S$. Since $\mathrm{Int}\, S$ is a subsemigroup, we have $\exp([n, n(1+\varepsilon)] \cdot Y) \subseteq \mathrm{Int}\, S$ for each $n \in \mathbb{N}$ and each $Y \in U$. But if $n \in \mathbb{N}$ with $n \geq \frac{1}{\varepsilon}$ then $\exp([n, n+1]) \cdot U) \subseteq \mathrm{Int}\, S$, so the assertion holds with $T = \frac{1}{\varepsilon} + 1$. ■

(iii) If S is an open subsemigroup of $\mathrm{Sl}(2,\mathbb{R})$ and $S \neq \mathrm{Sl}(2,\mathbb{R})$ then $\mathrm{Umb}(S) = \mathbb{R}^+ U_0(S)$.

We will now show that in the case of $G = \mathrm{Sl}(2,\mathbb{R})$ umbrella sets of subsemigroups have "nice" properties. We start with the remarkable fact that for open connected subsemigroups S the set $U_0(S)$ is a rectangular domain.

10.4. Theorem. *Let S be an open connected subsemigroup of $\mathrm{Sl}(2,\mathbb{R})$ with $S \neq \mathrm{Sl}(2,\mathbb{R})$. Then $S \subseteq \exp(\mathsf{Kill}^+)$ and $D = U_0(S)$ is a rectangular domain.*

Proof. The inclusion $S \subseteq \exp(\mathsf{Kill}^+)$ follows from 3.4(iii). Also, D is open, since S is open and exp is a homeomorphism on Kill^+, and D is connected, since S is connected and rlog is continuous. We know already from 9.4 that the closure $\overline{c(D)}$ of $c(D)$ in Hyp^- is a subsemigroup of the rectangular band Hyp^-. Thus $\overline{c(D)} = \overline{\mathsf{hor}(D)} \times \overline{\mathsf{vert}(D)}$. Furthermore, 8.9 implies that the diamond product $X \diamond Y$ exists for every $X, Y \in D$, hence $\overline{\mathsf{hor}(D)}$ and $\overline{\mathsf{vert}(D)}$ are two proper arcs. We conclude that $\mathrm{Int}\left(\overline{c(D)}\right) = \mathsf{hor}(D) \times \mathsf{vert}(D)$, so $\mathrm{Int}\left(\overline{c(D)}\right)$ contains no nilpotency points. This yields that $\mathrm{Int}(\overline{D}) = c^{-1}\big(\mathrm{Int}\left(\overline{c(D)}\right)\big)$ is a $\diamond$-semigroup.

Thus all we need to show is that D is the interior of $\overline{D}$. Suppose that there exists an interior point of $\overline{D}$ which does not belong to D. Applying a suitable inner automorphism of $\mathfrak{sl}(2,\mathbb{R})$ we enforce that this point is H.

Since H is an interior point of $\overline{D}$ the horizontal line $\mathsf{hor}(H) = H + \mathbb{R}P$ dissects $\overline{D}$, that is to say, $\overline{D} \setminus H + \mathbb{R}P$ is the union of two disjoint nonvoid open subsets. Since D is connected and dense in $\overline{D}$ this means that $D \cap \mathsf{hor}(H) \neq \emptyset$. Similarly, $D \cap \mathsf{vert}(H) \neq \emptyset$. Thus there exist points X and Y in D with $X \diamond Y = H$, we write $X = H + \beta P$, $Y = H + \gamma Q$. We claim that X and Y can be chooser so that β and γ have the same sign.

Indeed, if this is not the case then there exists a nonzero real number β such that $H + \beta P \in D$ but $H - \mathbb{R}^+\beta P \cap D = H + \mathbb{R}^+\beta Q \cap D = \emptyset$. This means, however, that D does not meet the union $H - \mathbb{R}^+\beta P \cup \{H\} \cup H + \mathbb{R}^+\beta Q$. Since the latter set also dissects $\overline{D}$ we therefore arrive at a contradiction.

The inner automorphism $\varphi = e^{\operatorname{ad} tH}$, with $t = \frac{1}{4}(\log|\gamma| - \log|\beta|)$ satisfies $\varphi(H) = H$, $\varphi(|\beta|P) = \sqrt{\beta\gamma}P$ and $\varphi(|\gamma|Q) = \sqrt{\beta\gamma}Q$. Applying φ we thus enforce that $\beta = \gamma$, so Y is the transpose of X. Since D is open the matrix

$$X_\varepsilon = X + \varepsilon(H + \frac{\beta}{2}P - \frac{2}{\beta}Q) = (1+\varepsilon)H + (1 + \frac{\varepsilon}{2})\beta P - \varepsilon\frac{2}{\beta}Q$$

and its transpose

$$Y_\varepsilon = Y + \varepsilon(H - \frac{2}{\beta}P + \frac{\beta}{2}Q) = (1+\varepsilon)H - \varepsilon\frac{2}{\beta}P + (1 + \frac{\varepsilon}{2})\beta Q$$

are also in D for all sufficiently small $\varepsilon \geq 0$. We know from 10.3 that there exist positive reals t_0 and ε_0 such that $\exp(tX_\varepsilon)$ and $\exp(tY_\varepsilon)$ lie in S for $t \geq t_0$ and $\varepsilon \in [0, \varepsilon_0]$. We now fix $t \geq t_0$ such that $\frac{\cosh(t)}{\sinh(t)} - 1 < \varepsilon_0$. A straightforward calculation using our formulas for exp shows that

$$\exp(tX_\varepsilon)\exp(tY_\varepsilon) = \begin{pmatrix} a & b \\ b & d \end{pmatrix}, \quad \text{where}$$

$$b = \beta\sinh^2(t)\Big\{\frac{-2\varepsilon}{\beta^2}\left(\frac{\cosh(t)}{\sinh(t)} + 1 + \varepsilon\right) + (1 + \frac{\varepsilon}{2})\left(\frac{\cosh(t)}{\sinh(t)} - 1 - \varepsilon\right)\Big\}.$$

If in this formula we put $\varepsilon = 0$ then b and β have the same sign, whereas for $\varepsilon = \frac{\cosh(t)}{\sinh(t)} - 1$ these numbers have the opposite sign. It follows that $b = 0$ for some ε in the interval $(0, \frac{\cosh(t)}{\sinh(t)} - 1)$, so $\exp\mathbb{R}^+H$ meets S and therefore $H \in D$, a contradiction. This finishes our proof. ■

10.5. Theorem. *Let* $S \neq \mathrm{Sl}(2,\mathbb{R})$ *be a subsemigroup of* $\mathrm{Sl}(2,\mathbb{R})$ *with dense and connected interior. Then the following assertions hold:*

(i) $\overline{\mathrm{Umb}}(S)$ *is a Lie semialgebra and its exponential image* $\exp(\overline{\mathrm{Umb}}(S))$ *is a closed semigroup;*

(ii) $\mathrm{Umb}(\mathrm{Int}\, S)$ *is the algebraic interior of* $\overline{\mathrm{Umb}}(S)$*;*

(iii) *The set* $S_* = \{s \in \mathrm{Sl}(2,\mathbb{R}) \mid s^n \in \operatorname{Int} S$ *for some power* $n \in \mathbb{N}\}$ *is an open semigroup. More specifically, we have*

$$S_* = \exp(\operatorname{Umb}(\operatorname{Int} S)) \cup -\exp(\operatorname{Umb}(\operatorname{Int} S)),$$

$$\overline{S_*} = \overline{\{s \in \mathrm{Sl}(2,\mathbb{R}) \mid \mathbf{1} \neq s^n \in S \textit{ for some power } n \in \mathbb{N}\}}.$$

Proof. We first note that $S \subseteq \exp(\overline{\mathsf{Kill}^+})$, by 3.4(iii).

(i) Since S has dense interior we see that $\operatorname{Umb}(\operatorname{Int} S)$ is dense in $\overline{\operatorname{Umb}}(S)$. Furthermore, $\operatorname{Umb}(\operatorname{Int} S) = \mathbb{R}^+ \operatorname{rlog}(\operatorname{Int} S)$ by 10.3(iii), and $\operatorname{rlog}(\operatorname{Int} S)$ is a rectangular domain by 10.4. Combining this with 5.5(i) we get assertion (i).

(ii) Since for a convex open subset C of a finite dimensional vector space we always have $\operatorname{Int}(\overline{C}) = C$ it follows that $\operatorname{Int}(\overline{\operatorname{Umb}}(S)) = \operatorname{Int}(\overline{\operatorname{Umb}}(\operatorname{Int} S)) = \operatorname{Umb}(\operatorname{Int} S)$.

(iii) We know from (i) and (ii) that $\exp(\operatorname{Umb}(\operatorname{Int} S)) \cup -\exp(\operatorname{Umb}(\operatorname{Int} S))$ is an open semigroup. Thus we are left to show that the two equalities hold. In order to prove the first one it suffices to show the inclusion $S_* \subseteq \exp(\operatorname{Umb}(\operatorname{Int} S)) \cup -\exp(\operatorname{Umb}(\operatorname{Int} S))$. Pick $s \in S_*$, and let $n \in \mathbb{N}$ with $s^n \in \operatorname{Int} S$. We suppose that $s \in \exp(\mathfrak{sl}(2,\mathbb{R}))$. By 2.2(ii) and since $(s \in S_*) \iff (-s \in S_*)$, this will inflict no loss of generality. Write $s = \exp(X)$. Then $s^n = \exp(nX) \in \operatorname{Int} S$, so by 10.3(ii) $X \in \operatorname{Umb}(\operatorname{Int} S)$, hence $s = \exp X \in \exp(\operatorname{Umb}(\operatorname{Int} S))$.

To show the second identity we only have to prove that

$$\{s \in \mathrm{Sl}(2,\mathbb{R}) \mid \mathbf{1} \neq s^n \in S \text{ for some power } n \in \mathbb{N}\} \subseteq \overline{S_*}.$$

Pick $s \in \mathrm{Sl}(2,\mathbb{R})$ such that $\mathbf{1} \neq s^n \in S$ for a suitable $n \in \mathbb{N}$. Observe that $(-s)^{2n} = s^{2n}$ also lies in $S \setminus \{\mathbf{1}\}$. As before we therefore may and do suppose that $s = \exp(X)$ with $0 \neq X \in \mathfrak{sl}(2,\mathbb{R})$. Now X cannot lie in Kill^- (since $S \subset \exp(\overline{\mathsf{Kill}^+})$ and $s^n \neq \mathbf{1}$), so $s = \exp(X) \in \exp(\overline{\mathsf{Kill}^+})$. Let U be an open neighborhood of s in $\exp(\overline{\mathsf{Kill}^+})$. Define $p_n\colon \exp(\overline{\mathsf{Kill}^+}) \to \exp(\overline{\mathsf{Kill}^+})$, $x \mapsto x^n$. Since the restriction of $\exp$ to $\overline{\mathsf{Kill}^+}$ is a homeomorphism, the map p_n is a homeomorphism, and therefore $p_n(U)$ is an open neighborhood of s^n in $\exp(\overline{\mathsf{Kill}^+})$. Since $\operatorname{Int} S$ is dense in S we therefore find an element $s_* \in U$ with $p_n(s_*) \in \operatorname{Int} S$. By definition, $s_* \in U \cap S_*$ and we conclude that $s_* \in \overline{S_*}$. The assertion follows. ∎

10.6. Corollary. *Let* S *be an open subsemigroup of* $\mathrm{Sl}(2,\mathbb{R})$ *with* $S \neq \mathrm{Sl}(2,\mathbb{R})$. *Then there are countably or finitely many Lie semialgebras* W_j, $j \in J$, *with nonvoid interior and contained in* $\overline{\mathsf{Kill}^+}$, *such that*

$$\operatorname{Umb}(S) = \bigcup_{j \in J} \operatorname{Int} W_j.$$

Proof. Note first that $\operatorname{Umb}(S) = \operatorname{Umb}(S \cap \exp(\mathsf{Kill}^+))$. By Proposition 3.5 we know that $S \cap \exp(\mathsf{Kill}^+)$ is the union of the semigroups

$$S_c(x) = \bigcup_{k \in \mathbb{N}} C(x^k),\ x \in S \cap \exp(\mathsf{Kill}^+),$$

where $C(x)$ denotes the connected component of x in S. (Since S is open, the family $\{C(x)\}$ is countable or finite.) Furthermore, for every $x \in S \cap \exp(\mathsf{Kill}^+)$ there is an index n such that $\mathrm{Umb}(S_c(x)) = \mathrm{Umb}(C(x^n))$. Now our assertion is an immediate consequence of Theorem 10.5. ∎

10.7. Remark. Let S be a subsemigroup of $\mathrm{Sl}(2,\mathbb{R})$ with dense interior and assume that $S \subset \overline{\exp(\mathsf{Kill}^+)}$. Then $\overline{\mathrm{Umb}}(S) = \overline{\mathrm{Umb}}(I)$ for every two sided ideal I in S. This can be seen from the following chain of inclusions and identities (see also Theorem 9.4)

$$\begin{aligned}\overline{\mathrm{Umb}}(S) \supseteq \overline{\mathrm{Umb}}(I) \supseteq \overline{\mathrm{Umb}}(\mathrm{Int}\, I) = \overline{\mathbb{R}^+ U_0(\mathrm{Int}\, I)} = \overline{\mathbb{R}^+ U(\mathrm{Int}\, I)} \\ = \overline{\mathbb{R}^+ U(S)} = \overline{\mathrm{Umb}}(S).\end{aligned}$$

Further generalizations to arbitrary subsemigroups $S \neq \mathrm{Sl}(2,\mathbb{R})$ with dense interior are left to the reader.

10.8. Theorem. *Let S be a Lie subsemigroup of $\mathrm{Sl}(2,\mathbb{R})$ with non-empty interior and let W be its Lie wedge. Then $\overline{\mathrm{Umb}}(S)$ is the semialgebra generated by W, i.e.,*

$$\overline{\mathrm{Umb}}(S) = \bigcap\{W_1 \subseteq \mathfrak{sl}(2,\mathbb{R}) \mid W \subseteq W_1 \textit{ and } W_1 \textit{ is a semialgebra}\}.$$

Proof. Denote the intersection

$$\bigcap\{W_1 \subseteq \mathfrak{sl}(2,\mathbb{R}) \mid W \subseteq W_1 \text{ and } W_1 \text{ is a semialgebra}\}$$

by $\widetilde{W}$. Obviously, $\mathrm{Umb}(S)$ contains W. We know by Theorem 10.5 that $\overline{\mathrm{Umb}}(S)$ is a semialgebra. Thus $\widetilde{W} \subseteq \overline{\mathrm{Umb}}(S)$. Since $\widetilde{W}$ is a Lie semialgebra contained in $\overline{\mathsf{Kill}^+}$, its exponential image $\exp \widetilde{W}$ is an exponential subsemigroup of $\mathrm{Sl}(2,\mathbb{R})$. On the other hand, the inclusion $W \subseteq \widetilde{W}$ implies that $S \subseteq \exp(\widetilde{W})$, hence $\mathrm{Umb}(S) \subseteq \mathrm{Umb}(\exp(\widetilde{W})) = \widetilde{W}$. Thus $\overline{\mathrm{Umb}}(S) \subseteq \widetilde{W}$. ∎

10.9. Remark. The Lie semialgebra V generated by a wedge W with $W \subseteq \overline{\mathsf{Kill}^+}$, and not contained in a single Borel algebra, can be computed comfortably as the intersection of all closed half spaces bounded by a Borel algebra which contain W. Alternatively: $V = \overline{\mathbb{R}^+ \cdot M}$, where

$$M = \{X \diamond Y \mid X, Y \in W \cap \mathsf{Hyp} \text{ and both } X \diamond Y \text{ and } Y \diamond X \text{ exist}\}.$$

To see this, notice that the algebraic interior W_0 of W lies in Kill^+ and that $W_0 \cap \mathsf{Hyp}$ is connected, hence $D = (W_0 \cap \mathsf{Hyp}) \diamond (W_0 \cap \mathsf{Hyp})$ is a rectangular domain. By 5.5(i) we know that $\overline{\mathbb{R}^+ D}$ is a Lie semialgebra. Since it contains W as well as M this implies the assertion.

10.10. Example. Let W be the wedge spanned by the three vectors $A = H+2P$, $B = H+2Q$, and H. Then W is not a semialgebra, since $[A,B] = 4H-4P-4Q$ does not belong to the span of A, B. The Lie semialgebra generated by W is spanned by the four vectors A, B, H, and $B \diamond A = P+Q$. This semialgebra is the intersection of $\mathfrak{sl}(2,\mathbb{R})^+$ with its conjugate $\varphi(\mathfrak{sl}(2,\mathbb{R})^+)$, where φ is the inner automorphism which induces a rotation of $\mathfrak{sl}(2,\mathbb{R})^+$ about the axis $\mathbb{R}(P-Q)$ by $\pi/2$.

10.11. Proposition. *Let W be the Lie wedge of a three dimensional Lie subsemigroup S of* $\mathrm{Sl}(2,\mathbb{R})$. *Then W,* $\mathrm{Umb}(S)$, *and* $\overline{\mathrm{Umb}}(S)$ *contain the same nilpotent matrices:*

$$\overline{\mathrm{Umb}}(S) \cap \mathsf{Kill}^0 = W \cap \mathsf{Kill}^0 = \mathrm{Umb}(S) \cap \mathsf{Kill}^0 .$$

Proof. Since $W \subseteq \mathrm{Umb}(S)$ it suffices to show the equality $\overline{\mathrm{Umb}}(S) \cap \mathsf{Kill}^0 = W \cap \mathsf{Kill}^0$. Also, by [5] V.4.23 (p.418) all Lie wedges in $\mathfrak{sl}(2,\mathbb{R})$ with nontrivial edge are conjugate to $\mathfrak{sl}(2,\mathbb{R})^+$, so our assertion is trivial if W is not pointed.

Thus let us assume that W is pointed.

Step 1. We first show that if W does not contain nonzero nilpotent elements then neither does $\overline{\mathrm{Umb}}(S)$. Assume W to be free of nonzero nilpotent elements. Then $W \setminus \{0\} \subseteq \mathsf{Kill}^+$. Let $g\colon \mathsf{Kill}^+ \to \mathsf{Hyp}$, $X \mapsto (1/\Delta(X))X$. The set $M \overset{\text{def}}{=} W \cap \mathsf{Hyp} = g(W \setminus \{0\})$ is compact (since $W \setminus \{0\} \subseteq \mathsf{Kill}^+$) and connected. Since $W \setminus \{0\} \subseteq \mathsf{Kill}^+$ we see that for any two points $X, Y \in M$ the line segment $\operatorname{conv}\{X, Y\}$ does not meet the light cone, so Theorem 8.9 implies that all diamond products $X \diamond Y$ with $X, Y \in M$ exist. Using Theorem 10.8 and Remark 10.9 we conclude that $\overline{\mathrm{Umb}}(S) = \overline{\mathbb{R}_0^+(M \diamond M)}$. Since M is compact we therefore see that $\overline{\mathrm{Umb}}(S) = \mathbb{R}_0^+(M \diamond M)$, and that $\overline{\mathrm{Umb}}(S)$ indeed contains no nonzero nilpotent elements.

Step 2. Next we suppose that $\overline{\mathrm{Umb}}(S)$ contains a nilpotent element N which is, however, not contained in W. To simplify our arguments we assume, not losing generality, that $W \subseteq \mathfrak{sl}(2,\mathbb{R})^+$ and $N = Q$. By 10.9 this means that there exist elements $X_n, Y_n \in W \cap \mathsf{Hyp}$ and numbers $\lambda_n \in \mathbb{R}^+$ such that $Q = \lim \lambda_n(X_n \diamond Y_n)$. Also, remembering step 1, we know that $P \in W$. By 4.10

$$(*) \qquad \lim_n c(X_n \diamond Y_n) = (\mathsf{hor}(-H), -\mathsf{hor}(-H)).$$

We claim that a suitable subsequence of $\langle X_n \rangle$ converges. If not then we can find a subsequence $\langle X_m \rangle$ of $\langle X_n \rangle$ and positive numbers λ_m with $\lambda_m \to 0$ such that $\langle \lambda_m X_m \rangle$ converges to a nonzero element. This element must be nilpotent and is contained in W, hence $\lim_m \lambda_m X_m = P$. Invoking 4.10 once more, we see that $\lim_m \mathsf{hor}(X_m) = \mathsf{hor}(H)$, a contradiction to $(*)$.

Thus $\langle X_n \rangle$ has a convergent subsequence $\langle X_k \rangle$, and, by the same argument, $\langle Y_n \rangle$ has a convergent subsequence $\langle Y_k \rangle$. By $(*)$ $\lim_k c(X_k, Y_k) = (\mathsf{hor}(-H), \mathsf{vert}(H))$, therefore

$$\mathsf{hor}(\lim_k X_k) = \mathsf{hor}(-H), \quad \mathsf{vert}(\lim_k Y_k) = \mathsf{vert}(H).$$

It follows that $A = \lim_k (X_k + Y_k) \in \mathbb{R}Q \cap W \subseteq \mathbb{R}_0^+ Q$. If $A = 0$ then $0 \neq \lim_k X_k = -\lim_k Y_k \in W$, a contradiction to our assumption that W is pointed. If $A \neq 0$ then $Q \in W$, contrary to our assumption in step 2. ∎

10.12. Example. We compute $\overline{\mathrm{Umb}}(S)$ and $\mathrm{Umb}(S)$ of the Lie semigroup $S = \mathrm{Sl}(2,\mathbb{R})^{++}$ of Example 6.5. The Lie wedge of S is $W = \mathbb{R}_0^+ H + \mathbb{R}_0^+ P + \mathbb{R}_0^+ Q$. Then Theorem 10.8 shows that $\overline{\mathrm{Umb}}(S)$ is the Lie semialgebra generated by W. Since W contains the linearly independent nilpotent elements P and Q we conclude that $\overline{\mathrm{Umb}}(S) = \mathfrak{sl}(2,\mathbb{R})^+$. For $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \exp(-\alpha H + \beta P)$ with $\alpha > 0, \beta \in \mathbb{R}$ we do not have $a \geq 1$, so $-H + \mathbb{R}P \cap \mathrm{Umb}(S) = \emptyset$. Similarly, $-H + \mathbb{R}Q \cap \mathrm{Umb}(S) = \emptyset$. Thus $\mathrm{Umb}(S) = \mathrm{Umb}(\mathrm{Int}\, S) \cup W$.

10.13. Example. Consider the Lie semigroup $S = \mathrm{Sl}(2,\mathbb{R})^{++,+-}$ of Example 6.6. The Lie wedge of S is $W = \mathbb{R}_0^+ P + \mathbb{R}_0^+ Q$. Since S has interior points Theorem 10.8 applies and shows that $\overline{\mathrm{Umb}}(S) = \mathfrak{sl}(2,\mathbb{R})^+$, which can be verified easily by direct computation as well. Thus every one parameter subsemigroup $\exp \mathbb{R}^+ X$ with $X \in \mathrm{algint}(\mathfrak{sl}(2,\mathbb{R})^+)$ meets the interior of S. However $H \notin \mathrm{Umb}(S)$, so $\mathrm{Umb}(S)$ is not closed. The only upper [lower] triangular matrices in S are the unipotent elements $\exp(tP)$ [$\exp(tQ)$], with $t \geq 0$. Thus $\mathrm{Umb}(S) = \mathrm{Umb}(\mathrm{Int}\, S) \cup W$. Note that W is only two dimensional; nevertheless the asymptotic behavior of the interior of $\mathrm{Sl}(2,\mathbb{R})^{++,+-}$ is the same as that of the interior of the apparently 'larger' semigroup $\mathrm{Sl}(2,\mathbb{R})^+$.

Note that, in accordance with Proposition 10.11, W and $\mathrm{Umb}(S)$ contain the same nilpotent elements.

References

[1] Breckner, B., "Compactifications of subsemigroups of Lie groups," PhD Thesis, Darmstadt 1998. Shaker Verlag 1998.

[2] Brumfiel, G. W., "Partially ordered rings and semi-algebraic geometry," Cambridge Univ. Press, 1979.

[3] Clifford, A. H., and G. B. Preston, "The algebraic Theory of semigroups I, II," AMS Mathematical surveys 7, 1961, 1967.

[4] Engelking, R., "General Topology," Polish Scientific Publishers, 1977.

[5] Hilgert, J., K. H. Hofmann, and J. D. Lawson, "Lie groups, Convex Cones, and Semigroups." Oxford University Press, 1989.

[6] Hilgert, J., and K.-H. Neeb, "Lie Semigroups and their Applications," Springer LNM **1552** (1993).

[7] —, *Maximality of compression semigroups,* Semigroup Forum **50** (1995), 205–222.

[8] Hofmann, K. H., and W. A. F. Ruppert, "Lie Groups and Subsemigroups with Surjective Exponential Function," Memoirs of the AMS **130** (1997), No. 618.

[9] Lawson, J. D., *Maximal semigroups of Lie groups that are total,* Proc. Edinburgh Math. Soc. **30** (1987), 479–501.

[10] —, *Maximal Ol'shanskiĭ Semigroups,* J. Lie Theory **4**(1994), 17–29.

[11] Mittenhuber, D., *Spacious Lie groups,* J. Lie Theory **5**(1995), 135–146.

[12] Mittenhuber, D., *The $\mathcal{L}_\infty$ problem for a special series of nilpotent Lie algebras,* Preprint (1996).

[13] —, *Compression semigroups in semisimple Lie groups: a direct approach,* in: Hilgert, J., Lawson, J.D., Neeb, K.H., and E.B. Vinberg (eds), "Positivity in Lie Theory: Open Problems" DeGreuyter Expositions in Mathematics **26** (1998), 133–145.

[14] Neeb, K.-H., "Invariant Subsemigroups of Lie groups." Memoirs of the AMS **104** (1993), No. 499.

[15] Ruppert, W. A. F., *Bohr compactifications of subsemigroups of non-abelian Lie groups,* Semigroup Forum **37** (1988), 325–342.

[16] Ruppert, W. A. F., *On open subsemigroups of connected groups,* Semigroup Forum **39** (1989), 347–362.

[17] Ruppert, W. A. F., and B. E. Breckner, *Bohr compactifications of subsemigroups of* $\mathrm{Sl}(2,\mathbb{R})$*,* (forthcoming).

[18] San Martin, L. A. B., *Maximal semigroups in semisimple Lie groups,* to appear.

[19] San Martin, L. A. B., and P. A. Tonelli, *Semigroup actions on homogeneous spaces,* Semigroup Forum **50** (1995), 59–88.

B. Breckner
Babeş-Bolyai University
Faculty of Mathematics and
Computer Science
Str. M. Kogălniceanu 1
RO-3400 Cluj-Napoca
Romania
brigitte@math.ubbcluj.ro

and

Institut für Mathematik und
Angewandte Statistik
Universität für Bodenkultur
Peter Jordanstr. 82
A-1190 Wien
Austria
breckner@edv1.boku.ac.at

W. A. F. Ruppert
Institut für Mathematik und
Angewandte Statistik
Universität für Bodenkultur
Peter Jordanstr. 82
A-1190 Wien
Austria
ruppert@edv1.boku.ac.at

Received August 7, 2000
and in final form March 8, 2001