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The Closure Diagrams for Nilpotent Orbits of Real Forms of $\mathbf{E}_6$

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Abstract. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be adjoint nilpotent orbits in a real semisimple Lie algebra. Write $\mathcal{O}_1 \geq \mathcal{O}_2$ if $\mathcal{O}_2$ is contained in the closure of $\mathcal{O}_1$. This gives a partial order on the set of such orbits, which is known as the closure ordering. We determine this ordering for the adjoint nilpotent orbits of the four noncompact real forms of the simple complex Lie algebra E_6.

1. Introduction

In this paper $\mathfrak{g}$ denotes a simple complex Lie algebra of type E_6 and $\mathfrak{g}_0$ one of its noncompact real forms. Let G be the adjoint group of $\mathfrak{g}$ and σ the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$. There is a unique anti-holomorphic involutory automorphism of G whose differential is σ. We denote it also by σ. The adjoint group G_0 of $\mathfrak{g}_0$ is the connected Lie subgroup of G corresponding to $\mathfrak{g}_0$. According to Matsumoto [14], $G_0 = G_{\mathbf{R}}$ where $G_{\mathbf{R}} = \{a \in G : \sigma(a) = a\}$ is the group of real points of G.

Fix a Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ of $\mathfrak{g}_0$ and denote by θ the corresponding Cartan involution of $\mathfrak{g}_0$: $\theta(X) = X$ for $X \in \mathfrak{k}_0$ and $\theta(X) = -X$ for $X \in \mathfrak{p}_0$. Let $\mathfrak{k}$ (resp. $\mathfrak{p}$) be the complexification of $\mathfrak{k}_0$ (resp. $\mathfrak{p}_0$). We extend θ to a complex linear map of $\mathfrak{g}$ and use the same letter θ to denote this extension. Furthermore we denote by θ also the corresponding involutory automorphism of G. Let K (resp. K_0) be the connected Lie subgroup of G with Lie algebra $\mathfrak{k}$ (resp. $\mathfrak{k}_0$). Then K_0 is a maximal compact subgroup of G_0, and $K = \{a \in G : \theta(a) = a\}$.

Let $\mathcal{N}$ be the nilpotent variety of $\mathfrak{g}$ and $\mathcal{N}/G$ the orbit space for the adjoint action of G, equipped with the quotient topology. The adjoint nilpotent orbits in $\mathfrak{g}$ were enumerated a long time ago by Dynkin. There are 21 such orbits (including the trivial one). Nowadays one uses the Bala–Carter symbols to label these orbits [3]. For any orbit $\mathcal{O} \in \mathcal{N}/G$ we denote by $\overline{\mathcal{O}}$ its closure in $\mathfrak{g}$. It is a union of $\mathcal{O}$ and some orbits of smaller dimension.

If $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{N}/G$ and $\mathcal{O}_2 \subset \overline{\mathcal{O}}_1$, then we write $\mathcal{O}_1 \geq \mathcal{O}_2$. If $\mathcal{O}_1 \geq \mathcal{O}_2$ and $\mathcal{O}_1 \neq \mathcal{O}_2$, then we write $\mathcal{O}_1 > \mathcal{O}_2$. If $\mathcal{O}_1 > \mathcal{O}_2$ and there is no G-orbit $\mathcal{O}$ such that

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$\mathcal{O}_1 > \mathcal{O} > \mathcal{O}_2$, then we write $\mathcal{O}_1 \to \mathcal{O}_2$. The topology of $\mathcal{N}/G$ can be represented by the so-called *closure diagram.* Each orbit $\mathcal{O} \in \mathcal{N}/G$ is represented by a dot and if $\mathcal{O}_1 \to \mathcal{O}_2$ then the dots corresponding to these two orbits are joined by a line. The dot for $\mathcal{O}_1$ is placed higher than the one for $\mathcal{O}_2$.

Since the closure diagram for $\mathcal{N}/G$ plays an essential role in the paper, we have reproduced it from [3] in Figure 1. Near each dot, the Bala–Carter symbol is displayed for the corresponding orbit $\mathcal{O}$, which may be followed by the Cartan symbols of other regular semisimple subalgebras of $\mathfrak{g}$ whose principal nilpotent orbit is contained in $\mathcal{O}$. On the left hand side of the diagram we indicate the complex dimensions of the orbits on each level.

Let $\mathcal{N}_{\mathbf{R}} = \mathcal{N} \cap \mathfrak{g}_0$ be the nilpotent variety of $\mathfrak{g}_0$ and $\mathcal{N}_1 = \mathcal{N} \cap \mathfrak{p}$ that of $\mathfrak{p}$. The corresponding orbit spaces $\mathcal{N}_{\mathbf{R}}/G_0$ and $\mathcal{N}_1/K$ are also equipped with their quotient topologies. Both of these spaces are finite. The Kostant–Sekiguchi correspondence establishes a bijection $\mathcal{N}_{\mathbf{R}}/G_0 \to \mathcal{N}_1/K$. For more details about this correspondence we refer the reader to the book [5]. Barbasch and Sepanski [1] have shown recently that this bijection is a homeomorphism.

The K-orbits in $\mathcal{N}_1$ were enumerated in our papers [6, 7]. For the sake of consistency, we use here the same enumeration. Our original enumeration is reproduced in [5] where the trivial orbit $\{0\}$ has been given the number 0. Up to G-conjugacy, $\mathfrak{g}$ has four noncompact real forms:

$$\mathrm{E\,I} = \mathrm{E}_{6(6)},\ \mathrm{E\,II} = \mathrm{E}_{6(2)},\ \mathrm{E\,III} = \mathrm{E}_{6(-14)},\ \mathrm{E\,IV} = \mathrm{E}_{6(-26)}$$

where the subscript k inside the parentheses is the so-called Cartan index

$$k = \dim(\mathfrak{p}_0) - \dim(\mathfrak{k}_0).$$

Our main result is an explicit description of the topology of $\mathcal{N}_{\mathbf{R}}/G_0$ (or, equivalently, $\mathcal{N}_1/K$): The closure diagrams for $\mathcal{N}_{\mathbf{R}}/G_0$ (or $\mathcal{N}_1/K$) are given by Figures 2 (p. 389), 3 (p. 397), and 5 (p.407). In the case E I we work directly with $\mathcal{N}_{\mathbf{R}}/G_0$ since this real form is of outer type. In the cases E II and E III, which are of inner type, it is more convenient to work with $\mathcal{N}_1/K$. The case E IV is rather trivial as it has only two nonzero nilpotent G_0-orbits.

The representatives of nilpotent G_0-orbits given in [8] are of special kind because they are embedded in real Cayley triples. Dropping that restriction, one can find simpler representatives. Table 3 below (p. 407) gives such representatives for the case E I.

Let us describe briefly the action of $\mathrm{Aut}(\mathfrak{g}_0)$ on $\mathcal{N}_{\mathbf{R}}/G_0$. We recall that $\mathrm{Aut}(\mathfrak{g}_0)/G_0 = Z_2$. (By Z_k we denote a cyclic group of order k.) If $\mathfrak{g}_0$ is of outer type (E I or E IV), then Z_2 acts trivially on $\mathcal{N}_{\mathbf{R}}/G_0$. If $\mathfrak{g}_0$ is of type E II, the generator of Z_2 interchanges the orbits 9 and 10, 12 and 13, 28 and 27, and 29 and 30. Finally, if $\mathfrak{g}_0$ is of type E III, the generator of Z_2 acts as the reflection in the vertical axis of symmetry of the E III diagram in Figure 5.

A few words are in order concerning the use of the computer. First of all we used it to compile most of our tables. Several of these tables can be easily verified by hand. Secondly, we often use the computer to determine the dimensions of various orbits and to analyze the orbit structure of some important prehomogeneous vector spaces.

A typical problem that we encounter is the following: Given a nilpotent element $X \in \mathfrak{g}_0$, decide to which G_0-orbit it belongs. As a rule, by using a computer, it is easy to find the dimension of the orbit $G_0 \cdot X$. If there is only one nilpotent G_0-orbit of that dimension, then the job is finished. Otherwise, for special types of elements X, we may be able to use the method developed in our paper [8]. On several occasions we had to resort to additional *ad hoc* arguments to finish the job. Two such cases occur in our justification of Table 3.

In addition to our own programs, we used extensively the software packages Maple and LiE (see [4, 16]).

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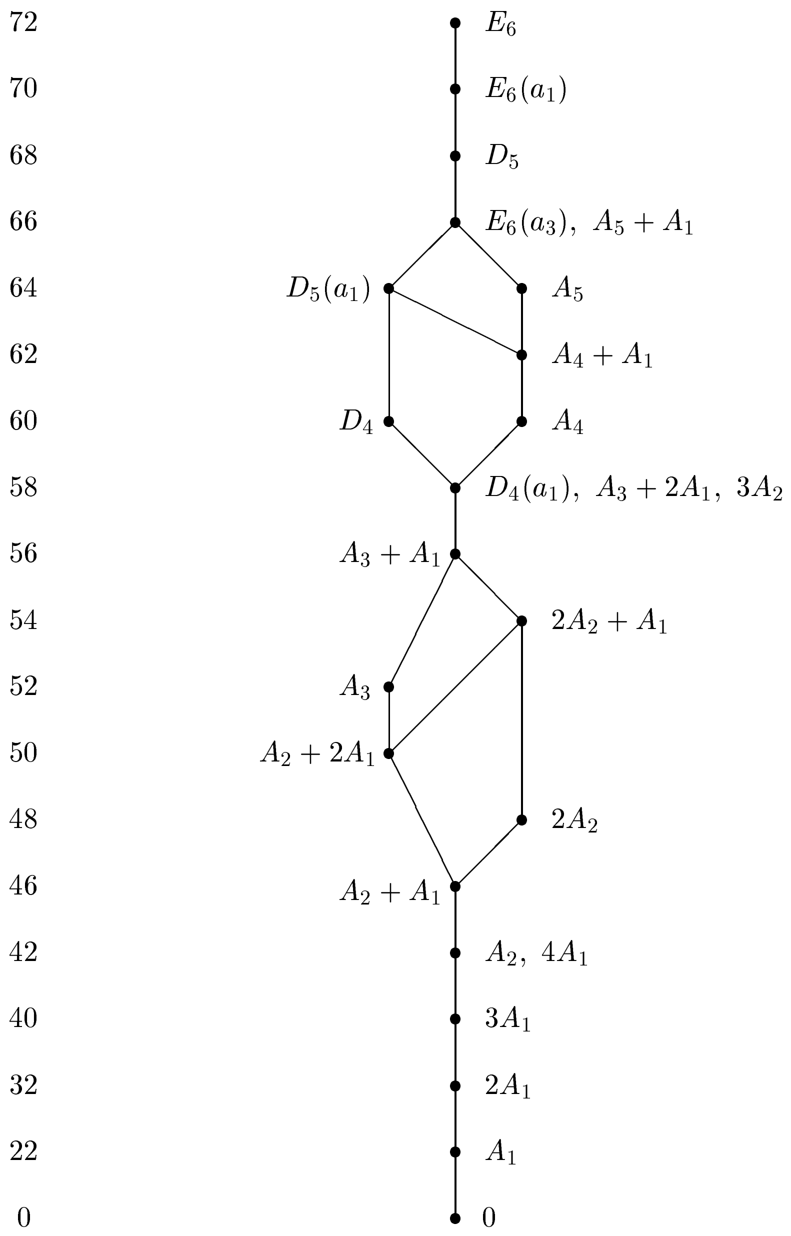

Figure 1: Closure diagram for E_6

2. Preliminaries

Fix a σ-stable Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Denote by R the root system of $(\mathfrak{g}, \mathfrak{h})$. Then $\mathfrak{h}_0 = \mathfrak{g}_0 \cap \mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}_0$. Choose a system of positive roots $R^+ \subset R$. Let α_i, $1 \leq i \leq 36$, be the enumeration of R^+ used in [8]. It is reproduced in the Appendix. In particular, $\{\alpha_1, \ldots, \alpha_6\}$ is a system of fundamental roots as in [2]. The negative root $-\alpha_i$ is also written as α_{-i}. Let $H_i \in \mathfrak{h}$ be the coroot corresponding to α_i. Then $H_{-i} = -H_i$. For $\alpha \in R$, $\mathfrak{g}^\alpha$ is the root space of α.

Let $X_i \in \mathfrak{g}^{\alpha_i}$ for $\pm i \in \{1, \ldots, 36\}$ be the root vectors chosen so that together with the H_i, $1 \leq i \leq 6$, they form a Chevalley basis of $\mathfrak{g}$. For the convenience of the reader, the Appendix includes the structure constants of $\mathfrak{g}$ from our paper [8, Table 13]. They are given here in a more extensive form and user-friendly way.

An ordered triple (E, H, F) of nonzero elements of $\mathfrak{g}$ is a *standard triple* if

$$[H, E] = 2E,\ [H, F] = -2F,\ [F, E] = H.$$

For instance, (X_i, H_i, X_{-i}) are standard triples for all i. A standard triple (E, H, F) is a *normal triple* if $H \in \mathfrak{k}$ and $E, F \in \mathfrak{p}$.

We enumerate the nonzero G-orbits in $\mathcal{N}$ as $\mathcal{O}^k$, $1 \leq k \leq 20$. We can choose a standard triple (E^k, H^k, F^k) with $E^k \in \mathcal{O}^k$, $H^k \in \mathfrak{h}$, and such that $\alpha_i(H^k) \geq 0$ for $1 \leq i \leq 6$. The nonzero G_0-orbits in $\mathcal{N}_{\mathbf{R}}$ will be enumerated as in our papers [6, 7] and in [5], and we denote them by $\mathcal{O}_0^i$ $(i = 1, 2, \ldots)$. The K-orbit in $\mathcal{N}_1$ that corresponds to $\mathcal{O}_0^i$ under the Kostant–Sekiguchi bijection is denoted by $\mathcal{O}_1^i$.

If nonempty, $\mathfrak{p} \cap \mathcal{O}^k$ is an equidimensional complex algebraic variety [12, Proposition 5] with

$$\dim_{\mathbf{C}}(\mathfrak{p} \cap \mathcal{O}^k) = \frac{1}{2} \dim_{\mathbf{C}}(\mathcal{O}^k).$$

(Recall that $\dim_{\mathbf{C}}(\mathcal{O}^k)$ is always even.) Each connected component of $\mathfrak{p} \cap \mathcal{O}^k$ is a single K-orbit. Similarly $\mathfrak{g}_0 \cap \mathcal{O}^k$ (if nonempty) is an equidimensional real algebraic variety with

$$\dim_{\mathbf{R}}(\mathfrak{g}_0 \cap \mathcal{O}^k) = \dim_{\mathbf{C}}(\mathcal{O}^k),$$

and each connected component of $\mathfrak{g}_0 \cap \mathcal{O}^k$ is a single G_0-orbit.

Furthermore

$$\mathcal{O}_0^i \subset \mathfrak{g}_0 \cap \mathcal{O}^k \iff \mathcal{O}_1^i \subset \mathfrak{p} \cap \mathcal{O}^k.$$

Hence $\mathfrak{g}_0 \cap \mathcal{O}^k$ and $\mathfrak{p} \cap \mathcal{O}^k$ have the same number of connected components. In particular

$$\mathfrak{g}_0 \cap \mathcal{O}^k \neq \emptyset \iff \mathfrak{p} \cap \mathcal{O}^k \neq \emptyset.$$

All topological notions (such as closure, connectedness, etc.) refer to the Euclidean topology in $\mathfrak{g}$ and the Lie group topology in G.

In Table 1 we enumerate the nonzero G-orbits $\mathcal{O}^k \subset \mathcal{N}$. Column 2 contains the Bala–Carter label of $\mathcal{O}^k$. In Column 3 we list the integers $\alpha_j(H^k)$, $1 \leq j \leq 6$. The next four columns show which orbits $\mathcal{O}_0^i$ (or, equivalently, $\mathcal{O}_1^i$) are contained in $\mathcal{O}^k$. This depends on the type of the real form $\mathfrak{g}_0$. For instance,

E I :	$\mathfrak{g}_0 \cap \mathcal{O}^4 = \mathcal{O}_0^4 \cup \mathcal{O}_0^5$,	$\mathfrak{p} \cap \mathcal{O}^4 = \mathcal{O}_1^4 \cup \mathcal{O}_1^5$;
E II :	$\mathfrak{g}_0 \cap \mathcal{O}^4 = \mathcal{O}_0^6 \cup \mathcal{O}_0^7 \cup \mathcal{O}_0^8$,	$\mathfrak{p} \cap \mathcal{O}^4 = \mathcal{O}_1^6 \cup \mathcal{O}_1^7 \cup \mathcal{O}_1^8$;
E III :	$\mathfrak{g}_0 \cap \mathcal{O}^4 = \mathcal{O}_0^6$,	$\mathfrak{p} \cap \mathcal{O}^4 = \mathcal{O}_1^6$;
E IV :	$\mathfrak{g}_0 \cap \mathcal{O}^4 = \emptyset$;	$\mathfrak{p} \cap \mathcal{O}^4 = \emptyset$.

The last column of this table records the complex dimension of $\mathcal{O}^k$.

Table 1: Nonzero nilpotent orbits in $\mathbf{E_6}$ and its real forms

k	Bala-Carter	$\alpha_j(H^k)$	E I	E II	E III	E IV	$\dim_{\mathbf{C}}(\mathcal{O}^k)$
1	A_1	0 1 0 0 0 0	1	1	1,2		22
2	$2A_1$	1 0 0 0 0 1	2	2,3	3,4,5	1	32
3	$3A_1$	0 0 0 1 0 0	3	4,5			40
4	A_2	0 2 0 0 0 0	4,5	6,7,8	6		42
5	$A_2 + A_1$	1 1 0 0 0 1	8	9,10	7,8		46
6	$2A_2$	2 0 0 0 0 2	6	11	9	2	48
7	$A_2 + 2A_1$	0 0 1 0 1 0	10	12,13,14			50
8	A_3	1 2 0 0 0 1	7	15,16	10,11		52
9	$2A_2 + A_1$	1 0 0 1 0 1	11	17			54
10	$A_3 + A_1$	0 1 1 0 1 0	15	18,19			56
11	$D_4(a_1)$	0 0 0 2 0 0	12,23	20,21,22			58
12	A_4	2 2 0 0 0 2	9	25,26	12		60
13	D_4	0 2 0 2 0 0	13	23,24			60
14	$A_4 + A_1$	1 1 1 0 1 1	16	27,28			62
15	$D_5(a_1)$	1 2 1 0 1 1	17	29,30			64
16	A_5	2 1 1 0 1 2	14	31			64
17	$E_6(a_3)$	2 0 0 2 0 2	19,22	32,33			66
18	D_5	2 2 0 2 0 2	21	34,35			68
19	$E_6(a_1)$	2 2 2 0 2 2	18	36			70
20	E_6	2 2 2 2 2 2	20	37			72

For $1 \leq k \leq 20$ and any integer j set

$$\mathfrak{g}(j,k) = \{X \in \mathfrak{g} : [H^k, X] = jX\}, \quad R(j,k) = \{\alpha \in R : \alpha(H^k) = j\}.$$

Then

$$\mathfrak{g}(0,k) = \mathfrak{h} + \sum_{\alpha \in R(0,k)} \mathfrak{g}^\alpha; \quad \mathfrak{g}(j,k) = \sum_{\alpha \in R(j,k)} \mathfrak{g}^\alpha, \quad j \neq 0.$$

Introduce the subalgebras

$$\mathfrak{q}(i,k) = \sum_{j \geq i} \mathfrak{g}(j,k), \quad i \geq 0,$$

and let $Q(k)$ be the parabolic subgroup of G corresponding to $\mathfrak{q}(0,k)$. The centralizer, $L(k)$, of H^k in G is a Levi factor of $Q(k)$ with Lie algebra $\mathfrak{g}(0,k)$.

The following theorem, due to Kostant (see [11, Theorem 4.3] or [5, Lemma 4.1.4]), is valid for arbitrary complex semisimple Lie algebras. For the last assertion of the theorem see [13, Satz 2, pp. 182–184].

Theorem 2.1. *Let* (E^k, H^k, F^k) *be a standard triple, as above, with* $E^k \in \mathcal{O}^k$. *Then*

$$\mathcal{O}^k \cap \mathfrak{g}(2,k) = L(k) \cdot E^k$$

is a dense open subset of $\mathfrak{g}(2,k)$, *and*

$$\mathcal{O}^k \cap \mathfrak{q}(2,k) = L(k) \cdot E^k + \mathfrak{q}(3,k) = Q(k) \cdot E^k$$

is a dense open subset of $\mathfrak{q}(2,k)$. *Moreover* $\overline{\mathcal{O}^k} = G \cdot \mathfrak{q}(2,k)$.

Table 2 lists the indices i of the roots α_i that belong to $R(j,k)$ for $j \geq 2$. Those for $R(2,k)$ are listed first and separated from the other (if any) by a semicolon.

Table 2: Indices of roots in $\mathbf{R(j,k)}$, $\mathbf{j \geq 2}$.

k	$R(2,k)$; $R(j,k)$, $j > 2$
1	36;
2	23 27 30 32 33 34 35 36;
3	24 26 28 29 30 31 32 33 34 ; 35 36
4	2 8 13 14 17 19 20 22 24 25 26 27 28 29 30 31 32 33 34 35 ; 36
5	17 20 22 23 25 26 28 29 31 ; 27 30 32 33 34 35 36
6	1 6 7 11 12 16 17 18 20 21 22 25 26 28 29 31 ; 23 27 30 32 33 34 35 36
7	15 18 19 21 22 23 24 25 26 27 28 30 ; 29 31 32 33 34 35 36
8	2 8 13 14 19 23 24 ; 17 20 22 25 26 27 28 29 30 31 32 33 34 35 36
9	12 16 17 18 20 21 22 24 25 ; 23 26 27 28 29 30 31 32 33 34 35 36
10	13 14 15 17 18 20 21 23 ; 19 22 24 25 26 27 28 29 30 31 32 33 34 35 36
11	4 8 9 10 12 13 14 15 16 17 18 19 20 21 22 23 25 27 ; 24 26 28 29 30 31 32 33 34 35 36
12	1 2 6 7 8 11 12 13 14 16 18 19 21 24 ; 17 20 22 23 25 26 27 28 29 30 31 32 33 34 35 36
13	2 4 9 10 12 15 16 18 21 23 ; 8 13 14 17 19 20 22 24 25 26 27 28 29 30 31 32 33 34 35 36
14	7 11 12 13 14 15 16 ; 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36
15	2 7 8 11 12 15 16 ; 13 14 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36
16	1 6 13 14 15 ; 7 11 12 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36
17	1 4 6 7 8 9 10 11 13 14 15 19 ; 12 16 17 18 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36
18	1 2 4 6 7 9 10 11 15 ; 8 12 13 14 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36
19	1 2 3 5 6 8 9 10 ; 7 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36
20	1 2 3 4 5 6 ; 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36

3. Type E I

In this section $\mathfrak{g}_0 = \mathrm{E\,I}$ is the split real form of $\mathfrak{g} = E_6$. Hence $\mathfrak{k}$ is of type C_4 and $K = \mathrm{Sp}_8/\mathrm{Z}_2$. Let $\mathfrak{h}'$ be a Cartan subalgebra of $\mathfrak{k}$ and $\{\beta_1, \beta_2, \beta_3, \beta_4\}$ a base of the root system of $(\mathfrak{k}, \mathfrak{h}')$ as in [2].

In Table 3 we list the nonzero G_0-orbits $\mathcal{O}_0^i \subset \mathcal{N}_{\mathbf{R}}$, $1 \le i \le 23$. In the first column of this table we give the unique integer k such that $\mathcal{O}_0^i \subset \mathcal{O}^k$ (see Table 1). We can choose a normal triple (E', H', F') such that $E' \in \mathcal{O}_1^i$, $H' \in \mathfrak{h}'$, and $\beta_j(H') \ge 0$ for $1 \le j \le 4$. The integers $\beta_j(H')$ are listed in the third column. They uniquely determine the orbit $\mathcal{O}_1^i$ (or $\mathcal{O}_0^i$).

Table 3: Nonzero nilpotent orbits in E I

k	i	$\beta_j(H')$	Representative $E \in \mathcal{O}_0^i$	Type of E
1	1	0 0 0 1	X_{36}	A_1
2	2	0 1 0 0	$(X_{23}) + (X_{36})$	$2A_1$
3	3	1 0 0 1	$(X_{24}) + (X_{30}) + (X_{34})$	$3A_1$
4	4	0 0 0 2	$X_2 + X_{35}$	A_2
			$(X_2) + (X_{24}) + (X_{30}) + (X_{34})$	$4A_1$
4	5	2 0 0 0	$(X_2) + (X_{24}) + (X_{30}) + (-X_{34})$	$4A_1$
5	8	0 1 0 1	$(X_{17} + X_{31}) + (X_{23})$	$A_2 + A_1$
6	6	0 2 0 0	$(X_1 + X_{31}) + (X_6 + X_{29})$	$2A_2$
7	10	1 0 1 0	$(X_{22} + X_{28}) + (X_{15}) + (X_{23})$	$A_2 + 2A_1$
8	7	0 1 0 2	$X_2 + X_{23} + X_{24}$	A_3
9	11	1 1 0 1	$(X_{12} + X_{25}) + (X_{16} + X_{22}) + (X_{24})$	$2A_2 + A_1$
10	15	1 0 1 1	$(X_{13} + X_{23} + X_{14}) + (X_{15})$	$A_3 + A_1$
11	12	2 0 0 2	$(X_{13} + X_{23} + X_{14}) + (X_4) + (X_{15})$	$A_3 + 2A_1$
			$(X_4 + X_{27}) + (X_{13} + X_{18}) + (X_{14} + X_{21})$	$3A_2$
11	23	0 0 2 0	$(X_{13} + X_{23} + X_{14}) + (X_4) + (-X_{15})$	$A_3 + 2A_1$
			$X_4 + X_{19} + X_{27} + X_{15}$	$D_4(a_1)$
12	9	0 2 0 2	$X_2 + X_{21} + X_1 + X_{24}$	A_4
13	13	2 0 0 4	$X_4 + X_2 + X_{15} + X_{23}$	D_4
14	16	1 1 1 1	$(X_{13} + X_{16} + X_7 + X_{14}) + (X_{15})$	$A_4 + A_1$
15	17	1 1 1 2	$X_{15} + X_8 + X_7 + X_{11} + X_{12}$	$D_5(a_1)$
16	14	1 2 1 1	$X_{13} + X_1 + X_{15} + X_6 + X_{14}$	A_5
17	19	2 2 0 2	$(X_{13} + X_1 + X_{15} + X_6 + X_{14}) + (X_4)$	$A_5 + A_1$
17	22	0 2 2 0	$(X_{13} + X_1 + X_{15} + X_6 + X_{14}) + (-X_4)$	$A_5 + A_1$
			$X_1 + X_4 + X_6 + X_7 + X_{11} + X_{19}$	$E_6(a_3)$
18	21	2 2 0 4	$X_4 + X_2 + X_{15} + X_1 + X_6$	D_5
19	18	2 2 2 2	$X_1 + X_2 + X_5 + X_6 + X_8 + X_9$	$E_6(a_1)$
20	20	4 2 2 4	$X_1 + X_2 + X_3 + X_4 + X_5 + X_6$	E_6

The fourth column gives a representative $E \in \mathcal{O}_0^i$. In most cases these representatives have been extracted (with some rescaling using the action of the identity component of the maximal torus of G_0) from our paper [8]. Only in the cases $i = 17,\ 18$ we are using new simpler representatives. Furthermore for $i \in \{4, 12, 22, 23\}$ we have included an additional representative of different type.

The representative of type $E_6(a_3)$ belongs to either $\mathcal{O}_0^{22}$ or $\mathcal{O}_0^{19}$ (see Table 1). By using the method of our paper [8] one can easily verify that the element

$$E = \sqrt{10}X_{19} + \sqrt{6}(X_1 + X_4 + X_6) + \sqrt{2}(X_7 - X_9 - X_{10} + X_{11})$$

belongs to $\mathcal{O}_0^{22}$. Consequently the same is true for the element

$$E_1 = X_{19} + X_1 + X_4 + X_6 + X_7 - X_9 - X_{10} + X_{11}.$$

By using Table 14 from the Appendix it is easy to check that:

$$\exp(\mathrm{ad}(-X_3 - X_5 + X_{-2}))(E_1) = X_1 + X_4 + X_6 + 2X_7 + 2X_{11} + X_{19}.$$

We conclude that the representative of type $E_6(a_3)$ indeed belongs to $\mathcal{O}_0^{22}$.

The last column gives the type of E. For instance, if $i = 23$ the first representative is $(X_{13} + X_{23} + X_{14}) + (X_4) + (-X_{15})$ and its type is $A_3 + 2A_1$. This means that $\{\alpha_{13}, \alpha_{23}, \alpha_{14}, \alpha_4, \alpha_{15}\}$ is a base for a closed root subsystem of type $A_3 + 2A_1$ with $\{\alpha_{13}, \alpha_{23}, \alpha_{14}\}$ being a base for the A_3 component.

The second representative of $\mathcal{O}_0^{23}$ is $E = X_4 + X_{19} + X_{27} + X_{15}$ of type $D_4(a_1)$, i.e., it is a subregular nilpotent element in a regular subalgebra of type D_4. Indeed $\{\alpha_4, \alpha_2, \alpha_{15}, \alpha_{23}\}$ is a base of a closed root subsystem of type D_4 with $\alpha_2 + \alpha_{15} = \alpha_{19}$ and $\alpha_2 + \alpha_{23} = \alpha_{27}$. Consequently $E \in \mathfrak{g}_0 \cap \mathcal{O}^{11} = \mathcal{O}_0^{22} \cup \mathcal{O}_0^{23}$. A more delicate argument is needed to show that in fact $E \in \mathcal{O}_0^{23}$. By using the method of my paper [8], one can check that the element $2X_4 + X_{15} - X_{23} + \sqrt{3}(X_{19} + X_{27})$ belongs to $\mathcal{O}_0^{23}$. Hence the same is true for $E_1 = X_4 + X_{15} - X_{23} + X_{19} + X_{27}$. By using Table 4 from the Appendix, we find that

$$\exp(-\mathrm{ad}\, X_{-2})(E_1) = X_4 + X_{19} + X_{27} + 2X_{15}.$$

This implies that also $E \in \mathcal{O}_0^{23}$.

Note that our enumeration of the orbits $\mathcal{O}_0^i$ has two obvious flaws: The dimensions of $\mathcal{O}_0^i$ do not increase with i, and there are two pairs $\mathcal{O}_0^i, \mathcal{O}_0^j \subset \mathcal{O}^k$ with $|i - j| > 1$ (for $k = 11$ and 17).

We now proceed to the proof of the main result of this section.

Theorem 3.1. *Let $\mathfrak{g}_0$ be of type* E I. *The closure diagram of the orbit space $\mathcal{N}_{\mathbf{R}}/G_0$ is as given in Figure 2. (The dotted horizontal lines in this diagram connect two G_0-orbits that are contained in the same G-orbit.)*

Proof. We claim that if i and j are two vertices in Figure 2, with i higher than j, which are connected by a solid line, then $\mathcal{O}_0^i > \mathcal{O}_0^j$. Let k be such that $\mathcal{O}_0^i \subset \mathcal{O}^k$.

Assume first that $k \neq 4, 11, 17$ or, equivalently, that $\mathfrak{g}_0 \cap \mathcal{O}^k = \mathcal{O}_0^i$. Then Theorem 2.1 implies that the intersection of $\mathcal{O}_0^i$ with

$$\mathfrak{q}_0(2, k) = \mathfrak{g}_0 \cap \mathfrak{q}(2, k)$$

is an open dense subset of $\mathfrak{q}_0(2, k)$. Hence in order to prove that $\mathcal{O}_0^i > \mathcal{O}_0^j$ it suffices to exhibit an element $E \in \mathfrak{q}_0(2, k) \cap \mathcal{O}_0^j$. Table 4 provides the list of such elements E for each pair i, j as above.

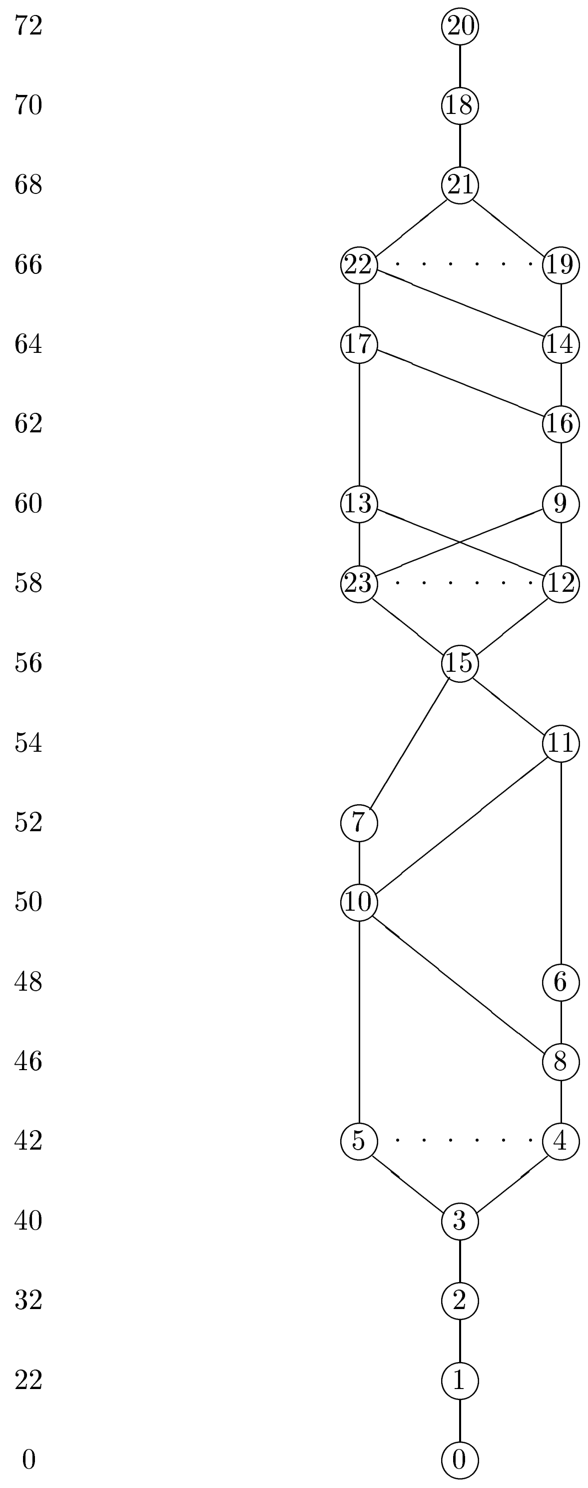

Figure 2: Closure diagram for E I

For instance, let $i = 15$ and $j = 11$. Then Table 4 gives the element

$$E = (X_{13} + X_{18}) + (X_{14} + X_{21}) + (X_{27})$$

of type $2A_2 + A_1$. Consulting Table 1 we see that $E \in \mathcal{O}^9$ (Col. 2) and $E \in \mathcal{O}_0^{11}$ (Col. 4). Since $k = 10$ and the roots $\alpha_{13}, \alpha_{18}, \alpha_{14}, \alpha_{21}, \alpha_{27}$ belong to the union of

$R(s,10)$ with $s \geq 2$ (see Table 2), we indeed have $E \in \mathfrak{q}_0(2,10)$. It follows that $\mathcal{O}_0^{15} > \mathcal{O}_0^{11}$.

Table 4: Elements $\mathbf{E} \in \mathfrak{q}_0(\mathbf{2},\mathbf{k}) \cap \overline{\mathcal{O}_0^{\mathbf{i}}} \cap \mathcal{O}_0^{\mathbf{j}}$

i	j	E	Type
20	18	$X_1+X_2+X_5+X_6+X_8+X_9$	$E_6(a_1)$
18	21	$X_{10}+X_2+X_9+X_1+X_{11}$	D_5
21	22	$(X_{13}+X_1-X_{15}+X_6+X_{14})+(X_4)$	A_5+A_1
21	19	$(X_{13}+X_1+X_{15}+X_6+X_{14})+(X_4)$	A_5+A_1
22	14	$X_{13}+X_1-X_{15}+X_6+X_{14}$	A_5
19	14	$X_{13}+X_1+X_{15}+X_6+X_{14}$	A_5
22	17	$X_{15}+X_8+X_7+X_{11}+X_{12}$	$D_5(a_1)$
14,17	16	$(X_{13}+X_{16}+X_7+X_{14})+(X_{15})$	A_4+A_1
17	13	$X_{15}+X_8+X_7+X_{11}$	D_4
16	9	$X_{13}+X_{16}+X_7+X_{14}$	A_4
9	12	$(X_1+X_{19}+X_{16})+(X_8)+(X_{32})$	A_3+2A_1
9	23	$(X_1+X_{19}+X_{16})+(-X_8)+(X_{32})$	A_3+2A_1
13	12	$(X_{13}+X_{23}+X_{14})+(X_4)+(X_{15})$	A_3+2A_1
13	23	$(X_{13}+X_{23}+X_{14})+(X_4)+(-X_{15})$	A_3+2A_1
12,23	15	$(X_{13}+X_{23}+X_{14})+(X_4)$	A_3+A_1
15	11	$(X_{13}+X_{18})+(X_{14}+X_{21})+(X_{27})$	$2A_2+A_1$
15	7	$X_{13}+X_{23}+X_{14}$	A_3
7	10	$(X_2+X_{23})+(X_{26})+(X_{28})$	A_2+2A_1
11	10	$(X_{12}+X_{25})+(X_{16})+(X_{24})$	A_2+2A_1
11	6	$(X_{12}+X_{25})+(X_{16}+X_{22})$	$2A_2$
10	8	$(X_{22}+X_{28})+(X_{15})$	A_2+A_1
6	8	$(X_1+X_{31})+(X_6)$	A_2+A_1
10	5	$(-X_{15})+(X_{22})+(X_{23})+(X_{25})$	$4A_1$
8	4	$X_{17}+X_{31}$	A_2
4,5	3	$(X_2)+(X_{24})+(X_{30})$	$3A_1$
3	2	$(X_{24})+(X_{30})$	$2A_1$
2	1	X_{36}	A_1
1	0	0	0

Now let $k=4$. Then from Table 3 (p. 387) we see that $i=4$ or 5 and $j=3$. Since

$$X_2+X_{24}+X_{30}+X_{34} \in \mathcal{O}_0^4,$$
$$X_2+X_{24}+X_{30}-X_{34} \in \mathcal{O}_0^5,$$
$$X_2+X_{24}+X_{30} \in \mathcal{O}_0^3,$$

it is clear that $\mathcal{O}_0^4, \mathcal{O}_0^5 > \mathcal{O}_0^3$. A similar argument shows that $\mathcal{O}_0^{12}, \mathcal{O}_0^{23} > \mathcal{O}_0^{15}$ (when $k=11$) and $\mathcal{O}_0^{19}, \mathcal{O}_0^{22} > \mathcal{O}_0^{14}$ (when $k=17$).

Now let $i=22$ and $j=17$. Then $k=17$ and Table 3 gives the representative

$$E = X_1+X_4+X_6+X_7+X_{11}+X_{19} \in \mathcal{O}_0^{22}$$

of type $E_6(a_3)$. Since the roots $\alpha_1, \alpha_4, \alpha_6, \alpha_7, \alpha_{11}$, and α_{19} are linearly independent, it is clear that the element

$$E_1 = X_1 + X_4 + X_6 + X_7 + X_{19}$$

belongs to the closure of $\mathcal{O}_0^{22}$. Since E_1 is of type $D_5(a_1)$, it must belong to $\mathcal{O}_0^{17}$, cf. Table 1. Hence $\mathcal{O}_0^{22} \to \mathcal{O}_0^{17}$. This completes the proof of our claim that if i and j are two vertices which are connected in Figure 2, with i above j, then $\mathcal{O}_0^i > \mathcal{O}_0^j$.

Now we are going to prove that there are no edges missing in Figure 2. Analyzing the graph in Figure 2, it turns out that it suffices to show

$$\mathcal{O}_0^6 \not\subset \overline{\mathcal{O}_0^7}, \quad \mathcal{O}_0^{13} \not\subset \overline{\mathcal{O}_0^{19}}, \quad \mathcal{O}_0^5 \not\subset \overline{\mathcal{O}_0^6}.$$

Since $\mathcal{O}_0^7 \subset \mathcal{O}^8$, $\mathcal{O}_0^6 \subset \mathcal{O}^6$, and $\mathcal{O}^6 \not\subset \overline{\mathcal{O}^8}$ (see Table 1 and Figure 1), it follows that $\mathcal{O}_0^6 \not\subset \overline{\mathcal{O}_0^7}$. It remains to prove that $\mathcal{O}_0^{13} \not\subset \overline{\mathcal{O}_0^{19}}$ and $\mathcal{O}_0^5 \not\subset \overline{\mathcal{O}_0^6}$.

Let $k = 17$. From Table 1 we see that

$$\mathfrak{g}_0 \cap \mathcal{O}^k = \mathcal{O}_0^{19} \cup \mathcal{O}_0^{22}.$$

We examine in more details the prehomogeneous vector spaces $(L(k), \mathfrak{g}(2,k))$ and $(Q(k), \mathfrak{q}(2,k))$. We have $L(k) = (\mathrm{GL}_2)^3/\langle(1,\zeta,\zeta^2)\rangle$ where the positive roots corresponding to the GL_2 factors are α_2, α_3, and α_5 and $\zeta = e^{2\pi i/3}$. As an $L(k)$-module, $\mathfrak{g}(2,k)$ is a direct sum of three simple modules V_1, V_2, V_3 whose bases are:

$$\begin{array}{lcl} V_1 & : & \{X_{13}, X_8, X_4, X_9, X_{19}, X_{14}, X_{10}, X_{15}\}, \\ V_2 & : & \{X_1, X_7\}, \\ V_3 & : & \{X_6, X_{11}\}. \end{array}$$

The action of $L(k)$ on $\mathfrak{g}(2,k)$ lifts to $(\mathrm{GL}_2)^3$. As a module for $(\mathrm{SL}_2)^3$, V_1 is the tensor product of the 2-dimensional simple modules for each of the three factors SL_2, while V_2 (resp. V_3) is the 2-dimensional simple module for the second (resp. third) factor SL_2 on which the other two factors SL_2 act trivially. The elements (t_1, t_2, t_3) of the central 3-dimensional torus $(\mathrm{GL}_1)^3$ of $(\mathrm{GL}_2)^3$ act on each V_i by scalar multiplications: as $(t_1t_2t_3)^{-1}$ on V_1, as $t_1^2t_2^3$ on V_2, and $t_1^2t_3^3$ on V_3. Note that the element $(1,\zeta,\zeta^2)$ indeed acts trivially. We mention that the prehomogeneous vector space $(L(k), \mathfrak{g}(2,k))$ has only finitely many orbits [10, Theorem 5.21].

Write an arbitrary $X \in \mathfrak{q}(2,k)$ as $X = X^{(1)} + X^{(2)} + X^{(3)} + X'$ where $X' \in \mathfrak{q}(3,k)$ and the $X^{(s)} \in V_s$ are written as:

$$\begin{array}{rcl} X^{(1)} & = & x_1X_{13} + x_2X_8 + x_3X_4 + x_4X_9 + y_1X_{19} + y_2X_{14} + y_3X_{10} + y_4X_{15}, \\ X^{(2)} & = & z_1X_1 + z_2X_7, \\ X^{(3)} & = & w_1X_6 + w_2X_{11}. \end{array}$$

Define the homogeneous polynomials $f_i : \mathfrak{q}(2,k) \to \mathbf{C}$ $(i = 1,2,3)$ by

$$\begin{array}{rcl} f_1(X) & = & 4(x_1x_3 + x_2x_4)(-y_1y_3 + y_2y_4) - (-x_1y_3 + x_2y_4 + x_3y_1 + x_4y_2)^2, \\ f_2(X) & = & (z_1x_4 - z_2x_3)(z_1y_1 + z_2y_2) - (z_1x_1 + z_2x_2)(z_1y_4 + z_2y_3), \\ f_3(X) & = & (w_1y_3 - w_2x_3)(w_1y_1 + w_2x_1) - (w_1y_2 + w_2x_2)(w_1y_4 + w_2x_4). \end{array}$$

It is tedious but straightforward to check that each f_i is a relative invariant for the action of $Q(k)$ on $\mathfrak{q}(2,k)$. The singular set S of this prehomogeneous vector space has three irreducible components: The three hypersurfaces S_i defined by the equations $f_i(X)=0$, respectively. The generic $Q(k)$-orbit in $\mathfrak{q}(2,k)$ is

$$\Omega = \mathfrak{q}(2,k) \setminus S = \mathfrak{q}(2,k) \cap \mathcal{O}^{17}.$$

By computing the dimension of the orbit $Q(k)\cdot E$, where E is the representative of $\mathcal{O}_0^{14}$ from Table 3, we conclude that $S_1 \cap \mathcal{O}^{16}$ is a dense open subset in S_1. Hence $S_1 \subset \overline{\mathcal{O}^{16}}$.

Let $\Omega_0 = \Omega \cap \mathfrak{q}_0(2,k)$ and set

$$\Gamma^+ = \{X \in \mathfrak{q}_0(2,k) : f_1(X) > 0\}, \quad \Gamma^- = \{X \in \mathfrak{q}_0(2,k) : f_1(X) < 0\}.$$

By analyzing the action of $Q(k)_{\mathbf{R}}$ on $\mathfrak{q}_0(2,k)$, we find that Ω_0 is the union of two $Q(k)_{\mathbf{R}}$-orbits Ω_0' and Ω_0''. As representatives of these orbits we can take the following elements, cf. Table 3:

$$\begin{aligned} E_1' &= (X_{13}+X_1+X_{15}+X_6+X_{14}) + (X_4) \in \Omega_0', \\ E_1'' &= (X_{13}+X_1+X_{15}+X_6+X_{14}) + (-X_4) \in \Omega_0''. \end{aligned}$$

One can show that $L(k)_{\mathbf{R}}$ has exactly 8 connected components. It is not hard to exhibit an element from a non-identity component of $L(k)_{\mathbf{R}}$ that fixes E_1', and similarly for E_1''. Hence each of the orbits Ω_0' and Ω_0'' has at most 4 connected components. Now consider the elements:

$$E' = x_1X_{13} + y_2X_{14} + y_4X_{15} + x_3X_4 + X_1 + X_6$$

where the coefficients x_1, y_2, y_4, x_3 are ± 1. All of them belong to Ω_0. We choose 8 of them by indicating the signs of the coefficients and compute the signs of the nonzero real numbers $f_i(E')$:

x_1	y_2	y_4	x_3	f_1	f_2	f_3
+	+	+	+	+	−	−
−	−	+	+	+	+	+
−	+	−	+	+	−	+
−	+	+	−	+	+	−
−	+	+	+	−	+	−
+	−	+	+	−	−	+
+	+	−	+	−	+	+
+	+	+	−	−	−	−

Since the sign patterns in the last 3 colums of the above table are all different, these 8 elements belong to different connected components of Ω_0. We conclude that Ω_0' and Ω_0'' have each exactly 4 connected components.

A computation shows that the representatives E' with $f_1(E') > 0$ belong to $\mathcal{O}_0^{19}$ and those with $f_1(E') < 0$ belong to $\mathcal{O}_0^{22}$. We obtain that

$$\mathfrak{q}_0(2,k) \cap \mathcal{O}_0^{19} = \Omega_0' \subset \Gamma^+, \quad \mathfrak{q}_0(2,k) \cap \mathcal{O}_0^{22} = \Omega_0'' \subset \Gamma^-.$$

We claim that if $X \in \Gamma^+ \cap S_2$, then $X^{(2)} = 0$, i.e., $z_1 = z_2 = 0$. Observe that $f_2(X)$ is a quadratic form in the variables z_1 and z_2 and that its discriminant is $-f_1(X)$. As we assume that $f_1(X) > 0$ and $f_2(X) = 0$, it follows that $z_1 = z_2 = 0$ as claimed. Another computation shows that the intersection of $\mathcal{O}^{16}$ with the codimension 2 subspace defined by $z_1 = z_2 = 0$ is dense in this subspace. Consequently, $\Gamma^+ \cap S_2 \subset \overline{\mathcal{O}^{16}}$. A similar argument shows that $\Gamma^+ \cap S_3 \subset \overline{\mathcal{O}^{16}}$. Since we have already shown that $S_1 \subset \overline{\mathcal{O}^{16}}$, it follows that the boundary, say Δ, of Ω_0' in $\mathfrak{q}_0(2,k)$ is contained in $\overline{\mathcal{O}^{16}}$. Consequently,

$$G_0 \cdot \Delta \subset \overline{\mathcal{O}^{16}}.$$

As $\mathcal{O}^{13} \not\subset \overline{\mathcal{O}^{16}}$ (see Figure 1 and Table 1) and $\mathcal{O}_0^{13} \subset \mathcal{O}^{13}$, we deduce that $\mathcal{O}_0^{13} \not\subset G_0 \cdot \Delta$. Since $Q(k)_{\mathbf{R}}$ is a parabolic subgroup of G_0, the homogeneous space $G_0/Q(k)_{\mathbf{R}}$ is compact. It follows that the set $G_0 \cdot \Delta$ is closed. Hence

$$\overline{\mathcal{O}_0^{19}} = \mathcal{O}_0^{19} \cup G_0 \cdot \Delta$$

and so $\mathcal{O}_0^{13} \not\subset \overline{\mathcal{O}_0^{19}}$.

Finally we show that $\mathcal{O}_0^5 \not\subset \overline{\mathcal{O}_0^6}$, so let $k = 6$. Then $\mathfrak{g}_0 \cap \mathcal{O}^k = \mathcal{O}_0^6$, and $L(k) = \mathrm{Spin}_8 \cdot T_2$. As an $L(k)$-module, $\mathfrak{g}(2,k)$ is a direct sum of two simple modules V_1 and V_2 with bases:

$$\begin{aligned} V_1: &\quad \{X_1, X_7, X_{12}, X_{17}, X_{18}, X_{22}, X_{26}, X_{29}\},\\ V_2: &\quad \{X_6, X_{11}, X_{16}, X_{20}, X_{21}, X_{25}, X_{28}, X_{31}\}. \end{aligned}$$

The space $V_3 = \mathfrak{g}(4,k)$ is also a simple $L(k)$-module. The three modules V_1, V_2, V_3 are pairwise non-isomorphic and $\mathfrak{q}(2,k) = V_1 \oplus V_2 \oplus V_3$.

Write any $X \in \mathfrak{q}(2,k)$ as $X = X^{(1)} + X^{(2)} + X^{(3)}$ with $X^{(s)} \in V_s$ and

$$\begin{aligned} X^{(1)} &= x_1X_1 + x_2X_7 + x_3X_{12} + x_4X_{17} + x_5X_{18} + x_6X_{22} + x_7X_{26} + x_8X_{29},\\ X^{(2)} &= y_1X_6 + y_2X_{11} + y_3X_{16} + y_4X_{20} + y_5X_{21} + y_6X_{25} + y_7X_{28} + y_8X_{31}. \end{aligned}$$

The quadratic forms $f_1, f_2 : \mathfrak{q}(2,k) \to \mathbf{C}$ defined by

$$\begin{aligned} f_1(X) &= x_1x_8 + x_2x_7 - x_3x_6 + x_4x_5,\\ f_2(X) &= y_1y_8 + y_2y_7 - y_3y_6 + y_4y_5, \end{aligned}$$

are relative invariants for the action of $Q(k)$ on $\mathfrak{q}(2,k)$. Let $S_i \subset \mathfrak{q}_0(2,k)$ be the hypersurface defined by $f_i(X) = 0$ $(i = 1,2)$. Let $S = S_1 \cup S_2$ and let $\Omega_0 = \mathfrak{q}_0(2,k) \backslash S$. Then $\Omega_0 = \mathfrak{q}_0(2,k) \cap \mathcal{O}_0^6$ and $\mathfrak{q}_0(2,k) \cap \mathcal{O}_0^8$ is a dense open subset of S. Consequently

$$\overline{\mathcal{O}_0^6} = \mathcal{O}_0^6 \cup \overline{\mathcal{O}_0^8}.$$

By applying [9, Theorem 4.1] to the adjoint module $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, one can show easily that $\mathcal{O}_1^8 \not> \mathcal{O}_1^5$. Indeed we have $d_1(2,8) = 29$ while $d_1(2,5) = 28$. (For the definition of the integers $d_i(j,k)$ see the next section.) It follows that $\mathcal{O}_0^8 \not> \mathcal{O}_0^5$ which entails that $\mathcal{O}_0^6 \not> \mathcal{O}_0^5$. ■

Table 5: Nonzero nilpotent orbits in $\mathfrak{p}$ ($\mathfrak{g}_0 = \mathbf{E\,II}$)

k	i	$\beta_j(H^i)$	E^i	Type of E^i
1	1	00100 1	X_{-2}	A_1
2	2	10001 2	$(X_{-13}) + (X_{-14})$	$2A_1$
2	3	01010 0	$(X_{35}) + (X_{-8})$	$2A_1$
3	4	00100 3	$(X_{-8}) + (X_{-22}) + (X_{-25})$	$3A_1$
3	5	10101 1	$(X_{35}) + (X_{-8}) + (X_{-19})$	$3A_1$
4	6	00000 4	$X_{-24} + X_{-27}$	A_2
			$(X_{-2}) + (X_{-24}) + (X_{-30}) + (X_{-34})$	$4A_1$
4	7	20002 0	$X_{34} + X_{-19}$	A_2
			$(X_{27}) + (X_{35}) + (X_{-8}) + (X_{-19})$	$4A_1$
4	8	00200 2	$(X_{35}) + (X_{-17}) + (X_{-19}) + X_{-20})$	$4A_1$
5	9	21001 1	$(X_{32} + X_{-13}) + (X_{-20})$	$A_2 + A_1$
5	10	10012 1	$(X_{33} + X_{-14}) + (X_{-17})$	$A_2 + A_1$
6	11	02020 0	$(X_{31} + X_{-20}) + (X_{32} + X_{-13})$	$2A_2$
7	12	30100 0	$(X_{33} + X_{-20}) + (X_{32}) + (X_{-19})$	$A_2 + 2A_1$
7	13	00103 0	$(X_{32} + X_{-17}) + (X_{33}) + (X_{-19)})$	$A_2 + 2A_1$
7	14	11011 2	$(X_{34} + X_{-19}) + (X_{-17}) + (X_{-20})$	$A_2 + 2A_1$
8	15	10201 4	$X_{35} + X_{-24} + X_{-27}$	A_3
8	16	01210 2	$X_{-22} + X_{34} + X_{-25}$	A_3
9	17	11111 1	$(X_{32} + X_{-17}) + (X_{33} + X_{-20}) + X_{-19}$	$2A_2 + A_1$
10	18	10301 1	$(X_{-22} + X_{34} + X_{-25}) + (X_{30})$	$A_3 + A_1$
10	19	11111 3	$(X_{-22} + X_{34} + X_{-25}) + (X_{-24})$	$A_3 + A_1$
11	20	00400 0	$X_{29} + X_{30} + X_{-17} + X_{-19}$	$D_4(a_1)$
			$(X_{32} + X_{-27} + X_{33}) + (X_{-8}) + (X_{-19})$	$A_3 + 2A_1$
			$(X_{24} + X_{-13}) + (X_{30} + X_{-20}) + (X_{34} + X_{-22})$	$3A_2$
11	21	02020 4	$X_{35} + X_{-19} + X_{-24} + X_{-30}$	$D_4(a_1)$
			$(X_{-26} + X_{35} + X_{-28}) + (X_{-19}) + (X_{-27})$	$A_3 + 2A_1$
11	22	20202 2	$(X_{-20} + X_{33} + X_{-22}) + (X_{32}) + (X_{-24})$	$A_3 + 2A_1$
12	25	40004 4	$X_{32} + X_{-25} + X_{-26} + X_{33}$	A_4
12	26	22022 0	$X_{30} + X_{-17} + X_{29} + X_{-24}$	A_4
13	23	00400 8	$X_{-29} + X_{35} + X_{-30} + X_{-31}$	D_4
13	24	20402 4	$X_{32} + X_{-27} + X_{33} + X_{-24}$	D_4
14	27	12113 1	$(X_{30} + X_{-20} + X_{31} + X_{-24}) + (X_{-22})$	$A_4 + A_1$
14	28	31121 1	$(X_{30} + X_{-17} + X_{29} + X_{-24}) + (X_{-25})$	$A_4 + A_1$
15	29	31310 4	$X_{33} + X_{-27} + X_{32} + X_{-24} + X_{29}$	$D_5(a_1)$
15	30	01313 4	$X_{32} + X_{-25} + X_{-26} + X_{31} + X_{-24}$	$D_5(a_1)$
16	31	13131 3	$X_{-22} + X_{29} + X_{-24} + X_{31} + X_{-25}$	A_5
17	32	22222 2	$(X_{-22} + X_{29} + X_{-24} + X_{31} + X_{-25}) + (X_{30})$	$A_5 + A_1$
17	33	04040 4	$X_{29} + X_{31} + X_{32} + X_{33} + X_{-24} + X_{-27}$	$E_6(a_3)$
			$(X_{-22} + X_{29} + X_{-24} + X_{31} + X_{-25}) + (X_{-30})$	$A_5 + A_1$
18	34	22422 4	$X_{30} + X_{-27} + X_{-24} + X_{29} + X_{31}$	D_5
18	35	40404 8	$X_{34} + X_{-27} + X_{30} + X_{-26} + X_{-28}$	D_5
19	36	44044 4	$X_{27} + X_{29} + X_{30} + X_{31} + X_{-22} + X_{-28}$	$E_6(a_1)$
20	37	44444 8	$X_{29} + X_{-27} + X_{-26} + X_{30} + X_{-28} + X_{31}$	E_6

4. Type E II

In this section $\mathfrak{g}_0 = \mathrm{E\,II}$ and so $\mathfrak{k}$ is of type $A_5 + A_1$ and $K = (\mathrm{SL}_6/Z_3 \times \mathrm{SL}_2)/Z_2$. As $\mathfrak{g}_0$ is of inner type, we may assume that $\mathfrak{h} \subset \mathfrak{k}$. The roots

$$\beta_1 = \alpha_1,\ \beta_2 = \alpha_3,\ \beta_3 = \alpha_4,\ \beta_4 = \alpha_5,\ \beta_5 = \alpha_6,\ \beta_6 = -\alpha_{36}$$

form a base for the root system of $(\mathfrak{k}, \mathfrak{h})$.

In Table 5 we list the nonzero K-orbits $\mathcal{O}_1^i \subset \mathcal{N}_1$, $1 \leq i \leq 37$. In the first column we give the integer k such that $\mathcal{O}_1^i \subset \mathcal{O}^k$. We choose a normal triple (E^i, H^i, F^i) such that $E^i \in \mathcal{O}_1^i$, $H^i \in \mathfrak{h}$, and $\beta_j(H^i) \geq 0$ for $1 \leq j \leq 6$. The integers $\beta_j(H^i)$ are listed in the third column. They uniquely determine the orbit $\mathcal{O}_1^i$. The fourth column gives the representative $E^i \in \mathcal{O}_1^i$ (in some cases several representatives are listed). The last column gives the type of E^i.

As in [9] we set

$$\mathfrak{g}_{H^i}(0, j) = \mathfrak{k} \cap \mathfrak{g}(j, H^i), \quad \mathfrak{g}_{H^i}(1, j) = \mathfrak{p} \cap \mathfrak{g}(j, H^i),$$

and

$$\mathfrak{p}_s(H^i) = \sum_{j \geq s} \mathfrak{g}_{H^i}(1, j).$$

By Q_{H^i} we denote the parabolic subgroup of K with Lie algebra.

$$\mathfrak{q}_{H^i} = \sum_{j \geq 0} \mathfrak{g}_{H^i}(0, j).$$

In Table 6 we list, for each i, the indices k of the roots α_k whose root space is contained in $\mathfrak{p}_2(H^i)$. We first list those k for which $\mathfrak{g}^{\alpha_k}$ is contained in $\mathfrak{g}_{H^i}(1, 2)$ and separate them by a semi-colon from the other indices (if any).

Table 6: Root spaces in $\mathfrak{g}_{\mathbf{H}^i}(\mathbf{1}, \mathbf{2})$ and $\mathfrak{p}_{\mathbf{3}}(\mathbf{H}^i)$

i	$\mathfrak{p}_2(H^i)$
1	$-2;$
2	$-2, -8, -13, -14, -19, -24;$
3	$-2, -8, 34, 35;$
4	$-8, -13, -14, -17, -19, -20, -22, -25, -27;\ -2$
5	$-8, -13, -14, -19, 35;\ -2$
6	$-2, -8, -13, -14, -17, -19, -20, -22, -24, -25, -26, -27, -28, -29, -30,$ $-31, -32, -33, -34, -35;$
7	$-2, -8, -13, -14, -19, -24, 27, 30, 32, 33, 34, 35;$
8	$-8, -13, -14, -17, -19, -20, -22, -25, -27, 35;\ -2$
9	$-13, -19, -20, -24, 32, 34, 35;\ -2, -8, -14$
10	$-14, -17, -19, -24, 33, 34, 35;\ -2, -8, -13$
11	$-13, -14, -17, -20, 29, 31, 32, 33;\ -2, -8, 34, 35$

Table 6: (continued)

i	$\mathfrak{p}_2(H^i)$
12	$-8, -13, -14, -19, -20, -25, 26, 29, 30, 32, 33, 34;\ -2, 35$
13	$-8, -13, -14, -17, -19, -22, 28, 30, 31, 32, 33, 34;\ -2, 35$
14	$-17, -19, -20, -24, 34, 35;\ -13, -14, -2, -8$
15	$-24, -27, 35;\ -17, -20, -22, -25, -8, -13, -14, -19, -2$
16	$-19, -22, -25, -27, 34;\ -13, -14, -17, -20, -8, 35, -2$
17	$-17, -19, -20, 32, 33;\ -13, -14, 34, -8, 35, -2$
18	$-17, -20, -22, -25, 30, 32, 33, 34;\ -8, -13, -14, -19, 35, -2$
19	$-22, -24, -25, 34;\ -17, -19, -20, 35, -13, -14, -8, -2$
20	$-8, -13, -14, -17, -19, -20, -22, 24, -25, 26, -27, 28, 29, 30, 31, 32, 33, 34;$
	$-2, 35$
21	$-19, -22, -24, -25, -26, -27, -28, -30, 34, 35;$
	$-13, -14, -17, -20, -2, -8$
22	$-17, -20, -22, -24, -25, 30, 32, 33, 34;\ -8, -13, -14, -19, 35, -2$
23	$-24, -26, -28, -29, -30, -31, -32, -33, -34, 35;$
	$-8, -13, -14, -17, -19, -20, -22, -25, -27, -2$
24	$-24, -27, 30, 32, 33, 34;\ -17, -20, -22, -25, -8, -13, -14, -19, 35, -2$
25	$-17, -20, -22, -25, -26, 27, -28, -29, 30, -31, 32, 33, 34, 35;$
	$-2, -8, -13, -14, -19, -24$
26	$-17, -19, -20, -24, 27, 29, 30, 31;\ -13, -14, 32, 33, -2, -8, 34, 35$
27	$-20, -22, -24, 30, 31;\ -17, -19, 33, -13, 32, -14, 34, -8, 35, -2$
28	$-17, -24, -25, 29, 30;\ -19, -20, 32, -14, 33, -13, 34, -8, 35, -2$
29	$-22, -24, -27, -28, 29, 32, 33;$
	$-17, 34, -19, -25, -13, -14, -20, 35, -8, -2$
30	$-24, -25, -26, -27, 31, 32, 33;$
	$-20, 34, -19, -22, -13, -14, -17, 35, -8, -2$
31	$-22, -24, -25, 29, 31;\ -19, 32, 33, -17, -20, -13, -14, 34, 35, -8, -2$
32	$-22, -24, -25, 29, 30, 31;\ -17, -19, -20, 32, 33, -13, -14, 34, -8, 35, -2$
33	$-19, -22, -24, -25, -26, -27, -28, 29, -30, 31, 32, 33;$
	$-13, -14, -17, -20, 34, 35, -2, -8$
34	$-24, -27, 29, 30, 31;$
	$-22, -25, 32, 33, -17, -19, -20, 34, -13, -14, -8, 35, -2$
35	$-26, -27, -28, -29, 30, -31, 32, 33, 34;$
	$-17, -20, -22, -24, -25, 35, -8, -13, -14, -19, -2$
36	$-22, -25, -26, 27, -28, 29, 30, 31;$
	$-17, -19, -20, -24, 32, 33, -13, -14, 34, 35, -2, -8$
37	$-26, -27, -28, 29, 30, 31;$
	$-22, -24, -25, 32, 33, -17, -19, -20, 34, -13, -14, 35, -8, -2$

Theorem 4.1. *Let $\mathfrak{g}_0$ be of type* E II. *The closure diagram of the orbit space $\mathcal{N}_1/K$ is as given in Figure 3. (The dotted horizontal lines in this diagram join the K-orbits that are contained in the same G-orbit.)*

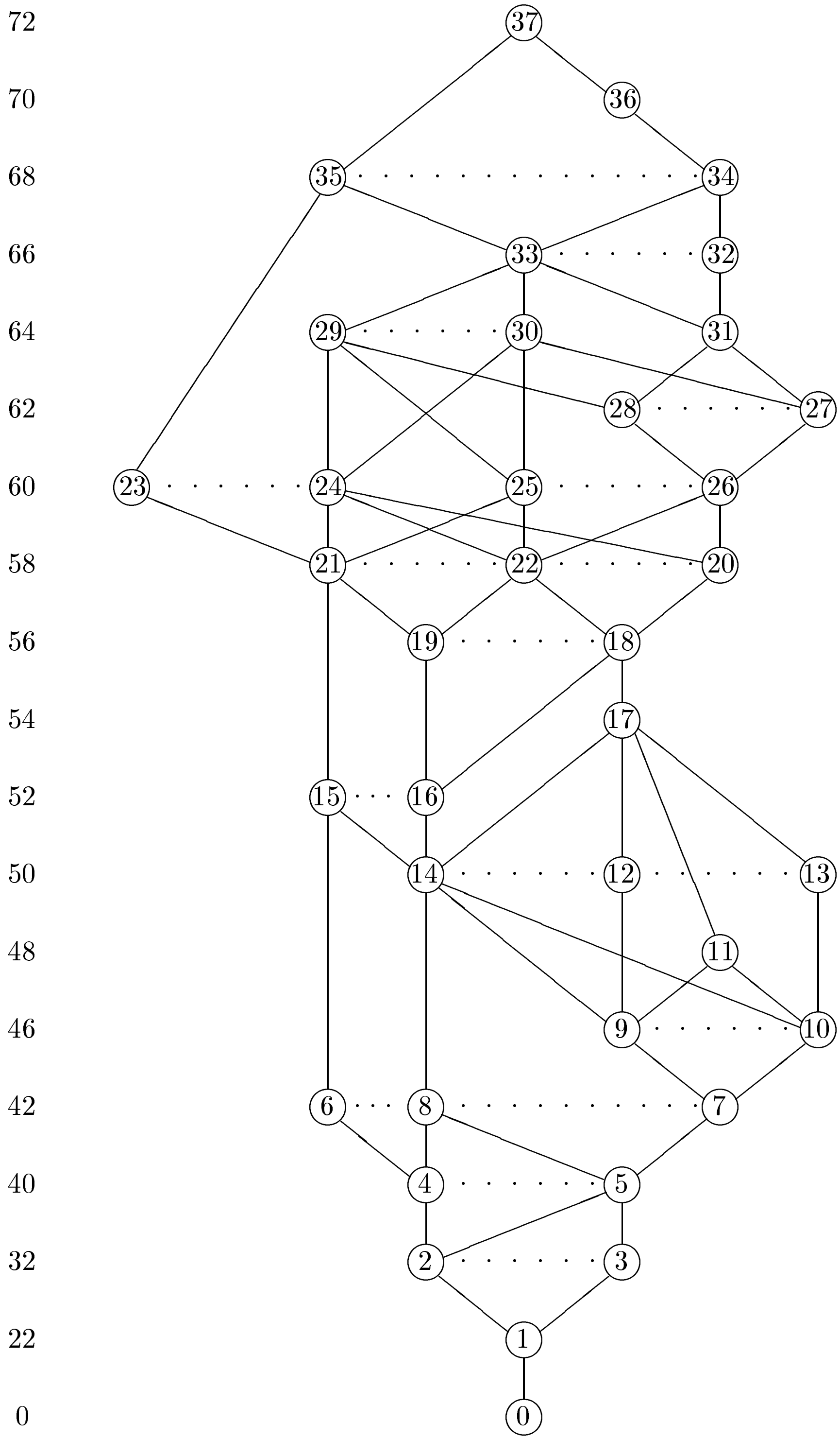

Figure 3: Closure diagram for E II

Proof. Let i and j be two vertices in the diagram of Figure 3 which are joined by a solid line with i being above j. In order to prove that $\mathcal{O}_1^i > \mathcal{O}_1^j$ it suffices, by [9, Theorem 3.1], to show that the orbit $\mathcal{O}_1^j$ meets the subspace $\mathfrak{p}_2(H^i)$. For each such pair (with $j \neq 0$) we have exhibited in Table 7 an element

$$E \in \mathfrak{p}_2(H^i) \cap \mathcal{O}_1^j.$$

The fact that $E \in \mathfrak{p}_2(H^i)$ can be verified using Table 6. The fact that $E \in \mathcal{O}_1^j$ follows from the observation that $E \in \mathfrak{g}_{H^j}(1,2)$. There are a few cases where this condition fails. In these cases we exhibit in the last column of the table an element w of the Weyl group W_0 of $(\mathfrak{k}, \mathfrak{h})$ such that $w(E) \in \mathfrak{g}_{H^j}(1,2)$. The element w is expressed as a product of reflections s_k (corresponding to the roots α_k).

Table 7: Elements $\mathbf{E} \in \mathfrak{p}_2(\mathbf{H}^i) \cap \mathcal{O}_1^j$

i	j	Type	E	w
37	36	$E_6(a_1)$	$X_{29}+X_{30}+X_{31}+X_{-25}+X_{-26}+X_{-28}$	
37	35	D_5	$X_{34}+X_{-27}+X_{30}+X_{-26}+X_{-28}$	
36	34	D_5	$X_{31}+X_{-28}+X_{-22}+X_{27}+X_{29}$	$s_5s_4s_1$
35	33	A_5+A_1	$(X_{-26}+X_{33}+X_{-27}+X_{32}+X_{-28})$ $+(X_{-19})$	
34	33	$E_6(a_3)$	$X_{29}+X_{31}+X_{-22}+X_{-24}+X_{-25}+X_{-27}$	
34	32	A_5+A_1	$(X_{-22}+X_{29}+X_{-24}+X_{31}+X_{-25})+(X_{30})$	
33	30	$D_5(a_1)$	$X_{32}+X_{-25}+X_{33}+X_{-27}+X_{-26}$	
33	29	$D_5(a_1)$	$X_{33}+X_{-22}+X_{32}+X_{-27}+X_{-28}$	
32,33	31	A_5	$X_{-22}+X_{29}+X_{-24}+X_{31}+X_{-25}$	
29,31	28	A_4+A_1	$(X_{33}+X_{-22}+X_{29}+X_{-24})+(X_{-25})$	s_5
30,31	27	A_4+A_1	$(X_{32}+X_{-25}+X_{31}+X_{-24})+(X_{-22})$	s_3
35	23	D_4	$X_{-27}+X_{34}+X_{-29}+X_{-31}$	s_4
29,30	24	D_4	$X_{32}+X_{-27}+X_{33}+X_{-24}$	
30	25	A_4	$X_{32}+X_{-25}+X_{-26}+X_{33}$	
29	25	A_4	$X_{32}+X_{-28}+X_{-22}+X_{33}$	
28	26	A_4	$X_{30}+X_{-17}+X_{29}+X_{-24}$	
27	26	A_4	$X_{30}+X_{-20}+X_{31}+X_{-24}$	
25	21	$D_4(a_1)$	$X_{34}+X_{-22}+X_{-25}+X_{-28}$	
24	21	$D_4(a_1)$	$X_{34}+X_{-19}+X_{-24}+X_{-27}$	
23	21	A_3+2A_1	$(X_{-26}+X_{35}+X_{-28})+(X_{-19})+(X_{-27})$	
24,25,26	22	A_3+2A_1	$(X_{-17}+X_{30}+X_{-20})+(X_{34})+(X_{-24})$	
26	20	$D_4(a_1)$	$X_{29}+X_{30}+X_{-17}+X_{-19}$	
24	20	$D_4(a_1)$	$X_{30}+X_{34}+X_{-8}+X_{-27}$	
21,22	19	A_3+A_1	$(X_{-22}+X_{34}+X_{-25})+(X_{-24})$	
20,22	18	A_3+A_1	$(X_{-22}+X_{34}+X_{-25})+(X_{30})$	
18	17	$2A_2+A_1$	$(X_{32}+X_{-17})+(X_{33}+X_{-20})+(X_{-19})$	
18,19	16	A_3	$X_{-22}+X_{34}+X_{-25}$	
21	15	A_3	$X_{35}+X_{-24}+X_{-27}$	
16,17	14	A_2+2A_1	$(X_{34}+X_{-19})+(X_{-17})+(X_{-20})$	
15	14	A_2+2A_1	$(X_{35}+X_{-24})+(X_{-17})+(X_{-20})$	
17	13	A_2+2A_1	$(X_{32}+X_{-17})+(X_{33})+(X_{-19})$	
17	12	A_2+2A_1	$(X_{33}+X_{-20})+(X_{32})+(X_{-19})$	

Table 7: (continued)

i	j	Type	E	w
17	11	$2A_2$	$(X_{32}+X_{-17})+(X_{33}+X_{-20})$	
13,14	10	A_2+A_1	$(X_{34}+X_{-19})+(X_{-17})$	
11	10	A_2+A_1	$(X_{33}+X_{-14})+(X_{-17})$	
12,14	9	A_2+A_1	$(X_{34}+X_{-19})+(X_{-20})$	
11	9	A_2+A_1	$(X_{32}+X_{-13})+(X_{-20})$	
14	8	$4A_1$	$(X_{35})+(X_{-17})+(X_{-19})+(X_{-20})$	
15	6	A_2	$X_{-24}+X_{-27}$	
9,10	7	A_2	$X_{34}+X_{-19}$	
7,8	5	$3A_1$	$(X_{35})+(X_{-8})+(X_{-19})$	
6,8	4	$3A_1$	$(X_{-8})+(X_{-22})+(X_{-25})$	
5	3	$2A_1$	$(X_{35})+(X_{-8})$	
4,5	2	$2A_1$	$(X_{-13})+(X_{-14})$	
2,3	1	A_1	X_{-2}	

In order to complete the proof of Theorem 4.1 it remains to show that $\mathcal{O}_1^i \not> \mathcal{O}_1^j$ when (i,j) is one of the following pairs:

$$\begin{array}{rrrrr} (6,3), & (12,4), & (13,4), & (12,10), & (13,9), \\ (17,16), & (20,19), & (29,27), & (30,28), & \\ (32,6), & (23,11), & (23,12), & (23,13), & \\ (35,32), & (11,4), & (25,20), & (36,23). & \end{array}$$

For the first nine pairs the assertion follows immediately from [9, Theorem 4.1] and Table 8 where we list the dimensions $d_i(j,k)$ for the Z_2-graded 27-dimensional simple $\mathfrak{g}$-module $V = V_0 \oplus V_1$ $(\dim V_0 = 15,\ \dim V_1 = 12)$. These dimensions are defined by

$$d_i(j,k) = \dim(V_i \cap \ker \rho(E)^j)$$

where (E,H,F) is a normal triple with $E \in \mathcal{O}_1^k$ and ρ is the representation of $\mathfrak{g}$ afforded by the module V. For each of the nine pairs (r,s) we give in Table 9 the reason why $\mathcal{O}_1^r \not> \mathcal{O}_1^s$: We have $d_i(j,r) > d_i(j,s)$ for suitably chosen i and j.

In order to deal with the remaining eight pairs (i,j), we need a more detailed description of the space $\mathfrak{p}$. As a module for $\mathrm{SL}_6 \times \mathrm{SL}_2$, $\mathfrak{p}$ is isomorphic to $V_0 \otimes V_1$ where V_0 is the fundamental 20-dimensional module for SL_6 (i.e. the third exterior power of the defining representation) and V_1 is the defining 2-dimensional module for SL_2. Let R_0 (resp. R_1) be the set of roots whose root spaces are contained in $\mathfrak{k}$ (resp. $\mathfrak{p}$). Set $R_1^+ = R_1 \cap R^+$ and $R_1^- = R_1 \backslash R_1^+$. Each of the sets R_1^+ and R_1^- consists of 20 roots. The subspaces

$$\mathfrak{p}^+ = \sum_{\alpha \in R_1^+} \mathfrak{g}^\alpha, \quad \mathfrak{p}^- = \sum_{\alpha \in R_1^-} \mathfrak{g}^\alpha$$

are simple SL_6-modules isomorphic to V_0 and $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$. An isomorphism $\varphi : \mathfrak{p}^+ \to \mathfrak{p}^-$ of SL_6-modules is given by $\varphi(X) = [X_{-36}, X]$. In Table 10 we give the φ-images of the basis of $\mathfrak{p}^+$ consisting of root vectors.

Table 8: The integers $d_i(j,k)$ for the module $V(\omega_1)$

k	$d_0(j,k),\ j \geq 1$	$d_1(j,k),\ j \geq 1$
1	12 15	9 12
2	10 15	7 11 12
3	10 14 15	7 12
4	9 15	6 9 12
5	9 13 15	6 11 12
6	9 15	6 6 12
7	9 11 15	6 10 12
8	9 12 15	6 9 12
9	8 11 14 15	4 10 12
10	7 11 15	5 10 11 12
11	6 9 14 14 15	3 8 11 12
12	8 10 13 15	3 10 12
13	6 10 15	5 10 10 12
14	7 11 14 15	4 9 11 12
15	7 10 12 15	4 6 9 11 12
16	7 9 12 14 15	4 7 9 12
17	6 9 13 14 15	3 8 10 12
18	6 9 12 13 15	3 7 9 12
19	6 9 12 14 15	3 7 9 11 12
20	6 9 12 12 15	3 6 9 12
21	6 9 12 14 15	3 6 9 10 12
22	6 8 12 13 15	3 7 9 11 12
23	6 9 9 12 12 15	3 3 6 6 9 9 12
24	6 7 9 10 12 13 15	3 5 6 8 9 11 12
25	5 7 10 11 14 15	2 5 7 10 11 11 12
26	5 6 10 11 14 14 15	2 6 7 10 11 12
27	4 6 10 11 14 14 15	2 6 7 10 10 12
28	5 6 10 11 13 14 15	1 6 7 10 11 12
29	5 6 9 10 12 13 14 15	1 5 6 8 9 11 12
30	4 6 9 10 12 13 15	2 5 6 8 9 11 11 12
31	4 5 8 9 12 13 14 14 15	1 4 5 8 9 11 11 12
32	4 5 8 9 12 12 14 14 15	1 4 5 8 9 11 11 12
33	4 5 8 9 12 13 14 14 15	1 4 5 8 9 10 11 12
34	4 5 7 8 10 10 12 12 14 14 15	1 3 4 6 7 9 9 11 11 12
35	4 5 7 8 10 11 12 13 14 15	1 3 4 6 7 8 9 10 11 11 12
36	3 3 6 6 9 9 11 11 13 13 14 14 15	0 3 3 6 6 8 8 10 10 11 11 12
37	3 3 5 5 7 7 9 9 11 11 12 12 13 13 14 14 15	0 2 2 4 4 6 6 8 8 9 9 10 10 11 11 12

We shall also need the subspaces

$$\mathfrak{p}_2^{\pm}(H^i) = \mathfrak{p}^{\pm} \cap \mathfrak{p}_2(H^i).$$

Note that

$$\mathfrak{p}_2(H^i) = \mathfrak{p}_2^{+}(H^i) \oplus \mathfrak{p}_2^{-}(H^i).$$

When viewed as a K-module, all the weights of $\mathfrak{p}$ are simple. The weight diagram of this module is exhibited in Figure 4 where a vertex with label i designates the root space of α_i. The subspaces $\mathfrak{p}^+$ and $\mathfrak{p}^-$ are clearly visible as the left and right half of that diagram. The arrows show how the root vectors of the simple roots $\beta_1, \ldots, \beta_6$ act on the weight spaces. For β_6 only two arrows are shown. The highest weight vector is X_{-2} and the lowest X_2.

Table 9: Some pairs (r,s) with $\mathcal{O}_1^r \not> \mathcal{O}_1^s$

r	s	$d_i(j,r)$	$d_i(j,s)$
6	3	$d_0(2,6)=15$	$d_0(2,3)=14$
12	4	$d_1(2,12)=10$	$d_1(2,4)=9$
13	4	$d_1(2,13)=10$	$d_1(2,4)=9$
12	10	$d_0(1,12)=8$	$d_0(1,10)=7$
13	9	$d_0(3,13)=15$	$d_0(3,9)=14$
17	16	$d_0(3,17)=13$	$d_0(3,16)=12$
20	19	$d_1(4,20)=12$	$d_1(4,19)=11$
29	27	$d_0(1,29)=5$	$d_0(1,27)=4$
30	28	$d_1(1,30)=2$	$d_1(1,28)=1$
32	23	$d_1(2,32)=4$	$d_1(2,23)=3$
33	23	$d_1(2,33)=4$	$d_1(2,23)=3$

Let $\pi : \mathfrak{p} \to \mathfrak{p}^+$ be the projector with kernel $\mathfrak{p}^-$. Note that π commutes with the action of SL_6. Any $Z \in \mathfrak{p}$ can be written uniquely as $Z = X + \varphi(Y)$ with $X, Y \in \mathfrak{p}^+$. The orbit $\mathrm{SL}_2 \cdot Z$ consists of all vectors

$$(aX + bY) + \varphi(cX + dY)$$

with $ad - bc = 1$. Hence

$$\pi(\mathrm{SL}_2 \cdot Z) = \langle X, Y\rangle \backslash \{0\}$$

where $\langle X, Y\rangle$ denotes the subspace of $\mathfrak{p}^+$ spanned by X and Y.

Table 10: The isomorphism φ

X	$\varphi(X)$	X	$\varphi(X)$	X	$\varphi(X)$	X	$\varphi(X)$
X_2	X_{-35}	X_{19}	$-X_{-30}$	X_{26}	$-X_{-25}$	X_{31}	X_{-17}
X_8	$-X_{-34}$	X_{20}	$-X_{-29}$	X_{27}	$-X_{-24}$	X_{32}	$-X_{-14}$
X_{13}	X_{-33}	X_{22}	X_{-28}	X_{28}	$-X_{-22}$	X_{33}	$-X_{-13}$
X_{14}	X_{-32}	X_{24}	X_{-27}	X_{29}	X_{-20}	X_{34}	X_{-8}
X_{17}	$-X_{-31}$	X_{25}	X_{-26}	X_{30}	X_{-19}	X_{35}	$-X_{-2}$

The pair $(\mathrm{SL}_6 \times \mathrm{GL}_1, \mathfrak{p}^+)$, where GL_1 is the maximal torus of the SL_2 factor of K leaving $\mathfrak{p}^+$ and $\mathfrak{p}^-$ invariant, is a regular prehomogeneous vector space (see [15, p. 145]). There are four nonzero orbits in this space:

$$\mathfrak{p}^+ \cap \mathcal{O}_1^6,\ \mathfrak{p}^+ \cap \mathcal{O}_1^4,\ \mathfrak{p}^+ \cap \mathcal{O}_1^2,\ \mathfrak{p}^+ \cap \mathcal{O}_1^1$$

with representatives

$$X_{25} + X_{26},\ X_8 + X_{22} + X_{25},\ X_2 + X_{24},\ X_2$$

and dimensions 20, 19, 15, 10, respectively. The singular set of this prehomogeneous vector space is the hypersurface $S = \mathfrak{p}^+ \backslash \mathcal{O}_1^6$.

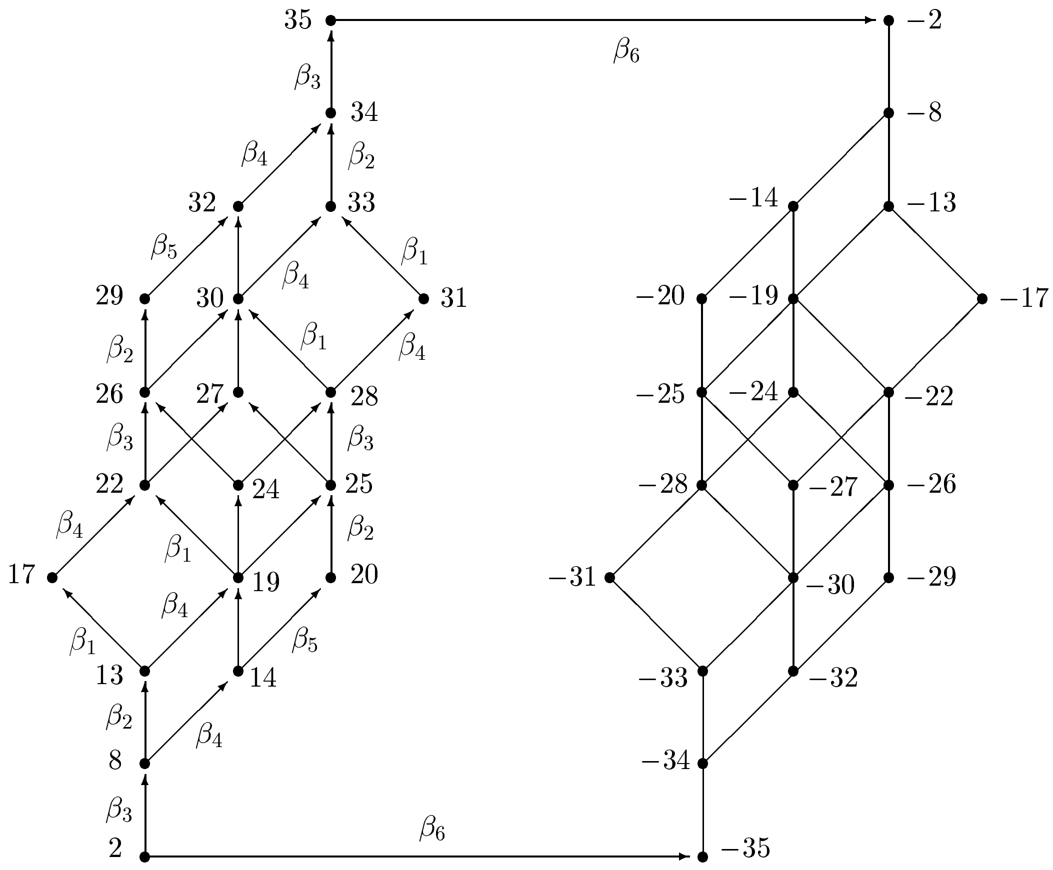

Figure 4: The weight diagram of $\mathfrak{p}$

We start with the pair $(32, 6)$. Note that $\mathcal{O}_1^{32} \not> \mathcal{O}_1^{23}$ follows immediately from Table 9, but the claim that $\mathcal{O}_1^{32} \not> \mathcal{O}_1^6$ that we have to prove is much stronger. Let $E \in \mathcal{O}_1^{32}$ be the representative from Table 5 and write $E = X + \varphi(Y)$ where

$$X = X_{29} + X_{30} + X_{31},\ Y = -X_{26} - X_{27} - X_{28}.$$

Since $\pi(\mathrm{SL}_2 \cdot E) \subset \langle X, Y\rangle \subset S$, we have $\pi(\mathcal{O}_1^{32}) \subset S$ and so $\mathcal{O}_1^{32} \subset S + \mathfrak{p}^-$. Since S is closed and $S \subset \mathfrak{p}^+$, it is clear that $S + \mathfrak{p}^-$ is closed. Hence $\overline{\mathcal{O}_1^{32}} \subset S + \mathfrak{p}^-$ and consequently $\mathcal{O}_1^6 \not\subset \overline{\mathcal{O}_1^{32}}$, i.e., $\mathcal{O}_1^{32} \not> \mathcal{O}_1^6$.

Next we consider the pairs $(23, 11)$, $(23, 12)$, $(23, 13)$. Let E be the representative from Table 5 of one of the orbits $\mathcal{O}_1^{11}, \mathcal{O}_1^{12}, \mathcal{O}_1^{13}$. Write $E = X + \varphi(Y)$ with $X, Y \in \mathfrak{p}^+$. It is easy to check that $\langle X, Y\rangle \subset \mathcal{O}_1^2 \cup \{0\}$. Consequently,

$$\pi(\mathrm{SL}_2 \cdot E) \subset \mathfrak{p}^+ \cap \mathcal{O}_1^2,$$

and

$$\pi(K\cdot E)=\mathrm{SL}_6\cdot\pi(\mathrm{SL}_2\cdot E)\subset\mathfrak{p}^+\cap\mathcal{O}_1^2.$$

Since

$$\mathfrak{p}_2(H^{23})=\mathfrak{p}_2^+(H^{23})\oplus\mathfrak{p}_2^-(H^{23})$$

and

$$\mathfrak{p}_2^+(H^{23})=\langle X_{35}\rangle\subset\mathcal{O}_1^1\cup\{0\},$$

we conclude that the orbit $K\cdot E$ does not meet $\mathfrak{p}_2(H^{23})$. Now [9, Theorem 3.1] implies that

$$\mathcal{O}_1^{23}\not\geq\mathcal{O}_1^{11},\mathcal{O}_1^{12},\mathcal{O}_1^{13}.$$

Let us now consider the pair $(35,32)$. The representative $E\in\mathcal{O}_1^{32}$ from Table 5 can be written as $E=X+\varphi(Y)$ where

$$X=X_{29}+X_{30}+X_{31},\quad Y=-X_{26}-X_{27}-X_{28}.$$

Hence if $E'\in\mathrm{SL}_2\cdot E$ then $\pi(E')$ is a nonzero linear combination of X and Y. It is easy to verify that all such elements $\pi(E')$ belong to $\mathcal{O}_1^4$. On the other hand

$$\mathfrak{p}_2^+(H^{35})=\langle X_{30},X_{32},X_{33},X_{34},X_{35}\rangle\subset\overline{\mathcal{O}_1^2}.$$

Since $\dim\mathcal{O}_1^4>\dim\mathcal{O}_1^2$, we conclude that $\mathcal{O}_1^4\cap\overline{\mathcal{O}_1^2}=\emptyset$ and so

$$\mathrm{SL}_6\cdot\pi(E')\cap\mathfrak{p}_2^+(H^{35})=\emptyset.$$

This implies that $\mathcal{O}_1^{32}=K\cdot E$ does not meet $\mathfrak{p}_2(H^{35})$. Hence $\mathcal{O}_1^{35}\not\geq\mathcal{O}_1^{32}$.

Next we consider the pair $(11,4)$. We have

$$\mathfrak{p}_2(H^{11})=V\oplus\varphi(V)$$

where

$$V=\langle X_{29},X_{31},X_{32},X_{33},X_{34},X_{35}\rangle\subset\mathfrak{p}^+.$$

It is easy to check that $V\cap\mathcal{O}_1^2$ is an open dense subset of V. As $\dim\mathcal{O}_1^4>\dim\mathcal{O}_1^2$, we conclude that

$$V\cap\mathcal{O}_1^4=\varphi(V)\cap\mathcal{O}_1^4=\emptyset.$$

The representative $E\in\mathcal{O}_1^4$ from Table 5 belongs to $\mathfrak{p}^-$. Hence

$$\mathrm{SL}_2\cdot E=\langle E,\varphi^{-1}(E)\rangle\backslash\{0\}.$$

Since

$$\mathrm{SL}_6\cdot\varphi^{-1}(E)\cap V\subset\mathcal{O}_1^4\cap V=\emptyset,$$

and

$$\mathrm{SL}_6\cdot E\cap\varphi(V)\subset\mathcal{O}_1^4\cap\varphi(V)=\emptyset,$$

we can conclude that $\mathcal{O}_1^4=K\cdot E$ does not meet $\mathfrak{p}_2(H^{11})$. Hence $\mathcal{O}_1^{11}\not\geq\mathcal{O}_1^4$.

We now turn to the pair $(25,20)$. The centralizer of H^{25} in K is $\mathrm{SL}_4\cdot T_3$ where T_3 is a 3-dimensional central torus. The space $\mathfrak{g}_{H^{25}}(1,2)$ is a direct sum of three simple modules for this centralizer:

$$\mathfrak{g}_{H^{25}}(1,2)=V_1\oplus V_2\oplus V_3.$$

The basis elements for these modules are:

$$\begin{aligned} V_1 &: \quad X_{27}, X_{30}, X_{32}, X_{33}, X_{34}, X_{35}; \\ V_2 &: \quad X_{-31}, X_{-28}, X_{-25}, X_{-20}; \\ V_3 &: \quad X_{-29}, X_{-26}, X_{-22}, X_{-17}. \end{aligned}$$

We write an arbitrary $X \in \mathfrak{p}_2(H^{25})$ as

$$X = X^{(1)} + X^{(2)} + X^{(3)} + X'$$

where $X' \in \mathfrak{p}_3(H^{25})$ and $X^{(i)} \in V_i$ are given by:

$$\begin{aligned} X^{(1)} &= x_1X_{27} + x_2X_{30} + x_3X_{32} + x_4X_{33} + x_5X_{34} + x_6X_{35}, \\ X^{(2)} &= y_1X_{-31} + y_2X_{-28} + y_3X_{-25} + y_4X_{-20}, \\ X^{(3)} &= z_1X_{-29} + z_2X_{-26} + z_3X_{-22} + z_4X_{-17}. \end{aligned}$$

The singular set S of $(Q_{H^{25}}, \mathfrak{p}_2(H^{25}))$ is the union of two hyperquadrics

$$\begin{aligned} S_1 &= \{X \in \mathfrak{p}_2(H^{25}) : \ f(X) = 0\}, \\ S_2 &= \{X \in \mathfrak{p}_2(H^{25}) : \ g(X) = 0\}, \end{aligned}$$

where f and g are the relative invariants defined by

$$\begin{aligned} f(X) &= x_1x_6 - x_2x_5 + x_3x_4, \\ g(X) &= y_1z_4 - y_2z_3 + y_3z_2 - y_4z_1. \end{aligned}$$

The intersection $S_1 \cap \mathcal{O}_1^{22}$ (resp. $S_2 \cap \mathcal{O}_1^{21}$) is a dense open subset of S_1 (resp. S_2). Since the orbits $\mathcal{O}_1^{20}, \mathcal{O}_1^{21}$, and $\mathcal{O}_1^{22}$ have the same dimension, we conclude that $\mathcal{O}_1^{20}$ does not meet $\mathfrak{p}_2(H^{25})$. This implies that $\mathcal{O}_1^{25} \not> \mathcal{O}_1^{20}$.

The last (and most difficult) pair we have to consider is $(36, 23)$. We shall need some facts about the prehomogeneous vector space $(Q_{H^{36}}, \mathfrak{p}_2(H^{36}))$. The centralizer of H^{36} in K has the form

$$Z_K(H^{36}) = \mathrm{SL}_2 \cdot T_5$$

where the positive root of this SL_2 is α_4 and T_5 is a 5-dimensional central torus. As a module for this centralizer, the space $\mathfrak{g}_{H^{36}}(1, 2)$ is a direct sum of the following five simple modules:

$$\begin{aligned} &V_1 = \langle X_{-22}, X_{-26}\rangle, \quad V_2 = \langle X_{27}, X_{30}\rangle, \quad V_3 = \langle X_{-25}, X_{-28}\rangle, \\ &V_4 = \langle X_{29}\rangle, \quad V_5 = \langle X_{31}\rangle. \end{aligned}$$

The torus T_5 acts on each of the V_i by scalar multiplications.

Write an arbitrary vector $X \in \mathfrak{p}_2(H^{36})$ as

$$X = x_1X_{-22} + x_2X_{-26} + y_1X_{27} + y_2X_{30} + z_1X_{-25} + z_2X_{-28} + uX_{29} + vX_{31} + X'$$

where $X' \in \mathfrak{p}_3(H^{36})$. The singular set S of this prehomogeneous vector space is the union of the three hyperquadrics:

$$S_1: \ y_1z_2 + y_2z_1 = 0; \quad S_2: \ x_1z_2 - x_2z_1 = 0; \quad S_3: \ x_1y_2 + x_2y_1 = 0;$$

and two hyperplanes:

$$S_4 :\ u = 0; \quad S_5 :\ v = 0.$$

Let

$$\begin{aligned}
Y_1 &= X_{27} + X_{29} + X_{31} + X_{-22} + X_{-28},\\
Y_2 &= (X_{31} + X_{-25} + X_{27} + X_{-22} + X_{29}) + (X_{-8}),\\
Y_3 &= X_{29} + X_{30} + X_{31} + X_{-22} + X_{-28},\\
Y_4 &= X_{27} + X_{30} + X_{31} + X_{-22} + X_{-28},\\
Y_5 &= X_{27} + X_{29} + X_{30} + X_{-22} + X_{-28}.
\end{aligned}$$

Then $Y_i \in S_i$ and a simple computation shows that the orbit $Q_{H^{36}} \cdot Y_i$ is a dense open subset of S_i. It is not hard to verify that

$$Y_1, Y_3 \in \mathcal{O}_1^{34}; \quad Y_2 \in \mathcal{O}_1^{32}; \quad Y_4 \in \mathcal{O}_1^{30}; \quad Y_5 \in \mathcal{O}_1^{29}.$$

It follows that $S \subset \overline{\mathcal{O}_1^{34}}$. The upshot of this argument is that the proof of $\mathcal{O}_1^{36} \not> \mathcal{O}_1^{23}$ is reduced to that of $\mathcal{O}_1^{34} \not> \mathcal{O}_1^{23}$.

We now turn to the pair $(34, 23)$. It is immediate from Table 9 that

$$\mathcal{O}_1^{32} \not> \mathcal{O}_1^{23}, \quad \mathcal{O}_1^{33} \not> \mathcal{O}_1^{23}.$$

The centralizer of H^{34} in K is just the maximal torus T_6 (with Lie algebra $\mathfrak{h}$). The space $\mathfrak{p}_2(H^{34})$ has dimension 18 while its subspace $\mathfrak{g}_{H^{34}}(1,2)$ has dimension 5 and a basis:

$$\{X_{30}, X_{-27}, X_{-24}, X_{29}, X_{31}\}.$$

Write an arbitrary vector $X \in \mathfrak{p}_2(H^{34})$ as

$$X = x_1 X_{30} + x_2 X_{-27} + x_3 X_{-24} + x_4 X_{29} + x_5 X_{31} + X'$$

where $X' \in \mathfrak{p}_3(H^{34})$. The singular set S of the prehomogeneous vector space $(Q_{H^{34}}, \mathfrak{p}_2(H^{34}))$ is the union of the five hyperplanes: $S_i :\ x_i = 0\}$.

Now let

$$\begin{aligned}
Y_1 &= X_{-27} + X_{-24} + X_{29} + X_{31} + X_{-22} + X_{-25},\\
Y_2 &= (X_{-22} + X_{29} + X_{-24} + X_{31} + X_{-25}) + (X_{30}),\\
Y_4 &= X_{30} + X_{-27} + X_{-24} + X_{31} + X_{32} + X_{-25},\\
Y_5 &= X_{30} + X_{-27} + X_{-24} + X_{29} + X_{32} + X_{33}.
\end{aligned}$$

Then $Y_i \in S_i$ and the orbit $Q_{H^{34}} \cdot Y_i$ is a dense open subset of S_i. It is not hard to verify that $Y_1 \in \mathcal{O}_1^{33}$ and observe that Y_2 is the representative $E^{32} \in \mathcal{O}_1^{32}$ listed in Table 5. It follows that

$$S_1 \cup S_2 \subset \overline{\mathcal{O}_1^{32}} \cup \overline{\mathcal{O}_1^{33}}.$$

We also have

$$S_4 \cup S_5 \subset \overline{\mathcal{O}_1^{33}}$$

because a computation shows that each of the orbits $G \cdot Y_4$ and $G \cdot Y_5$ has dimension 64.

If $Y \in S_3$ then the orbit $G \cdot Y$ has dimension ≤ 58, and so $S_3 \subset \overline{\mathcal{O}_1^{33}}$. As $\mathcal{O}_1^{23}$ is not contained in the union of $\overline{\mathcal{O}_1^{32}}$ and $\overline{\mathcal{O}_1^{33}}$, we infer that it does not meet $\mathfrak{p}_2(H^{34})$. We deduce that $\mathcal{O}_1^{34} \not> \mathcal{O}_1^{23}$.

This completes the proof of the theorem.

An interesting feature of the above prehomogeneous vector space is that the hyperplane S_3 contains infinitely many $Q_{H^{34}}$-orbits. Indeed $\dim S_3 = 17$ while each of the orbits in S_3 has dimension ≤ 15. ∎

5. Types E III and E IV

Let $\mathfrak{g}_0$ be of type E III and so $\mathfrak{k}$ is of type $D_5 + \mathbf{C}$ and $K = (\mathrm{Spin}_{10} \times \mathrm{GL}_1)/Z_4$. We may assume that $\mathfrak{h} \subset \mathfrak{k}$. The roots

$$\beta_1 = \alpha_1, \quad \beta_2 = \alpha_3, \quad \beta_3 = \alpha_4, \quad \beta_4 = \alpha_5, \quad \beta_5 = \alpha_2$$

form a base of the root system of $(\mathfrak{k}, \mathfrak{h})$. We also set $\beta_6 = \alpha_6$.

Table 11: Nonzero nilpotent orbits in $\mathfrak{p}$ ($\mathfrak{g}_0 =$ E III)

k	i	$\beta_j(H^i)$		E^i	Type of E^i
1	1	00001	0	X_{36}	A_1
1	2	00010	-2	X_{-6}	A_1
2	3	10000	1	$(X_{23}) + (X_{36})$	$2A_1$
2	4	10000	-2	$(X_{-6}) + (X_{-31})$	$2A_1$
2	5	00011	-2	$(X_{36}) + (X_{-6})$	$2A_1$
4	6	02000	-2	$X_{32} + X_{-6}$	A_2
5	7	11010	-2	$(X_{32} + X_{-6}) + (X_{33})$	$A_2 + A_1$
5	8	11001	-3	$(X_{36} + X_{-20}) + (X_{-21})$	$A_2 + A_1$
6	9	40000	-2	$(X_{23} + X_{-21}) + (X_{36} + X_{-20})$	$2A_2$
8	10	00013	-2	$X_{20} + X_{-6} + X_{32}$	A_3
8	11	00031	-6	$X_{-20} + X_{36} + X_{-32}$	A_3
12	12	02022	-6	$X_{-20} + X_{31} + X_{-21} + X_{32}$	A_4

Table 11 lists the nonzero K-orbits $\mathcal{O}_1^i \subset \mathcal{N}_1$, $1 \leq i \leq 12$. Its description is the same as that of Table 5 (except that $\beta_6(H^i)$ may be negative).

Theorem 5.1. *Let $\mathfrak{g}_0$ be of type* E III *or* E IV. *Then the closure diagram of the orbit space $\mathcal{N}_1/K$ is as given in Figure 5.*

Proof. Assume first that $\mathfrak{g}_0 =$ E IV. Then there are only two nonzero orbits in $\mathcal{N}_1$: $\mathcal{O}_1^1$ of dimension 32 and $\mathcal{O}_1^2$ of dimension 48. Since $\mathcal{N}_1$ is an equidimensional variety, in this case it is irreducible. Hence $\mathcal{O}_1^2 > \mathcal{O}_1^1$.

From now on let $\mathfrak{g}_0 =$ E III.

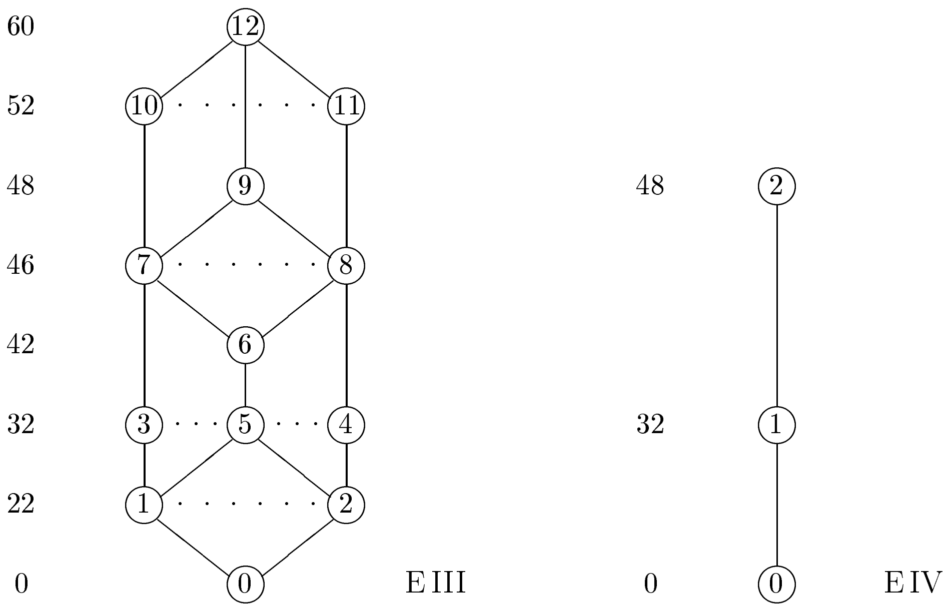

Figure 5: Closure diagrams for E III and E IV

Table 12: Elements $\mathbf{E} \in \mathfrak{p}_2(\mathbf{H}^{\mathbf{i}}) \cap \mathcal{O}_1^{\mathbf{j}}$

i	j	Type	E	w
12	11	A_3	$X_{-20} + X_{31} + X_{-21}$	s_{15}
12	10	A_3	$X_{31} + X_{-21} + X_{32}$	s_{17}
12	9	$2A_2$	$(X_{32} + X_{-21}) + (X_{33} + X_{-20})$	
9,11	8	$A_2 + A_1$	$(X_{36} + X_{-20}) + (X_{-21})$	
9,10	7	$A_2 + A_1$	$(X_{32} + X_{-6}) + (X_{33})$	
8	6	A_2	$X_{36} + X_{-20}$	
8	4	$2A_1$	$(X_{-20}) + (X_{-21})$	
7	6	A_2	$X_{32} + X_{-6}$	
7	3	$2A_1$	$(X_{32}) + (X_{33})$	
6	5	$2A_1$	$(X_{36}) + (X_{-6})$	
4,5	2	A_1	X_{-6}	
3,5	1	A_1	X_{36}	

Let (i, j) be a pair of vertices in the E III diagram in Figure 5 which are joined by a solid line with i above j. We show that $\mathcal{O}_1^i > \mathcal{O}_1^j$ by exhibiting an element

$$E \in \mathfrak{p}_2(H^i) \cap \mathcal{O}_1^j.$$

This is accomplished in Table 12 whose description is the same as that of Table 7.

It remains to show that $\mathcal{O}_1^i \not> \mathcal{O}_1^j$ when (i, j) is one of the four pairs: $(10, 4)$, $(11, 3)$, $(3, 2)$, and $(4, 1)$. In all four cases this can be verified by applying [9, Theorem 4.1] to the module $V(\omega_1)$. We omit the details. ■

6. Appendix

Here is our enumeration of the positive roots of E_6. (The simple roots are enumerated as in [2].)

Table 13: Positive roots of $\mathbf{E_6}$

i	α_i	i	α_i	i	α_i	i	α_i
1	100000	10	000110	19	011110	28	011211
2	010000	11	000011	20	010111	29	112210
3	001000	12	101100	21	001111	30	111211
4	000100	13	011100	22	111110	31	011221
5	000010	14	010110	23	101111	32	112211
6	000001	15	001110	24	011210	33	111221
7	101000	16	000111	25	011111	34	112221
8	010100	17	111100	26	111210	35	112321
9	001100	18	101110	27	111111	36	122321

Consider the structure constants $N(i,j)$ of E_6 defined by

$$[X_i, X_j] = N(i,j)X_k$$

where α_i and α_j are roots such that $\alpha_i+\alpha_j = \alpha_k$ is also a root. Then $N(i,j) = \pm 1$. Table 14 records the above relations for $i > 0$. The other such relations can be easily written down because $N(-i,-j) = N(i,j)$.

The relation displayed above is recorded in the table by inserting in the i-th box the entry $j : N(i,j)X_k$. For instance we have

$$[X_2, X_4] = -X_8, \quad [X_2, X_{-8}] = X_{-4}, \quad [X_2, X_{-13}] = -X_{-9}.$$

Table 14: Some defining relations of $\mathbf{E_6}$

1	$3 : X_7,\ 9 : X_{12},\ 13 : X_{17},\ 15 : X_{18},\ 19 : X_{22},\ 21 : X_{23},\ 24 : X_{26},\ 25 : X_{27},$ $28 : X_{30},\ 31 : X_{33},\ -7 : -X_{-3},\ -12 : -X_{-9},\ -17 : -X_{-13},\ -18 : -X_{-15},$ $-22 : -X_{-19},\ -23 : -X_{-21},\ -26 : -X_{-24}, -27 : -X_{-25},\ -30 : -X_{-28},$ $-33 : -X_{-31}$
2	$4 : -X_8,\ 9 : X_{13},\ 10 : X_{14},\ 12 : X_{17},\ 15 : X_{19},\ 16 : X_{20},\ 18 : X_{22},$ $21 : X_{25},\ 23 : X_{27},\ 35 : X_{36},\ -8 : X_{-4},\ -13 : -X_{-9},\ -14 : -X_{-10},$ $-17 : -X_{-12},\ -19 : -X_{-15},\ -20 : -X_{-16},\ -22 : -X_{-18},\ -25 : -X_{-21},$ $-27 : -X_{-23},\ -36 : -X_{-35}$
3	$1 : -X_7,\ 4 : -X_9,\ 8 : X_{13},\ 10 : X_{15},\ 14 : X_{19},\ 16 : X_{21},\ 20 : X_{25},$ $26 : X_{29},\ 30 : X_{32},\ 33 : X_{34},\ -7 : X_{-1},\ -9 : X_{-4},\ -13 : -X_{-8},$ $-15 : -X_{-10},\ -19 : -X_{-14},\ -21 : -X_{-16},\ -25 : -X_{-20},\ -29 : -X_{-26},$ $-32 : -X_{-30},\ -34 : -X_{-33}$
4	$2 : X_8,\ 3 : X_9,\ 5 : X_{10},\ 7 : X_{12},\ 11 : X_{16},\ 19 : X_{24},\ 22 : X_{26},\ 25 : X_{28},$ $27 : X_{30},\ 34 : X_{35},\ -8 : -X_{-2},\ -9 : -X_{-3},\ -10 : -X_{-5},\ -12 : -X_{-7},$ $-16 : -X_{-11},\ -24 : -X_{-19},\ -26 : -X_{-22},\ -28 : -X_{-25},\ -30 : -X_{-27},$ $-35 : -X_{-34}$
5	$4 : -X_{10},\ 6 : -X_{11},\ 8 : X_{14},\ 9 : X_{15},\ 12 : X_{18},\ 13 : X_{19},\ 17 : X_{22},$ $28 : X_{31},\ 30 : X_{33},\ 32 : X_{34},\ -10 : X_{-4},\ -11 : X_{-6},\ -14 : -X_{-8},$ $-15 : -X_{-9},\ -18 : -X_{-12},\ -19 : -X_{-13},\ -22 : -X-17,\ -31 : -X_{-28},$ $-33 : -X_{-30},\ -34 : -X_{-32}$
6	$5 : X_{11},\ 10 : X_{16},\ 14 : X_{20},\ 15 : X_{21},\ 18 : X_{23},\ 19 : X_{25},\ 22 : X_{27},$ $24 : X_{28},\ 26 : X_{30},\ 29 : X_{32},\ -11 : -X_{-5},\ -16 : -X_{-10},$ $-20 : -X_{-14},\ -21 : -X_{-15},\ -23 : -X_{-18},\ -25 : -X_{-19},\ -27 : -X_{-22},$ $-28 : -X_{-24},\ -30 : -X_{-26},\ -32 : -X_{-29}$
7	$4 : -X_{12},\ 8 : X_{17},\ 10 : X_{18},\ 14 : X_{22},\ 16 : X_{23},\ 20 : X_{27},\ 24 : -X_{29},$ $28 : -X_{32},\ 31 : -X_{34},\ -1 : X_3,\ -3 : -X_1,\ -12 : X_{-4},\ -17 : -X_{-8},$ $-18 : -X_{-10},\ -22 : -X_{-14},\ -23 : -X_{-16},\ -27 : -X_{-20},\ -29 : X_{-24},$ $-32 : X_{-28},\ -34 : X_{-31}$
8	$3 : -X_{13},\ 5 : -X_{14},\ 7 : -X_{17},\ 11 : -X_{20},\ 15 : X_{24},\ 18 : X_{26},\ 21 : X_{28},$ $23 : X_{30},\ 34 : -X_{36},\ -2 : -X_4,\ -4 : X_2,\ -13 : X_{-3},\ -14 : X_{-5},$ $-17 : X_{-7},\ -20 : X_{-11},\ -24 : -X_{-15},\ -26 : -X_{-18},\ -28 : -X_{-21},$ $-30 : -X_{-23},\ -36 : X_{-34}$
9	$1 : -X_{12},\ 2 : -X_{13},\ 5 : -X_{15},\ 11 : -X_{21},\ 14 : X_{24},\ 20 : X_{28},\ 22 : -X_{29},$ $27 : -X_{32},\ 33 : X_{35},\ -3 : -X_4,\ -4 : X_3,\ -12 : X_{-1},\ -13 : X_{-2},$ $-15 : X_{-5},\ -21 : X_{-11},\ -24 : -X_{-14},\ -28 : -X_{-20},\ -29 : X_{-22},$ $-32 : X_{-27},\ -35 : -X_{-33}$
10	$2 : -X_{14},\ 3 : -X_{15},\ 6 : -X_{16},\ 7 : -X_{18},\ 13 : X_{24},\ 17 : X_{26},\ 25 : -X_{31},$ $27 : -X_{33},\ 32 : X_{35},\ -4 : X_5 - 5 : -X_4,\ -14 : X_{-2},\ -15 : X_{-3},$ $-16 : X_{-6},\ -18 : X_{-7},\ -24 : -X_{-13},\ -26 : -X_{-17},\ -31 : X_{-25},$ $-33 : X_{-27},\ -35 : -X_{-32}$
11	$4 : -X_{16},\ 8 : X_{20},\ 9 : X_{21},\ 12 : X_{23},\ 13 : X_{25},\ 17 : X_{27},\ 24 : -X_{31},$ $26 : -X_{33},\ 29 : -X_{34},\ -5 : -X_6,\ -6 : X_5,\ -16 : X_{-4},\ -20 : -X_{-8},$ $-21 : -X_{-9},\ -23 : -X_{-12},\ -25 : -X_{-13},\ -27 : -X_{-17},\ -31 : X_{-24},$ $-33 : X_{-26},\ -34 : X_{-29}$

Table 14: (continued)

12	$2:-X_{17},\ 5:-X_{18},\ 11:-X_{23},\ 14:X_{26},\ 19:X_{29},\ 20:X_{30},\ 25:X_{32},$ $31:-X_{35},\ -1:X_9,\ -4:X_7,\ -7:-X_4,\ -9:-X_1,\ -17:X_{-2},$ $-18:X_{-5},\ -23:X_{-11},\ -26:-X_{-14},\ -29:-X_{-19},\ -30:-X_{-20},$ $-32:-X_{-25},\ -35:X_{-31}$
13	$1:-X_{17},\ 5:-X_{19},\ 10:-X_{24},\ 11:-X_{25},\ 16:-X_{28},\ 18:X_{29},\ 23:X_{32},$ $33:X_{36},\ -2:X_9,\ -3:X_8,\ -8:-X_3,\ -9:-X_2,\ -17:X_{-1},\ -19:X_{-5},$ $-24:X_{-10},\ -25:X_{-11},\ -28:X_{-16},\ -29:-X_{-18},\ -32:-X_{-23},$ $-36:-X_{-33}$
14	$3:-X_{19},\ 6:-X_{20},\ 7:-X_{22},\ 9:-X_{24},\ 12:-X_{26},\ 21:X_{31},\ 23:X_{33},$ $32:X_{36},\ -2:X_{10},\ -5:X_8,\ -8:-X_5,\ -10:-X_2,\ -19:X_{-3},$ $-20:X_{-6},\ -22:X_{-7},\ -24:X_{-9},\ -26:X_{-12},\ -31:-X_{-21},$ $-33:-X_{-23},\ -36:-X_{-32}$
15	$1:-X_{18},\ 2:-X_{19},\ 6:-X_{21},\ 8:-X_{24},\ 17:X_{29},\ 20:X_{31},\ 27:-X_{34},$ $30:-X_{35},\ -3:X_{10},\ -5:X_9,\ -9:-X_5,\ -10:-X_3,\ -18:X_{-1},$ $-19:X_{-2},\ -21:X_{-6},\ -24:X_{-8},\ -29:-X_{-17},\ -31:-X_{-20},$ $-34:X_{-27},\ -35:X_{-30}$
16	$2:-X_{20},\ 3:-X_{21},\ 7:-X_{23},\ 13:X_{28},\ 17:X_{30},\ 19:X_{31},\ 22:X_{33},$ $29:-X_{35},\ -4:X_{11},\ -6:X_{10},\ -10:-X_6,\ -11:-X_4,\ -20:X_{-2},$ $-21:X_{-3},\ -23:X_{-7},\ -28:-X_{-13},\ -30:-X_{-17},\ -31:-X_{-19},$ $-33:-X_{-22},\ -35:X_{-29}$
17	$5:-X_{22},\ 10:-X_{26},\ 11:-X_{27},\ 15:-X_{29},\ 16:-X_{30},\ 21:-X_{32},$ $31:-X_{36},\ -1:X_{13},\ -2:X_{12},\ -7:X_8,\ -8:-X_7,\ -12:-X_2,$ $-13:-X_1,\ -22:X_{-5},\ -26:X_{-10},\ -27:X_{-11},\ -29:X_{-15},$ $-30:X_{-16},\ -32:X_{-21},\ -36:X_{-31}$
18	$2:-X_{22},\ 6:-X_{23},\ 8:-X_{26},\ 13:-X_{29},\ 20:X_{33},\ 25:X_{34},\ 28:X_{35},$ $-1:X_{15},\ -5:X_{12},\ -7:X_{10},\ -10:-X_7,\ -12:-X_5,\ -15:-X_1,$ $-22:X_{-2},\ -23:X_{-6},\ -26:X_{-8},\ -29:X_{-13},\ -33:-X_{-20},$ $-34:-X_{-25},\ -35:-X_{-28}$
19	$1:-X_{22},\ 4:-X_{24},\ 6:-X_{25},\ 12:-X_{29},\ 16:-X_{31},\ 23:X_{34},\ 30:-X_{36},$ $-2:X_{15},\ -3:X_{14},\ -5:X_{13},\ -13:-X_5,\ -14:-X_3,\ -15:-X_2,$ $-22:X_{-1},\ -24:X_{-4},\ -25:X_{-6},\ -29:X_{-12},\ -31:X_{-16},$ $-34:-X_{-23},\ -36:X_{-30}$
20	$3:-X_{25},\ 7:-X_{27},\ 9:-X_{28},\ 12:-X_{30},\ 15:-X_{31},\ 18:-X_{33},$ $29:-X_{36},\ -2:X_{16},\ -6:X_{14},\ -8:-X_{11},\ -11:X_8,\ -14:-X_6,$ $-16:-X_2,\ -25:X_{-3},\ -27:X_{-7},\ -28:X_{-9},\ -30:X_{-12},$ $-31:X_{-15},\ -33:X_{-18},\ -36:X_{-29}$
21	$1:-X_{23},\ 2:-X_{25},\ 8:-X_{28},\ 14:-X_{31},\ 17:X_{32},\ 22:X_{34},\ 26:X_{35},$ $-3:X_{16},\ -6:X_{15},\ -9:-X_{11},\ -11:X_9,\ -15:-X_6,\ -16:-X_3,$ $-23:X_{-1},\ -25:X_{-2},\ -28:X_{-8},\ -31:X_{-14},\ -32:-X_{-17},$ $-34:-X_{-22},\ -35:-X_{-26}$
22	$4:-X_{26},\ 6:-X_{27},\ 9:X_{29},\ 16:-X_{33},\ 21:-X_{34},\ 28:X_{36},\ -1:X_{19},$ $-2:X_{18},\ -5:X_{17},\ -7:X_{14},\ -14:-X_7,\ -17:-X_5,\ -18:-X_2,$ $-19:-X_1,\ -26:X_{-4},\ -27:X_{-6},\ -29:-X_{-9},\ -33:X_{-16},$ $-34:X_{-21},\ -36:-X_{-28}$

Table 14: (continued)

23	$2:-X_{27},\ 8:-X_{30},\ 13:-X_{32},\ 14:-X_{33},\ 19:-X_{34},\ 24:-X_{35},$ $-1:X_{21},\ -6:X_{18},\ -7:X_{16},\ -11:X_{12},\ -12:-X_{11},\ -16:-X_7,$ $-18:-X_6,\ -21:-X_1,\ -27:X_{-2},\ -30:X_{-8},\ -32:X_{-13},$ $-33:X_{-14},\ -34:X_{-19},\ -35:X_{-24}$
24	$1:-X_{26},\ 6:-X_{28},\ 7:X_{29},\ 11:X_{31},\ 23:X_{35},\ 27:X_{36},\ -4:X_{19},$ $-8:X_{15},\ -9:X_{14},\ -10:X_{13},\ -13:-X_{10},\ -14:-X_9,\ -15:-X_8,$ $-19:-X_4,\ -26:X_{-1},\ -28:X_{-6},\ -29:-X_{-7},\ -31:-X_{-11},$ $-35:-X_{-23},\ -36:-X_{-27}$
25	$1:-X_{27},\ 4:-X_{28},\ 10:X_{31},\ 12:-X_{32},\ 18:-X_{34},\ 26:X_{36},\ -2:X_{21},$ $-3:X_{20},\ -6:X_{19},\ -11:X_{13},\ -13:-X_{11},\ -19:-X_6,\ -20:-X_3,$ $-21:-X_2,\ -27:X_{-1},\ -28:X_{-4},\ -31:-X_{-10},\ -32:X_{-12},$ $-34:X_{-18},\ -36:-X_{-26}$
26	$3:-X_{29},\ 6:-X_{30},\ 11:X_{33},\ 21:-X_{35},\ 25:-X_{36},\ -1:X_{24},\ -4:X_{22},$ $-8:X_{18},\ -10:X_{17},\ -12:X_{14},\ -14:-X_{12},\ -17:-X_{10},\ -18:-X_8,$ $-22:-X_4,\ -24:-X_1,\ -29:X_{-3},\ -30:X_{-6},\ -33:-X_{-11},$ $-35:X_{-21},\ -36:X_{-25}$
27	$4:-X_{30},\ 9:X_{32},\ 10:X_{33},\ 15:X_{34},\ 24:-X_{36},\ -1:X_{25},\ -2:X_{23},$ $-6:X_{22},\ -7:X_{20},\ -11:X_{17},\ -17:-X_{11},\ -20:-X_7,\ -22:-X_6,$ $-23:-X_2,\ -25:-X_1,\ -30:X_{-4},\ -32:-X_{-9},\ -33:-X_{-10},$ $-34:-X_{-15},\ -36:X_{-24}$
28	$1:-X_{30},\ 5:-X_{31},\ 7:X_{32},\ 18:-X_{35},\ 22:-X_{36},\ -4:X_{25},\ -6:X_{24},$ $-8:X_{21},\ -9:X_{20},\ -13:-X_{16},\ -16:X_{13},\ -20:-X_9,\ -21:-X_8,$ $-24:-X_6,\ -25:-X_4,\ -30:X_{-1},\ -31:X_{-5},\ -32:-X_{-7},$ $-35:X_{-18},\ -36:X_{-22}$
29	$6:-X_{32},\ 11:X_{34},\ 16:X_{35},\ 20:X_{36},\ -3:X_{26},\ -7:-X_{24},\ -9:-X_{22},$ $-12:X_{19},\ -13:X_{18},\ -15:X_{17},\ -17:-X_{15},\ -18:-X_{13},\ -19:-X_{12},$ $-22:X_9,\ -24:X_7,\ -26:-X_3,\ -32:X_{-6},\ -34:-X_{-11},\ -35:-X_{-16},$ $-36:-X_{-20}$
30	$3:-X_{32},\ 5:-X_{33},\ 15:X_{35},\ 19:X_{36},\ -1:X_{28},\ -4:X_{27},\ -6:X_{26},$ $-8:X_{23},\ -12:X_{20},\ -16:X_{17},\ -17:-X_{16},\ -20:-X_{12},\ -23:-X_8,$ $-26:-X_6,\ -27:-X_4,\ -28:-X_1,\ -32:X_{-3},\ -33:X_{-5},$ $-35:-X_{-15},\ -36:-X_{-19}$
31	$1:-X_{33},\ 7:X_{34},\ 12:X_{35},\ 17:X_{36},\ -5:X_{28},\ -10:-X_{25},\ -11:-X_{24},$ $-14:X_{21},\ -15:X_{20},\ -16:X_{19},\ -19:-X_{16},\ -20:-X_{15},\ -21:-X_{14},$ $-24:X_{11},\ -25:X_{10},\ -28:-X_5,\ -33:X_{-1},\ -34:-X_{-7},\ -35:-X_{-12},$ $-36:-X_{-17}$
32	$5:-X_{34},\ 10:-X_{35},\ 14:-X_{36},\ -3:X_{30},\ -6:X_{29},\ -7:-X_{28},$ $-9:-X_{27},\ -12:X_{25},\ -13:X_{23},\ -17:-X_{21},\ -21:X_{17},\ -23:-X_{13},$ $-25:-X_{12},\ -27:X_9,\ -28:X_7,\ -29:-X_6,\ -30:-X_3,\ -34:X_{-5},$ $-35:X_{-10},\ -36:X_{-14}$
33	$3:-X_{34},\ 9:-X_{35},\ 13:-X_{36},\ -1:X_{31},\ -5:X_{30},\ -10:-X_{27},$ $-11:-X_{26},\ -14:X_{23},\ -16:X_{22},\ -18:X_{20},\ -20:-X_{18},\ -22:-X_{16},$ $-23:-X_{14},\ -26:X_{11},\ -27:X_{10},\ -30:-X_5,\ -31:-X_1,\ -34:X_{-3},$ $-35:X_{-9},\ -36:X_{-13}$

Table 14: (continued)

34	$4:-X_{35},\ 8:X_{36},\ -3:X_{33},\ -5:X_{32},\ -7:-X_{31},\ -11:-X_{29},$ $-15:-X_{27},\ -18:X_{25},\ -19:X_{23},\ -21:X_{22},\ -22:-X_{21},\ -23:-X_{19},$ $-25:-X_{18},\ -27:X_{15},\ -29:X_{11},\ -31:X_{7},\ -32:-X_{5},\ -33:-X_{3},$ $-35:X_{-4},\ -36:-X_{-8}$
35	$2:-X_{36},\ -4:X_{34},\ -9:X_{33},\ -10:X_{32},\ -12:-X_{31},\ -15:-X_{30},$ $-16:-X_{29},\ -18:X_{28},\ -21:X_{26},\ -23:-X_{24},\ -24:X_{23},\ -26:-X_{21},$ $-28:-X_{18},\ -29:X_{16},\ -30:X_{15},\ -31:X_{12},\ -32:-X_{10},\ -33:-X_{9},$ $-34:-X_{4},\ -36:X_{-2}$
36	$-2:X_{35},\ -8:-X_{34},\ -13:X_{33},\ -14:X_{32},\ -17:-X_{31},\ -19:-X_{30},$ $-20:-X_{29},\ -22:X_{28},\ -24:X_{27},\ -25:X_{26},\ -26:-X_{25},\ -27:-X_{24},$ $-28:-X_{22},\ -29:X_{20},\ -30:X_{19},\ -31:X_{17},\ -32:-X_{14},\ -33:-X_{13},$ $-34:X_{8},\ -35:-X_{2}$

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