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Derivation Algebras of Certain Nilpotent Lie Algebras [1]

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Abstract. We compute the dimension of the algebra of derivations of any n-dimensional nilpotent real or complex Lie algebra whose Goze's invariant (or characteristic sequence) is $(n-3,1,\ldots,1)$.

1. Introduction

Let $\mathfrak{g}$ be a nilpotent Lie algebra of dimension n. The adjoint $\mathrm{ad}(X)$ of an element $X \in \mathfrak{g}$ is defined by $\mathrm{ad}(X)(Y) = [X,Y]$. For all $X \in \mathfrak{g} - [\mathfrak{g},\mathfrak{g}]$, $c(X) = (c_1(X), c_2(X), \ldots, 1)$ is the sequence, in decreasing order, of the dimensions of the characteristic subspaces of the *nilpotent operator* $\mathrm{ad}(X)$. The sequence $c(\mathfrak{g}) = \sup\{c(X) : X \in \mathfrak{g} - [\mathfrak{g},\mathfrak{g}]\}$ is called the *characteristic sequence* or *Goze's invariant* of $\mathfrak{g}$. The filiform, quasifiliform, and abelian Lie algebras of dimension n have as their Goze invariant $(n-1,1)$, $(n-2,1,1)$ and $(1,1,\ldots,1)$, respectively.

Definition 1.1. A nilpotent Lie algebra $\mathfrak{g}$, of dimension n, is called *p-filiform* if its Goze invariant is $(n-p,1,\ldots,1)$.

We will denote by

$$\mathcal{C}^0\mathfrak{g} = \mathfrak{g} \supseteq \mathcal{C}^1\mathfrak{g} = [\mathfrak{g},\mathfrak{g}] \supseteq \mathcal{C}^2\mathfrak{g} = \big[\mathfrak{g},[\mathfrak{g},\mathfrak{g}]\big] \supseteq \cdots \supseteq \mathcal{C}^{n-p}\mathfrak{g} = \{0\}, \quad 1 \le p < n,$$

the descending central series of $\mathfrak{g}$.

Remark 1.2. If $\mathfrak{g}$ is *p-filiform* then $\dim \mathcal{C}^k\mathfrak{g}/\mathcal{C}^{k+1}\mathfrak{g} \ge 1$, $1 \le k \le n-p-1$. Obviously, for a p-filiform algebra, the relation

$$n = \dim \frac{\mathfrak{g}}{[\mathfrak{g},\mathfrak{g}]} + \sum_{k=1}^{n-p-1} \frac{\mathcal{C}^k\mathfrak{g}}{\mathcal{C}^{k+1}\mathfrak{g}} \ge \dim \frac{\mathfrak{g}}{[\mathfrak{g},\mathfrak{g}]} + n - p - 1$$

implies $\dim \mathfrak{g}/[\mathfrak{g},\mathfrak{g}] \le p+1$.

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Remark 1.3. If $\mathfrak{g}$ is 1-filiform then $\dim \mathfrak{g}/[\mathfrak{g},\mathfrak{g}] = 2$ and $\dim \mathcal{C}^k\mathfrak{g}/\mathcal{C}^{k+1}\mathfrak{g} = 1$, $1 \le k \le n-2$. That is, $\dim \mathcal{C}^k\mathfrak{g} = n-k-1$, $1 \le k \le n-1$, and $\mathfrak{g}$ is a filiform Lie algebra.

Remark 1.4. We note that $(n-1)$-filiform algebras are abelian, and that an $(n-2)$-filiform algebra is a direct sum of a $(2k+1)$-dimensional Heisenberg algebra, $1 \le k \le [\frac{n-1}{2}]+1$, and an abelian algebra [9].

The next more general case among p-filiform algebras is that of $(n-3)$-filiform algebras, and this is the case we consider. The structure of these algebras was already clarified by the authors together with JIMÉNEZ-MERCHÁN in [9]. Here we use this information in order to compute for any $(n-3)$-filiform nilpotent Lie algebra $\mathfrak{g}$ the dimension of the derivation algebra $\operatorname{Der}\mathfrak{g}$.

If $\mathfrak{g}$ is an arbitrary Lie algebra, then the derivations of $\mathfrak{g}$ are precisely the 1-cocycles on $\mathfrak{g}$ with values in the adjoint module $\mathfrak{g}$, and the inner derivations $\operatorname{ad} x$, $x \in \mathfrak{g}$ are the 1-coboundaries. Hence the adjoint morphism ad is embedded into an exact sequence $0 \to \mathfrak{z}(\mathfrak{g}) \to \mathfrak{g} \stackrel{\mathrm{ad}}{\to} \operatorname{Der}(\mathfrak{g}) \to H^1(\mathfrak{g},\mathfrak{g}) \to 0$, where $\mathfrak{z}(\mathfrak{g})$ is the center of $\mathfrak{g}$. Accordingly, as an application of our calculation, we shall compute $\dim H^1(\mathfrak{g},\mathfrak{g})$ for any $(n-3)$-filiform nilpotent Lie algebra.

2. Some background and history

Nilpotent algebras over a field with characteristic 0 were characterized up to dimension 6 by MOROZOV in 1958 [17] and in dimension 7 by ANCOCHEA BERMÚDEZ and GOZE [3]. For dimensions beyond 7, classifications are known only for filiform algebras through the work of ANCOCHEA BERMÚDEZ and GOZE for dimension 8 [2], and of the second author and ECHARTE for dimension 9 [10]. Descriptions of the variety of nilpotent Lie algebras are available for dimension 7 [4] and dimension 8 [5]; however, beyond dimension 8, again one has information only for filiform algebras [1], [2], [4], [11], and [15]. It appears to be a general phenomenon that structural information on nilpotent Lie algebras without dimensional limitations is readily accessible only for the class of filiform algebras. For finite groups, the concept of "filiformity" was introduced by NORMAN BLACKBURN in [6] and discussed in the textbook literature [16], pp. 361–377.

The concept of a p-filiform nilpotent Lie algebra is more general than that of a filiform algebra; the role of filiform algebras in the structure theory of nilpotent Lie algebras motivates us to contribute information on this more general class of nilpotent Lie algebras—at least for low nilpotent class, i.e. large p, namely, that of $p = n-3$ [9], $p = n-4$ [8], and $p = n-5$ [7]. In this article we deal with $p = n-3$ exclusively.

3. Algebras of derivations: The results

In [9] it is shown that any n-dimensional nilpotent $(n-3)$-filiform Lie algebra $\mathfrak{g}$ over a ground field $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ with $n > 4$ has a basis $\{X_0, X_1, X_2, X_3\} \cup \{Y_1, \dots, Y_{n-4}\}$ such that $\mathfrak{g}$ is isomorphic to one of the following $n-2$ algebras, where $[m]$ is the largest integer not exceeding m:

$$\mathfrak{g}^{2q-1} \text{ for } q \in \{1, \dots, [\frac{n-2}{2}]\}, \quad \mathfrak{g}^{2q} \text{ for } q \in \{1, \dots, [\frac{n-3}{2}]\}, \quad \text{and } \mathfrak{g}^{n-2},$$

which are defined by the following equations:

$$\begin{array}{llll}
\mathfrak{g}^{2q-1}: & \begin{array}{l}[X_0,X_1]=X_2,\\ [X_0,X_2]=X_3,\end{array} & & [Y_{2k-1},Y_{2k}] \;=\; X_3, \quad 1\le k\le q-1,\\
\mathfrak{g}^{2q}: & \begin{array}{l}[X_0,X_1]=X_2,\\ [X_0,X_2]=X_3,\end{array} & [X_1,Y_{n-4}]=X_3, & [Y_{2k-1},Y_{2k}] \;=\; X_3, \quad 1\le k\le q-1,\\
\mathfrak{g}^{n-2}: & \begin{array}{l}[X_0,X_1]=X_2,\\ [X_0,X_2]=X_3\end{array}, & [X_1,X_2]=Y_{n-4}. &
\end{array}$$

With this notation we can formulate our main result.

Main Theorem. *For an n-dimensional nilpotent $(n-3)$-filiform Lie algebra $\mathfrak{g}$ over the field $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$, the dimension of the algebra of derivations is*

$$\begin{array}{lcll}
\dim\big(\mathrm{Der}(\mathfrak{g}^{2q-1})\big) & = & n^2-(3+2q)n+2q^2+3q+6, & q\in\{1,\dots,[\frac{n-2}{2}]\},\\
\dim\big(\mathrm{Der}(\mathfrak{g}^{2q}\big) & = & n^2-(4+2q)n+2q^2+5q+8, & q\in\{1,\dots,[\frac{n-3}{2}]\}, \quad \text{and}\\
\dim\big(\mathrm{Der}(\mathfrak{g}^{n-2})\big) & = & n^2-6n+15. &
\end{array}$$

The proof will be given in the following section.

4. Algebras of derivations: The proofs

In this section, we will determine the dimension of the algebra of derivations of each $(n-3)$-filiform Lie algebra of dimension n. However, we shall deal only with the algebra of derivations $\mathrm{Der}(\mathfrak{g}^{2q-1})$; the corresponding results for the other algebras are proved analogously.

For each q, the algebra $\mathfrak{g}^{2q-1}$ is the direct sum $\mathfrak{h}_1^{2q-1}\oplus\mathfrak{h}_2^{2q-1}$ of the subalgebra $\mathfrak{h}_1^{2q-1}=\langle X_0,X_1,X_2,X_3,Y_1,Y_2,\dots,Y_{2q-3},Y_{2q-2}\rangle$ and the central ideal $\mathfrak{h}_2^{2q-1}=\langle Y_{2q-1},Y_{2q},\dots,Y_{n-4}\rangle$. Any linear map $d\in\mathrm{Hom}(\mathfrak{g}^{2q-1},\mathfrak{g}^{2q-1})$ decomposes uniquely into a sum $d=\bar d_1+\bar d_2+\bar d_{12}+\bar d_{21}$ with $\bar d_i =\in \mathrm{Hom}(\mathfrak{h}_i^{2q-1},\mathfrak{h}_i^{2q-1})$, $i=1,2$, and $\bar d_{ij}\in\mathrm{Hom}(\mathfrak{h}_i^{2q-1},\mathfrak{h}_j^{2q-1})$, $(i,j)=(1,2)$, $(2,1)$. It is verified straightforwardly that each derivation $d\in\mathrm{Der}(\mathfrak{g}^{2q-1})$ maps $\mathfrak{h}_i^{2q-1}$ into itself for $=1,2$, whence $\bar d_i\in\mathrm{Der}\,\mathfrak{h}_i^{2q-1}$, $i=1,2$. Therefore, if $D(\mathfrak{h}_1^{2q-1},\mathfrak{h}_2^{2q-1})$ denotes that subspace of $\mathrm{Hom}(\mathfrak{h}_i^{2q-1},\mathfrak{h}_j^{2q-1})$ which is induced by $\mathrm{Der}(\mathfrak{g}^{2q-1})$, we have a direct sum decomposition

$$(*)\quad \mathrm{Der}(\mathfrak{g}^{2q-1})=\mathrm{Der}(\mathfrak{h}_1^{2q-1})\oplus\mathrm{Der}(\mathfrak{h}_2^{2q-1})\oplus D(\mathfrak{h}_1^{2q-1},\mathfrak{h}_2^{2q-1})\oplus D(\mathfrak{h}_2^{2q-1},\mathfrak{h}_1^{2q-1}),$$

such that each $d\in\mathrm{Der}(\mathfrak{g}^{2q-1})$ decomposes uniquely in the form $d=\bar d_1+\bar d_2+\bar d_{12}+\bar d_{21}$ with $\bar d_i\in\mathrm{Der}(\mathfrak{h}_i^{2q-1})$, $i=1,2$, $\bar d_{ij}\in D(\mathfrak{h}_i^{2q-1},\mathfrak{h}_i^{2q-1})$, $(i,j)=(1,2)$, $(2,1)$. Accordingly,

$$(**)\qquad \begin{array}{rcl}\dim\mathrm{Der}(\mathfrak{g}^{2q-1}) & = & \dim\mathrm{Der}(\mathfrak{h}_1^{2q-1})+\dim\mathrm{Der}(\mathfrak{h}_2^{2q-1})\\ & + & \dim D(\mathfrak{h}_1^{2q-1},\mathfrak{h}_2^{2q-1})+\dim D(\mathfrak{h}_2^{2q-1},\mathfrak{h}_1^{2q-1})\end{array}$$

and we have to determine the dimensions of the individual summands. It will be helpful to record here that $\bar d_{ij}\in D(\mathfrak{h}_i^{2q-1},\mathfrak{h}_j^{2q-1})$ implies

$$\bar d_{ij}(\mathfrak{h}_i^{2q-1})\subseteq\mathcal{Z}(\mathfrak{h}_j^{2q-1}),\quad \bar d_{ij}([\mathfrak{h}_i^{2q-1},\mathfrak{h}_i^{2q-1}])=\{0\}\quad\text{and } \bar d_{ij}(\mathfrak{h}_j^{2q-1})=\{0\}$$

for $(i,j)\in\{(1,2),(2,1)\}$.

Calculation of $\mathrm{Der}(\mathfrak{h}_2^{2q-1})$.
Since $\mathfrak{h}_2^{2q-1} = \langle Y_{2q-1}, Y_{2q}, \dots, Y_{n-4}\rangle$ is an abelian Lie algebra of dimension $n-2q-2$, it follows trivially that $\mathrm{Der}(\mathfrak{h}_2^{2q-1}) = \mathfrak{gl}(\mathfrak{h}_2^{2q-1})$. Thus $\dim(\mathrm{Der}(\mathfrak{h}_2^{2q-1})) = (n-2q-2)^2$.

Calculation of $D(\mathfrak{h}_1^{2q-1}, \mathfrak{h}_2^{2q-1})$.
Let $\bar{d}_{12} \in D(\mathfrak{h}_1^{2q-1}, \mathfrak{h}_2^{2q-1})$. From $\bar{d}_{12}([\mathfrak{h}_1^{2q-1}, \mathfrak{h}_1^{2q-1}]) = \{0\}$ we conclude $\bar{d}_{12}(X_2) = 0$ and $\bar{d}_{12}(X_3) = 0$, while $\bar{d}_{12}(\mathfrak{h}_1^{2q-1}) \subset \mathcal{Z}(\mathfrak{h}_2^{2q-1}) = \mathfrak{h}_2^{2q-1}$ implies $\bar{d}_{12}(X_i) = \displaystyle\sum_{k=2q-1}^{n-4} a_{ik}Y_k$ for $i = 0, 1$, and $\bar{d}_{12}(Y_j) = \displaystyle\sum_{k=2q-1}^{n-4} b_{jk}Y_k$ for $1 \le j \le 2q-2$. Finally, $\bar{d}_{12}(\mathfrak{h}_2^{2q-1}) = \{0\}$ gives us $\bar{d}_{12}(Y_k) = 0$, $2q-1 \le k \le n-4$. Thus we obtain a basis for $D(\mathfrak{h}_1^{2q-1}, \mathfrak{h}_2^{2q-1})$ in the form of $\{\bar{f}_{ik}, \bar{g}_{jk}\}$ where $\bar{f}_{ik}(X_i) = Y_k$, $\bar{g}_{jk}(Y_j) = Y_k$ for $0 \le i \le 1$, $1 \le j \le 2q-2$, and $2q-1 \le k \le n-4$. Accordingly, we find $\dim(D(\mathfrak{h}_1^{2q-1}, \mathfrak{h}_2^{2q-1})) = 2q.(n-2q-2)$.

Calculation of $D(\mathfrak{h}_2^{2q-1}, \mathfrak{h}_1^{2q-1})$
Let $\bar{d}_{21} \in D(\mathfrak{h}_2^{2q-1}, \mathfrak{h}_1^{2q-1})$. From $\bar{d}_{21}(\mathfrak{h}_2^{2q-1}) \subset \mathcal{Z}(\mathfrak{h}_1^{2q-1}) = \langle X_3\rangle$ it follows that $\bar{d}_{21}(Y_k) = c_k X_3$, $q-1 \le k \le n-4$. Further, $\bar{d}_{21}(\mathfrak{h}_1^{2q-1}) = \{0\}$ implies $\bar{d}_{21}(X_i) = 0$, $0 \le i \le 3$ and $\bar{d}_{21}(Y_j) = 0$, $1 \le j \le 2q-2$. Thus we get a basis of $D(\mathfrak{h}_2^{2q-1}, \mathfrak{h}_1^{2q-1})$ of the form $\{\bar{h}_k\}$, where $\bar{h}_k(Y_k) = X_3$, $2q-1 \le k \le n-4$. Hence we obtain $\dim(D(\mathfrak{h}_2^{2q-1}, \mathfrak{h}_1^{2q-1})) = n-2q-2$.

Calculation of $\mathrm{Der}(\mathfrak{h}_1^{2q-1})$
While the previous steps of the calculations were comparatively easy, the computation of the derivations of the non-abelian part of an $(n-3)$-filiform Lie algebra is difficult. However, the fact that these algebras can be graded helps in this task. Moreover, the computation will be easiest if the number of subspaces of the gradation is maximal. Therefore we shall now procede to construct a maximal gradation as follows:

$$\begin{array}{lcll}
\mathfrak{h}_1^{2q-1} &=& \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_k, & \mathbb{Z} = \{0, \pm1, \pm2, \dots\}, \text{ where}\\
\mathfrak{g}_{-k} &=& \{0\}, & q-2 < k,\\
\mathfrak{g}_{-k} &=& \langle Y_{-2k+2q-3}\rangle, & 0 \le k \le q-2,\\
\mathfrak{g}_k &=& \langle X_{k-1}\rangle, & 1 \le k \le 3,\\
\mathfrak{g}_4 &=& \langle X_3, Y_{2q-2}\rangle, & \\
\mathfrak{g}_k &=& \langle Y_{-2k+2q+6}\rangle, & 5 \le k \le q+2,\\
\mathfrak{g}_k &=& \{0\}, & q+2 < k.
\end{array}$$

Now let $\bar{d}_1 \in \mathrm{Der}(\mathfrak{h}_1^{2q-1})$. Then $\bar{d}_1 = \sum_{i\in Z} d_i$, where $d_i \in \mathrm{Der}(\mathfrak{h}_1^{2q-1})$ and $d_i(\mathfrak{g}_j) \subset \mathfrak{g}_{i+j}$. From $d_{2q}(\mathfrak{g}_{-q+2}) \subset \mathfrak{g}_{q+2}$ and $d_{-2q}(\mathfrak{g}_{q+2}) \subset \mathfrak{g}_{-q+2}$ we conclude that $d_i = 0$ for $i > 2q$ and $i < -2q$ and thus $\bar{d}_1 = \displaystyle\sum_{i=-2q}^{2q} d_i$. Now we will express each d_i, $-2q \le i \le 2q$, as a linear combination of some set B_i, $-2q \le i \le 2q$, of linearly independent derivations of $\mathfrak{h}_1^{2q-1}$ such that $\bigcup_{i=-2q}^{2q} B_i$ is a basis of $\mathrm{Der}(\mathfrak{h}_1^{2q-1})$. Consequently we shall have $\dim(\mathrm{Der}(\mathfrak{h}_1^{2q-1})) = \sum_{i=-2q}^{2q} \dim\langle B_i\rangle$. In the calculations we have to keep in mind that the ideals of central descending sequence:

$$\begin{array}{lcl}
\mathcal{C}^0(\mathfrak{h}_1^{2q-1}) &=& \mathfrak{h}_1^{2q-1}\\
\mathcal{C}^1(\mathfrak{h}_1^{2q-1}) &=& [\mathfrak{h}_1^{2q-1}, \mathcal{C}^0(\mathfrak{h}_1^{2q-1})] = \langle X_2, X_3\rangle\\
\mathcal{C}^2(\mathfrak{h}_1^{2q-1}) &=& [\mathfrak{h}_1^{2q-1}, \mathcal{C}^1(\mathfrak{h}_1^{2q-1})] = \langle X_3\rangle
\end{array}$$

are characteristic ideals, that is, they are stable under derivation. As an example, we will show the computation of d_{-j}, $q+3 \le j \le 2q$. Since $d_{-j}(\mathfrak{g}_t) \subset \mathfrak{g}_{t-j}$ for all t, we obtain

$$d_{-j}(Y_{2k}) = \beta^j_{2k}.Y_{4q-2j-2k+3},\ 1 \le k \le 2q-j+1.$$

Using that d_{-j} is a derivation, we get

$$\beta^j_{2k} = \beta^j_{4q-2j-2k+4},\ 1 \le k \le 2q-j+1.$$

Thus we have $[\frac{2q-j+2}{2}]$ free parameters. We define derivations δ^{-j}_k, $1 \le k \le [\frac{2q-j+2}{2}]$, by $\delta^{-j}_k(Y_{2k}) = Y_{4q-2j-2k+3}$, $\delta^{-j}_k(Y_{4q-2j-2k+4}) = Y_{2k-1}$, $1 \le k \le [\frac{2q-j+2}{2}]$ and set $B_{-j} = \{\delta^{-j}_k\}$, $1 \le k \le [\frac{2q-j+2}{2}]$. We then verify that $d_{-j} \in \langle B_{-j}\rangle$ and thus $\dim\langle B_{-j}\rangle = [\frac{2q-j+2}{2}]$, $q+3 \le j \le 2q$, $q \ge 3$.

This was already rather technical; however, the other calculations are even more difficult. Their results are summarized in the table in Section 6 at the end of the paper. As a consequence of this table, we obtain

$$\dim(\mathrm{Der}(\mathfrak{h}_1^{2q-1})) = \sum_{i=-2q}^{i=2q} \dim\langle B_i\rangle = 2q^2+q+4, \quad q \ge 1,$$

concluding the calculation of the summands in $(**)$. Indeed, the summary of our findings is

$$\begin{array}{lcl}
\dim(\mathrm{Der}(\mathfrak{h}_1^{2q-1})) & = & 2q^2+q+4 \\
\dim(\mathrm{Der}(\mathfrak{h}_2^{2q-1})) & = & (n-2q-2)^2 \\
\dim(D(\mathfrak{h}_1^{2q-1},\mathfrak{h}_2^{2q-1})) & = & 2q.(n-2q-2) \\
\dim(D(\mathfrak{h}_2^{2q-1},\mathfrak{h}_1^{2q-1})) & = & (n-2q-2).
\end{array}$$

Thus from $(**)$ we finally obtain

$$\dim(\mathrm{Der}(\mathfrak{g}^{2q-1})) = n^2-(3+2q)n+2q^2+3q+6 \ \text{ for } \ 1 \le q \le [\frac{n-2}{2}].$$

as asserted in the Main Theorem.

5. Applications to Cohomology

As we noted in the introduction, information on the algebra of derivations of a Lie algebra may be interpreted in terms of cohomology. Retaining the notation of the Main Theorem in Section 3 and the paragraph that precedes it we obtain the following result.

Theorem 5.1. *For an n-dimensional real or complex nilpotent $n-3$-filiform Lie algebra $\mathfrak{g}$ the dimension of the first cohomology group of $\mathfrak{g}$ over the adjoint module $\mathfrak{g}$ is*

$$\begin{array}{lcll}
\dim\left(H^1(\mathfrak{g}^{2q-1},\mathfrak{g}^{2q-1})\right) & = & n^2-(3+2q)n+2q^2+q+5, & q \in \{1,\ldots,[\frac{n-2}{2}]\}, \\
\dim\left(H^1(\mathfrak{g}^{2q},\mathfrak{g}^{2q})\right) & = & n^2-(4+2q)n+2q^2+3q+6, & q \in \{1,\ldots,[\frac{n-3}{2}]\}, \\
\dim\left(H^1(\mathfrak{g}^{n-2},\mathfrak{g}^{n-2})\right) & = & n^2-6n+12. &
\end{array}$$

6. Table

$$
\begin{array}{lcllr}
\dim\langle B_{-j}\rangle & = & [\frac{2q-j+2}{2}] & q+3\leq j\leq 2q & q\geq 3\\
\dim\langle B_{-j}\rangle & = & q-j+1+[\frac{j-2}{2}] & 4\leq j\leq q-3 & q\geq 7\\
\dim\langle B_{-q+2}\rangle & = & 3+[\frac{q-4}{2}] & & q\geq 6\\
\dim\langle B_{-q+1}\rangle & = & 1+[\frac{q-3}{2}] & & q\geq 5\\
\dim\langle B_{-q}\rangle & = & [\frac{q-2}{2}] & & q\geq 4\\
\dim\langle B_{-q-1}\rangle & = & [\frac{q-1}{2}] & & q\geq 3\\
\dim\langle B_{-q-2}\rangle & = & [\frac{q}{2}] & & q\geq 2\\
\dim\langle B_{-3}\rangle & = & \begin{cases} q-2 & \text{if} \quad q\geq 5\\ 1 & \text{if} \quad q=4\\ 0 & \text{if} \quad q\leq 3\end{cases} & & \\
\dim\langle B_{-2}\rangle & = & \begin{cases} q-1 & \text{if} \quad q\geq 4\\ 1 & \text{if} \quad q=3\\ 0 & \text{if} \quad q\leq 2\end{cases} & & \\
\dim\langle B_{-1}\rangle & = & \begin{cases} q & \text{if} \quad q\geq 3\\ 1 & \text{if} \quad q=2\\ 0 & \text{if} \quad q=1\end{cases} & & \\
\dim\langle B_{0}\rangle & = & \begin{cases} q+2 & \text{if} \quad q\geq 2\\ 2 & \text{if} \quad q=1\end{cases} & & \\
\dim\langle B_{1}\rangle & = & \begin{cases} q & \text{if} \quad q\geq 2\\ 2 & \text{if} \quad q=1\end{cases} & & \\
\dim\langle B_{2}\rangle & = & \begin{cases} q-1 & \text{if} \quad q\geq 3\\ 2 & \text{if} \quad 1\leq q\leq 2\end{cases} & & \\
\dim\langle B_{3}\rangle & = & \begin{cases} q-2 & \text{if} \quad q\geq 4\\ 2 & \text{if} \quad 2\leq q\leq 3\\ 1 & \text{if} \quad q=1\end{cases} & & \\
\dim\langle B_{j}\rangle & = & q-j+2+[\frac{j-4}{2}] & 4\leq j\leq q-2 \quad q\geq 6 & \\
\dim\langle B_{q-1}\rangle & = & 3+[\frac{q-5}{2}] & q\geq 5 & \\
\dim\langle B_{q}\rangle & = & 3+[\frac{q-4}{2}] & q\geq 4 & \\
\dim\langle B_{q+1}\rangle & = & 3+[\frac{q-3}{2}] & q\geq 3 & \\
\dim\langle B_{q+2}\rangle & = & 2+[\frac{q-2}{2}] & q\geq 2 & \\
\dim\langle B_{j}\rangle & = & [\frac{2q-j+2}{2}] \quad q+3\leq j\leq 2q \quad q\geq 3 & &
\end{array}
$$

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