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Subquotients in the Enveloping Algebra of a Nilpotent Lie Algebra

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Abstract. For any triple $(\mathfrak{g}, \mathfrak{h}, f)$ where $\mathfrak{g}$ is a nilpotent Lie algebra over a field $\mathbf{k}$ of characteristic zero, $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$, and f is a homomorphism of $\mathfrak{u}(\mathfrak{h})$ onto $\mathbf{k}$, a subquotient $\mathcal{D}(\mathfrak{g}, \mathfrak{h}, f)$ of $\mathfrak{u}(\mathfrak{g})$ is studied which generalizes the algebra of invariant differential operators on a nilpotent homogeneous space. A generalized version of a conjecture of Corwin and Greenleaf is formulated using geometry of $\exp(ad^*\mathfrak{h})$-orbits in the variety L_f of linear functionals in $\mathfrak{g}^*$ whose restriction to $\mathfrak{h}$ agree with f. Certain constructions lead to a procedure by which the question of non-commutativity of $\mathcal{D}(\mathfrak{g}, \mathfrak{h}, f)$ is reduced to a case where $(\mathfrak{g}, \mathfrak{h}, f)$ has a special structure. This reduction is then used to prove that the Corwin-Greenleaf conjecture about non-commutativity of $\mathcal{D}(\mathfrak{g}, \mathfrak{h}, f)$ holds in certain situations, in particular when the $\exp(ad^*\mathfrak{h})$-orbits in L_f have dimension no greater than one.

1. Introduction

Given a real nilpotent Lie group G, an analytic subgroup H of G, and a unitary character χ of H, it is known (see [4]) that the corresponding induced representation τ of G can be of two types: uniformly bounded multiplicity a.e. (abbrev. FM), or uniformly infinite multiplicity a.e. (abbrev. IM). In the FM case, it is known that the algebra of τ-invariant differential operators is commutative, and much evidence has accumulated that indicates that the converse is true, that is, that in the IM case, there are two invariant differential operators that do not commute with each other.

In this paper we study this conjecture from a purely algebraic/geometric standpoint. We begin with a nilpotent Lie algebra $\mathfrak{g}$ over a field $\mathbf{k}$ of characteristic zero, a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, and an algebra homomorphism $f : \mathfrak{u}(\mathfrak{h}) \to \mathbf{k}$. The spectral variety $L_f \subset \mathfrak{g}^*$ is the set of all linear functionals on $\mathfrak{g}$ whose restriction to $\mathfrak{h}$ agrees with f. The triple $(\mathfrak{g}, \mathfrak{h}, f)$ is defined to be an IM-triple if the generic dimension of the coadjoint orbits in $\mathfrak{g}^*$ that meet L_f exceeds twice the dimension of the $\exp(ad^*\mathfrak{h})$-orbits in L_f; of course this reduces to the well-known Corwin-Greenleaf condition when $\mathbf{k} = \mathbf{C}$ and f is the differential of a

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unitary character on the real analytic subgroup H. The algebra of "generalized" invariant differential operators is the subquotient $\mathcal{D}(\mathfrak{g},\mathfrak{h},f) = c(\mathfrak{h},\mathcal{I})/\mathcal{I}$ in the enveloping algebra $\mathfrak{u}(\mathfrak{g})$ where $\mathcal{I}$ is the left ideal in $\mathfrak{u}(\mathfrak{g})$ generated by the kernel of f and $c(\mathfrak{h},\mathcal{I}) = \{W \in \mathfrak{u}(\mathfrak{g}) : [W,\mathfrak{h}] \subset \mathcal{I}\}$. We make the following.

Conjecture 1. Let $(\mathfrak{g},\mathfrak{h},f)$ be an IM-triple with $\mathfrak{g}$ nilpotent. Then $\mathcal{D}(\mathfrak{g},\mathfrak{h},f)$ is not commutative.

A standard way to analyse the algebra $\mathcal{D}(\mathfrak{g},\mathfrak{h},f)$ has been to take a sequence of subalgebras $\mathfrak{h} = \mathfrak{h}_0 \subset \mathfrak{h}_1 \subset \cdots \subset \mathfrak{h}_m = \mathfrak{g}$ where dimensions increase one at a time, and in trying to prove Conjecture 1, one quickly reduces to the situation where $(\mathfrak{g},\mathfrak{h},f)$ is an IM-triple while $(\mathfrak{h}_{m-1},\mathfrak{h},f)$ is an FM-triple. Greenleaf in [7], and Fujiwara, Lion, and Mehdi in [6] have shown in this situation (and with $\mathbf{k} = \mathbf{C}$ and f the differential of a unitary character), that $\mathcal{D}(\mathfrak{g},\mathfrak{h},f) \neq \mathcal{D}(\mathfrak{h}_{m-1},\mathfrak{h},f)$ implies non-commutativity of $\mathcal{D}(\mathfrak{g},\mathfrak{h},f)$.

Our attack on Conjecture 1 also involves a reduction, although with a bit more algebraic information. First we introduce a class of triples $(\mathfrak{g},\mathfrak{h},f)$ of a special form: those for which there is an ideal $\mathfrak{z}$ in $\mathfrak{g}$ and an element $Y \in \mathfrak{g}$ with the properties that $[\mathfrak{g},\mathfrak{z}] \subset \mathcal{I}$, $[\mathfrak{g},Y] \subset \mathfrak{z}$, and that the centralizer $\mathfrak{g}^Y$ of Y modulo $\mathcal{I}$ is codimension one in $\mathfrak{g}$. We say that triples with a pair $(\mathfrak{z},Y)$ as just described admit a "Kirillov structure". What is the utility of this notion? First of all, this sort of structure occurs naturally: after laying out basic algebraic framework in Section 2, Section 3 is devoted to showing that for any triple $(\mathfrak{g},\mathfrak{h},f)$, one can construct an associated triple $(\hat{\mathfrak{g}},\hat{\mathfrak{h}},\hat{f})$ that admits a Kirillov structure, and for which all the necessary information about orbit dimensions and $\mathcal{D}(\mathfrak{g},\mathfrak{h},f)$ is preserved. The tradeoff is that the scaler field is extended (in fact to a subfield of the invariant rational functions on L_f) when passing to the associated triple; thus the necessity of considering arbitrary fields of characteristic zero. Secondly, suppose that we are given that a triple $(\mathfrak{g},\mathfrak{h},f)$ has a Kirillov structure, that $\mathfrak{h} \subset \mathfrak{g}^Y$, and that $(\mathfrak{g}^Y,\mathfrak{h},f)$ is an FM-triple. Then it is extremely easy to show that $\mathcal{D}(\mathfrak{g},\mathfrak{h},f) \neq \mathcal{D}(\mathfrak{g}^Y,\mathfrak{h},f)$ (or equivalently $c(\mathfrak{h},\mathcal{I}) \not\subset \mathfrak{u}(\mathfrak{g}^Y) + \mathcal{I}$) implies non-commutativity of $\mathcal{D}(\mathfrak{g},\mathfrak{h},f)$. Finally, and perhaps most significantly, suppose that we are given that a triple $(\mathfrak{g},\mathfrak{h},f)$ has a Kirillov structure and that $\mathfrak{h} \not\subset \mathfrak{g}^Y$. Setting $\mathfrak{h}^Y = \mathfrak{h} \cap \mathfrak{g}^Y$, a crucial result (Proposition 4.1.1) is that $\mathcal{D}(\mathfrak{g},\mathfrak{h},f)$ is commutative if and only if $\mathcal{D}(\mathfrak{g}^Y,\mathfrak{h}^Y,f)$ is commutative. Thus, in the presence of a Kirillov structure one can always pass to the triple $(\mathfrak{g}^Y,\mathfrak{h}^Y,f)$, whether $\mathfrak{h} \subset \mathfrak{g}^Y$ or not.

Now given an IM-triple $(\mathfrak{g},\mathfrak{h},f)$, the process of passing to an associated triple, and then to a codimension one centralizer, allows the construction of a sequence of triples $\{(\mathfrak{g}(j),\mathfrak{h}(j),f(j))\}_{j=0}^N$, where

(a) $(\mathfrak{g},\mathfrak{h},f) = (\mathfrak{g}(0),\mathfrak{h}(0),f(0))$,

(b) $(\mathfrak{g}(j),\mathfrak{h}(j),f(j))$ IM implies that $(\mathfrak{g}(j-1),\mathfrak{h}(j-1),f(j-1))$ is IM,

(c) $\mathcal{D}(\mathfrak{g}(j),\mathfrak{h}(j),f(j))$ non-commutative implies that $\mathcal{D}(\mathfrak{g}(j-1),\mathfrak{h}(j-1),f(j-1))$ is non-commutative, and

(d) the $\exp(ad^*(\mathfrak{g}))$-orbits that meet $L_{f(N)}$ are single points.

Such a sequence is called a reducing sequence. We emphasize that the scalar fields may get larger with the index j, and in particular, $\mathfrak{g}(j)$ is not necessarily a subalgebra of $\mathfrak{g}(j-1)$, although its dimension (over its scalar field) is necessarily smaller. Also, the construction of the reducing sequence does not require that the subalgebras $\mathfrak{h}(j)\,\hat{}$ be contained in the respective codimension one centralizers; as a result it is possible that the dimension of $\mathfrak{h}(j)$ will be smaller than that of $\mathfrak{h}(j-1)$.

In any case, there will necessarily be some m for which the triple $(\mathfrak{g}(m), \mathfrak{h}(m), f(m))$ is IM but $(\mathfrak{g}(m+1), \mathfrak{h}(m+1), f(m+1))$ is FM, and consideration of the associated triple allows one to reduce Conjecture 1 to the following.

Conjecture 2. Let $(\mathfrak{g}, \mathfrak{h}, f)$ be an IM triple with $\mathfrak{g}$ nilpotent. Assume that $(\mathfrak{g}, \mathfrak{h}, f)$ admits a Kirillov structure $(\mathfrak{z}, Y)$ for which $\mathfrak{h} \subset \mathfrak{g}^Y$ and for which $(\mathfrak{g}^Y, \mathfrak{h}, f)$ is FM. Then $c(\mathfrak{h}, \mathfrak{I}) \not\subset \mathfrak{u}(\mathfrak{g}^Y) + \mathfrak{I}$.

Specifically, we prove the following result in Section 4.

Theorem 4.2.1. *Let* $\{(\mathfrak{g}(j), \mathfrak{h}(j), f(j)) : 0 \leq j \leq k\}$ *be a reducing sequence for the IM-triple* $(\mathfrak{g}, \mathfrak{h}, f)$ *with* $\mathfrak{g}$ *nilpotent. Set*

$$m = max\{j : (\mathfrak{g}(j), \mathfrak{h}(j), f(j)) \text{ is an } IM\text{-triple}\},$$

and assume that Conjecture 2 *holds for* $((\mathfrak{g}(m)\,\hat{}, \mathfrak{h}(m)\,\hat{}, f(m)\,\hat{}\,)$. *Then* $\mathcal{D}(\mathfrak{g}, \mathfrak{h}, f)$ *is not commutative.*

In Section 5 we apply the above theorem to prove Conjecture 1 in two situations. First, we consider the case where all of the $\exp(ad^*\mathfrak{h})$-orbit dimensions in L_f are no greater than one. The result is that Conjecture 1 always holds in this case.

Theorem 5.1.4. *Let* $(\mathfrak{g}, \mathfrak{h}, f)$ *be an IM-triple with* $\mathfrak{g}$ *nilpotent and with the property that* $\delta(\mathfrak{g}, \mathfrak{h}, f) \leq 1$. *Then* $\mathcal{D}(\mathfrak{g}, \mathfrak{h}, f)$ *is not commutative.*

Secondly we consider the situation where $f = 0$ (or "essentially zero"; see Section 5.2) and a certain generic condition is satisfied. Recall that, given increasing sequences $\{\mathfrak{g}_i\}_{i=1}^n$ and $\{\mathfrak{h}_j\}_{j=1}^p$ of $\exp(ad^*\mathfrak{h})$-modules in $\mathfrak{g}$ and $\mathfrak{h}$ respectively, with $\dim(\mathfrak{g}_i) = i, 1 \leq i \leq n$ and $\dim(\mathfrak{h}_j) = j, 1 \leq j \leq p$, one has a definition of "generic" $\exp(ad^*\mathfrak{h})$-orbits.

Lemma 5.2.2. *Let* $(\mathfrak{g}, \mathfrak{h}, f)$ *be an IM-triple with* $\mathfrak{g}$ *nilpotent and with* $f = 0$. *Assume that* $(\mathfrak{g}, \mathfrak{h}, f)$ *admits a Kirillov structure* $(\mathfrak{z}, Y)$ *for which* $\mathfrak{h} \subset \mathfrak{g}^Y$ *and for which* $(\mathfrak{g}^Y, \mathfrak{h}, f)$ *is FM. Assume further that there are increasing sequences of* $\exp(ad^*\mathfrak{h})$*-modules in* $\mathfrak{g}$ *and* $\mathfrak{h}$ *as described above and with* $\mathfrak{g}_{n-1} = \mathfrak{g}^Y$, *such that* L_f *meets the corresponding set of generic* $\exp(ad^*\mathfrak{h})$*-orbits . Then* $c(\mathfrak{h}, \mathfrak{I}) \nsubseteq \mathfrak{u}(\mathfrak{g}^Y) + \mathfrak{I}$.

From Theorem 4.2.1 and Lemma 5.2.2, one obtains the following.

Theorem 5.2.3. *Let* $(\mathfrak{g}, \mathfrak{h}, f)$ *be an IM-triple with* $\mathfrak{g}$ *nilpotent and suppose that a reducing sequence for* $(\mathfrak{g}, \mathfrak{h}, f)$ *is given such that the hypothesis of Lemma 5.2.2 holds for* $(\mathfrak{g}(m)\,\hat{}, \mathfrak{h}(m)\,\hat{}, f(m)\,\hat{}\,)$. *Then* $\mathcal{D}(\mathfrak{g}, \mathfrak{h}, f)$ *is not commutative.*

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2. Basic Definitions

2.1 Let $\mathbf{k}$ be a field of characteristic zero, let $\mathfrak{g}$ be a Lie algebra over $\mathbf{k}$, and let $\mathfrak{u}(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$ over $\mathbf{k}$. Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$; we regard $\mathfrak{u}(\mathfrak{h})$ as a subalgebra of $\mathfrak{u}(\mathfrak{g})$ in the usual way. Let $f : \mathfrak{u}(\mathfrak{h}) \to \mathbf{k}$ be an associative algebra homomorphism, and let $\mathcal{I} = \mathcal{I}(\mathfrak{g}, \mathfrak{h}, f)$ be the left ideal in $\mathfrak{u}(\mathfrak{g})$ generated by the kernel of f.

We use multi-indices in the same way as [2]: given an ordered set of elements $X_1, X_2, \ldots, X_m$ in $\mathfrak{g}$ and $\alpha \in Z_+^m$, let X^α denote the element

$$X^\alpha = X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_m^{\alpha_m}$$

in $\mathfrak{u}(\mathfrak{g})$.

Proposition 2.1.1. *Let $\mathfrak{g}$ be any Lie algebra over a field $\mathbf{k}$ of characteristic zero, let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$, and let f be a $\mathbf{k}$-algebra homomorphism of $\mathfrak{u}(\mathfrak{h})$ onto $\mathbf{k}$. Let $Y_1, Y_2, \ldots, Y_p, X_1, X_2, \ldots, X_m$ be any basis for $\mathfrak{g}$ over $\mathbf{k}$ with*

$$\mathfrak{h} = \mathbf{k}\text{-}span\{Y_1, Y_2, \ldots, Y_p\}$$

and define $U_j = Y_j - f(Y)I, 1 \le j \le p$. Then one has the following.

(a) The monomials $X^\alpha U^\beta, \alpha \in \mathbf{Z}_+^m, \beta \in \mathbf{Z}_+^p$, form a basis for $\mathfrak{u}(\mathfrak{g})$ over $\mathbf{k}$.

(b) The left ideal $\mathcal{I}$ generated by the kernel of f is spanned by the monomials $X^\alpha U^\beta, |\beta| > 0$.

(c) Define $\mathfrak{a}(\mathfrak{h}, f) = \mathbf{k}\text{-}span\{Y - f(Y)1 : Y \in \mathfrak{h}\}$. Then $\mathcal{I} = \mathfrak{u}(\mathfrak{g})\mathfrak{a}(\mathfrak{h}, f)$.

Proof. We imitate the proof of [2, Lemma 4.2]. Let $\mathfrak{u}^r(\mathfrak{g})$ be the subspace spanned by the monomials $X^\alpha Y^\beta$, with $|\alpha| + |\beta| \le r$. For each α and β, we have

$$X^\alpha Y^\beta = \sum_{\gamma \preccurlyeq \beta} \binom{\beta}{\gamma} (f(Y))^{\beta-\gamma}\; X^\alpha V^\gamma$$

where $\gamma \preccurlyeq \beta$ means that $\gamma_j \le \beta_j$, $1 \le j \le p$, $\binom{\beta}{\gamma} = \prod_{j=1}^p \binom{\beta_j}{\gamma_j}$, and $f(Y)^{\beta-\gamma} = \prod_{j=1}^p f(Y_j)^{\beta_j - \gamma_j}$. This shows that $\{X^\alpha V^\beta \;: |\alpha| + |\beta| \le r\}$ spans $\mathfrak{u}^r(\mathfrak{g})$. Part (a) then follows by counting dimensions.

As for (b), let $\mathcal{K} = \ker(f) \subset \mathfrak{u}(\mathfrak{h})$. We claim that $\mathcal{K} = \mathbf{k}$-span$\{U^\beta : |\beta| > 0\}$. Since it is obvious that $\mathcal{K} \supset \mathbf{k}$-span$\{U^\beta : |\beta| > 0\}$, then we need only show that $\mathbf{k}$-span$\{U^\beta : |\beta| > 0\}$ has codimension one in $\mathfrak{u}(\mathfrak{h})$. This follows from considering the fact that for $r = 1, 2, 3, \ldots$, $\mathfrak{u}^r(\mathfrak{h}) = \mathbf{k}$-span$\{U^\beta : |\beta| = r\} + \mathfrak{u}^{r-1}(\mathfrak{h})$ and then using induction on r.

The claim being verified, we see that any monomial $U^\beta U^\gamma$, with $|\gamma| > 0$ can be written as a linear combination of monomials $U^\lambda, \lambda > 0$, and hence any monomial $X^\alpha U^\beta U^\gamma$ is a linear combination of monomials $X^\alpha U^\lambda, \lambda > 0$. But by part (a), $\mathfrak{u}(\mathfrak{g})$ is spanned by the monomials $X^\alpha U^\beta$, and (b) follows. Part (c) now follows immediately from part (b). ∎

Corollary 2.1.2. *Let $\mathfrak{g}$, $\mathfrak{h}$ and f be as in Proposition 2.1.1. Then $\mathcal{I}(\mathfrak{g},\mathfrak{h},f)\cap\mathfrak{g}=\mathfrak{h}\cap ker(f)$.*

Proof. From parts (a) and (b) above we see that $\mathcal{I}(\mathfrak{g},\mathfrak{h},f)\cap\mathfrak{g}\subset\mathbf{k}\text{-span}\{U_j : 1\le j\le p\}$. On the other hand, one computes that $\mathbf{k}\text{-span}\{U_j : 1\le j\le p\}\cap\mathfrak{g}\subset\mathfrak{h}\cap\ker(f)$: if $W\in\mathbf{k}\text{-span}\{U_j : 1\le j\le p\}$,

$$W=\sum_{j=1}^{p}a_jU_j=\sum_{j=1}^{p}a_jY_j-\left(\sum_{j=1}^{p}a_jf(Y_j)\right)I$$

can belong to $\mathfrak{g}$ only if $\sum_{j=1}^{p}a_jf(Y_j)=0$, showing that $W\in\mathfrak{h}$ and that $f(W)=0$. Hence $\mathcal{I}(\mathfrak{g},\mathfrak{h},f)\cap\mathfrak{g}\subset\mathfrak{h}\cap\ker(f)$. The reverse inclusion is obvious. ■

Corollary 2.1.3. *Let $\mathfrak{g}$, $\mathfrak{h}$ and f be as in Proposition 2.1.1 and suppose that $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$. Set $\mathfrak{h}_0=\mathfrak{h}\cap\mathfrak{k}$, and $f_0=f|_{\mathfrak{u}(\mathfrak{h}_0)}$. Then $\mathcal{I}(\mathfrak{g},\mathfrak{h},f)\cap\mathfrak{u}(\mathfrak{k})=\mathcal{I}(\mathfrak{k},\mathfrak{h}_0,f_0)$.*

Proof. Let $T_1,T_2,...,T_u$ be a basis of $\mathfrak{h}_0$, $Z_1,Z_2,...,Z_t$ a basis of a subspace of $\mathfrak{k}$ complementary to $\mathfrak{h}_0$, $Y_1,Y_2,...,Y_s$ a basis of a subspace of $\mathfrak{h}$ complementary to $\mathfrak{h}_0$, and $X_1,X_2,...,X_r$ a basis of a subspace of $\mathfrak{g}$ complementary to $\mathfrak{h}+\mathfrak{k}$. The union of these bases is obviously a basis of $\mathfrak{g}$. Set $V_i=T_i-f(T_i)I, 1\le i\le u$ and $U_i=Y_i-f(Y_i)I, 1\le i\le s$. From Proposition 2.1.1 (b) we have that the set $\{X^\alpha Z^\mu U^\beta V^\nu : |\beta|+|\nu|>0\}$ is a basis for $\mathcal{I}(\mathfrak{g},\mathfrak{h},f)$ and by Proposition 2.1.1 (a), that $\{Z^\mu V^\nu : \mu\in\mathbf{Z}_+^u, \nu\in\mathbf{Z}_+^t\}$ is a basis for $\mathfrak{u}(\mathfrak{k})$. The intersection of these two bases is $\{Z^\mu V^\nu : |\nu|>0\}$ which by part (b) of the Proposition is a basis for $\mathcal{I}(\mathfrak{k},\mathfrak{h}_0,f_0)$. ■

2.2 Let $\mathfrak{g}$ be a nilpotent Lie algebra over a field $\mathbf{k}$ of characteristic zero. For $\ell\in\mathfrak{g}^*$, define

$$\mathfrak{g}(\ell)=\{Z\in\mathfrak{g}:\ell([X,Z])=0 \text{ holds for all } X\in\mathfrak{g}\}$$

and set $d(\ell)=\dim(\mathfrak{g}/\mathfrak{g}(\ell))$. Let $\mathfrak{g}$ have the nilpotent group operation defined by the Campbell-Baker-Hausdorff formula. It is easily seen that for each $\ell\in\mathfrak{g}^*$, $\mathfrak{g}(\ell)$ is the stabilizer of ℓ in $\mathfrak{g}$ with respect to the coadjoint action: for X, Y in $\mathfrak{g}$,

$$(\exp(ad^*(X))\ell)(Y)=\ell(Y-[X,Y]+\frac{1}{2}[X,[X,Y]]-\frac{1}{6}[X,[X,[X,Y]]]+\cdots).$$

It is well-known that there is a finite partition $\mathcal{P}$ of $\mathfrak{g}^*$ with the properties

(i) each Ω in $\mathcal{P}$ is $\exp(ad^*(\mathfrak{g}))$-invariant,

(ii) there is a total ordering $<$ on $\mathcal{P}$ such that for each Ω, $\bigcup\{\Omega' : \Omega'\le\Omega\}$ is Zariski open in $\mathfrak{g}^*$,

(iii) d is constant on each Ω, and

(iv) $\Omega'<\Omega$ implies $d(\Omega')\ge d(\Omega)$.

The construction of the partition $\mathcal{P}$ depends only upon the choice of a Jordan-Holder sequence for $\mathfrak{g}$.

Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$, and set $\mathfrak{h}(\ell) = \mathfrak{h} \cap \mathfrak{g}(\ell), \ell \in \mathfrak{g}^*$. For each ℓ, $\mathfrak{h}(\ell)$ is the stabilizer of ℓ in $\mathfrak{h}$ with respect to the coadjoint action of $\mathfrak{h}$. Set $\delta(\ell) = \dim(\mathfrak{h}/\mathfrak{h}(\ell)), \ell \in \mathfrak{g}^*$. There is a similar partition $\mathcal{P}_{\mathfrak{h}}$ of $\mathfrak{g}^*$ having properties (i) - (iv) with $\exp(ad^*(\mathfrak{g}))$ replaced by $\exp(ad^*(\mathfrak{h}))$ and d replaced by δ.

Given $f : \mathfrak{u}(\mathfrak{h}) \to \mathbf{k}$ an associative algebra homomorphism, set

$$L_f = \{\ell \in \mathfrak{g}^* : \ell|_{\mathfrak{h}} = f|_{\mathfrak{h}}\}.$$

Lemma 2.2.1. *There is a Zariski open subset U of L_f on which the dimension functions d and δ are constant.*

Proof. From property (ii) of the partitions $\mathcal{P}$ and $\mathcal{P}_{\mathfrak{h}}$, we can choose $\Omega \in \mathcal{P}$ and $\Omega_{\mathfrak{h}} \in \mathcal{P}_{\mathfrak{h}}$ such that $\Omega \cap \Omega_{\mathfrak{h}} \cap L_f$ is a Zariski open subset of L_f; set $U = \Omega \cap \Omega_{\mathfrak{h}} \cap L_f$. ∎

We denote these "generic" values of d and δ by $d(\mathfrak{g}, \mathfrak{h}, f)$ and $\delta(\mathfrak{g}, \mathfrak{h}, f)$ respectively. (Note that from property (iv) above one has that $d(\mathfrak{g}, \mathfrak{h}, f) = \max\{d(\ell) : \ell \in L_f\}$ and $\delta(\mathfrak{g}, \mathfrak{h}, f) = \max\{\delta(\ell) : \ell \in L_f\}$, hence these values are independent of the choice of Jordan-Holder sequence.)

Lemma 2.2.2. *Let $\mathfrak{g}$, $\mathfrak{h}$, and f as above, and let $d(\mathfrak{g}, \mathfrak{h}, f)$ and $\delta(\mathfrak{g}, \mathfrak{h}, f)$ be the associated generic coadjoint orbit dimensions. Then $d(\mathfrak{g}, \mathfrak{h}, f) \geq 2 \cdot \delta(\mathfrak{g}, \mathfrak{h}, f)$.*

Proof. For any $\ell \in L_f$, $\mathfrak{h} + \mathfrak{g}(\ell)$ is a subspace of $\mathfrak{g}$ that is isotropic with respect to the alternate bilinear form defined by ℓ, so

$$d(\ell) = \dim(\mathfrak{g}/\mathfrak{g}(\ell)) \geq 2 \cdot \dim((\mathfrak{h} + \mathfrak{g}(\ell))/\mathfrak{g}(\ell)) = 2 \cdot \dim(\mathfrak{h}/\mathfrak{h}(\ell)). \quad ∎$$

A triple $(\mathfrak{g}, \mathfrak{h}, f)$, where $\mathfrak{g}$ is a nilpotent Lie algebra, $\mathfrak{h}$ is a Lie subalgebra, and $f : \mathfrak{u}(\mathfrak{h}) \to \mathbf{k}$ is an algebra homomorphism, will be called an "IM-triple" if $d(\mathfrak{g}, \mathfrak{h}, f) > 2 \cdot \delta(\mathfrak{g}, \mathfrak{h}, f)$. Otherwise $(\mathfrak{g}, \mathfrak{h}, f)$ will be called an "FM-triple."

3. Construction of an associated triple

3.1 Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbf{k}$ of characteristic zero, and let $\mathfrak{u}(\mathfrak{g})$ be its enveloping algebra. For any subsets $\mathcal{M}$ and $\mathcal{N}$ of $\mathfrak{u}(\mathfrak{g})$, set

$$c(\mathcal{M}, \mathcal{N}) = \{W \in \mathfrak{u}(\mathfrak{g}) : [W, \mathcal{M}] \subset \mathcal{N}\}.$$

Let $(\mathfrak{g}, \mathfrak{h}, f)$ be a triple with $\mathcal{I} = \mathcal{I}(\mathfrak{g}, \mathfrak{h}, f)$.

Lemma 3.1.1. *Let $\mathfrak{z}$ be an ideal in $\mathfrak{g}$ such that $\mathfrak{z} \subset c(\mathfrak{g}, \mathcal{I})$. Then $[\mathfrak{g}, \mathfrak{z}] \subset \mathfrak{z} \cap \mathfrak{h} \cap ker(f)$. In particular, $\mathfrak{z} \cap \mathfrak{h}$ is an ideal in $\mathfrak{g}$.*

Proof. By Corollary 2.1.2, $\mathfrak{z} \cap \mathcal{I} = \mathfrak{z} \cap \mathfrak{h} \cap \ker(f)$, and by definition of $\mathfrak{z}$, $[\mathfrak{g}, \mathfrak{z}] \subset \mathfrak{z} \cap \mathcal{I}$. ∎

An ideal $\mathfrak{z}$ in $\mathfrak{g}$ will be called an $\mathfrak{I}$-central ideal if $\mathfrak{z} \subset c(\mathfrak{g}, \mathfrak{I})$.

Definition. A Kirillov structure for $(\mathfrak{g}, \mathfrak{h}, f)$ is a pair $(\mathfrak{z}, Y)$ where $\mathfrak{z}$ is an $\mathfrak{I}$-central ideal in $\mathfrak{g}$ and Y is an element of $\mathfrak{g}$ that belongs to $c(\mathfrak{g}, \mathfrak{z})$ and for which $c(Y, \mathfrak{I}) \cap \mathfrak{g}$ is codimension one in $\mathfrak{g}$. The element Y will be called a Kirillov element for $(\mathfrak{g}, \mathfrak{h}, f)$.

Given a Kirillov structure for $(\mathfrak{g}, \mathfrak{h}, f)$, we shall employ the notation $\mathfrak{g}^Y = c(Y, \mathfrak{I}) \cap \mathfrak{g}$. If $\mathfrak{g}$ is nilpotent, then any triple $(\mathfrak{g}, \mathfrak{h}, f)$ obviously has a non-zero $\mathfrak{I}$-central ideal. However, even in cases where $\exp(ad^*(\mathfrak{h}))$ acts non-trivially on L_f, $(\mathfrak{g}, \mathfrak{h}, f)$ may not have a Kirillov element. The goal of this section is to give a construction for any triple $(\mathfrak{g}, \mathfrak{h}, f)$ with $\mathfrak{g}$ nilpotent and $\delta(\mathfrak{g}, \mathfrak{h}, f) > 0$, of an associated triple $(\hat{\mathfrak{g}}, \hat{\mathfrak{h}}, \hat{f})$ that does admit a Kirillov structure, and for which important aspects of the coadjoint orbits and quotient algebras $\mathfrak{D}(\mathfrak{g}, \mathfrak{h}, f)$ are preserved.

Choose an $\mathfrak{I}$-central ideal $\mathfrak{z}$, and set $\mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{z}$. Set $f_0 = f|_{\mathfrak{u}(\mathfrak{h}_0)}$, and let $\mathcal{J}$ be the left ideal in $\mathfrak{u}(\mathfrak{g})$ generated by $\ker(f_0)$, that is, $\mathcal{J} = \mathfrak{I}(\mathfrak{g}, \mathfrak{h}_0, f_0)$. Since $\mathfrak{h}_0$ is an ideal in $\mathfrak{g}$, $\mathcal{J}$ is a two-sided ideal in $\mathfrak{u}(\mathfrak{g})$.

Let $T_1, T_2, ..., T_u$ be a basis of $\mathfrak{h}_0$, $Z_1, Z_2, ..., Z_t$ a basis of a subspace of $\mathfrak{z}$ complementary to $\mathfrak{h}_0$, $Y_1, Y_2, ..., Y_s$ a basis of a subspace of $\mathfrak{h}$ complementary to $\mathfrak{h}_0$ and $X_1, X_2, ..., X_r$ a basis of a subspace of $\mathfrak{g}$ complementary to $\mathfrak{h} + \mathfrak{z}$. The set

$$B = \{T_1, T_2, \ldots, T_u, Z_1, Z_2, \ldots, Z_t, Y_1, Y_2, \ldots, Y_s, X_1, X_2, \ldots, X_r\}$$

is then a basis of $\mathfrak{g}$, where it is understood that if any of the above vector subspaces of $\mathfrak{g}$ are trivial, then the corresponding basis is the empty set. Set $V_i = T_i - f(T_i)I, 1 \le i \le u$ and $U_i = Y_i - f(Y_i)I, 1 \le i \le s$. From Proposition 2.1.1 and its corollaries we have the following.

(i) The monomials $X^\alpha Y^\beta Z^\mu V^\nu, \alpha \in \mathbf{Z}_+^r, \beta \in \mathbf{Z}_+^s, \mu \in \mathbf{Z}_+^t, \nu \in \mathbf{Z}_+^u$ form a basis for $\mathfrak{u}(\mathfrak{g})$ over $\mathbf{k}$,

(ii) $\mathcal{J}$ is spanned by the monomials $X^\alpha Y^\beta Z^\mu V^\nu, |\nu| > 0$,

(iii) the monomials $Z^\mu V^\nu, \mu \in \mathbf{Z}_+^t, \nu \in \mathbf{Z}_+^u$ form a basis of $u(\mathfrak{z})$ over $\mathbf{k}$,

(iv) $\mathfrak{u}(\mathfrak{z}) \cap \mathcal{J} = \mathfrak{I}(\mathfrak{z}, \mathfrak{h}_0, f_0)$, and

(v) $\mathfrak{u}(\mathfrak{z}) \cap \mathcal{J}$ is spanned by the monomials $\{Z^\mu V^\nu : |\nu| > 0\}$.

Proposition 3.1.2. *Let $\mathfrak{z}$ be an $\mathfrak{I}$-central ideal in $\mathfrak{g}$, let W be any element of $\mathfrak{u}(\mathfrak{g})$, and let T belong to $\mathfrak{u}(\mathfrak{z})$ such that $WT \in \mathfrak{I}$. Then either $W \in \mathfrak{I}$ or $T \in \mathfrak{I}$.*

Proof. In accordance with Proposition 2.1.1 (b) (applied to $\mathfrak{h}$), we write

$$W = \sum_{\alpha,\beta,\mu,\nu} a_{\alpha\beta\mu\nu} X^\alpha Z^\mu U^\beta V^\nu,$$

and similarly

$$T = \sum_{\sigma,\lambda,} b_{\sigma\lambda} Z^\sigma V^\lambda.$$

Now by Lemma 3.1.1, $\mathfrak{u}(\mathfrak{z}) \subset c(\mathfrak{u}(\mathfrak{g}), \mathcal{J})$, so for each set $\alpha, \beta, \mu, \nu, \sigma, \lambda$ of multi-indices, there is $Q \in \mathcal{J}$ such that

$$X^\alpha Z^\mu U^\beta V^\nu Z^\sigma V^\lambda = X^\alpha Z^{\mu+\sigma} U^\beta V^{\nu+\lambda} + Q.$$

Hence we can write

$$(*)\qquad WT = \sum_{\alpha,\beta,\phi,\psi}\Big(\sum_{\substack{\mu+\sigma=\phi\\ \nu+\lambda=\psi}} a_{\alpha\beta\mu\nu}b_{\sigma\lambda}\Big)X^\alpha Z^\phi U^\beta V^\psi + Q$$

where $Q \in \mathcal{J}$.

Now assume that niether W nor T belongs to $\mathcal{I}$. Then there are multi-indices $\alpha,\beta,\mu,\nu,\sigma,\lambda$ for which $a_{\alpha\beta\mu\nu} \neq 0$ and $|\beta|+|\nu| = 0$, and for which $b_{\sigma\lambda} \neq 0$ and $|\lambda| = 0$. Set $N = \{\mu \in \mathbf{Z}_+^t : a_{\alpha\beta\mu\nu} \neq 0 \text{ and } |\beta|+|\nu| = 0\}$ and $L = \{\sigma \in \mathbf{Z}_+^t : b_{\sigma\lambda} \neq 0 \text{ and } |\lambda| = 0\}$. We order $\mathbf{Z}_+^t$ lexographically, and set $\mu_0 = \max(N)$, $\sigma_0 = \max(L)$, and $\phi_0 = \mu_0 + \sigma_0$. If $a_{\alpha\beta\mu\nu}b_{\sigma\lambda} \neq 0$ and $|\beta|+|\nu|+|\lambda| = 0$ then $\mu+\sigma \le \phi_0$ with equality only if $\mu = \mu_0$ and $\sigma = \sigma_0$. Hence there is a term in $*$ whose monomial has the form $X^\alpha Z^{\phi_0}$ and whose coefficient is $a_{\alpha\beta\mu_0\nu}b_{\sigma_0\lambda} \neq 0$. By Proposition 2.1.1, this means that $WT \notin \mathcal{I}$. ∎

Lemma 3.1.3. *Let W be an element of $u(\mathfrak{g})$ of the form*

$$W = \sum_{\alpha,\beta} X^\alpha Y^\beta Q_{\alpha\beta}$$

where each $Q_{\alpha\beta}$ belongs to $u(\mathfrak{z})$. Then $W \in \mathcal{J}$ if and only if $Q_{\alpha\beta} \in u(\mathfrak{z}) \cap \mathcal{J}$ holds for all α and β.

Proof. The "if" part of the lemma is obvious, so let us suppose that W belongs to $\mathcal{J}$. By (ii) above we have

$$W = \sum_{\substack{\alpha,\beta,\mu,\nu\\ |\nu|>0}} a_{\alpha\beta\mu\nu}X^\alpha Y^\beta Z^\mu V^\nu.$$

Set

$$Q'_{\alpha\beta} = \sum_{\substack{\mu,\nu\\ |\nu|>0}} a_{\alpha\beta\mu\nu}Z^\mu V^\nu;$$

Now by [5, 2.2.7. Proposition], $\{X^\alpha Y^\beta : \alpha \in \mathbf{Z}_+^r, \beta \in \mathbf{Z}_+^s\}$ is a basis for $\mathfrak{u}(\mathfrak{g})$ as a right module over $\mathfrak{u}(\mathfrak{z})$. Hence for each α and β, $Q_{\alpha\beta} = Q'_{\alpha\beta}$, and $Q_{\alpha\beta} \in \mathcal{J}$. ∎

3.2 Let $(\mathfrak{g},\mathfrak{h},f)$ be a triple, with $\mathfrak{z}$ an $\mathcal{I}$-central ideal, and with $\mathfrak{h}_0$, f_0, and $\mathcal{J}$ as in the previous subsection. Set $\mathcal{A} = u(\mathfrak{g})/\mathcal{J}$, and let $\pi : u(\mathfrak{g}) \to \mathcal{A}$ be the canonical homomorphism. By Corollary 2.1.2, $\mathfrak{g} \cap \mathcal{J} = \mathfrak{h}_0 \cap \ker(f_0)$, and hence $\pi(\mathfrak{g}) \simeq \mathfrak{g}/(\mathfrak{h}_0 \cap \ker(f_0))$. Set $\mathcal{R} = \pi(u(\mathfrak{z}))$.

Lemma 3.2.1. *The subalgebra $\mathcal{R}$ of $\mathcal{A}$ is contained in the center of $\mathcal{A}$, and the elements of $\mathcal{R}$ are not divisors of zero in $\mathcal{A}$.*

Proof. The ideal $\mathfrak{h}_0 \cap \ker(f)$ generates a two-sided ideal $\mathcal{J}_0$ in $\mathfrak{u}(\mathfrak{g})$ that is contained in $\mathcal{J}$. Since $\mathfrak{z}/\mathfrak{h}_0 \cap \ker(f)$ is central in $\mathfrak{g}/\mathfrak{h}_0 \cap \ker(f)$, then the canonical image of $\mathfrak{u}(\mathfrak{z})$ in $\mathfrak{u}(\mathfrak{g})/\mathcal{J}_0 = \mathfrak{u}(\mathfrak{g}/\mathfrak{h}_0 \cap \ker(f))$ is central. It follows that $\mathcal{R}$ is central in $\mathcal{A}$. That elements of $\mathcal{R}$ are not divisors of zero follows immediately from Proposition 3.1.2. ∎

Lemma 3.2.2. *The monomials* $\{\pi(X^\alpha Y^\beta Z^\mu) : \alpha \in \mathbf{Z}_+^r, \beta \in \mathbf{Z}_+^s, \mu \in \mathbf{Z}_+^t\}$ *form a basis of* $\mathcal{A}$, *and* $\pi(\mathcal{I})$ *is spanned by the monomials* $\pi(X^\alpha U^\beta Z^\mu), |\beta| > 0$.

Proof. The first statement follows immediately from (ii) above. For the second, Lemma 3.2.1 shows that $\pi(X^\alpha U^\beta Z^\mu) = \pi(X^\alpha Z^\mu U^\beta)$. On the other hand by Proposition 2.1.1 (b), $\mathcal{I} = \mathbf{k}\text{-span}\{X^\alpha Z^\mu U^\beta V^\nu, |\beta| + |\nu| > 0\}$, and hence $\pi(\mathcal{I})$ is spanned by the monomials $\pi(X^\alpha Z^\mu U^\beta), |\beta| > 0$. ■

3.3 Retaining the setup and notation from the previous subsection, we now bring in the language and results from [5, Sect. 3.6] : by Lemma 3.2.1, the set $\mathcal{R}^* = \mathcal{R}\backslash\{0\}$ allows an arithmetic of fractions in $\mathcal{A}$, and we let $\mathcal{B} = \mathcal{A}_{\mathcal{R}^*}$ be the resulting fraction algebra. As in loc. cit., $\mathcal{B} = \mathcal{A} \times \mathcal{R}^* / \sim$ where $\sim$ is the equivalence relation on $\mathcal{A} \times \mathcal{R}^*$ defined by

$$(a, r) \sim (b, s) \text{ if and only if } as = br.$$

(Since $\mathcal{R}^* \subseteq \text{cent}(\mathcal{A})$, the definition of $\sim$ in loc. cit. reduces to the above.) We have the natural ring homomorphisms

$$u(\mathfrak{g}) \xrightarrow{\pi} \mathcal{A} \longrightarrow \mathcal{A} \times \mathcal{R}^* \xrightarrow{\epsilon} \mathcal{B}.$$

The above produces a canonical injection of $\mathcal{A}$ into $\mathcal{B}$, and we identify $\mathcal{A}$ with its image in $\mathcal{B}$. For an element of $\mathfrak{u}(\mathfrak{g})$ denoted by an upper case Roman letter, denote its image in $\mathcal{A}$ by the lower case counterpart; in particular denote $\pi(I)$ by i. For $X \in \mathfrak{u}(\mathfrak{g})$ and $Z \in \mathfrak{u}(\mathfrak{z})$, denote the equivalence class in $\mathcal{B}$ of (x, z) by x/z. In keeping with the above mentioned identification, denote x/i by $x, x \in \mathcal{A}$. For any subset $\mathcal{M}$ of $\mathcal{A}$, let $\sigma(\mathcal{M})$ denote the subset of fractions in $\mathcal{B}$ having numerators in $\mathcal{M}$, that is, $\sigma(\mathcal{M}) = \epsilon(\mathcal{M} \times \mathcal{R}^*)$, and in particular, $\mathcal{B} = \sigma(\mathcal{A})$.

Set $\mathbf{K} = \epsilon(\mathcal{R} \times \mathcal{R}^*)$. Since $\mathcal{R}$ is a commutative integral domain, $\mathbf{K}$ is a field of characteristic zero and $\mathcal{B}$ has in a natural way the structure of an associative $\mathbf{K}$-algebra. More generally, we have the following.

Lemma 3.3.1. *Let* $\mathcal{M}$ *be an* $\mathcal{R}$*-submodule of* $\mathcal{A}$ *with* $\mathcal{R}$*-basis* B. *Then* $\sigma(\mathcal{M})$ *is a* $\mathbf{K}$*-subspace of* $\mathcal{B}$ *with basis* B. *Moreover, if* $\mathcal{M}$ *is also a subalgebra of* $\mathcal{A}$, *then* $\sigma(\mathcal{M})$ *is a* $\mathbf{K}$*-subalgebra of* $\mathcal{B}$.

Proof. It is clear that $\sigma(\mathcal{M})$ is a $\mathbf{K}$-subspace of $\mathcal{B}$. It is spanned by B since, for $x/z \in \sigma(\mathcal{M})$, $x = \sum z_i x_i$ where $z_i \in \mathcal{R}$ and $x_i \in B$, so $x/z = \sum (z_i/z) x_i \in \mathbf{K}\text{-span}(B)$. Suppose that $\sum q_i x_i = 0$ with $q_i \in \mathbf{K}$ and $x_i \in B$. There is $p \in \mathcal{R}^*$ such that for each i, $pq_i \in \mathcal{R}$. Since B is an $\mathcal{R}$-basis, $\sum pq_i x_i = 0$ implies $pq_i = 0$ for all i, and hence $q_i = 0$ for all i. The "moreover" part follows from the fact that the canonical injection of $\mathcal{A}$ into $\mathcal{B}$ is a ring homomorphism.■

Define the Lie bracket in $\mathcal{B}$ as usual: for v and w belonging to $\mathcal{B}$, $[v, w] = vw - wv$. There is a connection between the Lie brackets in $\mathfrak{g}$, $\mathcal{A}$, and $\mathcal{B}$: let $X, Y \in \mathfrak{g}$, with $x = \pi(X), y = \pi(Y)$, and let $p, q \in \mathcal{R}, r, s \in \mathcal{R}^*$. Then

$$[(px/r), (qy/s)] = [(p/r)x, (q/s)y] = (pq/rs)[x, y] = (pq/rs)\pi([X, Y]). \tag{3.3.1}$$

This allows us to associate Lie subalgebras of $\mathfrak{g}$ to finite dimensional Lie algebras in $\mathcal{B}$.

Lemma 3.3.2. *Let $\mathfrak{k}$ be a subalgebra of $\mathfrak{g}$, and let $Y_1, Y_2, ..., Y_s$ be a basis of a subspace of $\mathfrak{k}$ complementary to $\mathfrak{k} \cap \mathfrak{z}$. Let $\mathfrak{u}(\mathfrak{z})\mathfrak{k}$ be the $\mathfrak{u}(\mathfrak{z})$-module generated by $\mathfrak{k}$, and set*

$$\hat{\mathfrak{k}} = \sigma(\pi(\mathfrak{u}(\mathfrak{z})\mathfrak{k}) + \mathcal{R}).$$

Then $\{i, y_1, y_2, ..., y_s\}$ is a $\mathbf{K}$-basis for $\hat{\mathfrak{k}}$. Moreover, $\hat{\mathfrak{k}}$ is a Lie algebra over $\mathbf{K}$, and is a Lie subalgebra of $\hat{\mathfrak{g}} = \sigma(\pi(\mathfrak{u}(\mathfrak{z})\mathfrak{g}) + \mathcal{R})$.

Proof. We need only show that $\{i, y_1, y_2, \ldots, y_s\}$ is an $\mathcal{R}$-basis for $\pi(\mathfrak{u}(\mathfrak{z})\mathfrak{k}) + \mathcal{R}$. Let $v + p \in \pi(\mathfrak{u}(\mathfrak{z})\mathfrak{k}) + \mathcal{R}$, with $v = \pi(V), V \in \mathfrak{u}(\mathfrak{z})\mathfrak{k}, p \in \mathcal{R}$. We have $V = \sum_{j=1}^{s} P_j Y_j + P_0$, where $P_j \in \mathfrak{u}(\mathfrak{z}), 0 \le j \le s$, so $v + p = \sum_{j=1}^{s} p_j y_j + p_0 + p \in \mathcal{R}$-span$\{i, y_1, y_2, ..., y_s\}$. Now suppose that $p_0, p_1, \ldots, p_s$ are elements of $\mathcal{R}$ for which $\sum_{j=1}^{s} p_j y_j + p_0 = 0$. Then we have $P_0, P_1, \ldots, P_s \in \mathfrak{u}(\mathfrak{z})$ for which $\sum_{j=1}^{s} Y_j P_j + P_0 \in \mathcal{J}$. By Lemma 3.1.3, we have $P_j \in \mathcal{J}$, hence $p_j = 0$ holds for each j.

That $\hat{\mathfrak{k}}$ is a Lie algebra over $\mathbf{K}$, and is a Lie subalgebra of $\hat{\mathfrak{g}}$, follows from the equations (3.3.1). ■

Let $\mathfrak{u}(\hat{\mathfrak{g}})$ be the enveloping algebra of $\hat{\mathfrak{g}}$ over $\mathbf{K}$; denote its identity element by $\hat{I}$.

Proposition 3.3.3. *The $\mathbf{K}$-algebra $\mathcal{B}$ is a canonical image of $\mathfrak{u}(\hat{\mathfrak{g}})$ with kernel $\mathfrak{u}(\hat{\mathfrak{g}})(i - \hat{I})$, and has as a $\mathbf{K}$-basis the set $\{x^\alpha y^\beta : \alpha \in \mathbf{Z}_+^r, \beta \in \mathbf{Z}_+^s\}$.*

Proof. By Proposition 2.1.1 (applied to the Lie subalgebra $\mathbf{K}i$ of $\hat{\mathfrak{g}}$), the set

$$\{x^\alpha y^\beta (i - \hat{I})^\mu : \alpha \in \mathbf{Z}_+^r, \beta \in \mathbf{Z}_+^s, \mu \in \mathbf{Z}_+\}$$

is a $\mathbf{K}$-basis of $\mathfrak{u}(\hat{\mathfrak{g}})$, and the ideal $\mathfrak{u}(\hat{\mathfrak{g}})(i - \hat{I})$ is spanned by the monomials $x^\alpha y^\beta (i - \hat{I})^\mu, \mu > 0$. The identity map from $\hat{\mathfrak{g}}$ to itself extends to a canonical homomorphism h from $\mathfrak{u}(\hat{\mathfrak{g}})$ into $\mathcal{B}$. The image of the above basis of $\mathfrak{u}(\hat{\mathfrak{g}})$ is the set $\{x^\alpha y^\beta : \alpha \in \mathbf{Z}_+^r, \beta \in \mathbf{Z}_+^s\}$ and (since $h(\hat{I}) = i$) the monomials $x^\alpha y^\beta (i - \hat{I})^\mu, \mu > 0$ are obviously contained in the kernel of h. On the other hand, it is immediate from Lemma 3.2.2 that $\{x^\alpha y^\beta : \alpha \in \mathbf{Z}_+^r, \beta \in \mathbf{Z}_+^s\}$ is an $\mathcal{R}$-basis for $\mathcal{A}$, and hence by Lemma 3.3.2, it's image in $\mathcal{B}$ is a $\mathbf{K}$-basis of $\mathcal{B}$. Hence the restriction of h to $\mathbf{K}$-span$\{x^\alpha y^\beta : \alpha \in \mathbf{Z}_+^r, \beta \in \mathbf{Z}_+^s\}$ in $\mathfrak{u}(\hat{\mathfrak{g}})$ is a bijection. The proposition follows. ■

3.4 Let $(\mathfrak{g}, \mathfrak{h}, f)$ be a triple, with $\mathfrak{z}$ an $\mathcal{I}$-central ideal, and with $\mathcal{A}$, $\mathcal{B}$ and $\hat{\mathfrak{g}}$ as above. A triple $(\hat{\mathfrak{g}}, \hat{\mathfrak{h}}, \hat{f})$ is associated with this data as follows. Let $\hat{\mathfrak{h}}$ be the Lie-subalgebra of $\hat{\mathfrak{g}}$ associated with $\mathfrak{h}$ via Lemma 3.3.2. An argument similar to that of Proposition 3.3.3 shows that $h(\mathfrak{u}(\hat{\mathfrak{h}}))$ is canonically isomorphic with $\sigma(\mathcal{R}\pi(\mathfrak{u}(\mathfrak{h})))$, and has the monomials $y^\beta, \beta \in \mathbf{Z}_+^s$ as a $\mathbf{K}$-basis. Let $B_{\mathfrak{h}} = \{T_1, T_2, \ldots, T_u, Y_1, Y_2, \ldots, Y_s\}$ be the basis of $\mathfrak{h}$ chosen in Section 3.1. By Lemma 3.3.2, a basis $B_{\hat{\mathfrak{h}}}$ for $\hat{\mathfrak{h}}$ over $\mathbf{K}$ is given by $\{i, y_1, \ldots, y_s\}$. Now f drops to a homomorphism $\bar{f}$ of $\pi(\mathfrak{u}(\mathfrak{h}))$, and $\pi(\mathfrak{u}(\mathfrak{h}))$ contains $B_{\hat{\mathfrak{h}}}$. Let $\hat{f}$ be the Lie algebra homomorphism from $\hat{\mathfrak{h}}$ onto $\mathbf{K}$ whose restriction to $B_{\hat{\mathfrak{h}}}$ coincides with $\bar{f}$, and extend $\hat{f}$ to $\mathfrak{u}(\hat{\mathfrak{h}})$ in the canonical way. The triple $(\hat{\mathfrak{g}}, \hat{\mathfrak{h}}, \hat{f})$ will be referred to as the triple associated with $(\mathfrak{g}, \mathfrak{h}, f)$ and $\mathfrak{z}$. Let $\hat{\mathcal{I}} = \mathcal{I}(\hat{\mathfrak{g}}, \hat{\mathfrak{h}}, \hat{f})$ be the left ideal in $\mathfrak{u}(\hat{\mathfrak{g}})$ generated by the kernel of $\hat{f}$.

Lemma 3.4.1. *Given any triple* $(\mathfrak{g}, \mathfrak{h}, f)$ *with associated triple* $(\hat{\mathfrak{g}}, \hat{\mathfrak{h}}, \hat{f})$*, we have* $h(\hat{\mathfrak{I}}) = \sigma(\pi(\mathfrak{I}))$ *and* $h(\hat{\mathfrak{I}}) \cap \mathcal{A} = \pi(\mathfrak{I})$.

Proof. By Proposition 2.1.1, $\hat{\mathfrak{I}}$ has as a $\mathbf{K}$-basis the set $\{x^\alpha u^\beta (i - \hat{I})^\mu : \alpha \in \mathbf{Z}_+^r, \beta \in \mathbf{Z}_+^s, \mu \in \mathbf{Z}_+\}$, and this set is mapped by h onto the set $\{x^\alpha u^\beta : \alpha \in \mathbf{Z}_+^r, \beta \in \mathbf{Z}_+^s\}$. By Lemma 3.2.2, this set is an $\mathcal{R}$-basis for $\pi(\mathfrak{I})$, and so by Lemma 3.3.1, it is a $\mathbf{K}$-basis for $\sigma(\pi(\mathfrak{I}))$. This proves the first part of the lemma.

Let $v/z = w$ belong to $h(\hat{\mathfrak{I}}) \cap \mathcal{A}$ with $v \in \pi(\mathfrak{I})$, $z \in \mathcal{R}^*$, and $w \in \mathcal{A}$. Taking preimages gives $V = WZ \in \mathfrak{I}$ with $Z \notin \mathfrak{I}$. By Proposition 3.1.2 we have $W \in \mathfrak{I}$. ∎

Proposition 3.4.2. *Let* $(\mathfrak{g}, \mathfrak{h}, f)$ *be a triple with associated triple* $(\hat{\mathfrak{g}}, \hat{\mathfrak{h}}, \hat{f})$. *Let* $\mathfrak{k}$ *be a subalgebra of* $\mathfrak{g}$ *with* $\hat{\mathfrak{k}}$ *the corresponding subalgebra of* $\hat{\mathfrak{g}}$. *Then* $h(c(\hat{\mathfrak{k}}, \hat{\mathfrak{I}})) = \sigma(\pi(c(\mathfrak{k}, \mathfrak{I})))$.

Proof. From the definitions of $\hat{\mathfrak{k}}$ and σ, and formula (3.3.1) above, in order to show that $\sigma(\pi(c(\mathfrak{k}, \mathfrak{I}))) \subset h(c(\hat{\mathfrak{k}}, \hat{\mathfrak{I}}))$, it is enough to show that $[\pi(c(\mathfrak{k}, \mathfrak{I})), \pi(\mathfrak{k})] \subset \pi(\mathfrak{I})$. But this is obvious since $[c(\mathfrak{k}, \mathfrak{I}), \mathfrak{k}] \subset \mathfrak{I}$.

On the other hand, let $w = \sum_{\alpha,\beta} q_{\alpha\beta} x^\alpha y^\beta$ be an element of $\mathcal{B}$ that belongs to $h(c(\hat{\mathfrak{k}}, \hat{\mathfrak{I}}))$. Choose some $p \in \mathcal{R}^*$ such that $x = pw \in \mathcal{A}$, and let $X = PW \in \mathfrak{u}(\mathfrak{g})$ be a pre-image of pw in $\mathfrak{u}(\mathfrak{g})$. I claim that $X \in c(\mathfrak{k}, \mathfrak{I})$. Let $Y \in \mathfrak{k}$. Then $y = \pi(Y) \in \hat{\mathfrak{k}}$, and it follows that $[x, y] \in h(\hat{\mathfrak{I}}) \cap \mathcal{A}$. By Lemma 3.4.1, this means $[x, y] \in \pi(\mathfrak{I})$, whence $[X, Y] \in \mathfrak{I}$ and the claim is proved. Now by definition of σ we have $w = x/p \in \sigma(\pi(c(\mathfrak{k}, \mathfrak{I})))$. ∎

Corollary 3.4.3. *Let* $(\mathfrak{g}, \mathfrak{h}, f)$ *be a triple with associated triple* $(\hat{\mathfrak{g}}, \hat{\mathfrak{h}}, \hat{f})$. *Then* $c(\mathfrak{h}, \mathfrak{I})/\mathfrak{I}$ *is commutative if and only if* $c(\hat{\mathfrak{h}}, \hat{\mathfrak{I}})/\hat{\mathfrak{I}}$ *is commutative.*

Proof. Clearly $c(\mathfrak{h}, \mathfrak{I})/\mathfrak{I}$ is commutative if and only if $\pi(c(\mathfrak{h}, \mathfrak{I}))/\pi(\mathfrak{I})$ is commutative, so we may as well assume that $\mathcal{J} = 0$. Set $\mathcal{M} = c(\mathfrak{h}, \mathfrak{I})$, $\bar{\mathcal{M}} = \mathcal{M}/\mathfrak{I}$, and let $\bar{\mathcal{R}}$ be the image of $\mathcal{R}$ in $\bar{\mathcal{M}}$. Then $\bar{\mathcal{R}}$ allows an arithmetic of fractions with respect to $\bar{\mathcal{M}}$, and $\bar{\mathcal{M}}_{\bar{\mathcal{R}}}$ is commutative if and only if $\bar{\mathcal{M}}$ is. Now the map $\phi : \bar{\mathcal{M}}_{\bar{\mathcal{R}}} \to \sigma(\mathcal{M})/\sigma(\mathfrak{I})$ defined by $\phi(\bar{m}/\bar{z}) = m/z + \sigma(\mathfrak{I})$ is easily seen to be an isomorphism. Thus we have that $\mathcal{M}/\mathfrak{I}$ is commutative if and only if $\sigma(\mathcal{M})/\sigma(\mathfrak{I})$ is commutative.

Now set $\hat{\mathcal{M}} = c(\hat{\mathfrak{h}}, \hat{\mathfrak{I}})$. Since both $\hat{\mathcal{M}}$ and $\hat{\mathfrak{I}}$ contain $\mathfrak{u}(\hat{\mathfrak{g}})(i - \hat{I})$, by Lemma 3.4.1 and Proposition 3.4.2,

$$\sigma(\mathcal{M})/\sigma(\mathfrak{I}) = h(\hat{\mathcal{M}})/h(\hat{\mathfrak{I}}) \simeq \hat{\mathcal{M}}/\hat{\mathfrak{I}}.$$

The Corollary follows. ∎

Finally we make good on our claim that if $\mathfrak{g}$ is nilpotent, then $(\hat{\mathfrak{g}}, \hat{\mathfrak{h}}, \hat{f})$ admits a Kirillov structure. Actually this is not necessarily true for any associated triple; indeed if we choose $\mathfrak{z} = (0)$, then the associated triple is the same as its progenitor $(\mathfrak{g}, \mathfrak{h}, f)$. The point is that $\mathfrak{z}$ must be sufficiently large.

Proposition 3.4.4. *Let* $(\mathfrak{g}, \mathfrak{h}, f)$ *be a triple with* $\mathfrak{g}$ *nilpotent, and let* $\mathfrak{z}$ *be maximal among all* $\mathfrak{I}$*-central ideals in* $\mathfrak{g}$. *If* $\mathfrak{z} \neq \mathfrak{g}$, *then* $(\hat{\mathfrak{g}}, \hat{\mathfrak{h}}, \hat{f})$ *admits a Kirillov structure.*

Proof. Since $\mathfrak{g}$ is nilpotent and $\mathfrak{z} \neq \mathfrak{g}$, $c(\mathfrak{g},\mathfrak{z})\cap\mathfrak{g} \neq \mathfrak{z}$; choose $Y \in c(\mathfrak{g},\mathfrak{z})\cap\mathfrak{g}$ but not in $\mathfrak{z}$. Then $y \in c(\hat{\mathfrak{g}}, \mathbf{K}i) \cap \hat{\mathfrak{g}}$, so the dimension of $\hat{\mathfrak{g}}/(c(y,\hat{\mathfrak{I}}) \cap \hat{\mathfrak{g}})$ over $\mathbf{K}$ is no greater than one. By maximality of $\mathfrak{z}$, we have $\mathfrak{g} \not\subseteq c(Y,\mathfrak{I})$, so if $X \in \mathfrak{g}\backslash c(Y,\mathfrak{I})$, then $x \in \hat{\mathfrak{g}}\backslash c(y,\hat{\mathfrak{I}})$. Thus $(\hat{\mathfrak{g}}, \hat{\mathfrak{h}}, \hat{f})$ admits the Kirillov structure $(\mathbf{K}i, y)$. ■

Note that if $\mathfrak{z} = \mathfrak{g}$, then for any $X \in \mathfrak{g}$, one has $[X,\mathfrak{g}] \subset \mathfrak{h} \cap \ker(f)$, so $X \in \mathfrak{g}(\ell)$ holds for all $\ell \in L_f$ and $d(\mathfrak{g},\mathfrak{h},f) = 0$. The converse is also true: if $d(\mathfrak{g},\mathfrak{h},f) = 0$ then $\mathfrak{g}$ is $\mathfrak{I}$-central.

3.5 Let $(\mathfrak{g},\mathfrak{h},f)$ be a triple with the associated triple $(\hat{\mathfrak{g}},\hat{\mathfrak{h}},\hat{f})$, let $S(\mathfrak{g})$ denote the symmetric algebra of $\mathfrak{g}$, and let $\omega : S(\mathfrak{g}) \to \mathfrak{u}(\mathfrak{g})$ be the symmetrization mapping. We have a natural surjective homomorphism $\tilde{\pi} : S(\mathfrak{g}) \to S(\pi(\mathfrak{g}))$, and in view of the inclusion $\pi(\mathfrak{g}) \hookrightarrow \hat{\mathfrak{g}}$, we have a natural injection $\iota : S(\pi(\mathfrak{g})) \to S(\hat{\mathfrak{g}})$.

Whenever convenient we regard $S(\mathfrak{g})$ as the algebra of polynomial functions on $\mathfrak{g}^*$ in the usual way. Set

$$L_{f_0} = \{\ell \in \mathfrak{g}^* : \ell|_{\mathfrak{h}_0} = f_0|_{\mathfrak{h}_0}\},$$

and let J be the ideal in $S(\mathfrak{g})$ of elements which vanish on L_{f_0}. Then J is the ideal generated by the elements in $S(\mathfrak{h}_0)$ for which f_0 is a zero, and $S(\mathfrak{g})/J$ is canonically isomorphic with the algebra $\mathbf{k}[L_{f_0}]$ of $\mathbf{k}$-valued polynomial functions on L_{f_0}. We denote this algebra by A and the canonical map from $S(\mathfrak{g})$ onto A by Π. For $P \in S(\mathfrak{g})$, denote $\Pi(P)$ by $\bar{P}$. Set $R = \Pi(S(\mathfrak{z}))$ and form the fraction ring $B = A_R$. We have a canonical injective homomorphism from A into B, and we thereby regard A as a subset of B. Let $p_{\mathfrak{z}} : \mathfrak{g}^* \to \mathfrak{z}^*$ be the restriction mapping. The field of fractions $\bar{\mathbf{K}} = \{\bar{Z}_1/\bar{Z}_2, \bar{Z}_1 \in R, \bar{Z}_2 \in R^*\}$ in B can be regarded as the field $\mathbf{k}[p(L_{f_0})]$ of $\mathbf{k}$-valued rational functions on $p(L_{f_0})$. On the other hand, the map $Z \to \pi(\omega(Z))$ from $S(\mathfrak{z})$ to $\mathcal{R}$ is surjective with kernel $S(\mathfrak{z}) \cap J$. Hence $R \simeq \mathcal{R}$ and $\bar{\mathbf{K}} \simeq \mathbf{K}$ in a canonical way, and B is an algebra over the field $\mathbf{K}$. More explicitly, let $P \in S(\mathfrak{g})$ and write

$$P = \sum_{\alpha,\beta,\mu,\nu} a_{\alpha\beta\mu\nu} X^\alpha Y^\beta Z^\mu T^\nu.$$

Then

$$\bar{P} = \sum_{\alpha,\beta} \left(\sum_{\mu,\nu} a_{\alpha\beta\mu\nu} f(T)^\nu \bar{Z}^\mu \right) \bar{X}^\alpha \bar{Y}^\beta.$$

This shows that the monomials $\bar{X}^\alpha\bar{Y}^\beta, \alpha \in \mathbf{Z}_+^r, \beta \in \mathbf{Z}_+^s$, span A as an R-module. In the same way as for the corresponding objects in $\mathfrak{u}(\mathfrak{g})$, we see that in fact the monomials $\bar{X}^\alpha\bar{Y}^\beta, \alpha \in \mathbf{Z}_+^r, \beta \in \mathbf{Z}_+^s$ constitute an R-basis for A, and hence a $\mathbf{K}$ basis for B.

In $\hat{\mathfrak{g}}^*$, set

$$L_{i^*} = \{\ell \in \hat{\mathfrak{g}}^* : \ell(i) = 1\}.$$

We regard $S(\hat{\mathfrak{g}})/S(\hat{\mathfrak{g}})(i-\hat{I})$ as the algebra $\mathbf{K}[L_{i^*}]$ of $\mathbf{K}$-valued polynomial functions on L_{i^*}, and we denote the image of an element $p \in S(\hat{\mathfrak{g}})$ in $S(\hat{\mathfrak{g}})/S(\hat{\mathfrak{g}})(i-\hat{I})$ by $\bar{p}$. Given $p \in S(\hat{\mathfrak{g}})$, write

$$p = \sum_{\alpha,\beta,\mu} q_{\alpha\beta\mu} x^\alpha y^\beta i^\mu;$$

then

$$\bar{p} = \sum_{\alpha,\beta} \left(\sum_{\mu} q_{\alpha\beta\mu} \right) \bar{x}^{\alpha} \bar{y}^{\beta}.$$

In the same way as Proposition 3.3.3, we find that the monomials $\bar{x}^{\alpha}\bar{y}^{\beta}, \alpha \in \mathbf{Z}_{+}^{r}, \beta \in \mathbf{Z}_{+}^{s}$, constitute a $\mathbf{K}$-basis for $S(\hat{\mathfrak{g}})/S(\hat{\mathfrak{g}})(i-\hat{I})$. Thus we have an obvious isomorphism between B and $S(\hat{\mathfrak{g}})/S(\hat{\mathfrak{g}})(i-\hat{I})$.

Now for $P \in S(\mathfrak{g})$, let $\tilde{P}$ be the restriction of P to L_f. In a similar way we find that $\mathbf{k}[L_f]$ has as a $\mathbf{k}$-basis the monomials $\tilde{X}^{\alpha}\tilde{Z}^{\mu}, \alpha \in \mathbf{Z}_{+}^{r}, \mu \in \mathbf{Z}_{+}^{t}$, and that it has the monomials $\tilde{X}^{\alpha}, \alpha \in \mathbf{Z}_{+}^{r}$ as an R-basis. Also the monomials $\tilde{x}^{\alpha}, \alpha \in \mathbf{Z}_{+}^{r}$ constitute a $\mathbf{K}$-basis for $\mathbf{K}[L_{\hat{f}}]$. Note that $p_{\mathfrak{z}}(L_f) = p_{\mathfrak{z}}(L_{f_0})$ and so for all $Z \in S(\mathfrak{z})$, $\tilde{Z} = \bar{Z}$. We have a mapping

$$\sum_{\alpha,\mu} a_{\alpha\mu} \tilde{Z}^{\mu} \tilde{X}^{\alpha} \longrightarrow \sum_{\alpha} \left(\sum_{\mu} a_{\alpha\mu} z^{\mu} \right) \tilde{x}^{\alpha}$$

which is an injective homomorphism of $\mathbf{k}[L_f]$ onto the subring of R-valued elements of $\mathbf{K}[L_{\hat{f}}]$. We sum up the preceding in the following commutative diagram where the vertical arrows are surjective restriction mappings, and the horozontal arrows are monomorphisms.

$$\begin{array}{ccc} S(\mathfrak{g}) & & \\ \tilde{\pi}\big\downarrow & & \\ S(\pi(\mathfrak{g})) & \xrightarrow{\iota} & S(\hat{\mathfrak{g}}) \\ \eta\big\downarrow & & \hat{\eta}\big\downarrow \\ \mathbf{k}[L_{f_0}] & \xrightarrow{\iota_0} & \mathbf{K}[L_{i^*}] \\ \theta\big\downarrow & & \hat{\theta}\big\downarrow \\ \mathbf{k}[L_f] & \xrightarrow{\epsilon} & \mathbf{K}[L_{\hat{f}}] \end{array} \tag{3.5.1}$$

3.6 Let $\mathfrak{g}$ be a nilpotent Lie algebra over $\mathbf{k}$ and let $\mathfrak{h}$ be a subalgebra. It is desirable to describe in explicit terms a scheme for layering the $\exp(ad^*(\mathfrak{h}))$-orbits in $\mathfrak{g}^*$. For simplicity of notation we set $\exp(ad^*(Y)) = \alpha(Y), Y \in \mathfrak{h}$. Let $(0) = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_n = \mathfrak{g}$ be a sequence of $\alpha(\mathfrak{h})$-modules in $\mathfrak{g}$ with $\dim(\mathfrak{g}_j) = j, 1 \leq j \leq n$, and $(0) = \mathfrak{h}_0 \subset \mathfrak{h}_1 \subset \cdots \subset \mathfrak{h}_p = \mathfrak{h}$ a Jordan-Holder sequence of ideals in $\mathfrak{h}$. Choose $X_j \in \mathfrak{g}_j \backslash \mathfrak{g}_{j-1}$ and $Y_j \in \mathfrak{h}_j \backslash \mathfrak{h}_{j-1}$.

Fix $\ell \in \mathfrak{g}^*$. Our scheme involves the definition of a pair $(\phi(\ell), \psi(\ell))$ of index sequences associated with ℓ and the $\alpha(\mathfrak{h})$-module sequences chosen above. If the $\alpha(\mathfrak{h})$-orbit of ℓ is just $\{\ell\}$, then set $\phi(\ell) = \emptyset$ and $\psi(\ell) = \emptyset$. Otherwise set $\phi_1(\ell) = \min\{j : \mathfrak{g}_j \not\subseteq \mathfrak{h}^{\ell}\}$, and $\psi_1(\ell) = \min\{j : \mathfrak{h}_j \not\subseteq (\mathfrak{g}_{\phi_1(\ell)})^{\ell}\}$. Define the codimension one ideal $\mathfrak{h}^1(\ell)$ by $\mathfrak{h}^1(\ell) = \mathfrak{h} \cap \mathfrak{g}^{\ell}_{\phi_1(\ell)}$; then $\dim(\alpha(\mathfrak{h})(\ell)) = 1$ if and only if $\mathfrak{h}^1(\ell) = \mathfrak{h}(\ell) = \mathfrak{h} \cap \mathfrak{g}^{\ell}$, and if $\dim(\alpha(\mathfrak{h})(\ell)) = 1$, then $\alpha(\mathfrak{h})(\ell) = \{\alpha(tY_{\psi_1(\ell)})(\ell) : t \in \mathbf{k}\}$, and we are done. If $\dim(\alpha(\mathfrak{h})(\ell)) > 1$, then set

$$\phi_2(\ell) = \min\{j : \mathfrak{g}_j \not\subseteq (\mathfrak{h}^1(\ell))^{\ell}\}$$

and
$$\psi_2(\ell) = \min\{j : \mathfrak{h}_j \cap \mathfrak{g}^\ell_{\phi_1(\ell)} \not\subseteq (\mathfrak{g}_{\phi_2(\ell)})^\ell\}.$$
Note that $\phi_1(\ell) < \phi_2(\ell)$. Define the subalgebra $\mathfrak{h}^2(\ell)$ by $\mathfrak{h}^2(\ell) = \mathfrak{h} \cap \mathfrak{g}^\ell_{\phi_2(\ell)}$; and we have $\dim(\alpha(\mathfrak{h})(\ell)) = 2$ if and only if $\mathfrak{h}^2(\ell) = \mathfrak{h}(\ell)$. If $\dim(\alpha(\mathfrak{h})(\ell)) = 2$, then $\alpha(\mathfrak{h})(\ell) = \{\alpha(t_1 Y_{\psi_1(\ell)})\alpha(t_2 Y_{\psi_2(\ell)}(\ell))(\ell) : t_1, t_2 \in \mathbf{k}\}$. Here $Y_{\psi_2(\ell)}(\ell)$ is the projection of $Y_{\psi_2(\ell)}$ into $\mathfrak{h}^1(\ell)$ parallel to $Y_{\psi_1(\ell)}$; explicitly,
$$Y_{\psi_2(\ell)}(\ell) = Y_{\psi_2(\ell)} - \frac{\ell([Y_{\psi_2(\ell)}, X_{\phi_1(\ell)}])}{\ell([Y_{\psi_1(\ell)}, X_{\phi_1(\ell)}])} Y_{\psi_1(\ell)}.$$

If $\dim(\alpha(\mathfrak{h})(\ell)) = 2$, then we stop here; otherwise continue the process until the following objects are obtained.
(a) Index sequences $\phi(\ell)$ and $\psi(\ell)$ where $\sharp(\phi(\ell)) = \sharp(\psi(\ell)) = \delta(\ell)$ and $\phi(\ell)$ is increasing.
(b) Subalgebras $\mathfrak{h}(\ell) = \mathfrak{h}^{\delta}(\ell) \subset \cdots \subset \mathfrak{h}^1(\ell) \subset \mathfrak{h}$, where $\mathfrak{h}^k(\ell)$ is codimension k in $\mathfrak{h}$. (Here $\delta = \delta(\ell)$.)
(c) Elements $Y_{\psi_1(\ell)} = Y_{\psi_1(\ell)}(\ell), Y_{\psi_2(\ell)}(\ell), \ldots, Y_{\psi_\delta(\ell)}(\ell)$ in $\mathfrak{h}$ with $Y_{\psi_k(\ell)}(\ell) \in \mathfrak{h}^{k-1}(\ell)\backslash\mathfrak{h}^k(\ell)$, and
$$\alpha(\mathfrak{h})(\ell) = \{\alpha(t_1 Y_{\psi_1(\ell)}))\alpha(t_2 Y_{\psi_2(\ell)}(\ell)) \cdots \alpha(t_\delta Y_{\psi_\delta(\ell)}(\ell))(\ell) : (t_1, t_2, \ldots, t_\delta) \in \mathbf{k}^\delta\}.$$
Let $\Phi = \{(\phi(\ell), \psi(\ell)) : \ell \in \mathfrak{g}^*\}$ and for each $(\phi, \psi) \in \Phi$, set
$$\Omega_{\phi,\psi} = \{\ell \in \mathfrak{g}^* : (\phi(\ell), \psi(\ell)) = (\phi, \psi)\}.$$
Obviously $\mathcal{P} = \{\Omega_{\phi,\psi} : (\phi, \psi) \in \Phi\}$ is a partition of $\mathfrak{g}^*$ and it is easily seen that each element of $\mathcal{P}$ is $\alpha(\mathfrak{h})$-invariant.

We shall now define an ordering $\prec$ on *all* pairs (ϕ, ψ) of index sequences with $1 \le \phi_1 < \phi_2 < \cdots < \phi_\delta \le n$ and $1 \le \psi_j \le p$. For two pairs (ϕ, ψ) and (ϕ', ψ') of index sequences as above, we say that $(\phi, \psi) \prec (\phi', \psi')$ if one of the following conditions hold:
(i) $\sharp(\phi) > \sharp(\phi')$, or
(ii) if $\sharp(\phi) = \sharp(\phi')$ and $k = \min\{j : (\phi_j, \psi_j) \neq (\phi'_j, \psi'_j)\}$, then $\phi_k < \phi'_k$, or
(iii) if $\sharp(\phi) = \sharp(\phi')$ and $k = \min\{j : (\phi_j, \psi_j) \neq (\phi'_j, \psi'_j)\}$, then $\phi_k = \phi'_k$ and $\psi_k < \psi'_k$.

Next we define polynomial functions $Q_{\phi,\psi}$ on $\mathfrak{g}^*$ that determine the layers $\Omega \in \mathcal{P}$. Fix an any index sequence pair (ϕ, ψ) with $\delta = \sharp(\phi)$, and for each $k, 1 \le k \le \delta$, set
$$Q^k_{\phi,\psi}(\ell) = \det\ [\ell\,([Y_{\psi_r}, X_{\phi_s}])]_{1 \le r,s \le k},$$
and define
$$Q_{\phi,\psi}(\ell) = Q^1_{\phi,\psi}(\ell) Q^2_{\phi,\psi}(\ell) \cdots Q^\delta_{\phi,\psi}(\ell).$$

Lemma 3.6.1. *Let (ϕ, ψ) be an index sequence pair with $\delta = \sharp(\phi)$. For any $\ell \in \mathfrak{g}^*$, if $Q_{\phi,\psi}(\ell) \neq 0$, then $\delta(\ell) \ge \delta$.*

Proof. Suppose that $Q_{\phi,\psi}(\ell) \neq 0$. Then the scheme above can be used to define δ elements
$$Y_{\psi_1}(\ell), Y_{\psi_2}(\ell), \ldots, Y_{\psi_\delta}(\ell)$$
in $\mathfrak{h}$ with the property that for each $1 \le k \le \delta$, $\ell([Y_{\psi_k}(\ell), X_{\phi_j}]) = 0$ if $j < k$ and $\ell([Y_{\psi_k}(\ell), X_{\phi_k}]) \neq 0$. It follows that $\delta(\ell) = \dim(\mathfrak{h}/\mathfrak{h}(\ell)) \ge \delta$. ∎

Lemma 3.6.2. *For any $\ell \in \mathfrak{g}^*$, we have*

$$(\phi(\ell), \psi(\ell)) = min\{(\phi, \psi) : Q_{\phi,\psi}(\ell) \neq 0\}.$$

Proof. Suppose that $(\phi, \psi) \prec (\phi(\ell), \psi(\ell))$; we show that $Q_{\phi,\psi}(\ell) = 0$. Set $\delta = \sharp(\phi)$. If $\delta > \delta(\ell)$, then by Lemma 3.6.1, $Q_{\phi,\psi}(\ell) = 0$. Suppose that $\delta = \delta(\ell)$. Set $k = \min\{j : (\phi_j, \psi_j) \neq (\phi_j(\ell), \psi_j(\ell))\}$. By definition of $\prec$, we have either $\phi_k < \phi_k(\ell)$, or $\phi_k = \phi_k(\ell)$ and $\psi_k < \psi_k(\ell)$. Let $Y_{\psi_k}(\ell)$ be the projection of Y_{ψ_k} into $\mathfrak{h}^{k-1}(\ell)$ parallel to $\mathbf{k}$-span$\{Y_{\psi_1}(\ell), Y_{\psi_2}(\ell), \dots, Y_{\psi_{k-1}}(\ell)\}$. If $\phi_k < \phi_k(\ell)$, then $X_{\phi_k} \in (\mathfrak{h}^{k-1}(\ell))^\ell$, so $\ell([Y_{\psi_k}(\ell), X_{\phi_k}]) = 0$. Otherwise $\phi_k = \phi_k(\ell)$ and $\psi_k < \psi_k(\ell)$, whence $Y_{\psi_k}(\ell) \in (\mathfrak{g}_{\phi_k})^\ell$ so again $\ell([Y_{\psi_k}(\ell), X_{\phi_k}]) = 0$. Now

$$Q^k_{\phi,\psi}(\ell) = \ell([Y_{\psi_1}(\ell), X_{\phi_1}(\ell)])\ell([Y_{\psi_2}(\ell), X_{\phi_2}(\ell)]) \cdots \ell([Y_{\psi_k}(\ell), X_{\phi_k}(\ell)]),$$

which shows that $Q_{\phi,\psi}(\ell) = 0$. The Lemma follows. ∎

The following description of the layers is immediate.

Proposition 3.6.3. *With all the data retained from the above, fix an index sequence pair $(\phi, \psi) \in \Phi$. Then*

$$\Omega_{\phi,\psi} = \{\ell \in \mathfrak{g}^* : Q_{\phi',\psi'}(\ell) = 0, \text{ for all } (\phi', \psi') \text{ with } (\phi', \psi') \prec (\phi, \psi) \text{ and } Q_{\phi,\psi}(\ell) \neq 0\}.$$

3.7 In this subsection we show that generic orbital dimensions in L_f are preserved when passing to $L_{\hat{f}}$. Let $(\mathfrak{g}, \mathfrak{h}, f)$ be a triple with $\mathfrak{g}$ nilpotent, let $\mathfrak{z}$ be an $\mathfrak{I}$-central ideal in $\mathfrak{g}$, and let $(\hat{\mathfrak{g}}, \hat{\mathfrak{h}}, \hat{f})$ be the associated triple. Choose a Jordan-Holder sequence $(0) = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_n = \mathfrak{g}$ with the following properties.

(a) For some $a, \mathfrak{g}_a = \mathfrak{z}$.

(b) For some $u, \mathfrak{g}_u = \mathfrak{h} \cap \mathfrak{z}$ (recall that $\mathfrak{h} \cap \mathfrak{z}$ must be an ideal in $\mathfrak{g}$.)

There are indices $1 \leq j_1 < j_2 < \cdots < j_p \leq n$ for which $\mathfrak{h} \cap \mathfrak{g}_{j_{k-1}} \subsetneq \mathfrak{h} \cap \mathfrak{g}_{j_k}$ and for which $(0) = \mathfrak{h}_0 \subset \mathfrak{h}_1 \subset \cdots \subset \mathfrak{h}_p = \mathfrak{h}$ is a Jordan Holder sequence for $\mathfrak{h}$

Choose $X'_j \in \mathfrak{g}_j \backslash \mathfrak{g}_{j-1}, 1 \leq j \leq n$ such that $X'_{j_k} \in \mathfrak{h}_k \backslash \mathfrak{h}_{k-1}, 1 \leq k \leq p$. Set $Y'_k = X'_{j_k}, 1 \leq k \leq p$. Denoting this basis by B, the first u elements of B are a basis for $\mathfrak{z} \cap \mathfrak{h}$, and the first a elements of B are a basis for $\mathfrak{z}$. With reordering if necessary, B can be written

$$B = \{T_1, T_2, \dots, T_u, Z_1, Z_2, \dots, Z_t, Y_1, Y_2, \dots, Y_s, X_1, X_2, \dots, X_r\}$$

as in Section 3.1 (a = t+u). There is a corresponding Jordan-Holder sequence

$$(0) = \hat{\mathfrak{g}}_0 \subset \hat{\mathfrak{g}}_1 \subset \hat{\mathfrak{g}}_2 \subset \cdots \subset \hat{\mathfrak{g}}_m = \hat{\mathfrak{g}}$$

where $\hat{\mathfrak{g}}_1 = \hat{\mathfrak{z}} = \mathbf{K}i$, $\hat{\mathfrak{g}}_2 = (\mathfrak{g}_{a+1})\hat{}$, etc., with a compatible basis $\{x'_l : 1 \leq l \leq m\}$ that can be written (again with reodering) as

$$\{i, y_1, y_2, \dots, y_s, x_1, x_2, \dots, x_r\}$$

(where $x'_1 = i$). Let $\mathcal{P}_{\mathfrak{h}}$ be the layering for the $\alpha(\mathfrak{h})$-orbits as defined in Section 3.6 relative to the sequences of $\alpha(\mathfrak{h})$-modules for $\mathfrak{g}$ and $\mathfrak{h}$ chosen above, and in the same way let $\mathcal{P}_{\hat{\mathfrak{h}}}$ be the layering for the $\alpha(\hat{\mathfrak{h}})$-orbits in $\hat{\mathfrak{g}}^*$ relative to the corresponding sequences of $\alpha(\hat{\mathfrak{h}})$-modules for $\hat{\mathfrak{g}}$ and $\hat{\mathfrak{h}}$. Now given any index sequence pair (ϕ, ψ) with the property that $\phi_1 > a$ and $\psi_j > u, 1 \le j \le p$, define $(\hat{\phi}, \hat{\psi})$ by $\hat{\phi}_j = \phi_j - a + 1$ and $\hat{\psi}_j = \psi_j - u + 1$. This obviously defines a bijection between the set of all index sequence pairs (ϕ, ψ) relative to the chosen $\alpha(\mathfrak{h})$-modules with $\phi_1 > a$ and $\psi_j > u, 1 \le j \le p$, and the set of all index sequence pairs relative to the $\alpha(\hat{\mathfrak{h}})$-modules whose values are greater than one. We denote elements of $S(\hat{\mathfrak{g}})$ by lower case letters.

Lemma 3.7.1. *Let (ϕ, ψ) be an index sequence pair relative the above $\alpha(\mathfrak{h})$-modules with the property that $\phi_1 > a$ and $\psi_j > u, 1 \le j \le p$. Then*

$$\hat{\eta}(q_{\hat{\phi},\hat{\psi}}) = \iota_0(\eta(\tilde{\pi}(Q_{\phi,\psi}))).$$

Proof. It is enough to show that

$$\hat{\eta}(q^k_{\hat{\phi},\hat{\psi}}) = \iota_0(\eta(\tilde{\pi}(Q^k_{\phi,\psi})))$$

holds for each $1 \le k \le \delta = \sharp(\phi)$. We have

$$Q^k_{\phi,\psi} = \sum_{\sigma \in S_k} sign(\sigma)[Y'_{\psi_1}, X'_{\phi_{\sigma(1)}}][Y'_{\psi_2}, X'_{\phi_{\sigma(2)}}] \cdots [Y'_{\psi_k}, X'_{\phi_{\sigma(k)}}]$$

and so

$$\begin{aligned}
&\iota\tilde{\pi}(Q^k_{\phi,\psi}) \\
&= \iota\left(\sum_{\sigma \in S_k} sign(\sigma)[\pi(Y'_{\psi_1}), \pi(X'_{\phi_{\sigma(1)}})][\pi(Y'_{\psi_2}), \pi(X'_{\phi_{\sigma(2)}})] \cdots [\pi(Y'_{\psi_k}), \pi(X'_{\phi_{\sigma(k)}})]\right) \\
&= \iota\left(\sum_{\sigma \in S_k} sign(\sigma)[y'_{\hat{\psi}_1}, x'_{\hat{\phi}_{\sigma(1)}}][y'_{\hat{\psi}_2}, x'_{\hat{\phi}_{\sigma(2)}}] \cdots [y'_{\hat{\psi}_k}, x'_{\hat{\phi}_{\sigma(k)}}]\right) \\
&= q^k_{\hat{\phi},\hat{\psi}}.
\end{aligned}$$

Now by commutativity of the diagram 3.5.1, we have

$$\iota_0(\eta(\tilde{\pi}(Q^k_{\phi,\psi}))) = \hat{\eta}(\iota(\tilde{\pi}(Q^k_{\phi,\psi}))) = \hat{\eta}(q^k_{\hat{\phi},\hat{\psi}}).$$

■

A version of the above lemma also holds for the index sequence pairs relative to the $\alpha(\mathfrak{g})$-action and the $\alpha(\hat{\mathfrak{g}})$-action. We omit the proof.

Lemma 3.7.2. *Let (ϕ, ψ) be an index sequence pair relative the $\alpha(\mathfrak{g})$-modules chosen above with the property that $\phi_1 > a$ and $\psi_j > u, 1 \le j \le p$. Then $\hat{\eta}(q_{\hat{\phi},\hat{\psi}}) = \iota_0(\eta(\tilde{\pi}(Q_{\phi,\psi})))$.*

Proposition 3.7.3. *Let* $(\mathfrak{g},\mathfrak{h},f)$ *be a triple with associated triple* $(\hat{\mathfrak{g}},\hat{\mathfrak{h}},\hat{f})$. *Then* $d(\mathfrak{g},\mathfrak{h},f)=d(\hat{\mathfrak{g}},\hat{\mathfrak{h}},\hat{f})$ *and* $\delta(\mathfrak{g},\mathfrak{h},f)=\delta(\hat{\mathfrak{g}},\hat{\mathfrak{h}},\hat{f})$.

Proof. We have $d(\mathfrak{g},\mathfrak{h},f)=\sharp(\phi_0)$, where (ϕ_0,ψ_0) is the minimum index sequence pair (ϕ,ψ) for which $\Omega_{\phi,\psi}\cap L_f\neq\emptyset$. By Proposition 3.6.3, this is equal to the minimum index sequence pair (ϕ,ψ) for which $\theta(\eta(\tilde{\pi}(Q_{\phi,\psi})))\neq 0$. A similar statement holds for $d(\hat{\mathfrak{g}},\hat{\mathfrak{h}},\hat{f})$. Applying Lemma 3.7.2 and diagram 3.5.1, we see that the minimum index sequence pair that determines $d(\hat{\mathfrak{g}},\hat{\mathfrak{h}},\hat{f})$ is $(\hat{\phi}_0,\hat{\psi}_0)$. Hence $d(\mathfrak{g},\mathfrak{h},f)=d(\hat{\mathfrak{g}},\hat{\mathfrak{h}},\hat{f})$. A parallel argument shows that $\delta(\mathfrak{g},\mathfrak{h},f)=\delta(\hat{\mathfrak{g}},\hat{\mathfrak{h}},\hat{f})$. ∎

Finally we examine the case where $(\mathfrak{g},\mathfrak{h},f)$ admits a Kirillov structure.

Proposition 3.7.4. *Suppose that* $(\mathfrak{g},\mathfrak{h},f)$ *is a triple with* $\mathfrak{g}$ *nilpotent, and that admits a Kirillov structure* $(\mathfrak{z},Y)$. *Set* $\mathfrak{g}^Y=c(Y,\mathfrak{I})\cap\mathfrak{g}$ *and* $\mathfrak{h}^Y=c(Y,\mathfrak{I})\cap\mathfrak{h}$. *Then* $d(\mathfrak{g}^Y,\mathfrak{h}^Y,f|_{\mathfrak{h}^Y})=d(\mathfrak{g},\mathfrak{h},f)-2$ *and* $\delta(\mathfrak{g}^Y,\mathfrak{h}^Y,f|_{\mathfrak{h}^Y})$ *is equal either to* $\delta(\mathfrak{g},\mathfrak{h},f)$ *or* $\delta(\mathfrak{g},\mathfrak{h},f)-1$.

Proof. Choose a Jordan-Holder sequence for $\mathfrak{g}$ that includes $\mathfrak{z}$, $\mathfrak{z}+\mathbf{k}Y$, and $\mathfrak{g}^Y$, and let $a=\dim(\mathfrak{z})$. Apply the constructions of Section 3.6 with $\mathfrak{h}=\mathfrak{g}$, and we find that for any $\ell\in L_f$, $\phi_1(\ell)>a$, and that $\phi_1(\ell)=a+1$ if and only if $\ell([X,Y])\neq 0$, and in this case $\psi_1(\ell)=n$, and $\mathfrak{g}^Y=\mathfrak{h}^1(\ell)$. It follows that if Ω_{ϕ^0,ψ^0} is the generic layer in $(\mathfrak{g}^Y)^*$ for $L_{f|_{\mathfrak{h}^Y}}$, then the generic layer in $\mathfrak{g}^*$ for L_f is $\Omega_{\phi,\psi}$, where $\phi_1=a+1,\phi_j=\phi^0_{j-1}$ and $\psi_1=n,\psi_j=\psi^0_{j-1},j>1$. This shows that $d(\mathfrak{g}^Y,\mathfrak{h}^Y,f|_{\mathfrak{h}^Y})=d(\mathfrak{g},\mathfrak{h},f)-2$.

Turning to the $\alpha(\mathfrak{h})$-action, a similar analysis shows that if $\mathfrak{h}\neq\mathfrak{h}^Y$, then $\phi_1(\ell)=a+1$ and $\psi_1(\ell)=n$ hold for generic $\ell\in L_f$, whence $\delta(\mathfrak{g}^Y,\mathfrak{h}^Y,f|_{\mathfrak{h}^Y})=\delta(\mathfrak{g},\mathfrak{h},f)-1$. On the other hand, if $\mathfrak{h}=\mathfrak{h}^Y$, then $L_{f|_{\mathfrak{h}^Y}}$ is just the set of restrictions of elements in L_f. For any ℓ, if ℓ_0 denotes its restriction to $\mathfrak{g}^Y$, then $\mathfrak{g}^\ell$ is codimension one in $(\mathfrak{g}^Y)^{\ell_0}$, and since $\mathfrak{h}(\ell)=\mathfrak{h}\cap\mathfrak{g}^\ell$ (and similarly for ℓ_0), the two possibilities for $\delta(\mathfrak{g}^Y,\mathfrak{h}^Y,f|_{\mathfrak{h}^Y})$ are evident. ∎

4. Non-commutativity of $\mathcal{D}(\mathfrak{g},\mathfrak{h},f)$

4.1 Let $\mathfrak{g}$ be a nilpotent Lie algebra over a field $\mathbf{k}$ of characteristic zero. Let $0=\mathfrak{u}_{-1}(\mathfrak{g})\subset\mathfrak{u}_0(\mathfrak{g})\subset\mathfrak{u}_1(\mathfrak{g})\subset\cdots\subset\mathfrak{u}(\mathfrak{g})$ be the standard filtration of $\mathfrak{u}(\mathfrak{g})$ and fix a strong Malcev basis $\{X_1,X_2,\ldots,X_n\}$ for $\mathfrak{g}$. Let $\mathcal{A}_m=\{(a_1,a_2,\ldots,a_m):a_i\in\{1,2,\ldots,n\},1\leq i\leq m,a_1\leq a_2\leq\cdots\leq a_m\}$. For each $a\in\mathcal{A}_m$, set $X^a=X_{a_m}X_{a_{m-1}}\ldots X_{a_1}$. Order $\mathcal{A}_m$ as follows: $a<a'$ if $a_k<a'_k$, where $k=\min\{j:a_j\neq a'_j\}$. Set

$$\mathfrak{u}^a_m(\mathfrak{g})=\mathfrak{u}_{m-1}(\mathfrak{g})\oplus\operatorname{span}\{X^{a'}:a'\leq a\}.$$

We have a total ordering on the set of pairs $P=\{(m,a):m\geq 0,a\in\mathcal{A}_m\}$ that has the following property. For $p=(m,a)\in P$ denote $\mathfrak{u}^a_m(\mathfrak{g})$ by $\mathfrak{u}_p$. If $Y\in\mathfrak{g}$, then $[Y,\mathfrak{u}_p]\subset\mathfrak{u}_{p-1}$, where $p-1$ is the predecessor of p.

Proposition 4.1.1. *Let $(\mathfrak{g}, \mathfrak{h}, f)$ be a triple that admits a Kirillov structure $(\mathfrak{z}, Y)$. Set $\mathfrak{g}^Y = c(Y, \mathcal{I}) \cap \mathfrak{g}$ and $\mathfrak{h}^Y = c(Y, \mathcal{I}) \cap \mathfrak{h}$. Assume that $\mathfrak{h} \neq \mathfrak{h}^Y$. Then*

(i) there is a monomorphism of $\mathcal{D}(\mathfrak{g}, \mathfrak{h}, f)$ into $\mathcal{D}(\mathfrak{g}^Y, \mathfrak{h}^Y, f|_{\mathfrak{h}^Y})$, and

(ii) $\mathcal{D}(\mathfrak{g}, \mathfrak{h}, f)$ is commutative if and only if $\mathcal{D}(\mathfrak{g}^Y, \mathfrak{h}^Y, f|_{\mathfrak{h}^Y})$ is commutative.

Proof. Choose $X \in \mathfrak{h} \backslash \mathfrak{g}^Y$, and set $\Xi = \mathfrak{u}(\mathfrak{g})(X - f(X)I)$. By Proposition 2.1.1, we have $\mathfrak{u}(\mathfrak{g}) = \mathfrak{u}(\mathfrak{g}^Y) \oplus \Xi$. Since $\Xi \subset c(\mathfrak{h}, \mathcal{I})$, it follows that $c(\mathfrak{h}, \mathcal{I}) = c(\mathfrak{h}, \mathcal{I}) \cap \mathfrak{u}(\mathfrak{g}^Y) \oplus \Xi$. Let $\mathcal{I}^Y$ be the left ideal in $\mathfrak{u}(\mathfrak{g}^Y)$ generated by the kernel of $f|_{\mathfrak{h}^Y}$, and $c(\mathfrak{h}^Y, \mathcal{I}^Y)$ the centralizer in $u(\mathfrak{g}^Y)$ of $\mathfrak{h}^Y$ modulo $\mathcal{I}^Y$. Then Corollary 2.1.3 gives $\mathcal{I}^Y = \mathcal{I} \cap \mathfrak{u}(\mathfrak{g}^Y)$, and again by Proposition 2.1.1, $\mathcal{I} = \mathcal{I}^Y \oplus \Xi$. Thus

$$\mathcal{D}(\mathfrak{g}, \mathfrak{h}, f) \simeq c(\mathfrak{h}, \mathcal{I}) \cap \mathfrak{u}(\mathfrak{g}^Y)/\mathcal{I}^Y.$$

But $c(\mathfrak{h}, \mathcal{I}) \cap \mathfrak{u}(\mathfrak{g}^Y) \subset c(\mathfrak{h}^Y, \mathcal{I}^Y)$. Thus $\mathcal{D}(\mathfrak{g}, \mathfrak{h}, f)$ injects into $\mathcal{D}(\mathfrak{g}^Y, \mathfrak{h}^Y, f|_{\mathfrak{h}^Y})$, and (i) is proved.

It is immediate from (i) that if $\mathcal{D}(\mathfrak{g}^Y, \mathfrak{h}^Y, f|_{\mathfrak{h}^Y})$ is commutative, then $\mathcal{D}(\mathfrak{g}, \mathfrak{h}, f)$ is also. Assume then that $\mathcal{D}(\mathfrak{g}^Y, \mathfrak{h}^Y, f|_{\mathfrak{h}^Y})$ is not commutative. Let

$$p_1 = \min\{p \in P : c(\mathfrak{h}^Y, \mathcal{I}^Y) \cap \mathfrak{u}_p(\mathfrak{g}^Y) \not\subseteq c(c(\mathfrak{h}^Y, \mathcal{I}^Y), \mathcal{I}^Y)\}$$

and let $W \in c(\mathfrak{h}^Y, \mathcal{I}^Y) \cap \mathfrak{u}_{p_1}(\mathfrak{g}^Y) \backslash c(c(\mathfrak{h}^Y, \mathcal{I}^Y), \mathcal{I}^Y)$. Now let

$$q_1 = \min\{q \in P : c(\mathfrak{h}^Y, \mathcal{I}^Y) \cap \mathfrak{u}_q(\mathfrak{g}^Y) \not\subseteq c(W, \mathcal{I}^Y)\}$$

and choose $V \in c(\mathfrak{h}^Y, \mathcal{I}^Y) \cap \mathfrak{u}_{q_1}(\mathfrak{g}^Y) \backslash c(W, \mathcal{I}^Y)$. Set

$$r = \max\{j : (adX)^j V \neq 0\} \text{ and } s = \max\{j : (adX)^j W \neq 0\}.$$

Now by definition of Kirillov structure, $[X, Y] \in \mathfrak{z} \backslash \mathcal{I}$. For any positive integer j we have $(adX)^j (Y^j) = j! [X, Y]^j$ modulo $\mathcal{I}$, hence $(adX)^j (Y^j) \notin \mathcal{I}$ and $(adX)^{j+1} (Y^j) \in \mathcal{I} \cap \mathfrak{u}(\mathfrak{z})$. Define

$$\begin{aligned}\rho(V) = V(adX)^r(Y^r) - adX(V)(adX)^{r-1}(Y^r) &+ (adX)^2(V)(adX)^{r-2}(Y^r) \\ &- \cdots + (-1)^r (adX)^r (V) Y^r\end{aligned}$$

and

$$\begin{aligned}\rho(W) = W(adX)^s(Y^s) - adX(W)(adX)^{s-1}(Y^s) &+ (adX)^2(W)(adX)^{s-2}(Y^s) \\ &- \cdots + (-1)^s (adX)^s (W) Y^s.\end{aligned}$$

Then it is easily seen that $[X, \rho(V)] = [X, \rho(W)] = 0$. Now I claim that $\rho(V)$ belongs to $c(\mathfrak{h}^Y, \mathcal{I})$. For this it is enough to show that for each j, $(adX)^j V$ belongs to $c(\mathfrak{h}^Y, \mathcal{I}^Y)$, since $(adX)^k (Y^r) \in \mathfrak{u}(\mathbf{k}Y + \mathfrak{z}) \subset c(\mathfrak{g}^Y, \mathcal{I}^Y)$ for all k. But this fact follows from an easy induction argument and the formula

$$[S, (adX)^j (V)] = [[S, X], (adX)^{j-1}(V)] + [X, [S, (adX)^{j-1}(V)]].$$

Therefore $\rho(V)$ and $\rho(W)$ belong to $c(\mathfrak{h}, \mathcal{I})$. It remains to show that $[\rho(V), \rho(W)]$ does not belong to $\mathcal{I}$. Since $(adX)^j (Y^r)$ belongs to $\mathfrak{u}(\mathbf{k}Y + \mathfrak{z})$, then $(adX)^j (Y^r)$

lies in the center of $\mathfrak{u}(\mathfrak{g}^Y)$ modulo $\mathcal{J}$, where $\mathcal{J}$ is the (two-sided) ideal in $\mathfrak{u}(\mathfrak{g})$ generated by $\mathfrak{I} \cap \mathfrak{u}(\mathfrak{z})$. Hence we can write

$$[\rho(V), \rho(W)] = [V, W]r!s![X, Y]^{r+s} + \sum_{j+k\geq 1} (-1)^{j+k}[(adX)^j(V), (adX)^k(W)](adX)^{r-j}(Y^r)(adX)^{s-k}(Y^s) \text{ modulo } \mathcal{J}.$$

Now by definition of p_1, $(adX)^k(W)$ lies in $c(c(\mathfrak{h}^Y, \mathfrak{I}^Y), \mathfrak{I}^Y)$ for all $k > 0$. Hence every term in the above sum with $k > 0$ lies in $\mathfrak{I}^Y$. On the other hand, if $j > 0$, then $adX^j(V)$ lies in $c(W, \mathfrak{I}^Y)$, so that the terms for which $j > 0$ and $k = 0$ belong to $\mathfrak{I}^Y$ also. Hence

$$[\rho(V), \rho(W)] = [V, W]r!s![X, Y]^{r+s} \text{ modulo } \mathfrak{I}^Y.$$

Now by choice of V and W, $[V, W] \notin \mathfrak{I}^Y$, and since $\mathfrak{I}^Y = \mathfrak{I} \cap \mathfrak{u}(\mathfrak{g}^Y)$, we have $[V, W] \notin \mathfrak{I}$. Since $[X, Y]^{r+s} \in \mathfrak{u}(\mathfrak{z})\backslash\mathfrak{I}$, Proposition 3.1.2 obtains that $[\rho(V), \rho(W)]$ is not in $\mathfrak{I}$. ■

4.2 Let $(\mathfrak{g}, \mathfrak{h}, f)$ be an triple with $\mathfrak{g}$ nilpotent. Thus far we have established two principles.

(1) There is an associated triple $(\hat{\mathfrak{g}}, \hat{\mathfrak{h}}, \hat{f})$ that admits a Kirillov structure, and for which $d(\mathfrak{g}, \mathfrak{h}, f) = d(\hat{\mathfrak{g}}, \hat{\mathfrak{h}}, \hat{f})$, $\delta(\mathfrak{g}, \mathfrak{h}, f) = \delta(\hat{\mathfrak{g}}, \hat{\mathfrak{h}}, \hat{f})$, and $\mathcal{D}(\mathfrak{g}, \mathfrak{h}, f)$ is commutative if and only if $\mathcal{D}(\hat{\mathfrak{g}}, \hat{\mathfrak{h}}, \hat{f})$ is commutative.

(2) If $(\mathfrak{g}, \mathfrak{h}, f)$ admits a Kirillov structure $(\mathfrak{z}, Y)$, and $\mathfrak{h} + \mathfrak{g}^Y = \mathfrak{g}$, then $\mathcal{D}(\mathfrak{g}, \mathfrak{h}, f)$ is commutative if and only if $\mathcal{D}(\mathfrak{g}^Y, \mathfrak{h}^Y, f|_{\mathfrak{h}^Y})$ is commutative.

What happens in the case where $(\mathfrak{g}, \mathfrak{h}, f)$ admits a Kirillov structure, and $\mathfrak{h} \subset \mathfrak{g}^Y$? At present, all we know in this case is that $\mathcal{D}(\mathfrak{g}^Y, \mathfrak{h}, f) \subset \mathcal{D}(\mathfrak{g}, \mathfrak{h}, f)$ (this is easily seen using Corollary 2.1.3 with $\mathfrak{k} = \mathfrak{g}^Y$), and so if $\mathcal{D}(\mathfrak{g}^Y, \mathfrak{h}, f)$ is not commutative, then $\mathcal{D}(\mathfrak{g}, \mathfrak{h}, f)$ is not commutative.

Based upon the work in [2] and [7], we conjecture that if $\mathfrak{h} \subset \mathfrak{g}^Y$, then

(3) if $\delta(\mathfrak{g}^Y, \mathfrak{h}, f) = \delta(\mathfrak{g}, \mathfrak{h}, f) - 1$, then $\mathcal{D}(\mathfrak{g}^Y, \mathfrak{h}, f) = \mathcal{D}(\mathfrak{g}, \mathfrak{h}, f)$, while

(4) if $\delta(\mathfrak{g}^Y, \mathfrak{h}, f) = \delta(\mathfrak{g}, \mathfrak{h}, f)$, then $\mathcal{D}(\mathfrak{g}^Y, \mathfrak{h}, f) \subsetneq \mathcal{D}(\mathfrak{g}, \mathfrak{h}, f)$.

In any case the above facts (1) and (2) suggest the following reduction procedure. Let $(\mathfrak{g}, \mathfrak{h}, f)$ be a triple with $\mathfrak{g}$ nilpotent and with $d(\mathfrak{g}, \mathfrak{h}, f) > 0$. Choose $\mathfrak{z}$ an $\mathfrak{I}$-central ideal in $\mathfrak{g}$ for which the associated triple $(\hat{\mathfrak{g}}, \hat{\mathfrak{h}}, \hat{f})$ admits a Kirillov structure $(\mathfrak{z}(1), y(1))$. Define the triple $(\mathfrak{g}(1), \mathfrak{h}(1), f(1))$ by $\mathfrak{g}(1) = (\hat{\mathfrak{g}})^{\,y(1)}$, $\mathfrak{h}(1) = (\hat{\mathfrak{h}})^{\,y(1)}$, and $f(1) = \hat{f}\,|_{\mathfrak{h}(1)}$. Note that by Propositions 3.7.3 and 3.7.4, $d(\mathfrak{g}(1), \mathfrak{h}(1), f(1)) = d(\mathfrak{g}, \mathfrak{h}, f) - 2$, and by the preceding, if $\mathcal{D}(\mathfrak{g}(1), \mathfrak{h}(1), f(1))$ is not commutative, then $\mathcal{D}(\mathfrak{g}, \mathfrak{h}, f)$ is not commutative also. Now provided $d(\mathfrak{g}(1), \mathfrak{h}(1), f(1)) > 0$ we can apply the reduction procedure to $(\mathfrak{g}(1), \mathfrak{h}(1), f(1))$ to produce another triple $(\mathfrak{g}(2), \mathfrak{h}(2), f(2))$, and so on. Thus if $\{(\mathfrak{g}(j), \mathfrak{h}(j), f(j)) : 0 \leq j \leq k\}$ is any sequence of triples so obtained, then $\mathfrak{g}(j) = (\mathfrak{g}(j-1)\hat{\ })^{\,y_j}$ where y_j is a Kirillov element in $\mathfrak{g}(j-1)\hat{\ }$, $\mathfrak{h}(j) = (\mathfrak{h}(j-1))\hat{\ } \cap \mathfrak{g}(j)$, and $f(j) = (f(j-1))\hat{\ }|_{\mathfrak{h}(j)}, 1 \leq j \leq k$.

Definition. Let $(\mathfrak{g},\mathfrak{h},f)$ be a triple. A sequence of triples $\{(\mathfrak{g}(j),\mathfrak{h}(j),f(j)) : 0 \le j \le k\}$ will be called a reducing sequence for $(\mathfrak{g},\mathfrak{h},f)$ if

(i) $(\mathfrak{g}(0),\mathfrak{h}(0),f(0)) = (\mathfrak{g},\mathfrak{h},f)$,

(ii) for each $j \ge 1$, $(\mathfrak{g}(j),\mathfrak{h}(j),f(j))$ is obtained from $(\mathfrak{g}(j-1),\mathfrak{h}(j-1),f(j-1))$ by the reduction procedure described above, and

(iii) $(\mathfrak{g}(k),\mathfrak{h}(k),f(k))$ is an FM-triple whose associated coadjoint orbits have zero dimension.

Let $d(j) = d(\mathfrak{g}(j),\mathfrak{h}(j),f(j))$ and $\delta(j) = \delta(\mathfrak{g}(j),\mathfrak{h}(j),f(j))$, $0 \le j \le k$. From Propositions 3.7.3 and 3.7.4, we have $d(j) = d(j-1) - 2, 1 \le j \le k$, and that $\delta(j) = \delta(j-1)$ or $\delta(j-1)-1$. From this and Proposition 3.4.4 the following is immediate.

Proposition 4.2.1. *Let $(\mathfrak{g},\mathfrak{h},f)$ be a triple with $\mathfrak{g}$ nilpotent. Then $(\mathfrak{g},\mathfrak{h},f)$ has a reducing sequence.*

It seems plausible that a determination of the conjectures (3) and (4) above in complete detail, together with the results contained herein, would provide a complete description of $\mathcal{D}(\mathfrak{g},\mathfrak{h},f)$.

In this paper we are concerned with the application of the above to establish non-commutativity of $\mathcal{D}(\mathfrak{g},\mathfrak{h},f)$ in certain situations. Note that if $(\mathfrak{g},\mathfrak{h},f)$ is an IM-triple with $\mathfrak{g}$ nilpotent, then any reducing sequence that terminates with a FM-triple has a minimal term that is IM; that is, there is $m \ge 0$ such that $(\mathfrak{g}(j),\mathfrak{h}(j),f(j))$ is an IM-triple for $0 \le j \le m$, and $(\mathfrak{g}(j),\mathfrak{h}(j),f(j))$ is an FM-triple for $j > m$. The following shows that proof of non-commutativity of $\mathcal{D}(\mathfrak{g},\mathfrak{h},f)$ hinges on Conjecture 2.

Theorem 4.2.2. *Let $\{(\mathfrak{g}(j),\mathfrak{h}(j),f(j)) : 0 \le j \le k\}$ be a reducing sequence for the IM-triple $(\mathfrak{g},\mathfrak{h},f)$ with $\mathfrak{g}$ nilpotent. Set*

$$m = max\{j : (\mathfrak{g}(j),\mathfrak{h}(j),f(j)) \textit{ is an IM-triple}\},$$

and assume that Conjecture 2 holds for $((\mathfrak{g}(m)\hat{\ },\mathfrak{h}(m)\hat{\ },f(m)\hat{\ })$. Then $\mathcal{D}(\mathfrak{g},\mathfrak{h},f)$ is not commutative.

Proof. From Corollary 3.4.3 and Propostion 4.2.1, it is enough to show that the algebra $\mathcal{D}((\mathfrak{g}(m))\hat{\ },(\mathfrak{h}(m))\hat{\ },(f(m))\hat{\ })$ is not commutative, which is to say, we may assume that $m=0$ and $\hat{\mathfrak{g}} = \mathfrak{g}$.

Let $(\mathfrak{z},Y)$ be a Kirillov structure for $(\mathfrak{g},\mathfrak{h},f)$. Now from Conjecture 2, we have an element $V \in c(\mathfrak{h},\mathcal{I}) \setminus \mathfrak{u}(\mathfrak{g}^Y) + \mathcal{I}$. Now a standard algebraic argument (see for example Proposition 2.1 of [7]) shows that we may assume that V has the form $V = PX + Q$ where $X \in \mathfrak{g}\backslash\mathfrak{g}^Y$, $P \in \mathfrak{u}(\mathfrak{g}^Y) \setminus \mathcal{I}$, and $Q \in \mathfrak{u}(\mathfrak{g}^Y)$. It is clear that $Y \in c(\mathfrak{h},\mathcal{I})\backslash\mathcal{I}$, and that $[V,Y] = P[X,Y]$. Since $P \notin \mathcal{I}$ and $[X,Y] \in \mathfrak{z}\backslash\mathcal{I}$, then Proposition 3.1.2 shows that $[V,Y] \notin \mathcal{I}$. The Theorem follows. ∎

5. Two Applications

5.1 Let $(\mathfrak{g},\mathfrak{h},f)$ be a triple with $\mathfrak{g}$ nilpotent and $\delta(\mathfrak{g},\mathfrak{h},f) = 1$. As before we denote $\exp(ad^*(Y))$ by $\alpha(Y)$. Let $(0) = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_n = \mathfrak{g}$ be a sequence of $\alpha(\mathfrak{h})$-modules in $\mathfrak{g}$ with $\dim(\mathfrak{g}_j)=j, 1\leq j\leq n$. Set $a=\{j:\mathfrak{g}_j\subset\mathfrak{g}_{j-1}+\mathfrak{h}\}$, and write $a=\{a_1<a_2<\cdots<a_p\}$; for $1\leq k\leq p$, set $\mathfrak{h}_k=\mathfrak{h}\cap\mathfrak{g}_{a_k}$. Then $\{\mathfrak{h}_k\}$ is a Jordan-Holder sequence for $\mathfrak{h}$. Choose $X_j\in\mathfrak{g}_j\setminus\mathfrak{g}_{j-1}$, with $X_j\in\mathfrak{h}$ for $j\in a$. Let $\mathcal{P}$ be the partition of $\mathfrak{g}^*$ relative to the chosen $\alpha(\mathfrak{h})$-modules, and let (ϕ,ψ) be the minimal index pair for which $\Omega_{\phi,\psi}$ meets L_f. For ease of notation set $b=\phi_1, c=\psi_1$, $\Omega=\Omega_{\phi,\psi}$, $X=X_b$, $S=X_{a_c}\in\mathfrak{h}_c\setminus\mathfrak{h}_{c-1}$, and $p(\ell)=\ell([S,X])$, $\ell\in\mathfrak{g}^*$. Then

$$\Omega\cap L_f=\{\ell\in L_f:p(\ell)\neq 0\}.$$

We now make some helpful observations, based on the simple fact that $\cap\{\ker(\ell):\ell\in L_f\}=\mathfrak{h}\cap\ker(f)$. Only (4) requires the assumption that $\delta(\mathfrak{g},\mathfrak{h},f)=1$.

(1) If $\mathfrak{z}$ is an ideal in $\mathfrak{g}$ such that $\mathfrak{z}\subset\cap\{\mathfrak{h}^\ell:\ell\in L_f\}$, then $\mathfrak{z}\subset c(\mathfrak{h},\mathcal{I})$,
(2) $\cap\{\mathfrak{h}(\ell):\ell\in L_f\}\subset c(\mathfrak{g},\mathcal{I})$,
(3) $b=min\{j:\mathfrak{g}_j\not\subset c(\mathfrak{h},\mathcal{I})\}$, and
(4) $c=min\{k:\mathfrak{h}_k\not\subset c(\mathfrak{g},\mathcal{I})\}$.

Detailed proofs are omitted here, but we sketch an argument for (4): if $\mathfrak{h}_{c-1}\not\subset c(\mathfrak{g},\mathcal{I})$, then by observation (2), there would be $X_0\in\mathfrak{g}$ and $Y_0\in\mathfrak{h}_{c-1}$ such that $[Y_0,X_0]\notin\mathcal{I}$. Since $\ell([Y_0,X])=0$ for all $\ell\in L_f$, then the polynomial function $Q=[S,X][Y_0,X_0]-[S,X_0][Y_0,X]$ would not vanish on L_f. Lemma 3.6.1 would then imply that $\delta(\mathfrak{g},\mathfrak{h},f)>1$.

The well-known description of the collective $\alpha(\mathfrak{h})$-orbit structure due to Pukanszky and summarized in [3] has the following form. For each $j, 1\leq j\leq n$, there is a rational function $F_j:\mathbf{k}\times\Omega\to\mathbf{k}$ such that the collection $\{F_j\}$ has the properties

(i) for $1\leq j\leq b-1$, $F_j(z,\ell)=\ell_j$,
(ii) $F_b(z,\ell)=z$,
(iii) for $b<j\leq n$, $F_j(z,\ell)=\ell_j+F_j(z,\ell_1,\ell_2,\ldots,\ell_{j-1})$.

Let $K(\mathfrak{g})$ be the field of fractions of $S(\mathfrak{g})$. We explicitly compute elements $R_j\in K(\mathfrak{g})$, , $j\neq b, 1\leq j\leq n$, with the property that for any $\ell\in\Omega\cap L_f$, we have $R_j(\ell)=F_j(0,\ell)$. It is obvious that for $j<b$, one takes $F_j=X_j$. Fix $\ell\in\Omega\cap L_f$; then the H-orbit of ℓ is $\mathcal{O}_\ell=\{\alpha(tS)\ell:t\in\mathbf{k}\}$; for each $t\in\mathbf{k}$, $(\alpha(tS)\ell)_b=\ell_b-t\ell([S,X])$. The functions $F_j(z,\ell)$ are obtained by making the substitution

$$t=\frac{\ell_b-z}{p(\ell)}$$

into the functions $(\alpha(tS)\ell)_j, b<j\leq n$. The result is that we obtain

$$\begin{aligned}R_j&=X_j-\left(\frac{X}{[S,X]}\right)[S,X_j]+\frac{1}{2}\left(\frac{X}{[S,X]}\right)^2[S,[S,X_j]]-\cdots\\&=\sum_{r\geq 0}(-1)^r\frac{1}{r!}\left(\frac{X_b}{[S,X]}\right)^r ad(S)^r(X_j).\end{aligned}$$

Note that the form of the functions R_j shows that for some non-negative integer N, p^NR_j belongs to $S(\mathfrak{g})$. Set $I(L_f)=\{W\in S(\mathfrak{g}):W\text{ vanishes on }L_f\}$.

Lemma 5.1.1. *Let* $(\mathfrak{g}, \mathfrak{h}, f)$ *be a triple with* $\mathfrak{g}$ *nilpotent and with* $\delta(\mathfrak{g}, \mathfrak{h}, f) \leq 1$. *Set* $\mathfrak{m} = \mathfrak{g}_{b-1} + \mathfrak{g}_{a_c-1}$. *Then* $\omega(S(\mathfrak{m}) \cap I(L_f)) \subset \mathfrak{I}$.

Proof. Set $\mathfrak{m}_0 = \mathfrak{m} \cap I(L_f) = \mathfrak{m} \cap \mathfrak{h} \cap \ker(f)$. I claim that $\mathfrak{m}_0$ is an ideal in $\mathfrak{m}$. Let $X \in \mathfrak{m}$ and $Y \in \mathfrak{m}_0$. If $a_c > b$, then $\mathfrak{m}_0$ is a subspace of $\mathfrak{h}_{c-1}$ and so $\mathfrak{m}_0$ is an ideal in $\mathfrak{g}$. On the other hand if $a_c < b$, then we have $\mathfrak{m} \subset c(\mathfrak{h}, \mathfrak{I})$; in either case the claim follows.

Let $\eta : \mathfrak{m} \to \mathfrak{m}/\mathfrak{m}_0$ be the canonical map. We have maps $\eta_S : S(\mathfrak{m}) \to S(\mathfrak{m}/\mathfrak{m}_0)$ and $\eta_u : \mathfrak{u}(\mathfrak{m}) \to \mathfrak{u}(\mathfrak{m}/\mathfrak{m}_0)$ so that the following diagram

$$\begin{array}{ccc} S(\mathfrak{m}) & \xrightarrow{\eta_S} & S(\mathfrak{m}/\mathfrak{m}_0) \\ {\scriptstyle\omega}\big\downarrow & & {\scriptstyle\omega'}\big\downarrow \\ \mathfrak{u}(\mathfrak{m}) & \xrightarrow{\eta_u} & \mathfrak{u}(\mathfrak{m}/\mathfrak{m}_0) \end{array}$$

commutes. Now for any $q \in S(\mathfrak{m})$, $Y \in \mathfrak{m} \cap \mathfrak{h}$, $\eta_S(Y - f(Y)I)$ is central in $S(\mathfrak{m}/\mathfrak{m}_0)$, so

$$\eta_u(\omega(q(Y - f(Y)I)) = \omega'(\eta_S(q))\omega'(\eta_S(Y - f(Y)I)) = \eta_u(\omega(q))\eta_u(Y - f(Y)I).$$

Hence

$$\omega(q(Y - f(Y)I)) = \omega(q)(Y - f(Y)I) + W$$

where $W \in u(\mathfrak{m})\mathfrak{m}_0 \subset \mathfrak{I}$. Therefore $\omega(q(Y - f(Y)I))$ belongs to $\mathfrak{I}$. Since every element in $S(\mathfrak{m}) \cap I(L_f)$ is a sum of elements of the form $q(Y - f(Y)I)$, the lemma follows. ∎

Proposition 5.1.2. *Let* $(\mathfrak{g}, \mathfrak{h}, f)$ *be a triple with* $\mathfrak{g}$ *nilpotent and* $\delta(\mathfrak{g}, \mathfrak{h}, f) = 1$. *Then for each* $j, 1 \leq j \leq n, j \neq b$, $\omega(p^N R_j)$ *belongs to* $c(\mathfrak{h}, \mathfrak{I})$.

Proof. Fix $j, 1 \leq j \leq n, j \neq b$, and fix $Y \in \mathfrak{h}$. We compute that

$$[Y, p^N R_j] = P_0 + \sum_{m \geq 0} \frac{(-1)^m}{m!} \cdot p^{N-m-1} q_m X^m$$

where $q_m = (adY(adS)^m X_j) \cdot p - ((adS)^{m+1} X_j) \cdot [Y, X], m = 0, 1, 2, \ldots,$ and where P_0 has $[Y, p]$ as a factor. Now by observation (3), $p = [S, X]$ belongs to $c(\mathfrak{h}, \mathfrak{I})$, and hence by observation (4), $[Y, p]$ belongs to $c(\mathfrak{g}, \mathfrak{I}) \cap \mathfrak{I}$ (we regard $\mathfrak{g}$ as a subset both of $S(\mathfrak{g})$ and $\mathfrak{u}(\mathfrak{g})$, of course.) It follows that $\omega(P_0) \in \mathfrak{I}$.

Turning our attention now to the second term of the above, we begin by showing that $\omega(p^{N-1} q_0) \in \mathfrak{I}$. Since $\delta(\mathfrak{g}, \mathfrak{h}, f) = 1$, then linear algebra shows that $q_0 \in I(L_f)$. Now each of p, $[S, X_j]$, and $[Y, X]$ belong to $\mathfrak{m}$. Therefore we must have

$$[Y, X_j] = Y_0 + Z$$

where $Y_0 \in \mathfrak{h}$ and $Z \in \mathfrak{m}$. Hence

$$p^{N-1} q_0 = p^N U_0 + W$$

where $U_0 = Y_0 - f(Y_0)I$ and $W \in S(\mathfrak{m})$. Since $p \in \mathfrak{g}_{b-1}$, one gets $\omega(p^N U_0) \in \mathfrak{I}$. Since $p^{N-1}q_0$ and $p^N U_0$ both belong to $I(L_f)$, then $W \in S(\mathfrak{m}) \cap I(L_f)$. Lemma 5.1.1 now gives $\omega(W) \in \mathfrak{I}$, and linearity of ω now gives $\omega(p^{N-1}q_0) \in \mathfrak{I}$.

Next we turn to $T = \sum_{m>0} \frac{(-1)^m}{m!} \cdot p^{N-m-1} q_m X^m$. To show that $\omega(T) \in \mathfrak{I}$, observe that if $a_c > b$,then $T \in S(\mathfrak{m}) \cap I(L_f)$, so again by Lemma 5.1.1 we get $\omega(T) \in \mathfrak{I}$. Suppose then that $a_c < b$. Set $T_m = p^{N-m-1} q_m X^m, m \geq 1$. Since $[S, X_j] \in c(\mathfrak{h}, \mathfrak{I})$ (observation (3)), then for $m \geq 1$, we have $adY(adS)^m X_j$ and $(adS)^{m+1} X_j$ both belong to $\mathfrak{h} \cap \ker(f)$. In fact by choice of S, they lie in $\mathfrak{h}_{c-1}$. Hence

$$T_m = p^{N-m} X^m Y_0 + p^{N-m-1}[Y, X] X^m Y_1$$

where Y_0 and Y_1 belong to $\mathfrak{h}_{c-1} \cap \ker(f)$. Now by definition of c and observation (4) above, Y_0 and Y_1 belong to $c(\mathfrak{g}, \mathfrak{I}) \cap \mathfrak{I}$. It is immediate from this that $\omega(T_m) \in \mathfrak{I}$ holds for each m, so $\omega(T) \in \mathfrak{I}$. ∎

Corollary 5.1.3. *Let $(\mathfrak{g}, \mathfrak{h}, f)$ be an IM-triple with $\mathfrak{g}$ nilpotent and $\delta(\mathfrak{g}, \mathfrak{h}, f) \leq 1$. Then Conjecture 2 holds for $(\mathfrak{g}, \mathfrak{h}, f)$.*

Proof. To prove that Conjecture 2 holds for $(\mathfrak{g}, \mathfrak{h}, f)$, we assume that the IM-triple $(\mathfrak{g}, \mathfrak{h}, f)$ admits a Kirillov structure $(\mathfrak{z}, Y)$ with $\mathfrak{h} \subset \mathfrak{g}^Y$, and with $(\mathfrak{g}^Y, \mathfrak{h}, f)$ an FM-triple. Choose $X \in \mathfrak{g} \setminus \mathfrak{g}^Y$. If $\delta(\mathfrak{g}, \mathfrak{h}, f) = 0$, then $\mathfrak{g} \subset c(\mathfrak{h}, \mathfrak{I})$, and we have that X and Y belong to $c(\mathfrak{h}, \mathfrak{I})$ and $[X, Y] \notin \mathfrak{I}$. Hence Conjecture 2 holds in this case.

Suppose then that $\delta(\mathfrak{g}, \mathfrak{h}, f) = 1$. Choose a sequence $\{\mathfrak{g}_j\}$ of $Ad(\mathfrak{h})$-modules as above with $\mathfrak{g}_{n-1} = \mathfrak{g}^Y$. Since $(\mathfrak{g}^Y, \mathfrak{h}, f)$ an FM-triple, we have $\delta(\mathfrak{g}^Y, \mathfrak{h}, f) = 1$ and so $n \neq b$. Set $V = \omega(p^N R_n)$; then it is easily seen that $V = PX + Q$, where P and Q belong to $\mathfrak{u}(\mathfrak{g}^Y)$, and where P is of the form p^N. Now it follows from Proposition 2.1.1, Corollary 2.1.2, and the fact that $p \in \mathfrak{g} \setminus (\mathfrak{h} \cap \ker(f))$, that every power of p does not lie in $\mathfrak{I}$. Thus Conjecture 2 is proved. ∎

Theorem 5.1.4. *Let $(\mathfrak{g}, \mathfrak{h}, f)$ be an IM-triple with $\mathfrak{g}$ nilpotent and with the property that $\delta(\mathfrak{g}, \mathfrak{h}, f) \leq 1$. Then $\mathcal{D}(\mathfrak{g}, \mathfrak{h}, f)$ is not commutative.*

Proof. Let $\{(\mathfrak{g}(j), \mathfrak{h}(j), f(j)) : 0 \leq j \leq k\}$ be a reducing sequence for $(\mathfrak{g}, \mathfrak{h}, f)$ and define the index m as in Theorem 4.2.1. Then by Corollary 5.1.3, Conjecture 2 holds for $((\mathfrak{g}(m)\hat{}, \mathfrak{h}(m)\hat{}, f(m)\hat{})$. Hence by Theorem 4.2.2, Conjecture 1 holds for $(\mathfrak{g}, \mathfrak{h}, f)$. ∎

5.2 Let $(\mathfrak{g}, \mathfrak{h}, f)$ be a triple with $\mathfrak{g}$ nilpotent. Let us say that f is "essentially zero" if there are ideals $\mathfrak{h}_1$ and $\mathfrak{g}_0$ in $\mathfrak{g}$ such that $\mathfrak{g}$ is the direct sum of $\mathfrak{g}_0$ and $\mathfrak{h}_1$, $\mathfrak{h}_1 \subset c(\mathfrak{g}, \mathfrak{I})$, and such that f vanishes on $\mathfrak{h} \cap \mathfrak{g}_0$.

Lemma 5.2.1. *Let $(\mathfrak{g}, \mathfrak{h}, f)$ be a triple for which f is essentially zero. Let $P \in S(\mathfrak{g}_0)$ be a homogeneous element for which $\omega(P) \in \mathfrak{I}$. Then P vanishes on L_f.*

Proof. Choose a basis of $\mathfrak{g}_0$ that passes through $\mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{g}_0$, say:

$$\{Y_1, Y_2, \ldots, Y_p, X_1, X_2, \ldots, X_m\}$$

with $\{Y_1, Y_2, \dots, Y_p\}$ a basis of $\mathfrak{h}_0$. Using multi-indices in the usual way, let $P_{\alpha\beta} = X^\alpha Y^\beta \in S(\mathfrak{g}_0)$ and write P as a finite sum

$$P = \sum_{\alpha+\beta=d} a_{\alpha\beta} P_{\alpha\beta}$$

where d is the degree of P. Now let $\tilde{P}_{\alpha\beta}$ be the element in $\mathfrak{u}(\mathfrak{g}_0)$ defined by the same expression: $\tilde{P}_{\alpha\beta} = X^\alpha Y^\beta$. Then

$$\omega(P) = \sum_{\alpha+\beta=d} a_{\alpha\beta} \tilde{P}_{\alpha\beta} + Z$$

where $Z \in \mathfrak{u}_{d-1}(\mathfrak{g}_0) = \mathbf{k}\text{-span}\{X^\mu Y^\nu : |\mu| + |\nu| \leq d-1\}$. Now a basis of $\mathfrak{I} \cap \mathfrak{u}(\mathfrak{g}_0)$ is given by the monomials $X^\mu Y^\nu$ with $|\nu| > 0$. Since $\omega(P) \in \mathfrak{I} \cap \mathfrak{u}(\mathfrak{g}_0)$, it follows that $Z \in \mathfrak{I} \cap \mathfrak{u}_{d-1}(\mathfrak{g}_0)$ and that $a_{\alpha\beta} \neq 0$ only if $|\beta| > 0$. Thus P vanishes on L_f. ∎

We recall the constructions of Section 3.6. Given an increasing sequence $\{\mathfrak{g}_j : 1 \leq j \leq n\}$ of $\alpha(\mathfrak{h})$-modules in $\mathfrak{g}$ with $\dim(\mathfrak{g}_j) = j$, and a Jordan-Holder sequence for $\mathfrak{h}$, one has a partition of $\mathfrak{g}^*$ into the $\alpha(\mathfrak{h})$-invariant algebraic subsets as defined in Section 3.6. Let Ω be the minimal element of the partition, that is, $\Omega = \Omega_{\phi,\psi}$ where (ϕ, ψ) is the smallest index pair for which $\Omega_{\phi,\psi} \neq \emptyset$. Then Ω is Zariski open in $\mathfrak{g}^*$.

Lemma 5.2.2. *Let $(\mathfrak{g}, \mathfrak{h}, f)$ be an IM-triple with $\mathfrak{g}$ nilpotent and with f essentially zero. Assume that $(\mathfrak{g}, \mathfrak{h}, f)$ admits a Kirillov structure $(\mathfrak{z}, Y)$ for which $\mathfrak{h} \subset \mathfrak{g}^Y$ and for which $(\mathfrak{g}^Y, \mathfrak{h}, f)$ is FM. Assume further that there is an increasing sequence $\{\mathfrak{g}_j\}$ of $\alpha(\mathfrak{h})$-modules in $\mathfrak{g}$ with $\mathfrak{g}_{n-1} = \mathfrak{g}^Y$ and a Jordan-Holder sequence for $\mathfrak{h}$, such that $L_f \cap \Omega \neq \emptyset$. Then $c(\mathfrak{h}, \mathfrak{I}) \not\subseteq \mathfrak{u}(\mathfrak{g}^Y) + \mathfrak{I}$.*

Proof. Let $X \in \mathfrak{g} \setminus \mathfrak{g}^Y$. It is well-known that there is an $\mathfrak{h}$-invariant element $P \in S(\mathfrak{g})$ of the form $P = QX + R$ where Q and R belong to $S(\mathfrak{g}^Y)$, and where Q is $\mathfrak{h}$-invariant and never zero on Ω. From the definition of essentially zero and the construction of P we see that $Q \in S(\mathfrak{g}_0)$. Now $\omega(P) \in c(\mathfrak{h}, \mathfrak{I})$ since P is $\mathfrak{h}$-invariant, and it is easily seen that $\omega(P)$ has the form

$$\omega(P) = \omega(Q)X + Z$$

where $Z \in \mathfrak{u}(\mathfrak{g}^Y)$. Hence $[\omega(P), Y] = \omega(Q)[X, Y]$ modulo $\mathfrak{I}$. Since Q is never zero on Ω, then Q is non-vanishing on L_f, and Lemma 5.2.1 allows us to conclude that $\omega(Q) \notin \mathfrak{I}$. Now from Proposition 3.1.2 we have $[\omega(P), Y] \notin \mathfrak{I}$. Since $Y \in c(\mathfrak{h}, \mathfrak{I})$ it follows that $\omega(P) \in c(\mathfrak{h}, \mathfrak{I}) \setminus \mathfrak{u}(\mathfrak{g}^Y) + \mathfrak{I}$. ∎

The following is now immediate from Theorem 4.2.2.

Theorem 5.2.3. *Let $(\mathfrak{g}, \mathfrak{h}, f)$ be an IM-triple with $\mathfrak{g}$ nilpotent and suppose that a reducing sequence for $(\mathfrak{g}, \mathfrak{h}, f)$ is given with $(\mathfrak{g}(m), \mathfrak{h}(m), f(m))$ defined as in Theorem 4.2.2. Assume that the hypothesis of Lemma 5.2.2 holds for $(\mathfrak{g}(m)\hat{\ }, \mathfrak{h}(m)\hat{\ }, f(m)\hat{\ })$. Then $\mathcal{D}(\mathfrak{g}, \mathfrak{h}, f)$ is not commutative.*

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