Journal of Lie Theory
Volume **11** (2001) 305–337

Fusion Algebras for N = 1 Superconformal Field Theories through Coinvariants, II: N = 1 Super-Virasoro-Symmetry

Kenji Iohara and Yoshiyuki Koga*

Communicated by K.-H. Neeb

Abstract. We calculate fusion algebras of the $N = 1$ super-minimal models for the Neveu-Schwarz sector. We also state conjectures on the fusion rule and give some supporting evidences for the Ramond sector. Fusion algebras for the Neveu-Schwarz sector coincide with Verlinde algebras.

0. Introduction

This is the part II of the series of our paper [13]. In [13], we calculated the fusion algebras for $o\hat{s}p(1|2)$, and here we compute those for the $N = 1$ super Virasoro algebras by a similar argument. The $N = 1$ super Virasoro algebra is the ($N = 1$) super-extension of the Virasoro algebra and it has 2-sectors, called the Neveu-Schwarz sector and the Ramond sector (NS and R in short resp.). To be concrete, we obtain the complete description of the fusion $\mathrm{NS} \times \mathrm{NS} \to \mathrm{NS}$ (Theorem 3.4 and 3.5) and partial results of that $\mathrm{R} \times \mathrm{NS} \to \mathrm{R}$ and $\mathrm{R} \times \mathrm{R} \to \mathrm{NS}$ (Proposition 6.5) with the conjectures for general cases (Conjecture 3.6). Let us add a few words for vertex operator superalgebras (SVOA for short). It is known that for each central charge, the vacuum of the $N = 1$ super minimal model has a SVOA structure [16] and the rationality holds for the NS-sector of the $N = 1$ super minimal models [1]. Our results suggest that it might be more natural to include R-sector as 'NS SVOA'-module in order to describe the whole theory.

Let us briefly explain, why this paper can be regarded as the continuation of the part I [13] and what motivates us to work on these subjects. For non-super case, there are the functors, called the quantized Drinfel'd-Sokolov functor [8], from the full subcategory of the category $\mathcal{O}$ of an affine Lie algebra $\hat{\mathfrak{g}}$ consisting of off-critical representations to the category $\mathcal{O}$ of the corresponding $\mathcal{W}$-algebra $\mathcal{W}(\mathfrak{g})$. Via these functors, some classes of admissible representations of $\hat{\mathfrak{s}l}_2$ correspond to the minimal series representations of the Virasoro algebra (Vir for short). What is remarkable is that by these functors, integrable $\hat{\mathfrak{s}l}_2$-modules do not correspond to any minimal Vir-modules! It is widely believed that such correspondence generalizes to some classes of admissible representations of $\hat{\mathfrak{g}}$ (called non-degenerate

* Yukawa Fellow

ISSN 0949–5932 / $2.50

principal admissible representations [11]) and the minimal series representations of $\mathcal{W}(\mathfrak{g})$. On the other hand, the fusion algebras for $\hat{\mathfrak{sl}}_2$ are partially computed in [3] and completely by [9]. The fusion algebras for Vir are also computed in [7] and the relations between the fusion algebras for $\hat{\mathfrak{sl}}_2$ and that for Vir are investigated in [10]. It is quite natural to consider the generalizations of these theories to the other cases (e.g. the higher rank cases). But unfortunately, some technical difficulties arise and higher rank generalizations are still unclear until now. But for super case, there are still computable examples other than $\hat{\mathfrak{sl}}_2$ and the Virasoro algebra, that is *the affine Lie super algebra* $\hat{osp}(1|2)$ *and the* $N=1$ *super Virasoro algebra.* In fact, Kimura showed that these algebras are related, at least at the level of the algebras, via Hamiltonian reductions [14]. We expect that these reduction can be lifted to the functors as in [8] and the series of our papers ([13] and this paper) is an attempt toward understanding of these structures.

This paper is organized as follows: In Section 1, we recall the definition of the $N=1$ super Virasoro algebras and Verma modules over them. We also recall embedding diagrams of Verma modules and resolutions for the minimal series representations. In Section 2, we define fusion algebras through coinvariants of Lie superalgebras generated by tensor fields on $\mathbb{C}P^1$. In section 3, we state our main result. We consider not only the minimal models but also the case of generic central charge. In both cases, we give the multiplication rules of the fusion algebras for the NS sector in terms of the finite dimensional representations of $\mathfrak{sl}_2$ (Theorem 3.4 and 3.5). For the fusion rules for both NS and R sectors, we give a conjecture at the end of this section (Conjecture 3.6). In Section 4, we calculate a modular transformation of the (super-) characters of the minimal series representations, and show an isomorphism between the fusion algebras and Verlinde algebras for the NS sector. In Section 5, our main results Theorem 3.4 and 3.5 are proved, and in Section 6, some supporting evidences of Conjecture 3.6 and partial results are stated. In Section 7, we prove two propositions related with the rationality of the minimal models.

Acknowledgement. We would like to thank Dr. C. Shweigert for useful comments. Especially, we would like to thank Prof. B. Feigin for discussions. We also would like to thank the referee for his careful reading and suggestions.

1. $N=1$ super Virasoro algebra

In this section, we recall some basic facts about the $N=1$ super Virasoro algebras.

The $N=1$ super Virasoro algebra Vir_ϵ ($\epsilon=\frac{1}{2},0$) is the Lie superalgebra over $\mathbb{C}$ with the basis $\{L_i, G_j, C | i\in\mathbb{Z}, j\in\epsilon+\mathbb{Z}\}$. These generators have the $\mathbb{Z}_2$-gradation

$$\deg L_i = \deg C = 0,\ \deg G_i = 1,$$

and satisfy the commutation relations

$$\begin{aligned}
&[L_i, L_j] = (i-j)L_{i+j} + \frac{1}{12}(i^3-i)\delta_{i+j,0}C,\\
&[G_i, L_j] = (i-\frac{1}{2}j)G_{i+j},\\
&[G_i, G_j] = 2L_{i+j} + \frac{1}{3}(i^2-\frac{1}{4})\delta_{i+j,0}C,\\
&[C, Vir_\epsilon] = 0.
\end{aligned} \tag{1}$$

The $N=1$ super Virasoro algebras $Vir_{\frac{1}{2}}$ and Vir_0 are called the Neveu-Schwarz and Ramond algebras respectively. Below we abbreviate them as NS algebra and R algebra.

We fix a triangular decomposition of Vir_ϵ as follows:

$$Vir_\epsilon = Vir_\epsilon^+ \oplus Vir_\epsilon^0 \oplus Vir_\epsilon^-,$$

where

$$\begin{aligned} Vir_\epsilon^+ &= \bigoplus_{i>0} \mathbb{C}L_i \oplus \bigoplus_{i>0} \mathbb{C}G_i, \\ Vir_\epsilon^0 &= \begin{cases} \mathbb{C}L_0 \oplus \mathbb{C}C & \text{if } \epsilon = \frac{1}{2}, \\ \mathbb{C}L_0 \oplus \mathbb{C}G_0 \oplus \mathbb{C}C & \text{if } \epsilon = 0, \end{cases} \\ Vir_\epsilon^- &= \bigoplus_{i<0} \mathbb{C}L_i \oplus \bigoplus_{i<0} \mathbb{C}G_i. \end{aligned}$$

We also set

$$Vir_\epsilon^{\geq} = Vir_\epsilon^+ \oplus Vir_\epsilon^0.$$

First we introduce the Verma module $M_\epsilon(c,h)$ over Vir_ϵ with highest weight $(c,h) \in \mathbb{C}^2$. For $\epsilon = \frac{1}{2}$, let $V_{c,h}^{\frac{1}{2}} = \mathbb{C}\mathbf{1}_{c,h}^{\frac{1}{2}}$ be the 1-dimensional representation of $Vir_{\frac{1}{2}}^{\geq}$ given by

$$\begin{aligned} x.\mathbf{1}_{c,h}^{\frac{1}{2}} &= 0 \qquad \text{if } x \in Vir_{\frac{1}{2}}^+, \\ C.\mathbf{1}_{c,h}^{\frac{1}{2}} &= c\mathbf{1}_{c,h}^{\frac{1}{2}}, \\ L_0.\mathbf{1}_{c,h}^{\frac{1}{2}} &= h\mathbf{1}_{c,h}^{\frac{1}{2}}. \end{aligned}$$

For $\epsilon = 0$, let $V_{c,h}^0 = \bigoplus_{\alpha=0,1} \mathbb{C}\mathbf{1}_{c,h:\alpha}^0$ be the 2-dimensional representation of $Vir_0^{\geq}$ given by

$$\begin{aligned} x.\mathbf{1}_{c,h:\alpha}^0 &= 0 \qquad \text{if } x \in Vir_0^+, \\ C.\mathbf{1}_{c,h:\alpha}^0 &= c\mathbf{1}_{c,h:\alpha}^0, \\ L_0.\mathbf{1}_{c,h:\alpha}^0 &= h\mathbf{1}_{c,h:\alpha}^0, \\ G_0.\mathbf{1}_{c,h:0}^0 &= \mathbf{1}_{c,h:1}^0. \end{aligned}$$

We define the Verma module $M_\epsilon(c,h)$ with highest weight (c,h) by

$$M_\epsilon(c,h) = \mathrm{Ind}_{U(Vir_\epsilon^{\geq})}^{U(Vir_\epsilon)} V_{c,h}^\epsilon.$$

The value c is called central charge. In the sequel, we denote the highest weight vectors $1 \otimes \mathbf{1}_{c,h}^{\frac{1}{2}}$ and $1 \otimes \mathbf{1}_{c,h:0}^0$ by $|c,h\rangle_{\frac{1}{2}}$ and $|c,h\rangle_0$. Since the choice of the parity of a highest weight vector is not essential for our fusion algebras, we choose it appropriately. Every highest weight module with highest weight (c,h) is a quotient of the Verma module $M_\epsilon(c,h)$. In particular, let us denote the irreducible quotient by $L_\epsilon(c,h)$.

For our calculation, we need the following two induced modules over the R algebra. From now on, for any $\xi \in \mathbb{C}$, let us fix the argument of ξ such that $0 \leq \arg \xi < 2\pi$. Let $W_{c,h}^0 = \bigoplus_{\alpha=0,1} \mathbb{C}\tilde{\mathbf{1}}_{c,h:\alpha}^0$ be the 2-dimensional representation of $Vir_0^{\geq}$ defined by

$$\begin{aligned} x.\tilde{\mathbf{1}}_{c,h:\alpha}^0 &= 0 \qquad \text{if } x \in Vir_0^+, \\ C.\tilde{\mathbf{1}}_{c,h:\alpha}^0 &= c\tilde{\mathbf{1}}_{c,h:\alpha}^0, \\ L_0.\tilde{\mathbf{1}}_{c,h:\alpha}^0 &= h\tilde{\mathbf{1}}_{c,h:\alpha}^0, \\ G_0.\tilde{\mathbf{1}}_{c,h:\alpha}^0 &= (h - \tfrac{1}{24}c)^{\frac{1}{2}}\tilde{\mathbf{1}}_{c,h:1-\alpha}^0. \end{aligned}$$

We set
$$M_0'(c,h)=\mathrm{Ind}_{U(Vir_0^{\geq})}^{U(Vir_0)} W_{c,h}^0.$$
Notice that $M_0'(c,h)\simeq M_0(c,h)$ if and only if $h\neq \frac{1}{24}c$. Further we define $\tilde{M}_0^{\pm}(c,h)$ as follows. Notice that $\mathbb{C}_{c,h}^{\pm}=\mathbb{C}\{\tilde{\mathbf{1}}_{c,h:1}^0\pm\tilde{\mathbf{1}}_{c,h:0}^0\}$ are $Vir_0^{\geq}$-submodules of $W_{c,h}^0$. For $\tau=\pm$, we set
$$\tilde{M}_0^{\tau}(c,h)=\mathrm{Ind}_{U(Vir_0^{\geq})}^{U(Vir_0)}\mathbb{C}_{c,h}^{\tau},$$
and call it the half Verma module with highest weight $(c,h:\tau)$. It is clear that $M_0'(c,h)=\bigoplus_{\tau=\pm}\tilde{M}_0^{\tau}(c,h)$.

Here we remark on the injectivity of homomorphisms between Verma modules. A non-trivial homomorphism between Verma modules over the $N=1$ super Virasoro algebras is not always injective. To be precise, for the NS algebra, such a homomorphism is always injective, but for the R algebra, there exists a non-zero homomorphism which is not injective. Even though all non-trivial homomorphism between Verma modules are not necessarily injective, we could construct Bernstein-Gel'fand-Gel'fand type resolutions for these algebras [12].

Next, we recall embedding diagrams of Verma modules and BGG type resolutions, which are necessary for our purpose. For $t\in\mathbb{C}\setminus\{0\}$ and $a,b\in\mathbb{Z}_{\geq 0}$, set
$$\begin{aligned} c(t)&=\frac{15}{2}-3(t+t^{-1}),\\ h_{a,b}^{\epsilon}(t)&=\frac{1}{8}(a^2-1)t-\frac{1}{4}(ab-1)+\frac{1}{8}(b^2-1)t^{-1}+\frac{1}{16}(1-2\epsilon). \end{aligned}\tag{2}$$
Note that if the Verma module $M_\epsilon(c,h)$ with central charge $c=c(t)$ is reducible then $h=h_{a,b}^{\epsilon}(t)$ for some $a,b\in\mathbb{Z}_{\geq 0}$ such that $a-b\in 2\epsilon+1+2\mathbb{Z}$. In this paper, we consider fusion algebras in the following two cases:

Case 1 Generic central charge i.e. $t\in\mathbb{R}_{>0}\setminus\mathbb{Q}_{>0}$.

Case 2 Minimal central charge i.e. $t=\frac{p}{q}$ for $p,q\in\mathbb{Z}_{>0}$ such that $p,q\geq 2$, $p-q\in 2\mathbb{Z}$, $p\neq q$ and $(\frac{p-q}{2},q)=1$.

Case 1. We describe resolutions of irreducible highest weight representations with generic (not rational) central charge.

Proposition 1.1. *Suppose that $t\in\mathbb{R}_{>0}\setminus\mathbb{Q}_{>0}$. If there exist $a,b\in\mathbb{Z}_{>0}$ such that $a-b\in 2\epsilon+1+2\mathbb{Z}$, $c=c(t)$ and $h=h_{a,b}^{\epsilon}(t)$, then the following sequence:*
$$0\longrightarrow M_\epsilon(c,h+\frac{1}{2}ab)\longrightarrow M_\epsilon(c,h)\longrightarrow L_\epsilon(c,h)\longrightarrow 0$$
is exact.

Case 2. We state embedding diagrams of Verma modules and resolutions of the irreducible representations $L_\epsilon(c(\frac{p}{q}),h_{r,s}^{\epsilon}(\frac{p}{q}))$ with $r,s\in\mathbb{Z}_{>0}$. For our calculation, it is enough to consider the following two cases:

Case 2.1 $r - s \in 2\epsilon + 1 + 2\mathbb{Z}$ and

$$\begin{cases} (1 \leq r < q) \wedge (1 \leq s < p) & \text{if } \epsilon = \frac{1}{2}, \\ (1 \leq r < q) \wedge (1 \leq s < p) \wedge (r,s) \neq (\frac{q}{2}, \frac{p}{2}) & \text{if } \epsilon = 0. \end{cases}$$

Case 2.2

$$r = \frac{1}{2}q \wedge s = \frac{1}{2}p \wedge (p,q) = 2 \wedge \epsilon = 0.$$

For simplicity, set

$$h_{i:\epsilon} := \begin{cases} h^{\epsilon}_{iq+r,s}(\frac{p}{q}) & i \in 2\mathbb{Z} \\ h^{\epsilon}_{iq+r,p-s}(\frac{p}{q}) & i \in 2\mathbb{Z}+1 \end{cases}, \quad h'_{i:\epsilon} := \begin{cases} h^{\epsilon}_{iq+(q-r),p-s}(\frac{p}{q}) & i \in 2\mathbb{Z} \\ h^{\epsilon}_{iq+(q-r),s}(\frac{p}{q}) & i \in 2\mathbb{Z}+1 \end{cases}.$$

Definition 1.2. [15] $L_\epsilon(c(\frac{p}{q}), h^\epsilon_0)$ in Case 2.1 and Case 2.2 are called minimal series representations.

Now we recall embedding diagrams of Verma modules in Case 2.1.

Proposition 1.3. *There exist the following embedding diagrams of Verma modules:*

$$\begin{tikzcd} \cdots \arrow[r] \arrow[dr] & M_\epsilon(c,h_{3:\epsilon}) \arrow[r] \arrow[dr] & M_\epsilon(c,h_{2:\epsilon}) \arrow[r] \arrow[dr] & M_\epsilon(c,h_{1:\epsilon}) \arrow[dr] & \\ \cdots \arrow[r] \arrow[ur] & M_\epsilon(c,h'_{3:\epsilon}) \arrow[r] \arrow[ur] & M_\epsilon(c,h'_{2:\epsilon}) \arrow[r] \arrow[ur] & M_\epsilon(c,h'_{1:\epsilon}) \arrow[ur] & M'_\epsilon(c,h_{0:\epsilon}). \end{tikzcd}$$

Next we state the resolution for the irreducible highest weight module $L_\epsilon(c,h_0)$.

Theorem 1.4. [12] *The following resolutions exist:*

Case 2.1

$$\cdots \to M_\epsilon(c,h_{k:\epsilon}) \oplus M_\epsilon(c,h'_{k:\epsilon}) \to \cdots$$

$$\cdots \to M_\epsilon(c,h_{1:\epsilon}) \oplus M_\epsilon(c,h'_{1:\epsilon}) \to M_\epsilon(c,h_{0:\epsilon}) \to L_\epsilon(c,h_{0:\epsilon}) \to 0,$$

Case 2.2

$$\cdots \to M_0(c,h_{k:\epsilon})^{\oplus 2} \to \cdots$$

$$\cdots \to M_0(c,h_{1:\epsilon})^{\oplus 2} \to M'_0(c,h_{0:\epsilon}) \to L_0(c,h_{0:\epsilon}) \oplus \Pi L_0(c,h_{0:\epsilon}) \to 0,$$

where Π *denotes the parity shift.*

2. Fusion algebras

In this section, we define fusion algebras through coinvariants.

We denote by $\mathbb{C}[[t]]$ and $\mathbb{C}((t))$ the completions of $\mathbb{C}[t]$ and $\mathbb{C}[t,t^{-1}]$ by the positive powers of t.

To compute fusion rules explicitly, the following loop module is important. Let θ be the Grassmann variable i.e. $\theta^2 = 0$. For $\epsilon = \frac{1}{2}, 0$, $\sigma = 0, 1$ and λ, $\mu \in \mathbb{C}$, let

$$\mathcal{F}^{\epsilon,\sigma}_{\lambda,\mu} = (\bigoplus_{i \in \mathbb{Z}+\sigma\epsilon} \mathbb{C}F_i) \oplus (\bigoplus_{i \in \mathbb{Z}+(1-\sigma)\epsilon} \mathbb{C}F_i\theta)$$

be a Vir_ϵ-module with basis F_i and $F_i\theta$, on which Vir_ϵ acts via

$$\begin{aligned}
&C.F_i\theta^\gamma = 0, \\
&L_n.F_i\theta^\gamma = \{-i + \mu + (n-1)\lambda + \tfrac{1}{2}n\gamma(-1)^{2\epsilon}\}F_{i+n}\theta^\gamma, \\
&G_m.F_i\theta^\gamma = \begin{cases} \delta_{\gamma,1}F_{i+m} + \delta_{\gamma,0}\{-i + \mu + (2m-1)\lambda\}F_{i+m}\theta & \text{if } \epsilon = \frac{1}{2}, \\ \delta_{\gamma,0}F_{i+m}\theta + \delta_{\gamma,1}\{-i + m + \mu + (2m-1)\lambda\}F_{i+m} & \text{if } \epsilon = 0, \end{cases}
\end{aligned} \tag{3}$$

for $\gamma = 0, 1$, $n \in \mathbb{Z}$ and $m \in \mathbb{Z} + \epsilon$. Remark that $\mathcal{F}^{0,0}_{\lambda,\mu} = \mathcal{F}^{0,1}_{\lambda,\mu}$.

Next, we introduce Verma modules and loop modules attached to a point on $\mathbb{C}P^1$. For $w \in \mathbb{C}P^1$, let

$$z_w = \begin{cases} z - w & \text{if } w \in \mathbb{C}, \\ z^{-1} & \text{if } w = \infty, \end{cases}$$

be a local coordinate arround w. For $f(z_w)$, $g(z_w) \in \mathbb{C}((z_w))$, let

$$f(z_w)\partial_{z_w} \quad \text{and} \quad g(z_w)z_w^{\frac{1}{2}-\epsilon}\partial_{z_w}^{\frac{1}{2}},$$

be tensor fields over the formal neighborhood of w which satisfy the following commutation relations:

$$\begin{aligned}
&[p(z_w)\partial_{z_w}, q(z_w)\partial_{z_w}] = \{p(z_w)q'(z_w) - p'(z_w)q(z_w)\}\partial_{z_w}, \\
&[p(z_w)\partial_{z_w}, q(z_w)\partial_{z_w}^{\frac{1}{2}}] = \{p(z_w)q'(z_w) - \tfrac{1}{2}p'(z_w)q(z_w)\}\partial_{z_w}^{\frac{1}{2}}, \\
&[p(z_w)\partial_{z_w}^{\frac{1}{2}}, q(z_w)\partial_{z_w}^{\frac{1}{2}}] = 2p(z_w)q(z_w)\partial_{z_w},
\end{aligned}$$

where $p'(z_w) = \frac{d}{dz_w}p(z_w)$. We define the $\mathbb{Z}_2$-gradation of these fields by

$$\deg(f(z_w)\partial_{z_w}) = 0, \quad \deg(g(z_w)z_w^{\frac{1}{2}-\epsilon}\partial_{z_w}^{\frac{1}{2}}) = 1.$$

Here let us introduce $\mathcal{L}_{w;\epsilon}$ as the Lie superalgebra of the tensor fields over the formal neighborhood of w spanned by the above tensor fields.

Moreover, we consider a central extension of the Lie superalgebras $\mathcal{L}_{w;\epsilon}$. Let us define the cocycle $\langle , \rangle_{w;\epsilon}$ of $\mathcal{L}_{w;\epsilon}$ by

$$\begin{aligned}
&\langle p(z_w)\partial_{z_w}, q(z_w)\partial_{z_w}\rangle_{w;\epsilon} = \frac{1}{12}\mathrm{Res}_{z_w=0}(p'''(z_w)q(z_w))dz_w, \\
&\langle p(z_w)\partial_{z_w}, q(z_w)\partial_{z_w}^{\frac{1}{2}}\rangle_{w;\epsilon} = 0, \\
&\langle p(z_w)\partial_{z_w}^{\frac{1}{2}}, q(z_w)\partial_{z_w}^{\frac{1}{2}}\rangle_{w;\epsilon} = \frac{1}{3}\mathrm{Res}_{z_w=0}(p'(z_w)q'(z_w))dz_w,
\end{aligned}$$

and let $\hat{\mathcal{L}}_{w;\epsilon} = \mathcal{L}_{w;\epsilon} \oplus \mathbb{C}C$ be the 1-dimensional central extension defined by the cocycle $\langle , \rangle_{w;\epsilon}$. Remark that for $n \in \mathbb{Z}$ and $m \in \mathbb{Z} + \epsilon$,

$$C \mapsto C, \quad L_n \mapsto -z_w^{n+1}\partial_{z_w}, \quad G_m \mapsto \sqrt{-1}z_w^{m+\frac{1}{2}}\partial_{z_w}^{\frac{1}{2}} \tag{4}$$

give an isomorphism $Vir_\epsilon \simeq \hat{\mathcal{L}}_{w;\epsilon}$. In the sequel, for $w \in \mathbb{C}P^1$, we denote

$$L_n^{(w)} = -z_w^{n+1}\partial_{z_w},\ G_m^{(w)} = \sqrt{-1}z_w^{m+\frac{1}{2}}\partial_{z_w}^{\frac{1}{2}}. \tag{5}$$

Now we define the Verma module $M_\epsilon(c,h)(w)$ attached to $w \in \mathbb{C}P^1$. Set

$$\hat{\mathcal{L}}_{w;\epsilon}^{\geq} = \mathbb{C}[[z_w]]z_w\partial_{z_w} \oplus \mathbb{C}[[z_w]]z_w^{\frac{1}{2}+\epsilon}\partial_{z_w}^{\frac{1}{2}} \oplus \mathbb{C}C.$$

For $(c,h) \in \mathbb{C}^2$, let $V_{c,h}^{\frac{1}{2}}(w) = \mathbb{C}\mathbf{1}_{c,h}^{\frac{1}{2}:w}$ be the 1-dimensional $\hat{\mathcal{L}}_{w;\frac{1}{2}}^{\geq}$-module defined by

$$\begin{aligned}
&z_w^{n+1}\partial_{z_w}.\mathbf{1}_{c,h}^{\frac{1}{2}:w} = 0 && \text{for } n \in \mathbb{Z}_{>0},\\
&z_w^{m+\frac{1}{2}}\partial_{z_w}^{\frac{1}{2}}.\mathbf{1}_{c,h}^{\frac{1}{2}:w} = 0 && \text{for } m \in \mathbb{Z}_{>0} - \tfrac{1}{2},\\
&-z_w\partial_{z_w}.\mathbf{1}_{c,h}^{\frac{1}{2}:w} = h\mathbf{1}_{c,h}^{\frac{1}{2}:w},\\
&C.\mathbf{1}_{c,h}^{\frac{1}{2}:w} = c\mathbf{1}_{c,h}^{\frac{1}{2}:w},
\end{aligned}$$

and let $V_{c,h}^0(w) = \bigoplus_{\alpha=0,1}\mathbb{C}\mathbf{1}_{c,h:\alpha}^{0:w}$ be the 2-dimensional $\hat{\mathcal{L}}_{w;0}^{\geq}$-module defined by

$$\begin{aligned}
&z_w^{n+1}\partial_{z_w}.\mathbf{1}_{c,h:\alpha}^{0:w} = 0 && \text{for } n \in \mathbb{Z}_{>0},\\
&z_w^{m+\frac{1}{2}}\partial_{z_w}^{\frac{1}{2}}.\mathbf{1}_{c,h:\alpha}^{0:w} = 0 && \text{for } m \in \mathbb{Z}_{>0},\\
&-z_w\partial_{z_w}.\mathbf{1}_{c,h:\alpha}^{0:w} = h\mathbf{1}_{c,h:\alpha}^{0:w},\\
&\sqrt{-1}z_w^{\frac{1}{2}}\partial_{z_w}^{\frac{1}{2}}.\mathbf{1}_{c,h:0}^{0:w} = \mathbf{1}_{c,h:1}^{0:w},\\
&C.\mathbf{1}_{c,h:\alpha}^{0:w} = c\mathbf{1}_{c,h:\alpha}^{0:w}.
\end{aligned}$$

We define the Verma module $M_\epsilon(c,h)(w)$ attached to $w \in \mathbb{C}P^1$ by

$$M_\epsilon(c,h)(w) = \mathrm{Ind}_{U(\hat{\mathcal{L}}_{w;\epsilon}^{\geq})}^{U(\hat{\mathcal{L}}_{w;\epsilon})} V_{c,h}^\epsilon(w).$$

We denote the irreducible quotient of $M_\epsilon(c,h)(w)$ by $L_\epsilon(c,h)(w)$.

We similarly define the induced module $M_\epsilon(c,h)(w)$ attached to $w \in \mathbb{C}P^1$. For $(c,h) \in \mathbb{C}^2$, let $W_{c,h}^0(w) = \bigoplus_{\alpha=0,1}\mathbb{C}\tilde{\mathbf{1}}_{c,h:\alpha}^{0:w}$ be the following 2-dimensional representation of $\hat{\mathcal{L}}_{w;0}^{\geq}$:

$$\begin{aligned}
&z_w^{n+1}\partial_{z_w}.\tilde{\mathbf{1}}_{c,h:\alpha}^{0:w} = 0 && \text{for } n \in \mathbb{Z}_{>0},\\
&z_w^{m+\frac{1}{2}}\partial_{z_w}^{\frac{1}{2}}.\tilde{\mathbf{1}}_{c,h:\alpha}^{0:w} = 0 && \text{for } m \in \mathbb{Z}_{>0},\\
&-z_w\partial_{z_w}.\tilde{\mathbf{1}}_{c,h:\alpha}^{0:w} = h\tilde{\mathbf{1}}_{c,h:\alpha}^{0:w},\\
&\sqrt{-1}z_w^{\frac{1}{2}}\partial_{z_w}^{\frac{1}{2}}.\tilde{\mathbf{1}}_{c,h:\alpha}^{0:w} = (h - \tfrac{1}{24}c)^{\frac{1}{2}}\tilde{\mathbf{1}}_{c,h:1-\alpha}^{0:w},\\
&C.\tilde{\mathbf{1}}_{c,h:\alpha}^{0} = c\tilde{\mathbf{1}}_{c,h:\alpha}^{0:w}.
\end{aligned}$$

We define the induced module $M'_0(c,h)(w)$ attached to $w \in \mathbb{C}P^1$ by

$$M'_0(c,h)(w) = \mathrm{Ind}_{U(\hat{\mathcal{L}}_{w;0}^{\geq})}^{U(\hat{\mathcal{L}}_{w;0})} W_{c,h}^0(w).$$

Half Verma modules attached to $w \in \mathbb{C}P^1$ are defined as follows: If we set $\mathbb{C}_{c,h}^{\pm}(w) = \mathbb{C}\{\tilde{\mathbf{1}}_{c,h:1}^{0:w} \pm \tilde{\mathbf{1}}_{c,h:0}^{0:w}\}$, then $\mathbb{C}_{c,h}^{\pm}(w)$ is a submodule of $W_{c,h}^0(w)$. We define $\tilde{M}_0^\tau(c,h)(w)$ by

$$\tilde{M}_0^\tau(c,h)(w) = \mathrm{Ind}_{U(\hat{\mathcal{L}}_{w;0}^{\geq})}^{U(\hat{\mathcal{L}}_{w;0})} \mathbb{C}_{c,h}^\tau(w),$$

and call it the half Verma module with highest weight $(c, h : \tau)$ attached to w. For each Verma module $M_\epsilon(c,h)(w)$ attached to w and $j \in (1-\epsilon)\mathbb{Z}$, we denote by $M_\epsilon(c,h)(w)_j$ the subspace of level j i.e.

$$M_\epsilon(c,h)(w)_j = \{v \in M_\epsilon(c,h)(w) | \, (-z_w \partial_{z_w}).v = (h+j)v\}. \tag{6}$$

For the induced modules $M'_0(c,h)(w)$ and $\tilde{M}_0^\tau(c,h)(w)$, we sometimes use similar notation.

Now we define a loop module attached to $w \in \mathbb{C}P^1$. To introduce it, we set $\mathcal{F}^{\epsilon,\sigma}_{\lambda,\mu}(w) = (\bigoplus_{i \in \mathbb{Z}+\sigma\epsilon} \mathbb{C}F_i) \oplus (\bigoplus_{i \in \mathbb{Z}+(1-\sigma)\epsilon} \mathbb{C}F_i \theta)$ and define the $\hat{\mathcal{L}}_{w;\epsilon}$-module structure on this space by using the isomorphism (4) and the action (3). If we set

$$\hat{\mathcal{L}}^<_{w;\epsilon} = \mathbb{C}[[z_w^{-1}]]\partial_{z_w} \oplus \mathbb{C}[[z_w^{-1}]] z_w^{\epsilon - \frac{1}{2}} \partial_{z_w}^{\frac{1}{2}},$$

and

$$\mathcal{H}^{\epsilon,\sigma}_{\lambda,\mu}(w) = (\bigoplus_{i<0} \mathbb{C}F_i) \oplus (\bigoplus_{i<0} \mathbb{C}F_i \theta),$$

then $\mathcal{H}^{\epsilon,\sigma}_{\lambda,\mu}(w)$ is a $\hat{\mathcal{L}}^<_{w;\epsilon}$-submodule of $\mathcal{F}^{\epsilon,\sigma}_{\lambda,\mu}(w)$. We regard the quotient space

$$\mathcal{G}^{\epsilon,\sigma}_{\lambda,\mu}(w) = \mathcal{F}^{\epsilon,\sigma}_{\lambda,\mu}(w) \,/\, \mathcal{H}^{\epsilon,\sigma}_{\lambda,\mu}(w) \tag{7}$$

as a $\hat{\mathcal{L}}^<_{w;\epsilon}$-module.

Now, we construct fusion algebras for the $N=1$ super Virasoro algebras by using coinvariants of a Lie superalgebra of tensor fields on $\mathbb{C}P^1$, following [7]. For a finite subset $E = \{(w_1, \epsilon_1), \cdots, (w_m, \epsilon_m)\}$ of $\mathbb{C}P^1 \times \{\frac{1}{2}, 0\}$ such that $w_1, \cdots, w_m$ are distinct, let $\mathcal{L}_E$ be the direct sum $\bigoplus_{(w,\epsilon) \in E} \mathcal{L}_{w;\epsilon}$, and $\hat{\mathcal{L}}_E = \mathcal{L}_E \oplus \mathbb{C}C$ be the central extension of $\mathcal{L}_E$ by the cocycle $\langle , \rangle_E = \sum_{(w,\epsilon) \in E} \langle , \rangle_{w,\epsilon}$. Let $\mathcal{L}(E)$ be the Lie superalgebra consists of the multi-valued meromorphic tensor fields $F(z)$ which satisfy the following conditions:

(i) $F(z)$ is holomorphic outside $\{w_i | (w_i, \epsilon_i) \in E\}$.

(ii) For any w such that $(w, \epsilon) \in E$, $F(z)$ has the following Laurent expansion:

$$F(z) = f(z_w)\partial_{z_w} + g(z_w) z_w^{\frac{1}{2} - \epsilon} \partial_{z_w}^{\frac{1}{2}},$$

where $f(z_w), \; g(z_w) \in \mathbb{C}((z_w))$.

Remark that, by the Riemann-Hurwitz formula, $\mathcal{L}(E)$ is trivial if $\sharp\{i | \epsilon_i = 0\} \not\equiv 0 \pmod 2$. By the Laurent expansion, we have the Lie superalgebra homomorphism $\mathcal{L}(E) \longrightarrow \mathcal{L}_E$. It is easy to see that this lifts to a homomorphism

$$\mathcal{L}(E) \longrightarrow \hat{\mathcal{L}}_E. \tag{8}$$

Let $W_\epsilon(w)$ be a highest weight representation of $\hat{\mathcal{L}}_{w;\epsilon}$. We regard the tensor product $\bigotimes_{(w,\epsilon) \in E} W_\epsilon(w)$ as a $\hat{\mathcal{L}}_E$-module via the homomorphism (8). We define fusion algebras using the following coinvariants

$$H_0(\mathcal{L}(E), \bigotimes_{(w,\epsilon) \in E} W_\epsilon(w)) = \bigotimes_{(w,\epsilon) \in E} W_\epsilon(w) \Big/ \mathcal{L}(E)(\bigotimes_{(w,\epsilon) \in E} W_\epsilon(w)) \,.$$

Let us fix a positive number $t \in \mathbb{R}_{>0}$ and set $c = c(t)$. Let $\mathcal{S}_c$ be a subset (at most countable) of $\mathbb{C} \times \{\frac{1}{2}, 0\}$, and $\mathcal{A}_c$ the $\mathbb{Z}$-free module with a basis $\{l_h^\epsilon | (h, \epsilon) \in \mathcal{S}_c\}$. Set $\mathcal{A}_c^{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{A}_c$. First we introduce symmetric forms on $\mathcal{A}_c^{\mathbb{Q}}$. For a set of distinct points $\{w_1, \cdots, w_m\}$ of $\mathbb{C}P^1$, we define a $\mathbb{Q}$-multilinear symmetric form $\Phi_m : \bigotimes^m \mathcal{A}_c^{\mathbb{Q}} \to \mathbb{Q}$ by

$$\Phi_m(l_{h_1}^{\epsilon_1}, \cdots, l_{h_m}^{\epsilon_m}) = \left(\frac{1}{\sqrt{2}}\right)^{\sharp\{i|\epsilon_i = 0\}} \dim H_0(\mathcal{L}(E), \bigotimes_{i=1}^m L'_{\epsilon_i}(c, h_i)(w_i)),$$

where $E = \{(w_1, \epsilon_1), \cdots, (w_m, \epsilon_m)\}$ and

$$L'_\epsilon(c,h)(w) = \begin{cases} L_0(c,h)(w) \oplus \Pi L_0(c,h)(w) & \text{if } \epsilon = 0 \text{ and } h = \frac{1}{24}c, \\ L_\epsilon(c,h)(w) & \text{otherwise .} \end{cases}$$

Note that Φ_m is well-defined as a form on $\mathcal{A}_c^{\mathbb{Q}}$, since $\mathcal{L}(E) = \{0\}$ if $\sharp\{i|\epsilon_i = 0\} \not\equiv 0$ (mod 2).

Remark 2.1. **(i)** Roughly speaking, the symbol l_h^ϵ corresponds to a primary field as follows:

$$l_h^\epsilon \Leftrightarrow \begin{cases} \phi_{L_{\frac{1}{2}}(c,h)} & \text{if } \epsilon = \frac{1}{2}, \\ \dfrac{1}{\sqrt{2}}\phi_{L_0(c,h)} & \text{if } \epsilon = 0 \text{ and } h \neq \frac{1}{24}c, \\ \dfrac{1}{\sqrt{2}}\left(\phi_{L_0(c,h)} + \phi_{\Pi L_0(c,h)}\right) & \text{if } \epsilon = 0, \text{ and } h = \frac{1}{24}c, \end{cases}$$

where $\phi_{L_\epsilon(c,h)}$ denotes the primary field attached to the irreducible representation $L_\epsilon(c,h)$.

(ii) In the R sector, the fields which have physical meaning correspond to $L'_0(c,h)$ rather than $L_0(c,h)$. Mathematically, $L'_0(c,h)$ naturally appears in the ($\mathbb{Z}_2$-graded) BGG type resolution (see Theorem 1.4).

For the form Φ_2, we have the following lemma.

Lemma 2.2. *Let w_1, w_2 be distinct points in $\mathbb{C}P^1$. Suppose that ϵ_1, $\epsilon_2 \in \{\frac{1}{2}, 0\}$ and $E = \{(w_1, \epsilon_1), (w_2, \epsilon_2)\}$. For $c = c(t)$ such that $t \in \mathbb{R}_{>0}$ and h_1, $h_2 \in \mathbb{C}$, we have*

$$\left(\frac{1}{\sqrt{2}}\right)^{\sharp\{i|\epsilon_i = 0\}} \dim H_0(\mathcal{L}(E), L'_{\epsilon_1}(c,h_1)(w_1) \otimes L'_{\epsilon_2}(c,h_2)(w_2))$$

$$= \delta_{h_1,h_2}\delta_{\epsilon_1,\epsilon_2} \times \begin{cases} 1 & \text{if } \epsilon_1 = \frac{1}{2}, \\ 2^{\delta_{h_1,\frac{1}{24}c}} & \text{if } \epsilon_1 = 0. \end{cases}$$

Remark that the form Φ_2 is non-degenerate, but unlike the Virasoro algebra case, the basis $\{l_h^\epsilon | (h, \epsilon) \in \mathcal{S}_c\}$ is not self-dual with respect to the form Φ_2. Hence we define a fusion algebra on $\mathcal{A}_c$ as follows: We first introduce an algebraic structure on $\mathcal{A}_c^{\mathbb{Q}}$ by using the form $\Phi_3 : (\mathcal{A}_c^{\mathbb{Q}})^{\otimes^3} \to \mathbb{Q}$ and the isomorphism $\mathcal{A}_c^{\mathbb{Q}} \simeq (\mathcal{A}_c^{\mathbb{Q}})^*$

(restricted dual) defined by Φ_2. More precisely, the multiplication in $\mathcal{A}_c^{\mathbb{Q}}$ is given by

$$l_{h_1}^{\epsilon_1} \circ l_{h_2}^{\epsilon_2} = \sum_{(h_3,\epsilon_3)\in\mathcal{S}_c} N_{(h_1,\epsilon_1),(h_2,\epsilon_2)}^{(h_3,\epsilon_3)} l_{h_3}^{\epsilon_3},$$

where

$$\begin{aligned} N_{(h_1,\epsilon_1),(h_2,\epsilon_2)}^{(h_3,\epsilon_3)} &= \frac{\Phi_3(l_{h_1}^{\epsilon_1}, l_{h_2}^{\epsilon_2}, l_{h_3}^{\epsilon_3})}{\Phi_2(l_{h_3}^{\epsilon_3}, l_{h_3}^{\epsilon_3})} \\ &= \Phi_3(l_{h_1}^{\epsilon_1}, l_{h_2}^{\epsilon_2}, l_{h_3}^{\epsilon_3}) \times \begin{cases} 1 & \text{if } \epsilon_3 = \frac{1}{2}, \\ 2^{-\delta_{h_3,\frac{1}{24}c}} & \text{if } \epsilon_3 = 0. \end{cases} \end{aligned}$$

A priori, we do not see whether $N_{(h_1,\epsilon_1),(h_2,\epsilon_2)}^{(h_3,\epsilon_3)} \in \mathbb{Z}_{\geq 0}$ or not. As a consequence of Theorem 3.4, Theorem 3.5, Remark 6.4 and Proposition 6.5, we see that $N_{(h_1,\epsilon_1),(h_2,\epsilon_2)}^{(h_3,\epsilon_3)} \in \mathbb{Z}_{\geq 0}$. Therefore, the multiplication $\circ$ restricts to an operation on the $\mathbb{Z}$-module $\mathcal{A}_c$ and we call it fusion algebra. Here we set

$$\mathcal{A}_c^{NS} = \bigoplus_{(h,\frac{1}{2})\in\mathcal{S}_c} \mathbb{Z} l_h^{\frac{1}{2}} \quad \text{and} \quad \mathcal{A}_c^{R} = \bigoplus_{(h,0)\in\mathcal{S}_c} \mathbb{Z} l_h^{0}$$

and call them the NS and R sectors respectively. By the definition, $\mathcal{A}_c^{NS}$ is a subalgebra of the fusion algebra $\mathcal{A}_c$ and $\mathcal{A}_c^{R}$ becomes an $\mathcal{A}_c^{NS}$-module.

3. Main results

In this section, we will calculate fusion rules explicitly for two classes, a minimal case and a generic case.

Case 1 Generic central charge i.e. $c = c(t)$ and $t \in \mathbb{R}_{>0} \setminus \mathbb{Q}_{>0}$. Recall that the Verma module $M_\epsilon(c,h)$ is reducible if and only if $h \in \Lambda_\epsilon^c$, where the set Λ_ϵ^c is given by

$$\Lambda_\epsilon^c = \left\{ h_{a,b}^\epsilon(t) \,\middle|\, (a,b) \in (\mathbb{Z}_{>0})^2,\ a-b \equiv (1-2\epsilon) \pmod 2 \right\}.$$

Case 2 Minimal central charge i.e. $c = c(t)$ and $t = \frac{p}{q} \in \mathbb{Q}_{>0}$ for $p, q \in \mathbb{Z}_{>0}$ such that $p, q \geq 2$, $p - q \in 2\mathbb{Z}$, $p \neq q$ and $(\frac{p-q}{2}, q) = 1$. In this case, we define the set Λ_ϵ^c by

$$\Lambda_\epsilon^c = \left\{ h_{a,b}^\epsilon(t) \,\middle|\, (a,b) \in \mathbb{Z}^2, 1 \leq a < q,\ 1 \leq b < p,\ a-b \equiv (1-2\epsilon) \pmod 2 \right\}.$$

Remark 3.1. By the definition 1.2, $L_\epsilon(c,h)$ is a minimal series representation if and only if $c = c(\frac{p}{q})$ satisfies the condition in Case 2 and $h \in \Lambda_\epsilon^c$.
For each case, the following theorem characterize the set $\mathcal{S}_c$. We first state for Case 1. Set $E = \{(w_1,\epsilon_1),(w_2,\epsilon_2),(w_3,\epsilon_3)\}$ with distinct points $w_1, w_2, w_3 \in \mathbb{C}P^1$ and $\epsilon_1, \epsilon_2, \epsilon_3 \in \{0, \frac{1}{2}\}$ such that $\sharp\{i|\epsilon_i = 0\} \equiv 0 \pmod 2$.

Theorem 3.2. *Suppose that $c = c(t)$ satisfies the condition in Case 1. For fixed $h_1, h_2 \in \mathbb{C}$,*

(i) $\sharp\{h_3|\Phi_3(l_{h_1}^{\epsilon_1}, l_{h_2}^{\epsilon_2}, l_{h_3}^{\epsilon_3}) \neq 0\} < \infty$ if and only if $h_i \in \Lambda_{\epsilon_i}^c$ $(i = 1,2)$.

(ii) If there exists $i \in \{1,2\}$ such that $h_i \notin \Lambda_{\epsilon_i}^c$, then $\sharp\{h_3|\Phi_3(l_{h_1}^{\epsilon_1}, l_{h_2}^{\epsilon_2}, l_{h_3}^{\epsilon_3}) \neq 0\}$ is uncountable.

Next for Case 2, set $E_\epsilon = \{(w_1, \frac{1}{2}), (w_2, \epsilon), (w_3, \epsilon)\}$ $(\epsilon = \frac{1}{2}, 0)$. Then the following theorem holds

Theorem 3.3. *Suppose that $c = c(\frac{p}{q})$ satisfies the condition in Case 2. If*

$$H_0(\mathcal{L}(E), L_{\frac{1}{2}}(c,0)(w_1) \otimes L_\epsilon(c,h_2)(w_2) \otimes L_\epsilon(c,h_3)(w_3)) \neq 0,$$

then $h_2 = h_3$ and $h_2 \in \Lambda_\epsilon^c$.

Case 1 (The fusion algebra at a generic central charge) First we fix the central charge $c = c(t)$, where $t \in \mathbb{R}_{>0} \setminus \mathbb{Q}_{>0}$. We take

$$\{(h,\epsilon)|h \in \Lambda_\epsilon^c \text{ for } \epsilon = 0, \frac{1}{2}\}$$

as the index set $\mathcal{S}_c$ of the fusion algebra $\mathcal{A}_c$. For $j \in \mathbb{Z}_{\geq 0}$, let V_j be the irreducible $(j+1)$-dimensional representation of sl_2. If $h = h_{a,b}^\epsilon(t) \in \Lambda_\epsilon^c$, then we assign the symbol (V_{a-1}, V_{b-1}) to $l_h^\epsilon \in \mathcal{A}_c$. Further we extend this notation by $\mathbb{Z}$-bilinearlity. Note that this is a one-to-one correspondence between the set $\{l_h^\epsilon|(h,\epsilon) \in \mathcal{S}_c\}$ and the set of the symbols $\{(V_i, V_j)|i,j \in \mathbb{Z}_{\geq 0}\}$. Remark that if $i \equiv j \pmod 2$ (resp. $i \not\equiv j \pmod 2$) then (V_i, V_j) corresponds to an element of the NS sector (resp. R sector). The multiplication rule of the NS sector $\mathcal{A}_c^{NS}$ can be explicitly described as follows:

Theorem 3.4. *For the NS sector (i.e. in the case of $i_\alpha \equiv j_\alpha \pmod 2$ for $\alpha = 0, 1$) the structure of the fusion algebra at a generic central charge $c = c(t)$ can be written as follows:*

$$(V_{i_0}, V_{j_0}) \circ (V_{i_1}, V_{j_1}) = (V_{i_0} \otimes V_{i_1}, V_{j_0} \otimes V_{j_1}).$$

Case 2 (The fusion algebra of minimal models) We fix a central charge $c = c(\frac{p}{q})$, where $p,\ q \in \mathbb{Z}_{>0}$ such that $p, q \geq 2$, $p - q \in 2\mathbb{Z}$, $p \neq q$ and $(\frac{p-q}{2}, q) = 1$. In this case, we take $\{(h,\epsilon)|h \in \Lambda_\epsilon^c \text{ for } \epsilon = 0, \frac{1}{2}\}$ as the index set $\mathcal{S}_c$. Define an equivalence relation $\approx$ on the set of (V_i, V_j) as follows (cf. [FM2]):

$$(V_i, V_j) \approx (V_{-i+q-2}, V_{-j+p-2}) \quad \text{for } 0 \leq i \leq q-2,\ 0 \leq j \leq p-2.$$

Let us denote the equivalence class of (V_i, V_j) by $[(V_i, V_j)]$. We assign $[(V_{a-1}, V_{b-1})]$ such that $a - b \equiv (1 - 2\epsilon) \pmod 2$ to l_h^ϵ with $h = h_{a,b}^\epsilon(\frac{p}{q})$. This is a one-to-one correspondence between the set $\{l_h^\epsilon|(h,\epsilon) \in \mathcal{S}_c\}$ and the set of the equivalence classes $\{(V_i, V_j)|0 \leq i \leq q-2,\ 0 \leq j \leq p-2\}/\approx$. By using the following Kazhdan-Lusztig fusion functor:

$$V_i \dot{\otimes}_k V_j \simeq V_{|i-j|} \oplus V_{|i-j|+2} \oplus V_{|i-j|+4} \oplus \cdots \oplus V_{\min\{2k-i-j,\, i+j\}}, \tag{9}$$

we can state one of the main theorems of this paper in the following way:

Theorem 3.5. *For the NS sector (i.e. in the case of $i_\alpha \equiv j_\alpha \pmod 2$ for $\alpha = 0, 1$) the structure of the fusion algebra of the minimal model with central charge $c = c(\frac{p}{q})$ can be written as follows:*

$$[(V_{i_0}, V_{j_0})] \circ [(V_{i_1}, V_{j_1})] = [(V_{i_0} \dot{\otimes}_{q-2} V_{i_1}, V_{j_0} \dot{\otimes}_{p-2} V_{j_1})].$$

Conjecture 3.6. Theorem 3.4 and 3.5 are valid for any sectors (i.e. without restriction of the parity of $i_\alpha - j_\alpha$)

4. Verlinde algebras

In this section we list up the characters of minimal series for Vir_ϵ and calculate their modular transformations. As an application, we calculate Verlinde's algebra [21]. Consequently we see that the Verlinde's algebra coincides with the fusion algebra defined by coinvariants in the previous section.

To write down the characters of the minimal series for Vir_ϵ, we recall some holomorphic modular forms of weight $\frac{1}{2}$. Let $\eta(\tau)$ be the Dedekind η-function

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n \in \mathbb{Z}_{>0}} (1-q^n),$$

and $\Theta_{n,m}(\tau)$ be the classical theta function defined for $m \in \frac{1}{2}\mathbb{Z}_{>0}$ and $n \in \mathbb{Z}/2m\mathbb{Z}$ by

$$\Theta_{n,m}(\tau) = \sum_{k \in \mathbb{Z}+\frac{n}{2m}} e^{2\pi i m k^2 \tau},$$

where $q = e^{2\pi i \tau}$.

Through this section, let us fix $c = c(\frac{p}{q})$ as in Case 2 (minimal central charge) and set $h = h_{r,s} = h_{r,s}(\frac{p}{q})$. Since by definition

$$h_{q-r,p-s} = h_{r,s},$$

it is reasonable to define the sets $\Delta^{NS}_{p,q}$ and $\Delta^{R}_{p,q}$, which parameterize the minimal series, as follows. For $\epsilon = \frac{1}{2}, 0$, we set

$$KT^\epsilon_{p,q} = \{(r,s) \in \mathbb{Z}^2 | 1 \le r < q,\ 1 \le s < p,\ r - s \equiv 1 - 2\epsilon \pmod 2\},$$

and define the equivalence relation $\sim$ on $KT^\epsilon_{p,q}$ by

$$(r,s) \sim (r',s') \text{ if and only if } r + r' = q \text{ and } s + s' = p.$$

We sometimes denote $KT^{NS}_{p,q}$ (resp. $KT^{R}_{p,q}$) for $KT^{\frac{1}{2}}_{p,q}$ (resp. $KT^{0}_{p,q}$). Let us define $\Delta^{NS}_{p,q}$ and $\Delta^{R}_{p,q}$ as the set of the equivalence classes with respect to $\sim$ i.e.

$$\Delta^*_{p,q} = KT^*_{p,q}/\sim, \qquad \text{for} \quad * \in \{NS, R\},$$

and let π be the canonical projection

$$\pi : KT^{NS}_{p,q} \coprod KT^{R}_{p,q} \longrightarrow \Delta^{NS}_{p,q} \coprod \Delta^{R}_{p,q}.$$

For $(r,s) \in KT^\epsilon_{p,q}$, we set

$$[r,s] := \pi\{(r,s)\}.$$

Let

$$\chi_\epsilon(c,h)(\tau) = \frac{1}{2(1-\epsilon)} \operatorname{tr}_{L_\epsilon(c,h)} q^{L_0 - \frac{1}{24}c} = q^{-\frac{1}{24}c} \operatorname{ch} L_\epsilon(c,h),$$
$$\tilde{\chi}_\epsilon(c,h)(\tau) = \frac{1}{2(1-\epsilon)} \operatorname{str}_{L_\epsilon(c,h)} q^{L_0 - \frac{1}{24}c} = q^{-\frac{1}{24}c} \operatorname{sch} L_\epsilon(c,h),$$

be the normalized (super-)character of the irreducible representation $L_\epsilon(c,h)$. For simplicity, we set $\chi^{NS}_{[r,s]}(\tau) = \chi_{\frac{1}{2}}(c,h_{r,s})(\tau)$, $\tilde{\chi}^{NS}_{[r,s]}(\tau) = \tilde{\chi}_{\frac{1}{2}}(c,h_{r,s})(\tau)$ and $\chi^{R}_{[r,s]}(\tau) = (1 + \delta_{h,\frac{c}{24}})\chi_0(c,h_{r,s})(\tau)$. The following character formulae are immediate consequences of Theorem 1.4.

Lemma 4.1.

1. **Characters for the NS algebra**

$$\chi^{NS}_{[r,s]}(\tau) = \frac{\eta(\tau)}{\eta(\frac{1}{2}\tau)\eta(2\tau)} \left\{ \Theta_{\frac{rp-sq}{2},\frac{pq}{2}}(\tau) - \Theta_{\frac{rp+sq}{2},\frac{pq}{2}}(\tau) \right\}.$$

2. **Characters for the R algebra**

$$\chi^{R}_{[r,s]}(\tau) = \frac{\eta(2\tau)}{\eta(\tau)^2} \left\{ \Theta_{\frac{rp-sq}{2},\frac{pq}{2}}(\tau) - \Theta_{\frac{rp+sq}{2},\frac{pq}{2}}(\tau) \right\}.$$

3. **Super-characters for the NS algebra**

$$\begin{aligned}\tilde{\chi}^{NS}_{[r,s]}(\tau) = \frac{\eta(\frac{1}{2}\tau)}{\eta(\tau)^2} &\{ (\Theta_{rp-sq,2pq}(\tau) - \Theta_{rp-sq+2pq,2pq}(\tau)) \\ &-(-1)^{rs} \left(\Theta_{rp+sq,2pq}(\tau) - \Theta_{rp+sq-2pq,2pq}(\tau)\right)\},\end{aligned}$$

if $p \equiv q \equiv 1 \pmod 2$, *and*

$$\tilde{\chi}^{NS}_{[r,s]}(\tau) = \frac{\eta(\frac{1}{2}\tau)}{\eta(\tau)^2} \left\{ \Theta_{\frac{rp-sq}{2},\frac{pq}{2}}(\tau) - (-1)^{rs}\Theta_{\frac{rp+sq}{2},\frac{pq}{2}}(\tau) \right\},$$

if $p \equiv q \equiv 0 \pmod 2$.

Next, we will show how the characters of the minimal series transform under the action of $SL(2,\mathbb{Z})$. By using the following modular transformation laws

$$\begin{aligned} &\eta(\tau+1) = e^{\frac{\pi i}{12}}\eta(\tau), && \eta(-\frac{1}{\tau}) = \sqrt{\frac{\tau}{i}}\eta(\tau), \\ &\Theta_{n,m}(\tau+1) = e^{\frac{\pi i n^2}{2m}}\Theta_{n,m}(\tau), && \Theta_{n,m}(-\frac{1}{\tau}) = \sqrt{\frac{\tau}{2mi}} \textstyle\sum_{n' \in \mathbb{Z}/2m\mathbb{Z}} e^{-\frac{\pi i n n'}{m}} \Theta_{n',m}(\tau), \end{aligned}$$

$$\text{for} \quad m \in \mathbb{Z}_{>0}, \ n \in \mathbb{Z}/2m\mathbb{Z},$$

we obtain the following generalization of a result in [17].

Proposition 4.2. *Modular transformations of the (super)characters of the minimal series are given as follows:*

1. *For* $\tau \mapsto \tau + 1$, *we have*

$$\chi^{NS}_{[r,s]}(\tau+1) = e^{\pi i\{\frac{(rp-sq)^2}{4pq} - \frac{1}{8}\}} \tilde{\chi}^{NS}_{[r,s]}(\tau), \ \tilde{\chi}^{NS}_{[r,s]}(\tau+1) = e^{\pi i\{\frac{(rp-sq)^2}{4pq} - \frac{1}{8}\}} \chi^{NS}_{[r,s]}(\tau),$$
$$\chi^{R}_{[r,s]}(\tau+1) = e^{\pi i\frac{(rp-sq)^2}{4pq}} \chi^{R}_{[r,s]}(\tau).$$

2. *For $\tau \mapsto -\frac{1}{\tau}$, we have*

$$\begin{aligned}
\chi^{NS}_{[r,s]}(-\tfrac{1}{\tau}) &= \sum_{[r',s'] \in \Delta^{NS}_{p,q}} S^{NS,NS}_{[r,s],[r',s']} \chi^{NS}_{[r',s']}(\tau), \\
\sqrt{2}\chi^{R}_{[r,s]}(-\tfrac{1}{\tau}) &= \sum_{[r',s'] \in \Delta^{NS}_{p,q}} S^{R,NS}_{[r,s],[r',s']} \tilde{\chi}^{NS}_{[r',s']}(\tau), \\
\tilde{\chi}^{NS}_{[r,s]}(-\tfrac{1}{\tau}) &= \sum_{[r',s'] \in \Delta^{R}_{p,q}} S^{NS,R}_{[r,s],[r',s']} \sqrt{2}\chi^{R}_{[r',s']}(\tau),
\end{aligned}$$

where

$$S^{NS,NS}_{[r,s],[r',s']} = \frac{4}{\sqrt{pq}} \sin\frac{\pi r r'}{2q}(p-q) \sin\frac{\pi s s'}{2p}(p-q),$$

$$S^{R,NS}_{[r,s],[r',s']} = (-1)^{\frac{1}{2}(r'-s')} \frac{4}{\sqrt{pq}} \sin\frac{\pi r r'}{2q}(p-q) \sin\frac{\pi s s'}{2p}(p-q),$$

$$S^{NS,R}_{[r,s],[r',s']} = \begin{cases} (-1)^{\frac{1}{2}(r-s)} \dfrac{2}{\sqrt{pq}} \sin\dfrac{\pi r r'}{2q}(p-q) \sin\dfrac{\pi s s'}{2p}(p-q) & \textit{if } [r',s'] = [\frac{q}{2}, \frac{p}{2}], \\ (-1)^{\frac{1}{2}(r-s)} \dfrac{4}{\sqrt{pq}} \sin\dfrac{\pi r r'}{2q}(p-q) \sin\dfrac{\pi s s'}{2p}(p-q) & \textit{otherwise.} \end{cases}$$

We remark that by the definition of the matrices $(S_{[r,s],[r',s']}^{*,\ *'})$ $(*, *' \in \{NS, R\})$, the following formulae hold:

$$\textbf{Unitarity} \qquad \sum_{[r',s'] \in \Delta^{*'}_{p,q}} S_{[r,s],[r',s']}^{*,\ *'} S_{[r',s'],[r'',s'']}^{*',\ *} = \delta_{[r,s],[r'',s'']}. \tag{10}$$

Now we define the Verlinde's algebra of $N = 1$ super minimal series as follows. Let $\mathcal{V}_{p,q}$ be the $\mathbb{Z}$-free module generated by $\{\phi^{NS}_{[r,s]}\}_{[r,s] \in \Delta^{NS}_{p,q}}$ and $\{\phi^{R}_{[r,s]}\}_{[r,s] \in \Delta^{R}_{p,q}}$. We introduce the multiplication $\circ$ on $\mathcal{V}_{p,q}$ by

$$\phi_{[r_1,s_1]}^{*_1} \circ \phi_{[r_2,s_2]}^{*_2} := \sum_{[r_3,s_3] \in \Delta^{*_3}_{p,q}} N_{[r_1,s_1],[r_2,s_2]}^{[r_3,s_3]} \phi_{[r_3,s_3]}^{*_3} \qquad \text{for} \quad *_1, *_2, *_3 \in \{NS, R\},$$

where $*_i$ $(i = 1, 2, 3)$ are so chosen that $\sharp\{i | *_i = R\} \equiv 0 \mod 2$ and

$$N_{[r_1,s_1],[r_2,s_2]}^{[r_3,s_3]} = \sum_{[r',s'] \in \Delta^{NS}_{p,q}} \frac{S_{[r_1,s_1],[r',s']}^{*_1,\ NS} S_{[r_2,s_2],[r',s']}^{*_2,\ NS} S_{[r',s'],[r_3,s_3]}^{NS,\ *_3}}{S_{[1,1],[r',s']}^{NS,\ NS}}.$$

These formulae have already been discussed in [5] and [19], and their explicit values are computed in a few examples. We call the above formulae modified Verlinde's formulae and the algebra $\mathcal{V}_{p,q}$ equipped with the multiplication $\circ$ Verlinde's algebra. In the sequel, we compute the concrete value of $N_{[r_1,s_1],[r_2,s_2]}^{[r_3,s_3]}$. For $(r_i, s_i) \in KT^{*_i}_{p,q}$ $(i = 1, 2)$, set

$$I_{(r_1,s_1),(r_2,s_2)} := \left\{ (r,s) \in KT^{*_3}_{p,q} \;\middle|\; \begin{array}{r} |r_1 - r_2| < r < \min\{r_1 + r_2, 2q - (r_1 + r_2)\}, \\ r \not\equiv |r_1 - r_2| \mod 2, \\ |s_1 - s_2| < s < \min\{s_1 + s_2, 2p - (s_1 + s_2)\}, \\ s \not\equiv |s_1 - s_2| \mod 2 \end{array} \right\}.$$

We have the following expression.

Lemma 4.3. *The following formulae hold:*

$$N_{[r_1,s_1],[r_2,s_2]}^{[r_3,s_3]} = \sharp\{\pi^{-1}([r_3,s_3]) \cap I_{(r_1,s_1),(r_2,s_2)}\}.$$

Proof. Note that from the explicit formulae of the matrices $(S_{[r,s],[r',s']}^{*,\ *'})$ $(*,*' \in \{NS, R\})$, one can prove

$$\frac{S_{[r_1,s_1],[r',s']}^{*_1,\ NS} S_{[r_2,s_2],[r',s']}^{*_2,\ NS}}{S_{[1,1],[r',s']}^{NS,\ NS}} = \sum_{(r,s)\in I_{(r_1,s_1),(r_2,s_2)}} S_{[r,s],[r',s']}^{*_3,\ NS}.$$

Now the results follow from the unitarity of the matrices $(S_{[r,s],[r',s']}^{*,\ *'})$ (cf. see equations (10)). ■

Let $\mathcal{V}_{p,q}^{NS}$ be the $\mathbb{Z}$-submodule of $\mathcal{V}_{p,q}$ generated by $\{\phi_{[r,s]}^{NS} | (r,s) \in \Delta_{p,q}^{NS}\}$. By Lemma 4.3 and Theorem 3.5, it is easy to see that the following theorem holds.

Theorem 4.4. *The correspondence* $\phi_{[r,s]}^{*} \mapsto [(V_{r-1}, V_{s-1})]$ *gives rise to the isomorphism*

$$\mathcal{V}_{p,q}^{NS} \cong \mathcal{A}_c^{NS}.$$

Conjecture 4.5. One can extend the above isomorphism to an isomorphism

$$\mathcal{V}_{p,q} \cong \mathcal{A}_c.$$

5. Proof of Theorem 3.4 and 3.5

In this section, we give a sketch of a proof of Theorem 3.4 and 3.5, since the proof is similar to the case of $\hat{osp}(1|2)$ [13]. Namely, we will calculate the dimension of the coinvariant

$$H_0(\mathcal{L}(E),\ L'_{\epsilon_1}(c,h_1)(w_1) \otimes L'_{\epsilon_2}(c,h_2)(w_2) \otimes L'_{\epsilon_3}(c,h_3)(w_3)),$$

where $E = \{(w_1,\epsilon_1),(w_2,\epsilon_2),(w_3,\epsilon_3)\}$. From now on, let us fix $w_1 = 0$, $w_2 = 1$ and $w_3 = \infty$ for simplicity, since the dimension of this coinvariant does not depend on the choice of points on $\mathbb{C}P^1$. For $\epsilon = \frac{1}{2}, 0$, we set $E_\epsilon = \{(0,\epsilon),(1,\frac{1}{2}),(\infty,\epsilon)\}$. Note that it suffices to consider the Lie superalgebras $\mathcal{L}(E_\epsilon)$ in our proof.

To prove our main results, we need explicit forms of the action of singular vectors on the loop modules. If $c = c(t)$ and $h = h_{a,b}^{\frac{1}{2}}(t)$ for some a, $b \in \mathbb{Z}_{>0}$ such that $a-b \in 2\mathbb{Z}$, then there exists $S_{a,b} \in U(Vir_{\frac{1}{2}}^-) \setminus \{0\}$ (unique up to scalar) such that $S_{a,b}|c,h\rangle_{\frac{1}{2}}$ is a singular vector of level $\frac{1}{2}ab$. Set

$$\delta_{i,j}^{(2)} = \begin{cases} 1 & \text{if } i \equiv j \pmod 2, \\ 0 & \text{if } i \not\equiv j \pmod 2. \end{cases}$$

$S_{a,b}$ acts on the element $F_0\theta^\gamma \in \mathcal{F}_{\lambda,\mu}^{\frac{1}{2},\sigma}$ as follows:

Lemma 5.1.
$$S_{a,b}.F_0\theta^\gamma = P^\gamma_{a,b}(\lambda,\mu,t)_{\frac{1}{2}}F_{-\frac{1}{2}ab}\theta^{\delta^{(2)}_{ab,1-\gamma}},$$
where
$$\left\{P^\gamma_{a,b}(\lambda,\mu,t)_{\frac{1}{2}}\right\}^2 = \prod_{\substack{0\le k\le a-1\\ 0\le l\le b-1\\ k-l\equiv\gamma \pmod 2}} Q^{k,l}_{a,b}(\lambda,\mu,t)_{\frac{1}{2}}$$
and
$$\begin{aligned} &Q^{k,l}_{a,b}(\lambda,\mu,t)_{\frac{1}{2}}\\ &=\left[(\mu-2\lambda)-\frac{1}{2}(kt^{\frac{1}{2}}-lt^{-\frac{1}{2}})\{(a-k)t^{\frac{1}{2}}-(b-l)t^{-\frac{1}{2}}\}\right]\\ &\times\left[(\mu-2\lambda)-\frac{1}{2}\{(k+1)t^{\frac{1}{2}}-(l+1)t^{-\frac{1}{2}}\}\{(a-k-1)t^{\frac{1}{2}}-(b-l-1)t^{-\frac{1}{2}}\}\right]\\ &+\frac{1}{2}\left\{(a-1-2k)t^{\frac{1}{2}}-(b-1-2l)t^{-\frac{1}{2}}\right\}^2\lambda \end{aligned} \tag{11}$$

Proof. We can show this lemma in the same way as in [7] by using embedding diagrams of Verma modules (see Proposition 1.3). ∎

From now on, we will prove Theorem 3.5. We reduce our calculation of coinvariants to computation of cohomology of a nilpotent subalgebra. Let $\mathfrak{a}_{\frac{1}{2}}$ be the following nilpotent subalgebra of $\mathcal{L}(E_{\frac{1}{2}})$:

$$\mathfrak{a}_{\frac{1}{2}} = \{f(z)\in\mathcal{L}(E_{\frac{1}{2}})|f(z)\in[\hat{\mathcal{L}}^{\geq}_{w;\frac{1}{2}},\hat{\mathcal{L}}^{\geq}_{w;\frac{1}{2}}] \quad \text{for } w=0,1\}.$$

It is easy to see that

$$\mathfrak{a}_{\frac{1}{2}} = z^2(z-1)^2\mathbb{C}[z]\partial_z\oplus z(z-1)\mathbb{C}[z]\partial_z^{\frac{1}{2}}.$$

We set
$$\ell^{(0)}_0 = -z(z-1)^2\partial_z \quad\text{and}\quad \ell^{(1)}_0 = -z^2(z-1)\partial_z.$$

First we show a preliminary lemma, since the idea of the proof is useful in the remainder of this paper.

Lemma 5.2. *Suppose that $c=c(t)$ satisfies the condition in Case 1 or Case 2. For any h_0, h_1, $h_\infty\in\mathbb{C}$, we have*

$$\dim H_0(\mathcal{L}(E_{\frac{1}{2}}),M_{\frac{1}{2}}(c,h_0)(0)\otimes M_{\frac{1}{2}}(c,h_1)(1)\otimes M_{\frac{1}{2}}(c,h_\infty)(\infty))_\sigma = 1,$$

where the subscript $\sigma=0,1$ specifies the even i.e. $\sigma=0$ (resp. odd i.e. $\sigma=1$) subspaces.

Proof. Shapiro's lemma and Poincaré duality imply that

$$\begin{aligned}&\dim H_0(\mathcal{L}(E_{\frac{1}{2}}),M_{\frac{1}{2}}(c,h_0)(0)\otimes M_{\frac{1}{2}}(c,h_1)(1)\otimes M_{\frac{1}{2}}(c,h_\infty)(\infty))\\ &=\dim H^0(\mathfrak{a}_{\frac{1}{2}},M_{\frac{1}{2}}(c,h_\infty)(\infty)^*)^{(h_0,h_1)}.\end{aligned}$$

Here V^* stands for the full dual of V and the superscripts (h_0, h_1) signify the simultaneous eigenspace of the operators $\ell_0^{(0)}$ and $\ell_0^{(1)}$ with the eigenvalues h_0 and h_1 respectively. Let us calculate the dimension of the space

$$H^0(\mathfrak{a}_{\frac{1}{2}}, M_{\frac{1}{2}}(c, h_\infty)(\infty)^*)^{(h_0,h_1)}$$

in the following way: For $\Phi \in M_{\frac{1}{2}}(c, h_\infty)(\infty)^*$ and $j \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, we denote by Φ_j the restriction $\Phi|_{M_{\frac{1}{2}}(c,h_\infty)(\infty)_j}$. Then we have

Fact 5.1. *For $\Phi \in M_{\frac{1}{2}}(c, h_\infty)(\infty)^*$, $\Phi \in H^0(\mathfrak{a}_{\frac{1}{2}}, M_{\frac{1}{2}}(c, h_\infty)(\infty)^*)^{(h_0,h_1)}$ if and only if*

$$\begin{aligned} L_{-n}^{(\infty)}.\Phi_j &= \{-h_\infty + h_0 - n(h_1 - 1) - j\}\Phi_{j-n}, \\ G_{-m}^{(\infty)}.\Phi_j &= G_{-\frac{1}{2}}^{(\infty)}.\Phi_{j-m+\frac{1}{2}} \end{aligned}$$

for any $n \in \mathbb{Z}_{>0}$ and $m \in \mathbb{Z}_{>0} - \frac{1}{2}$.

Using this fact, we can show

Lemma 5.3. *Set $V = \bigoplus_{j=0,\frac{1}{2}} M_{\frac{1}{2}}(c, h_\infty)(\infty)_j$. Then the map*

$$\begin{array}{ccc} H^0(\mathfrak{a}_{\frac{1}{2}}, M_{\frac{1}{2}}(c, h_\infty)(\infty)^*)^{(h_0,h_1)} & \to & V^* \\ \Phi & \mapsto & \Phi|_V \end{array} \tag{12}$$

gives an isomorphism of a $\mathbb{Z}_2$-graded vector space.

Lemma 5.2 is an immediate consequence of the last lemma. ■

Till the end of the proof of Theorem 3.5, we assume that $c = c(\frac{p}{q})$ satisfies the condition in Case 2.

Theorem 5.4. *Suppose that $c = c(\frac{p}{q})$ satisfies the condition in Case 2 and h_0, h_1, $h_\infty \in \Lambda_{\frac{1}{2}}^c$. Then we have*

$$\begin{aligned} &\dim H_0(\mathcal{L}(E_{\frac{1}{2}}), L_{\frac{1}{2}}(c, h_0)(0) \otimes L_{\frac{1}{2}}(c, h_1)(1) \otimes L_{\frac{1}{2}}(c, h_\infty)(\infty)) \\ &\quad = \dim H_0(\mathfrak{a}_{\frac{1}{2}}, L_{\frac{1}{2}}(c, h_\infty)(\infty))^{(-h_0,-h_1)}, \end{aligned}$$

where $H_0(\mathfrak{a}_{\frac{1}{2}}, L_{\frac{1}{2}}(c, h_\infty)(\infty))^{(-h_0,-h_1)}$ is the simultaneous eigenspace of $\ell_0^{(0)}$ and $\ell_0^{(1)}$.

From now on, let us examine

$$H_0(\mathfrak{a}_{\frac{1}{2}}, L_{\frac{1}{2}}(c, h_\infty)(\infty))^{(-h_0,-h_1)}.$$

For h_0, h_1, $h_\infty \in \Lambda_{\frac{1}{2}}^c$, let $[(V_{i_0}, V_{j_0})]$, $[(V_{i_1}, V_{j_1})]$ and $[(V_{i_\infty}, V_{j_\infty})]$ be the equivalence classes corresponding to $l_{h_0}^{\frac{1}{2}}$, $l_{h_1}^{\frac{1}{2}}$ and $l_{h_\infty}^{\frac{1}{2}}$ respectively.

Theorem 5.5. *Suppose that $c = c(\frac{p}{q})$ satisfies the condition in Case 2.*

(i) $\dim H_0(\mathfrak{a}_{\frac{1}{2}}, L_{\frac{1}{2}}(c, h_\infty)(\infty)) < \infty$, *if and only if* $h_\infty \in \Lambda^c_{\frac{1}{2}}$.

(ii) *If* $h_\infty \in \Lambda^c_{\frac{1}{2}}$, *then we have*

$$\dim H_0(\mathfrak{a}_{\frac{1}{2}}, L_{\frac{1}{2}}(c, h_\infty)(\infty))^{(-h_0,-h_1)} = \begin{cases} 2 & \text{if } h_0, h_1 \in \Lambda^c_{\frac{1}{2}} \text{ and both } C^0 \text{ and } C^1 \text{ hold,} \\ 1 & \text{if } h_0, h_1 \in \Lambda^c_{\frac{1}{2}} \text{ and only one of } C^0 \text{ and } C^1 \text{ holds,} \\ 0 & \text{otherwise,} \end{cases}$$

where the conditions C^0 and C^1 are defined as follows:

$$P = \left\{ (i,j) \in (\mathbb{Z})^2 \;\middle|\; \begin{array}{l} |i_0 - i_1| \le i \le \min(i_0 + i_1, 2q - i_0 - i_1 - 4), \\ |j_0 - j_1| \le j \le \min(j_0 + j_1, 2p - j_0 - j_1 - 4), \\ i_0 + i_1 + i \equiv 0, \; j_0 + j_1 + j \equiv 0 \pmod 2, \end{array} \right\}$$

and

$$C^0 \Leftrightarrow [(i_\infty, j_\infty) \in P], \quad C^1 \Leftrightarrow [(-i_\infty + q - 2, -j_\infty + p - 2) \in P].$$

Proof. Since the proof of Theorem 5.5 is very similar to that of Theorem 4.4 in [13], here we only give an outline of the proof. To prove the theorem, we identify the invariant

$$H^0(\mathfrak{a}_{\frac{1}{2}}, M_{\frac{1}{2}}(c, h_\infty)(\infty)^*)^{(h_0,h_1)} \tag{13}$$

with the quotient module $\mathcal{G}^{\frac{1}{2},\sigma}_{\lambda,\mu}(\infty)$ and further, by using the formulae of the action of singular vectors on the loop modules given in Lemma 5.7, we determine when a form $\Phi \in H^0(\mathfrak{a}_{\frac{1}{2}}, M_{\frac{1}{2}}(c, h_\infty)(\infty)^*)^{(h_0,h_1)}$ induces a non-trivial element of $H^0(\mathfrak{a}_{\frac{1}{2}}, L_{\frac{1}{2}}(c, h_\infty)(\infty)^*)^{(h_0,h_1)}$. First we state how one can identify the invariant (13) with the quotient modules. It follows from Lemma 5.2 that

$$\dim H^0(\mathfrak{a}_{\frac{1}{2}}, M_{\frac{1}{2}}(c, h_\infty)(\infty)^*)^{(h_0,h_1)}_\sigma = 1,$$

for each $\sigma = 0, 1$. Let $\Phi^\sigma \in H^0(\mathfrak{a}_{\frac{1}{2}}, M_{\frac{1}{2}}(c, h_\infty)(\infty)^*)^{(h_0,h_1)}_\sigma$ such that $\Phi^\sigma \neq 0$. For $j \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, we denote by Φ^σ_j the restriction $\Phi^\sigma|_{M_{\frac{1}{2}}(c,h_\infty)(\infty)_j}$. If we set

$$\varphi^\sigma_j = \begin{cases} \Phi^\sigma_j & \text{if } j \in \frac{1}{2}\sigma + \mathbb{Z}, \\ G^{(\infty)}_{-\frac{1}{2}}.\Phi^\sigma_{j+\frac{1}{2}} & \text{if } j \in \frac{1}{2}(1-\sigma) + \mathbb{Z}, \end{cases}$$

then from the formula (12) we obtain the following lemma.

Lemma 5.6. *Let $\mu = -h_\infty + h_0 + h_1 - \frac{1}{2}$ and $\lambda = h_1 - \frac{1}{2}$. For each $\sigma = 0, 1$,*

$$\bigoplus_{j \in \frac{1}{2}\mathbb{Z}_{\geq 0}} \mathbb{C}\varphi^\sigma_j \simeq \mathcal{G}^{\frac{1}{2},1-\sigma}_{\lambda,\mu}(\infty)$$

as a $\hat{\mathcal{L}}^<_{\infty;\frac{1}{2}}$-module, via

$$\varphi^\sigma_j \mapsto F_j \theta^{\delta^{(2)}_{2j,\sigma}} \quad \text{for } j \in \frac{1}{2}\mathbb{Z}_{\geq 0}.$$

For the action of the singular vector $S^{(w)}_{a,b}|c, h_\infty\rangle^{(w)}_{\frac{1}{2}}$, we have the following lemma that is an analog of Lemma 5.1:

Lemma 5.7. *For $F_{\frac{1}{2}ab}\theta^{\gamma} \in \mathcal{F}^{\frac{1}{2},\sigma}_{\lambda,\mu}(w)$, we have*

$$a(S^{(w)}_{a,b}).F_{\frac{1}{2}ab}\theta^{\gamma} = \prod_{\substack{0 \le k \le a-1 \\ 0 \le l \le b-1 \\ k-l \equiv \gamma \pmod 2}} Q^{k,l}_{a,b}(\lambda, \mu - \frac{1}{2}ab, -t)_{\frac{1}{2}} F_0 \theta^{\delta^{(2)}_{ab,1-\gamma}},$$

where a denotes the antipode and the polynomial $Q^{k,l}_{a,b}(\lambda,\mu,t)_{\frac{1}{2}}$ is defined as (11).
Using the isomorphism in Lemma 5.6 and the formulae in Lemma 5.7, we obtain the condition that the form Φ^{σ} vanishes on the singular vectors of $M_{\frac{1}{2}}(c,h_{\infty})(\infty)$. Therefore, we have proved Theorem 5.5. ■

To complete the proof of Theorem 3.5, we need to show Theorem 5.4. The following lemma implies Theorem 5.4.

Lemma 5.8. *Let $c = c(\frac{p}{q})$ satisfy the condition in Case 2, and h_0, h_1, $h_{\infty} \in \Lambda^c_{\frac{1}{2}}$. Then we have*

(i)

$$\begin{aligned} &H_0(\mathcal{L}(E_{\frac{1}{2}}), M_{\frac{1}{2}}(c,h_0)(0) \otimes M_{\frac{1}{2}}(c,h_1)(1) \otimes L_{\frac{1}{2}}(c,h_{\infty})(\infty)) \\ &= H_0(\mathcal{L}(E_{\frac{1}{2}}), L_{\frac{1}{2}}(c,h_0)(0) \otimes L_{\frac{1}{2}}(c,h_1)(1) \otimes L_{\frac{1}{2}}(c,h_{\infty})(\infty)), \end{aligned}$$

(ii)

$$\begin{aligned} &H_0(\mathcal{L}(E_{\frac{1}{2}}), M_{\frac{1}{2}}(c,h_0)(0) \otimes M_{\frac{1}{2}}(c,h_1)(1) \otimes L_{\frac{1}{2}}(c,h_{\infty})(\infty)) \\ &= H_0(\mathfrak{a}_{\frac{1}{2}}, L_{\frac{1}{2}}(c,h_{\infty})(\infty))^{(-h_0,-h_1)}. \end{aligned}$$

Proof. We can prove this lemma in the same way as in [13] (Lemma 4.5). ■

From now on, we will consider Case 1 i.e. Theorem 3.4. Here we assume that $c = c(t)$ for $t \in \mathbb{R}_{>0} \setminus \mathbb{Q}_{>0}$. Similarly to Case 2, we introduce a nilpotent subalgebra, which plays the same role as $\mathfrak{a}_{\frac{1}{2}}$ in this case.

$$\begin{aligned} \mathfrak{c}_{\frac{1}{2}} &= \{f(z) \in \mathcal{L}(E_{\frac{1}{2}}) | f(z) \in [\hat{\mathcal{L}}^{\geq}_{w;\frac{1}{2}}, \hat{\mathcal{L}}^{\geq}_{w;\frac{1}{2}}] \quad \text{at } w = \infty\} \\ &= \mathbb{C}[z^{-1},(1-z)^{-1}]\partial_z \oplus \mathbb{C}[z^{-1},(1-z)^{-1}]\partial_z^{\frac{1}{2}}. \end{aligned}$$

We set

$$\ell^{(\infty)}_0 = z\partial_z.$$

Theorem 5.9. *Suppose that $c = c(t)$ satisfies the condition in Case 1 and h_0, h_1, $h_{\infty} \in \Lambda^c_{\frac{1}{2}}$, then we have*

$$\begin{aligned} &\dim H_0(\mathcal{L}(E_{\frac{1}{2}}), L_{\frac{1}{2}}(c,h_0)(0) \otimes L_{\frac{1}{2}}(c,h_1)(1) \otimes L_{\frac{1}{2}}(c,h_{\infty})(\infty)) \\ &= \dim H_0(\mathfrak{c}_{\frac{1}{2}}, L_{\frac{1}{2}}(c,h_0)(0) \otimes L_{\frac{1}{2}}(c,h_1)(1))^{(-h_{\infty})}, \end{aligned}$$

where $H_0(\mathfrak{c}_{\frac{1}{2}}, L_{\frac{1}{2}}(c,h_0)(0) \otimes L_{\frac{1}{2}}(c,h_1)(1))^{(-h_{\infty})}$ denotes the eigenspace of the operator $\ell^{(\infty)}_0$ with the eigenvalue $-h_{\infty}$.

To prove Theorem 3.4, we calculate the right hand side of Theorem 5.9. For h_0, h_1, $h_{\infty} \in \Lambda^c_{\frac{1}{2}}$, let (V_{i_0}, V_{j_0}), (V_{i_1}, V_{j_1}) and $(V_{i_{\infty}}, V_{j_{\infty}})$ be the pairs of two irreducible representations of sl_2 which correspond to $l^{\frac{1}{2}}_{h_0}$, $l^{\frac{1}{2}}_{h_1}$ and $l^{\frac{1}{2}}_{h_{\infty}}$ respectively. We say that a triple (h_0,h_1,h_{∞}) is proper if $(V_{i_{\infty}}, V_{j_{\infty}})$ is a component of $(V_{i_0} \otimes V_{i_1}, V_{j_0} \otimes V_{j_1})$.

Theorem 5.10. *We assume that $c = c(t)$ satisfies the condition in Case 1 and $h_0,\ h_1 \in \Lambda^c_{\frac{1}{2}}$, then we have*

$$\dim H_0(\mathfrak{c}_{\frac{1}{2}}, L_{\frac{1}{2}}(c,h_0)(0) \otimes L_{\frac{1}{2}}(c,h_1)(1))^{(-h_\infty)}$$

$$= \begin{cases} 1 & \text{if } h_\infty \in \Lambda^c_{\frac{1}{2}} \text{ and } (h_0,h_1,h_\infty)\text{:proper}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Here we only identify (co-)invariants with the loop modules. It follows from Lemma 5.2 that

$$\dim H^0(\mathfrak{c}_{\frac{1}{2}}, (M_{\frac{1}{2}}(c,h_0)(0) \otimes M_{\frac{1}{2}}(c,h_1)(1))^*)^{(h_\infty)}_\sigma = 1,$$

for any $\sigma = 0,\ 1$. For a non-zero element

$$\Psi^\sigma \in H^0(\mathfrak{c}_{\frac{1}{2}}, (M_{\frac{1}{2}}(c,h_0)(0) \otimes M_{\frac{1}{2}}(c,h_1)(1))^*)^{(h_\infty)}_\sigma,$$

we introduce functionals $\Psi^{\sigma,w}$ $(w = 0,1)$ on $M_{\frac{1}{2}}(c,h_w)(w)$ by

$$\Psi^{\sigma,0}(u_0) = \Psi^\sigma(u_0 \otimes |c,h_1\rangle^{(1)}_{\frac{1}{2}}) \text{ and } \Psi^{\sigma,1}(u_1) = \Psi^\sigma(|c,h_0\rangle^{(0)}_{\frac{1}{2}} \otimes u_1),$$

where $u_w \in M_{\frac{1}{2}}(c,h_w)(w)$. Let $\Psi^{\sigma,w}_j$ be the restriction $\Psi^{\sigma,w}|_{M_{\frac{1}{2}}(c,h_w)(w)_j}$. We set

$$\psi^{\sigma,w}_j = \begin{cases} \Psi^{\sigma,w}_j & \text{if } j \in \frac{1}{2}\sigma + \mathbb{Z}, \\ G^{(w)}_{-\frac{1}{2}}.\Psi^{\sigma,w}_{j+\frac{1}{2}} & \text{if } j \in \frac{1}{2}(1-\sigma) + \mathbb{Z}, \end{cases}$$

then we have the following lemma.

Lemma 5.11.

(i) *Let $\mu = h_\infty - h_0 + h_1 - \frac{1}{2}$ and $\lambda = h_1 - \frac{1}{2}$. For each $\sigma = 0,\ 1$,*

$$\bigoplus_{j \in \frac{1}{2}\mathbb{Z}_{\geq 0}} \mathbb{C}\psi^{\sigma,0}_j \simeq \mathcal{G}^{\frac{1}{2},1-\sigma}_{\lambda,\mu}(0)$$

as a $\hat{\mathcal{L}}^<_{0;\frac{1}{2}}$-module, via $\psi^{\sigma,0}_j \mapsto F_j\theta^{\delta^{(2)}_{2j,\sigma}}$ ($j \in \frac{1}{2}\mathbb{Z}_{\geq 0}$).

(ii) *Let $\mu = h_\infty - h_1 + h_0 - \frac{1}{2}$ and $\lambda = h_0 - \frac{1}{2}$. For each $\sigma = 0,\ 1$,*

$$\bigoplus_{j \in \frac{1}{2}\mathbb{Z}_{\geq 0}} \mathbb{C}\psi^{\sigma,1}_j \simeq \mathcal{G}^{\frac{1}{2},1-\sigma}_{\lambda,\mu}(1)$$

as a $\hat{\mathcal{L}}^<_{1;\frac{1}{2}}$-module, via $\psi^{\sigma,1}_j \mapsto F_j\theta^{\delta^{(2)}_{2j,\sigma}}$ ($j \in \frac{1}{2}\mathbb{Z}_{\geq 0}$).

Since the other parts of the proof of Theorem 5.10 are the very analogous to those of Theorem 4.4 in [13], we omit them. ■

From the following lemma, we obtain Theorem 5.9 and thus we complete the proof of Theorem 3.4.

Lemma 5.12. *Assume that $c = c(t)$ satisfies the condition in Case 1 and h_0, h_1, $h_\infty \in \Lambda^{c}_{\frac{1}{2}}$. Then we have*

(i)

$$\begin{aligned}&H_0(\mathcal{L}(E_{\frac{1}{2}}), L_{\frac{1}{2}}(c,h_0)(0) \otimes L_{\frac{1}{2}}(c,h_1)(1) \otimes M_{\frac{1}{2}}(c,h_\infty)(\infty))\\ &\quad = H_0(\mathcal{L}(E_{\frac{1}{2}}), L_{\frac{1}{2}}(c,h_0)(0) \otimes L_{\frac{1}{2}}(c,h_1)(1) \otimes L_{\frac{1}{2}}(c,h_\infty)(\infty)),\end{aligned}$$

(ii)

$$\begin{aligned}&H_0(\mathcal{L}(E_{\frac{1}{2}}), L_{\frac{1}{2}}(c,h_0)(0) \otimes L_{\frac{1}{2}}(c,h_1)(1) \otimes M_{\frac{1}{2}}(c,h_\infty)(\infty))\\ &\quad = H_0(\mathfrak{c}_{\frac{1}{2}}, L_{\frac{1}{2}}(c,h_0)(0) \otimes L_{\frac{1}{2}}(c,h_1)(1))^{(-h_\infty)}.\end{aligned}$$

6. On Conjecture 3.6

Here we state two propositions, which support Conjecture 3.6. To prove the first proposition, we identify coinvariants with the quotient module $\mathcal{G}^{0,0}_{\lambda,\mu}$ and calculate the action of singular vectors. We need a formulae of the action of singular vectors on the loop module $\mathcal{F}^{0,0}_{\lambda,\mu}$. For $\tau = \pm$, let $|\widetilde{c, h : \tau}\rangle_0$ be a highest weight vector of the half Verma module $\tilde{M}^{\tau}_0(c,h)$. Suppose that $c = c(t)$ for $t \in \mathbb{R}_{>0}$ and $h = h^0_{a,b}(t)$ for $a, b \in \mathbb{Z}_{>0}$ such that $a - b \equiv 1 \pmod 2$. Then there exists an even element $S^0_{a,b}$ (resp. odd element $S^1_{a,b}$) of $U(Vir^-_0) \setminus \{0\}$ such that a singular vector of the half Verma module $M^{\tau}_0(c,h)$ can be written as

$$(S^0_{a,b} + \tau S^1_{a,b})|\widetilde{c, h : \tau}\rangle_0.$$

For $\alpha, \beta \in \{0,1\}$, let $P^{\alpha,\beta}_{a,b}(\lambda,\mu,t)$ be polynomials of t defined by

$$\begin{aligned}S^0_{a,b}.F_0 &= P^{0,0}_{a,b}(\lambda,\mu,t)_0 F_{-\frac{1}{2}ab},\\ S^0_{a,b}.F_0\theta &= P^{1,1}_{a,b}(\lambda,\mu,t)_0 F_{-\frac{1}{2}ab}\theta,\\ S^1_{a,b}.F_0 &= P^{1,0}_{a,b}(\lambda,\mu,t)_0 F_{-\frac{1}{2}ab}\theta,\\ S^1_{a,b}.F_0\theta &= P^{0,1}_{a,b}(\lambda,\mu,t)_0 F_{-\frac{1}{2}ab},\end{aligned}$$

where $F_i, F_i\theta \in \mathcal{F}^{0,0}_{\lambda,\mu}$. Set

$$\mathbf{P}_{a,b}(\lambda,\mu,t)_0 = \begin{pmatrix} P^{0,0}_{a,b}(\lambda,\mu,t)_0 & P^{0,1}_{a,b}(\lambda,\mu,t)_0 \\ P^{1,0}_{a,b}(\lambda,\mu,t)_0 & P^{1,1}_{a,b}(\lambda,\mu,t)_0 \end{pmatrix}. \tag{14}$$

Lemma 6.1.

$$\{\det \mathbf{P}_{a,b}(\lambda,\mu,t)_0\}^2 = \prod_{\substack{0 \le k \le a-1 \\ 0 \le l \le b-1}} Q^{k,l}_{a,b}(\lambda,\mu,t)_0,$$

where

$$\begin{aligned}&Q^{k,l}_{a,b}(\lambda,\mu,t)_0\\ &= \left[(\mu - 2\lambda) + \frac{1}{2}ab - \frac{1}{2}(kt^{\frac{1}{2}} + lt^{-\frac{1}{2}})\{(a-k)t^{\frac{1}{2}} + (b-l)t^{-\frac{1}{2}}\}\right]\\ &\quad \times \left[(\mu - 2\lambda) + \frac{1}{2}ab - \frac{1}{2}\{(k+1)t^{\frac{1}{2}} + (l+1)t^{-\frac{1}{2}}\}\{(a-k-1)t^{\frac{1}{2}} + (b-l-1)t^{-\frac{1}{2}}\}\right]\\ &\quad + \frac{1}{2}\left\{(a-1-2k)t^{\frac{1}{2}} + (b-1-2l)t^{-\frac{1}{2}}\right\}^2 \lambda.\end{aligned}$$

Remark 6.2. At present, only the determinant of the matrix $\mathbf{P}_{a,b}(\lambda,\mu,t)_0$ can be calculated and explicit forms of the matrix elements $P_{a,b}^{\alpha,\beta}(\lambda,\mu,t)_0$ are not obtained except for some special cases.

From now on, we give two supporting evidences of Conjecture 3.6. First one is a necessary condition that the coinvariants

$$H_0(\mathcal{L}(E_0), L_0(c,h_0)(0)\otimes L_{\frac{1}{2}}(c,h_1)(1)\otimes L_0(c,h_\infty)(\infty)) \tag{15}$$

do not vanish. In the sequel we suppose that $c=c(t)$ satisfies the condition in Case 1 or Case 2 and $h_0,h_\infty\in\Lambda_0^c$, $h_1\in\Lambda_{\frac{1}{2}}^c$. In Case 1, let (V_{i_0},V_{j_0}), (V_{i_1},V_{j_1}) and $(V_{i_\infty},V_{j_\infty})$ be the pairs of two irreducible representations of sl_2 which correspond to $l_{h_0}^0$, $l_{h_1}^{\frac{1}{2}}$ and $l_{h_\infty}^0$ respectively. In Case 2, let $[(V_{i_0},V_{j_0})]$, $[(V_{i_1},V_{j_1})]$ and $[(V_{i_\infty},V_{j_\infty})]$ be the equivalence classes which correspond to $l_{h_0}^0$, $l_{h_1}^{\frac{1}{2}}$ and $l_{h_\infty}^0$ respectively. We say that a triple (h_0,h_1,h_∞) is proper if $(V_{i_\infty},V_{j_\infty})$ (resp. $[(V_{i_\infty},V_{j_\infty})]$) is a component of $(V_{i_0}\otimes V_{i_1},V_{j_0}\otimes V_{j_1})$ (resp. $[(V_{i_0}\dot{\otimes}_{q-2}V_{i_1},V_{j_0}\dot{\otimes}_{p-2}V_{j_1})]$).

Proposition 6.3. *Suppose that $c=c(t)$ satisfies one of the conditions in Case 1 and Case 2. If at least two of $(h_0,0)$, $(h_1,\frac{1}{2})$ and $(h_\infty,0)$ are elements of $\mathcal{S}_c$ and*

$$\dim H_0(\mathcal{L}(E_0), L_0(c,h_0)(0)\otimes L_{\frac{1}{2}}(c,h_1)(1)\otimes L_0(c,h_\infty)(\infty))\neq 0, \tag{16}$$

then $(h_0,0),(h_1,\frac{1}{2}),(h_\infty,0)\in\mathcal{S}_c$ and the triple (h_0,h_1,h_∞) is proper.

Proof. Similarly to $NS\times NS\to NS$ case, by using Shapiro's lemma and Poincaré duality, we can reduce calculation of coinvariants (16) to computation of coinvariants of nilpotent subalgebras. Furthermore in the case of $h_0,h_\infty\neq\frac{1}{24}c$, these coinvariants of nilpotent subalgebras can be identified with the quotient module $\mathcal{G}_{\lambda,\mu}^{0,0}$. By using Lemma 6.1 and its variants, we can show this proposition by an argument similar to the proof of Theorem 5.4 and 5.5. In particular, we will calculate the coinvariants (16) explicitly for $h_0=\frac{1}{24}c$ or $h_\infty=\frac{1}{24}c$ in Proposition 6.7. ■

Remark 6.4. The above coinvariant

$$H_0(\mathcal{L}(E_0), L_0(c,h_0)(0)\otimes L_{\frac{1}{2}}(c,h_1)(1)\otimes L_0(c,h_\infty)(\infty))$$

has even dimension. In fact, we see that

$$\begin{aligned}&H_0(\mathcal{L}(E_0), L_0(c,h_0)(0)\otimes L_{\frac{1}{2}}(c,h_1)(1)\otimes L_0(c,h_\infty)(\infty))\\ &\quad= H_0(\tilde{\mathfrak{a}}_0, L_0(c,h_0)(0)\otimes L_0(c,h_\infty)(\infty))^{(-h_1)},\end{aligned}$$

where

$$\begin{aligned}\tilde{\mathfrak{a}}_0 &= \{f(z)\in\mathcal{L}(E_0)\,|\,f(z)\in[\mathcal{L}_{1:\frac{1}{2}}^{\geq},\mathcal{L}_{1:\frac{1}{2}}^{\geq}]\}\\ &= \mathbb{C}[z,z^{-1}](z-1)^2\partial_z\oplus\mathbb{C}[z,z^{-1}](z-1)z^{\frac{1}{2}}\partial_z^{\frac{1}{2}}\end{aligned}$$

and $H_0(\tilde{\mathfrak{a}}_0, L_0(c,h_0)(0)\otimes L_0(c,h_\infty)(\infty))^{(-h_1)}$ denotes the eigenspace of the operator $-(z-1)\partial_z$ with the eigenvalue $-h_1$. The parity shift Π acts on the coinvariant

$$H_0(\tilde{\mathfrak{a}}_0, L_0(c,h_0)(0)\otimes L_0(c,h_\infty)(\infty))^{(-h_1)}$$

as an isomorphism, since

$$\Pi(L_0(c,h_0)(0)\otimes L_0(c,h_\infty)(\infty)) = L_0(c,h_0)(0)\otimes L_0(c,h_\infty)(\infty).$$

This implies the desired result.

Next we give the second supporting evidence (Proposition 6.5). Consider the following subcase of Case 2.

Case 2' $c = c(\frac{p}{q})$ for $p, q \in 2\mathbb{Z}_{>0}$, $p \neq q$ and $(\frac{1}{2}(p-q), q) = 1$.

We prove

Proposition 6.5. *If $c = c(\frac{p}{q})$ satisfies the condition in Case 2' and $h_0 = \frac{1}{24}c$ or $h_\infty = \frac{1}{24}c$, then Conjecture 3.6 holds.*

For the proof, we have to calculate

$$\dim H_0(\mathcal{L}(E_0), L_0(c,h_0)(0)\otimes L_{\frac{1}{2}}(c,h_1)(1)\otimes L_0(c,h_\infty)(\infty)) \tag{17}$$

for $h_1 \in \Lambda^c_{\frac{1}{2}}$ and $h_0, h_\infty \in \Lambda^c_0$ such that $h_0 = \frac{1}{24}c$ or $h_\infty = \frac{1}{24}c$. To examine the above coinvariants, we introduce the following nilpotent subalgebra $\mathfrak{a}_0$:

$$\begin{aligned}\mathfrak{a}_0 &= \left\{ f(z)\in\mathcal{L}(E_0) \,\middle|\, f(z)\in[\hat{\mathcal{L}}^{\geq}_{w,\epsilon}, \hat{\mathcal{L}}^{\geq}_{w,\epsilon}] \text{ for } (w,\epsilon) = (0,0),(\infty,0)\right\}\\ &= z^2(z-1)^{-2}\mathbb{C}[(z-1)^{-1}]\partial_z \oplus z^{\frac{3}{2}}(z-1)^{-2}\mathbb{C}[(z-1)^{-1}]\partial_z^{\frac{1}{2}}.\end{aligned} \tag{18}$$

We set

$$\bar{\mathfrak{a}}_0 = \mathfrak{a}_0 \bigoplus_{w=0,\infty} (\mathbb{C}\ell_0^{(w)} \oplus \mathbb{C}g_0^{(w)}), \tag{19}$$

where

$$\begin{aligned} \ell_0^{(0)} &= -z(z-1)^{-2}\partial_z, & \ell_0^{(\infty)} &= z^2(z-1)^{-1}\partial_z,\\ g_0^{(0)} &= -\sqrt{-1}z^{\frac{1}{2}}(z-1)^{-1}\partial_z^{\frac{1}{2}}, & g_0^{(\infty)} &= z^{\frac{3}{2}}(z-1)^{-1}\partial_z^{\frac{1}{2}}.\end{aligned}$$

To reduce our calculation by using Sapiro's lemma, we need the following representations of $\bar{\mathfrak{a}}_0$: For $w = 0,\infty$, let $W^w_{c,h_w} = \bigoplus_{\alpha=0,1}\mathbb{C}\mathbf{1}^w_{c,h_w:\alpha}$ be the 2-dimensional representation of $\bar{\mathfrak{a}}_0$ defined by

$$\begin{array}{ll} x.\mathbf{1}^w_{c,h_w:\alpha} = 0 & \text{if } x\in\mathfrak{a}_0\\ \ell_0^{(w')}.\mathbf{1}^w_{c,h_w:\alpha} = \delta_{w',w}h_w\mathbf{1}^w_{c,h_w:\alpha} & \text{for } w' = 0,\infty,\\ g_0^{(w')}.\mathbf{1}^w_{c,h_w:\alpha} = (h_w - \frac{1}{24}c)^{\frac{1}{2}}\mathbf{1}^w_{c,h_w:1-\alpha} & \text{for } w' = 0,\infty,\\ \deg \mathbf{1}^w_{c,h_w:\alpha} = \alpha. & \end{array} \tag{20}$$

Now we have

Proposition 6.6. *Suppose that $c = c(\frac{p}{q})$ satisfies the condition in Case 2' and $h_0, h_\infty \in \Lambda^c_0$, $h_1 \in \Lambda^c_{\frac{1}{2}}$. If $h_0 = \frac{1}{24}c$ or $h_\infty = \frac{1}{24}c$ then*

$$\begin{aligned}&\dim H_0(\mathcal{L}(E_0), L_0(c,h_0)(0)\otimes L_{\frac{1}{2}}(c,h_1)(1)\otimes L_0(c,h_\infty)(\infty))\\ &= \dim H_0(\bar{\mathfrak{a}}_0, W^0_{c,h_0}\otimes L_{\frac{1}{2}}(c,h_1)(1)\otimes W^\infty_{c,h_\infty}).\end{aligned}$$

Proof. Using Shapiro's lemma, the resolution for $L_\epsilon(c,h_w)(w)$ and Theorem 6.6, we can prove this theorem by the same argument as in the proof of Lemma 4.5 in [13]. ∎

To show Conjecture 3.6 in Case 2', we calculate

$$\dim H_0(\bar{\mathfrak{a}}_0, W^0_{c,h_0} \otimes L_{\frac{1}{2}}(c,h_1)(1) \otimes W^\infty_{c,h_\infty}).$$

For the sake of the commutativity of the tensor products in the coinvariant (17), we can assume that $h_\infty = \frac{1}{24}c$ without loss of generality. We show the following proposition.

Proposition 6.7. *Suppose that $c = c(\frac{p}{q})$ satisfies the condition in Case 2' and $h_1 = h^{\frac{1}{2}}_{r_1,s_1}(\frac{p}{q}) \in \Lambda^c_{\frac{1}{2}}$. If $h_\infty = \frac{1}{24}c$ then*

$$\begin{aligned} &H_0(\bar{\mathfrak{a}}_0, W^0_{c,h_0} \otimes L_{\frac{1}{2}}(c,h_1)(1) \otimes W^\infty_{c,h_\infty}) \\ &\quad = \begin{cases} 4 & \text{if } h_0 = h^0_{r,s}(\frac{p}{q}) \text{ for some } (r,s) \in \mathbf{E}_{\min(r_1,q-r_1),\min(s_1,p-s_1)}, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

where, for $a, b \in \mathbb{Z}_{>0}$ such that $a \equiv b \pmod 2$, the set $\mathbf{E}_{a,b}$ is defined by

$$\mathbf{E}'_{a,b} = \left\{ (r,s) \in (\mathbb{Z}_{>0} \times \mathbb{Z}) \sqcup (\{0\} \times \mathbb{Z}_{>0}) \,\middle|\, \begin{array}{l} r \le a-1 \wedge |s| \le b-1 \\ r \equiv a-1, \ s \equiv b-1 \pmod 2 \end{array} \right\}, \tag{21}$$

and

$$\mathbf{E}_{a,b} = \begin{cases} \mathbf{E}'_{a,b} \sqcup \{(0,0)\} & \text{if } a \equiv b \equiv 1 \pmod 2, \\ \mathbf{E}'_{a,b} & \text{if } a \equiv b \equiv 0 \pmod 2. \end{cases} \tag{22}$$

Proof. By Poincaré duality we have

$$\begin{aligned} &\dim H_0(\bar{\mathfrak{a}}_0, W^0_{c,h_0} \otimes L_{\frac{1}{2}}(c,h_1)(1) \otimes W^\infty_{c,h_\infty}) \\ &\quad = \dim H^0(\bar{\mathfrak{a}}_0, (W^0_{c,h_0} \otimes L_{\frac{1}{2}}(c,h_1)(1) \otimes W^\infty_{c,h_\infty})^*). \end{aligned} \tag{23}$$

In the sequel, we calculate the invariant in the right hand side of (23). By using the following part of BGG type resolution:

$$N_{\frac{1}{2}}(c,h_1)(1) \to M_{\frac{1}{2}}(c,h_1)(1) \to L_{\frac{1}{2}}(c,h_1)(1) \to 0,$$

we obtain

$$\begin{array}{ll} H^0(\bar{\mathfrak{a}}_0, (W^0_{c,h_0} \otimes N_{\frac{1}{2}}(c,h_1)(1) \otimes W^\infty_{c,h_\infty})^*) & \xleftarrow{\xi} \\ H^0(\bar{\mathfrak{a}}_0, (W^0_{c,h_0} \otimes M_{\frac{1}{2}}(c,h_1)(1) \otimes W^\infty_{c,h_\infty})^*) & \leftarrow \\ H^0(\bar{\mathfrak{a}}_0, (W^0_{c,h_0} \otimes L_{\frac{1}{2}}(c,h_1)(1) \otimes W^\infty_{c,h_\infty})^*) & \leftarrow \quad 0. \end{array}$$

Therefore, we see that

$$H^0(\bar{\mathfrak{a}}_0, (W^0_{c,h_0} \otimes L_{\frac{1}{2}}(c,h_1)(1) \otimes W^\infty_{c,h_\infty})^*) \simeq \ker \xi.$$

To calculate $\dim \ker \xi$, we first determine how the nilpotent subalgebra $\hat{\mathcal{L}}^<_{1,\frac{1}{2}}$ acts on the invariant:

$$H^0(\bar{\mathfrak{a}}_0, (W^0_{c,h_0} \otimes M_{\frac{1}{2}}(c,h_1)(1) \otimes W^\infty_{c,h_\infty})^*). \tag{24}$$

Let us consider the following functionals on the Verma module $M_{\frac{1}{2}}(c,h_1)(1)$. For $\Psi \in H^0(\bar{\mathfrak{a}}_0,(W^0_{c,h_0}\otimes M_{\frac{1}{2}}(c,h_1)(1)\otimes W^\infty_{c,h_\infty})^*)$ and $\alpha,\beta\in\{0,1\}$, we define $\Psi^{\alpha,\beta}\in M_{\frac{1}{2}}(c,h_1)(1)^*$ by

$$\Psi^{\alpha,\beta}(v_1)=\Psi(\mathbf{1}^0_{c,h_0:\alpha}\otimes v_1\otimes \mathbf{1}^\infty_{c,h_\infty:\beta}) \quad \text{for} \quad v_1\in M_{\frac{1}{2}}(c,h_1)(1), \tag{25}$$

and set

$$\psi_j^{\alpha,\beta}=\Psi^{\alpha,\beta}|_{M_{\frac{1}{2}}(c,h_1)(1)_j} \quad \text{for} \quad j\in \frac{1}{2}\mathbb{Z}_{\geq 0}. \tag{26}$$

By direct calculation, we can show that the subalgebra $\mathcal{L}^<_{1,\frac{1}{2}}\oplus\mathbb{C}L_0^{(1)}\oplus\mathbb{C}C$ acts on the functionals $\psi_j^{\alpha,\beta}$ as follows:

Lemma 6.8. *For $j\in\frac{1}{2}\mathbb{Z}_{\geq 0}$ and $m,n\in\mathbb{Z}_{>0}$,*

$$\begin{aligned}
&(i) && C.\varphi_j^{\alpha,\beta}=-c\varphi_j^{\alpha,\beta},\\
&(ii) && L^{(1)}_{-n}.\varphi_j^{\alpha,\beta}=(-1)^{n-1}(n(h_0-1)+h_1-h_\infty+j)\varphi_{j-n}^{\alpha,\beta},\\
& && L^{(1)}_{0}.\varphi_j^{\alpha,\beta}=-(h_1+j)\varphi_j^{\alpha,\beta}\\
&(iii) && G^{(1)}_{-m+\frac{1}{2}}.\varphi^{\alpha,\beta}_{m-\frac{1}{2}+j}=\left(-\frac{1}{4}\right)^{m-1}\begin{pmatrix}2m-2\\ m-1\end{pmatrix}\\
& && \qquad\times\left\{(-1)^{2j+\alpha}(2m-1)K_0\varphi_j^{1-\alpha,\beta}+K_\infty\varphi_j^{\alpha,1-\beta}\right\}\\
& && \qquad+\left(-\frac{1}{4}\right)^{m}\begin{pmatrix}2m\\ m\end{pmatrix}\sum_{k=0}^{[j-\frac{1}{2}]}\begin{pmatrix}-\frac{1}{2}\\ k\end{pmatrix}\frac{m}{m+k}G^{(1)}_{k+\frac{1}{2}}.\varphi^{\alpha,\beta}_{j-k-\frac{1}{2}},
\end{aligned}$$

where $K_0=(h_0-\frac{1}{24}c)^{\frac{1}{2}}$, $K_\infty=\sqrt{-1}(h_\infty-\frac{1}{24}c)^{\frac{1}{2}}$ and $[*]$ stands for the Gaussian symbol.

Using this lemma, we can obtain the dimension of the invariant (24). Indeed, if we set

$$V_0=W^0_{c,h_0}\otimes M_{\frac{1}{2}}(c,h_1)(1)_0\otimes W^\infty_{c,h_\infty}, \tag{27}$$

then we have

Lemma 6.9. *The map*

$$\begin{array}{rcl} H^0(\bar{\mathfrak{a}}_0,(W^0_{c,h_0}\otimes M_{\frac{1}{2}}(c,h_1)(1)\otimes W^\infty_{c,h_\infty})^*) & \to & (V_0)^* \\ \Psi & \mapsto & \Psi|_{V_0}\end{array} \tag{28}$$

gives an isomorphism of a vector space.

In particular, we have

$$\dim H^0(\bar{\mathfrak{a}}_0,(W^0_{c,h_0}\otimes M_{\frac{1}{2}}(c,h_1)(1)\otimes W^\infty_{c,h_\infty})^*)=\dim V_0=4.$$

Remark 6.10. In the above case, it follows that

$$\dim H_0(\mathcal{L}(E_0), M_0(c,h_0)(0)\otimes M_{\frac{1}{2}}(c,h_1)(1)\otimes M_0(c,h_\infty)(\infty)) = 4,$$

since Shapiro's lemma implies

$$\begin{aligned} &H_0(\mathcal{L}(E_0), M_0(c,h_0)(0)\otimes M_{\frac{1}{2}}(c,h_1)(1)\otimes M_0(c,h_\infty)(\infty)) \\ &\simeq H_0(\bar{\mathfrak{a}}_0, W^0_{c,h_0}\otimes M_{\frac{1}{2}}(c,h_1)(1)\otimes W^\infty_{c,h_\infty}). \end{aligned}$$

Next we calculate the action of singular vectors on functionals $\{\psi_j^{\alpha,\beta}\}$. Suppose that $c=c(t)$ for $t\in\mathbb{R}_{>0}$ (not necessarily rational) and $h_1 = h^{\frac{1}{2}}_{r_1,s_1}(t)$. Let $S^{(1)}_{r_1,s_1}|c,h_1\rangle^{(1)}_{\frac{1}{2}}$ be a singular vector of level $N_1=\frac{1}{2}r_1s_1$, and $\mathbf{A}(h_0,h_\infty;t)$ be the 4×4 matrix defined by

$$a(S^{(1)}_{r_1,s_1}) : \{\psi^{0,0}_{N_1},\psi^{1,1}_{N_1},\psi^{0,1}_{N_1},\psi^{1,0}_{N_1}\}\mapsto\{\psi^{0,0}_0,\psi^{1,1}_0,\psi^{0,1}_0,\psi^{1,0}_0\}\mathbf{A}(h_0,h_\infty;t),$$

By Lemma 6.8, $\mathbf{A}(h_0,h_\infty;t)$ become the direct sum of the following 2×2 matrices

$$\begin{aligned} a(S^{(1)}_{r_1,s_1}) &: \{\psi^{0,0}_{N_1},\psi^{1,1}_{N_1}\}\mapsto\{\psi^{0,\beta}_0,\psi^{1,1-\beta}_0\}\mathbf{A}'(h_0,h_\infty;t),\\ a(S^{(1)}_{r_1,s_1}) &: \{\psi^{0,1}_{N_1},\psi^{1,0}_{N_1}\}\mapsto\{\psi^{0,1-\beta}_0,\psi^{1,\beta}_0\}\mathbf{A}''(h_0,h_\infty;t), \end{aligned}$$

where β is given by

$$\beta = \begin{cases} 0 & \text{if } N_1\in\mathbb{Z},\\ 1 & \text{if } N_1\in\mathbb{Z}+\frac{1}{2}. \end{cases}$$

Lemma 6.11. *Suppose that $c=c(\frac{p}{q})$ satisfies the condition in Case 2' and $h_1 = h_{r_1,s_1}(\frac{p}{q})\in\Lambda^c_{\frac{1}{2}}$. If $h_\infty=\frac{1}{24}c$ then*

$$\begin{aligned} &\mathbf{A}'(h_0,h_\infty;t) = \mathbf{A}''(h_0,h_\infty;t)\\ &= \begin{cases} \begin{pmatrix} 0 & -X_{r_1,s_1}\\ X_{r_1,s_1} & 0\end{pmatrix} & \textit{if } r_1\equiv s_1\equiv 1 \pmod 2,\\ \begin{pmatrix} Y_{r_1,s_1} & 0\\ 0 & Y_{r_1,s_1}\end{pmatrix} & \textit{if } r_1\equiv s_1\equiv 0 \pmod 2, \end{cases} \end{aligned}$$

where

$$\begin{aligned} X_{r_1,s_1} &= (-1)^{\sharp\mathbf{E}'_{r_1,s_1}}(h_0-h^0_{0,0}(t))^{\frac{1}{2}}\prod_{(a,b)\in\mathbf{E}'_{r_1,s_1}}(h_0-h^0_{a,b}(t)),\\ Y_{r_1,s_1} &= \prod_{(a,b)\in\mathbf{E}'_{r_1,s_1}}(h_0-h^0_{a,b}(t)), \end{aligned}$$

Remark 6.12. For $c=c(\frac{p}{q})$ which satisfies the condition in Case 2', and $r_1,s_1\in\mathbb{Z}_{>0}$ such that $h^{\frac{1}{2}}_{r_1,s_1}(\frac{p}{q})\in\Lambda^c_{\frac{1}{2}}$,

$$\begin{aligned} &\left\{h^0_{r,s}(\tfrac{p}{q})\,\middle|\,(r,s)\in\mathbf{E}_{\min(r_1,q-r_1),\min(s_1,p-s_1)}\right\}\\ &= \left\{h^0_{r,s}(\tfrac{p}{q})\,\middle|\,\begin{array}{l} |\frac{q}{2}-r_1|+1\le r\le\min\{\frac{3q}{2}-r_1,\frac{q}{2}+r_1\}-1\\ |\frac{p}{2}-s_1|+1\le s\le\min\{\frac{3p}{2}-s_1,\frac{q}{2}+s_1\}-1\\ r\equiv\frac{q}{2}-r_1+1,\quad s\equiv\frac{p}{2}-s_1+1 \pmod 2\end{array}\right\}. \end{aligned}$$

Using this lemma, we can show Proposition 6.7. The maximal proper submodule $N_{\frac{1}{2}}(c,h_1)(1)$ of the Verma module $M_{\frac{1}{2}}(c,h_1)(1)$ is generated by the singular vectors $S^{(1)}_{r_1,s_1}|c,h_1\rangle^{(1)}_{\frac{1}{2}}$ and $S^{(1)}_{q-r_1,p-s_1}|c,h_1\rangle^{(1)}_{\frac{1}{2}}$ of level $\frac{1}{2}r_1s_1$ and $\frac{1}{2}(q-r_1)(p-s_1)$. We set $S_1=S^{(1)}_{r_1,s_1}$, $S_2=S^{(1)}_{q-r_1,p-s_1}$, $N_1=\frac{1}{2}r_1s_1$ and $N_2=\frac{1}{2}(q-r_1)(p-s_1)$. For $i=1,2$, let $\mathbf{A}'_i$ and $\mathbf{A}''_i$ be the above 2×2 matrices $\mathbf{A}'(h_0,h_\infty;t)$ and $\mathbf{A}''(h_0,h_\infty;t)$ defined by the singular vector $S_i|c,h_1\rangle^{(1)}_{\frac{1}{2}}$. Set

$$\beta_i=\begin{cases}0 & \text{if } N_i\in\mathbb{Z},\\ 1 & \text{if } N_i\in\mathbb{Z}+\frac{1}{2}.\end{cases}$$

Notice that for $\Psi\in H^0(\bar{\mathfrak{a}}_0,(W^0_{c,h_0}\otimes M_{\frac{1}{2}}(c,h_1)(1)\otimes W^\infty_{c,h_\infty})^*)$,

$$\begin{aligned}\Psi\in\ker\xi &\Leftrightarrow \psi^{\alpha,\beta}_{N_i}(S_i|c,h_1\rangle^{(1)}_{\frac{1}{2}})=0 \ \text{ for any } \ \alpha,\beta\in\{0,1\} \text{ and } i=1,2,\\ &\Leftrightarrow \{\psi^{0,\beta_i}_0,\psi^{1,1-\beta_i}_0\}\mathbf{A}'_i=0 \ \text{ and } \ \{\psi^{0,1-\beta_i}_0,\psi^{1,\beta_i}_0\}\mathbf{A}''_i=0 \ \text{ for } \ i=1,2.\end{aligned}$$

Therefore, from Lemma 6.9, 6.11 and Remark 6.12, we obtain Proposition 6.7.

Remark 6.13. Combining Theorem 6.6 and 6.7, we have proved Conjecture 3.6 in Case 2', since we have

$$\begin{aligned}&\Phi_3(l^0_{h_0},l^{\frac{1}{2}}_{h_1},l^0_{h_\infty})\\ &=\frac{1}{2}\dim H_0(\mathcal{L}(E_0),L_0(c,h_0)(0)\otimes L_{\frac{1}{2}}(c,h_1)(1)\otimes L_0(c,h_\infty)(\infty))\end{aligned}$$

and $\Phi_2(l^0_{h_\infty},l^0_{h_\infty})=2$.

Remark 6.14. From the proof of Proposition 6.7, it seems that we have to determine the matrix elements of the matrix (14) to prove Conjecture 3.6 along this line. At the moment, we can do it only in the following two cases:

1. $h_0=\frac{1}{24}c$ or $h_\infty=\frac{1}{24}c$,
2. $h_0=h_\infty$ (See Lemma 7.4).

7. Rationality of SVOA $L_{\frac{1}{2}}(c,0)$

In this section, we prove Theorems 3.2 and 3.3, which are relevant to the rationality of the SVOA $L_{\frac{1}{2}}(c,0)$.

It follows from Theorem 5.4 and 5.9 that the sufficiency in Theorem 3.2 (i) and Theorem 3.3 for $\epsilon=\frac{1}{2}$ is a corollary of Theorem 5.5 and 5.10. Similarly, the sufficiency in Theorem 3.2 (i) and Theorem 3.3 are immediate consequences of Proposition 6.3, that is stated only with a sketch of a proof. The necessity in Theorem 3.2 (i) follows from Theorem 3.2 (ii). So we indicate how one can prove Theorem 3.2 (ii) for both $\epsilon=\frac{1}{2},0$. Since Theorem 3.3 is important for its relation with the rationality of the minimal models, we give an alternative proof of Theorem 3.3.

First we comment on the proof of Theorem 3.2 (ii) briefly. Notice that for a generic central charge c, if $h \notin \Lambda_\epsilon^c$ then $M_\epsilon(c,h)(w)$ is irreducible. In the case of $h_1 \notin \Lambda_{\epsilon_1}^c$ and $h_2 \notin \Lambda_{\epsilon_2}^c$,

$$\dim H_0(\mathcal{L}(E), M_{\epsilon_1}(c,h_1)(w_1) \otimes M_{\epsilon_1}(c,h_2)(w_2) \otimes M_{\epsilon_3}(c,h_3)(w_3)) \neq 0,$$

for any $h_3 \notin \Lambda_{\epsilon_3}^c$ (See Remark 6.10). On the other hand, in the case of $h_1 \in \Lambda_{\epsilon_1}^c$ and $h_2 \notin \Lambda_{\epsilon_2}^c$, the Verma module $M_{\frac{1}{2}}(c,h)(0)$ ($\epsilon = \frac{1}{2}$) or the half Verma module $\tilde{M}_0^\tau(c,h)(0)$ ($\epsilon = 0$) has unique (up to scalar) singular vector. As in Theorem 5.5, Theorem 5.10 and Proposition 6.3, this singular vector defines lines where (r_3, s_3) such that $h_3 = h_{r_3,s_3}^{\epsilon_3}(t)$ should lie. Combining these equations with Theorem 5.5, Theorem 5.10 and Proposition 6.3, we obtain the result. To prove Theorem 3.3, we first show the following proposition.

Proposition 7.1. *Suppose that $c = c(t)$ for $t \in \mathbb{R}_{>0}$. If*

$$H_0(\mathcal{L}(E_0), L_0'(c,h_0)(0) \otimes L_{\frac{1}{2}}(c,0)(1) \otimes L_0'(c,h_\infty)(\infty)) \neq 0,$$

then $h_0 = h_\infty$.

Proof. Notice that $M_{\frac{1}{2}}(c,0)(1)$ has the singular vector $G_{-\frac{1}{2}}^{(1)}|c,0\rangle_{\frac{1}{2}}^{(1)}$. We set $N_{\frac{1}{2}}'(c,0)(1) = U(\hat{\mathcal{L}}_{1:\frac{1}{2}}^{<})G_{-\frac{1}{2}}^{(1)}|c,0\rangle_{\frac{1}{2}}^{(1)}$ and $V_{\frac{1}{2}}(c,0)(1) = M_{\frac{1}{2}}(c,0)(1)/N'(c,0)(1)$. To prove Proposition 7.1, it is enough to show that

$$\dim H_0(\mathcal{L}(E_0), M_0'(c,h_0)(0) \otimes V_{\frac{1}{2}}(c,0)(1) \otimes M_0'(c,h_\infty)(\infty)) = 2\delta_{h_0,h_\infty} 2^{\delta_{h_0,\frac{1}{24}c}}. \tag{29}$$

Indeed, if we can prove (29), then by using the exact sequences

$$V_{\frac{1}{2}}(c,0)(1) \to L_{\frac{1}{2}}(c,0)(1) \to 0$$

and

$$M_0'(c,h_w)(w) \to L_0'(c,h_w)(w) \to 0 \quad (w = 0, \infty),$$

we obtain Proposition 7.1 by an argument similar to Lemma 4.5 in [13].

By Shapiro's lemma and Poincaré duality, we have

$$\begin{aligned} &\dim H_0(\mathcal{L}(E_0), M_0'(c,h_0)(0) \otimes V_{\frac{1}{2}}(c,0)(1) \otimes M_0'(c,h_\infty)(\infty)) \\ &\quad = \dim H^0(\bar{\mathfrak{a}}_0, (W_{c,h_0}^0 \otimes V_{\frac{1}{2}}(c,0)(1) \otimes W_{c,h_\infty}^\infty)^*). \end{aligned} \tag{30}$$

Then we need to calculate the right hand side of (30). Using the exact sequence

$$N_{\frac{1}{2}}'(c,0)(1) \to M_{\frac{1}{2}}(c,0)(1) \to V_{\frac{1}{2}}(c,0)(1) \to 0,$$

we have

$$\begin{array}{rl} H^0(\bar{\mathfrak{a}}_0, (W_{c,h_0}^0 \otimes N_{\frac{1}{2}}'(c,0)(1) \otimes W_{c,h_\infty}^\infty)^*) & \stackrel{\xi}{\leftarrow} \\ H^0(\bar{\mathfrak{a}}_0, (W_{c,h_0}^0 \otimes M_{\frac{1}{2}}(c,0)(1) \otimes W_{c,h_\infty}^\infty)^*) & \leftarrow \\ H^0(\bar{\mathfrak{a}}_0, (W_{c,h_0}^0 \otimes V_{\frac{1}{2}}(c,0)(1) \otimes W_{c,h_\infty}^\infty)^*) & \leftarrow \quad 0, \end{array}$$

and then

$$H^0(\bar{\mathfrak{a}}_0, (W_{c,h_0}^0 \otimes V_{\frac{1}{2}}(c,0)(1) \otimes W_{c,h_\infty}^\infty)^*) \simeq \ker \xi.$$

Note that for $\Psi \in H^0(\bar{\mathfrak{a}}_0, (W^0_{c,h_0} \otimes M_{\frac{1}{2}}(c,0)(1) \otimes W^\infty_{c,h_\infty})^*)$,

$$\Psi \in \ker \xi \Leftrightarrow G^{(1)}_{-\frac{1}{2}}.\psi^{\alpha,\beta}_{\frac{1}{2}} = 0 \quad \text{for any} \quad \alpha, \beta \in \{0,1\},$$

where $\psi^{\alpha,\beta}_j$ is defined as (26). By Lemma 6.8, we have

$$\left(G^{(1)}_{-\frac{1}{2}}.\psi^{\alpha,\beta}_{\frac{1}{2}}, G^{(1)}_{-\frac{1}{2}}.\psi^{1-\alpha,1-\beta}_{\frac{1}{2}}\right) = \left(\psi^{\alpha,1-\beta}_0, \psi^{1-\alpha,\beta}_0\right) \begin{pmatrix} K_\infty & (-1)^{\alpha+1}K_0 \\ (-1)^\alpha K_0 & K_\infty \end{pmatrix},$$

and

$$\det \begin{pmatrix} K_\infty & (-1)^{\alpha+1}K_0 \\ (-1)^\alpha K_0 & K_\infty \end{pmatrix} = h_0 - h_\infty.$$

Therefore,

$$\dim\{\ \Psi|_{V_0} \mid \Psi \in \ker \xi\} = \begin{cases} 0 & \text{if } h_0 \neq h_\infty, \\ 2 & \text{if } h_0 = h_\infty \neq \frac{1}{24}c, \\ 4 & \text{if } h_0 = h_\infty = \frac{1}{24}c, \end{cases}$$

where V_0 is given in (27). Combining isomorphism (28), we obtain

$$\dim \ker \xi = 2\delta_{h_0,h_\infty} 2^{\delta_{h_0,\frac{1}{24}c}},$$

and we have proved Proposition 7.1. ∎

From now on, we will prove Theorem 3.3. First we remark that, in the minimal central charge case, Theorem 3.3 holds in the following strong form:

Theorem 7.2. *Let $\epsilon = \frac{1}{2}, 0$. Suppose that c satisfies the condition in Case 2. Then we have*

$$H_0(\mathcal{L}(E_0), L_\epsilon(c,h_0)(0) \otimes L_{\frac{1}{2}}(c,0)(1) \otimes L_\epsilon(c,h_\infty)(\infty)) \neq 0$$

if and only if $h_0 = h_\infty$ and $h_0 \in \Lambda^c_\epsilon$.

Proof. For $\epsilon = \frac{1}{2}$, this theorem is a corollary of Theorem 5.5. Now let us prove for $\epsilon = 0$. To do so, it is enough to show that

$$\begin{aligned} &H_0(\mathcal{L}(E_0), M_0(c,h_0)(0) \otimes L_{\frac{1}{2}}(c,0)(1) \otimes M'_0(c,h_\infty)(\infty)) \neq 0 \\ &\Leftrightarrow h_0 = h_\infty \quad \text{and} \quad h_0 \in \Lambda^c_0. \end{aligned} \tag{31}$$

Indeed, by an argument similar to Lemma 4.5 in [13], we see that (31) implies

$$H_0(\mathcal{L}(E_0), L_0(c,h_0)(0) \otimes L_{\frac{1}{2}}(c,0)(1) \otimes L_0(c,h_\infty)(\infty)) \neq 0.$$

For the proof, we use Shapiro's lemma in the following way: Set

$$\bar{\mathfrak{a}}'_0 = \mathfrak{a}_0 \oplus \mathbb{C}\ell^{(0)}_0 \oplus \mathbb{C}\ell^{(\infty)}_0 \oplus \mathbb{C}g^{(\infty)}_0.$$

Let ${W'}^0_{c,h_0} = \mathbb{C}\mathbf{1}^0_{c,h_0}$ be the 1-dimensional representation of $\bar{\mathfrak{a}}'_0$ given by

$$\begin{aligned} &x.\mathbf{1}^0_{c,h_0} = 0 \quad \text{if } x \in \mathfrak{a}_0, \\ &l^{(w')}_0.\mathbf{1}^0_{c,h_0} = \delta_{w',0}h_0\mathbf{1}^0_{c,h_0} \quad \text{for} \quad w' = 0, \infty, \\ &g^{(\infty)}_0.\mathbf{1}^0_{c,h_0} = 0, \\ &\deg \mathbf{1}^0_{c,h_0} = 0 \end{aligned}$$

and let W^{∞}_{c,h_∞} be the 2-dimensional representation of $\bar{\mathfrak{a}}'_0$ defined in (20). For $\sigma = \pm$, we set $\tilde{\mathbf{1}}^{\infty}_{c,h_\infty:\sigma} = \frac{1}{2}\{\mathbf{1}^{\infty}_{c,h_\infty:0} + \sigma \mathbf{1}^{\infty}_{c,h_\infty:1}\}$. By Shapiro's lemma and Poincaré duality, we have

$$\begin{aligned}&\dim H_0(\mathcal{L}(E_0), M_0(c,h_0)(0) \otimes L_{\frac{1}{2}}(c,0)(1) \otimes M'_0(c,h_\infty)(\infty))\\ &\quad = \dim H^0(\bar{\mathfrak{a}}'_0, ({W'}^0_{c,h_0} \otimes L_{\frac{1}{2}}(c,0)(1) \otimes W^{\infty}_{c,h_\infty})^*).\end{aligned}$$

From the following part of the BGG type resolution for $L_{\frac{1}{2}}(c,0)(1)$:

$$N_{\frac{1}{2}}(c,0)(1) \to M_{\frac{1}{2}}(c,0)(1) \to L_{\frac{1}{2}}(c,0)(1) \to 0,$$

we obtain

$$\begin{array}{rcl} H^0(\bar{\mathfrak{a}}'_0, ({W'}^0_{c,h_0} \otimes N_{\frac{1}{2}}(c,0)(1) \otimes W^{\infty}_{c,h_\infty})^*) & \overset{\xi'}{\leftarrow} & \\ H^0(\bar{\mathfrak{a}}'_0, ({W'}^0_{c,h_0} \otimes M_{\frac{1}{2}}(c,0)(1) \otimes W^{\infty}_{c,h_\infty})^*) & \leftarrow & \\ H^0(\bar{\mathfrak{a}}'_0, ({W'}^0_{c,h_0} \otimes L_{\frac{1}{2}}(c,0)(1) \otimes W^{\infty}_{c,h_\infty})^*) & \leftarrow & 0. \end{array}$$

Then we have

$$H^0(\bar{\mathfrak{a}}'_0, ({W'}^0_{c,h_0} \otimes L_{\frac{1}{2}}(c,0)(1) \otimes W^{\infty}_{c,h_\infty})^*) \simeq \ker \xi'.$$

Next we determine when $\Phi \in H^0(\bar{\mathfrak{a}}'_0, ({W'}^0_{c,h_0} \otimes M_{\frac{1}{2}}(c,0)(1) \otimes W^{\infty}_{c,h_\infty})^*)$ vanishes on the singular vectors of $M_{\frac{1}{2}}(c,0)(1)$. For $\sigma = \pm$ we set

$$\Phi^\sigma(v_1) = \Phi(\mathbf{1}^0_{c,h_0} \otimes v_1 \otimes \tilde{\mathbf{1}}^{\infty}_{c,h_\infty:\sigma}) \ \text{ for } v_1 \in M_{\frac{1}{2}}(c,0)(1)$$

and

$$\varphi^\sigma_j = \Phi^\sigma|_{M_{\frac{1}{2}}(c,0)(1)_j} \ \text{ for } j \in \frac{1}{2}\mathbb{Z}_{\geq 0}.$$

Since the maximal proper submodule $N_{\frac{1}{2}}(c,0)(1)$ of $M_{\frac{1}{2}}(c,0)(1)$ is generated by the singular vectors

$$S^{(1)}_{q-1,p-1}|c,0\rangle^{(1)}_{\frac{1}{2}} \text{ and } G^{(1)}_{-\frac{1}{2}}|c,0\rangle^{(1)}_{\frac{1}{2}},$$

we see that

$$\Phi^\sigma \in \ker \xi' \ \Leftrightarrow\ a(S^{(1)}_{q-1,p-1}).\varphi^\sigma_{\frac{1}{2}(q-1)(p-1)} = 0 \text{ and } a(G^{(1)}_{-\frac{1}{2}}).\varphi^\sigma_{\frac{1}{2}} = 0.$$

Therefore, we have to know how the above singular vectors act on the functionals φ^σ_j. We need the following two lemmata: The first one is

Lemma 7.3. *For $j \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, $m \in \mathbb{Z}_{>0}$ and $n \in \mathbb{Z}_{\geq 0}$,*

$$\begin{aligned}
&(i) \quad C.\varphi^\sigma_j = -c\varphi^\sigma_j,\\
&(ii) \quad L^{(1)}_{-m}.\varphi^\sigma_j = (-1)^{m-1}(m(h_0-1) + h_1 - h_\infty + j)\varphi^\sigma_{j-m},\\
&\qquad L^{(1)}_0.\varphi^\sigma_j = -(h_1 + j)\varphi^\sigma_j\\
&(iii) \quad G^{(1)}_{-n-\frac{3}{2}}.\varphi^\sigma_{n+j+\frac{3}{2}} = \left(-\frac{1}{4}\right)^n (2n+1)\begin{pmatrix} 2n \\ n \end{pmatrix}\\
&\qquad\qquad \times \left\{ \sigma K_\infty \varphi^\sigma_j - \frac{2n+3}{3}\sum_{k=1}^{[j+\frac{3}{2}]} \begin{pmatrix} \frac{3}{2} \\ k \end{pmatrix} \frac{n}{n+k} G_{k-\frac{3}{2}}.\varphi^\sigma_{j-k+\frac{3}{2}} \right\},
\end{aligned}$$

where $K_\infty = \sqrt{-1}(h_\infty - \frac{1}{24}c)^{\frac{1}{2}}$.

To state the second lemma that is a key of our proof, we assume that $c = c(t)$ for $t \in \mathbb{R}_{>0}$ (not necessarily rational) and $h_1 = h^{\frac{1}{2}}_{r,s}(t)$ for some $r, s \in \mathbb{Z}_{>0}$ such that $r - s \in 2\mathbb{Z}$. For a singular vector $S^{(1)}_{r,s}|c, h_1\rangle^{(1)}_{\frac{1}{2}}$ of level $N = \frac{1}{2}rs$, we define a 2×2-matrix $M^{\sigma}_{r,s}(h_0, h_\infty; t)$ by

$$a(S^{(1)}_{r,s}) : \{\varphi^{\sigma}_N, G^{(1)}_{-\frac{1}{2}}.\varphi^{\sigma}_{N+\frac{1}{2}}\} \mapsto \{\varphi^{\sigma}_0, G^{(1)}_{-\frac{1}{2}}.\varphi^{\sigma}_{\frac{1}{2}}\} M^{\sigma}_{r,s}(h_0, h_\infty; t),$$

where a denotes the antipode of $\hat{\mathcal{L}}^{<}_{1,\frac{1}{2}}$.

Lemma 7.4. *If $h = h_0 = h_\infty$, then we have*

$$M^{\sigma}_{r,s}(h, h; t) = \begin{pmatrix} X^{(0)}_{r,s} Y^{(1)}_{r,s} & -2\tau X^{(0)}_{r,s} Y^{(1)}_{r,s} \\ \frac{1}{2}\sigma K^{-1}_{\infty}(X^{(1)}_{r,s} Y^{(0)}_{r,s} - X^{(0)}_{r,s} Y^{(1)}_{r,s}) & X^{(1-\tau)}_{r,s} Y^{(\tau)}_{r,s} \end{pmatrix} \begin{pmatrix} (-1)^{\frac{1}{2}(r-s)\tau} & 0 \\ 0 & 1 \end{pmatrix},$$

where $\tau = \tau_{r,s}$ such that $r \equiv s \equiv \tau_{r,s} \pmod 2$ and $X^{(\gamma)}_{r,s}$, $Y^{(\gamma)}_{r,s}$ are given by

$$X^{(\gamma)}_{r,s} = \prod_{(k,l) \in \tilde{\mathbf{E}}^{\gamma}_{r,s}} 2^{-3}\{(r - 1 - 2k)t^{\frac{1}{2}} - (s - 1 - 2l)t^{-\frac{1}{2}}\},$$

$$Y^{(\gamma)}_{r,s} = \prod_{(k,l) \in \tilde{\mathbf{E}}^{\gamma}_{r,s}} 4\left\{(h_{\frac{1}{2}(r-1-2k), \frac{1}{2}(s-1-2l)}(t) - \frac{1}{24}c(t)) - (h - \frac{1}{24}c(t))\right\}.$$

and

$$\tilde{\mathbf{E}}^{\gamma}_{r,s} = \left\{ (k, l) \in \mathbb{Z}^2 \;\middle|\; \begin{array}{l} 0 \le k \le r-1 \;\wedge\; 0 \le l \le s-1 \\ r - 1 - 2k > 0 \;\vee\; (r - 1 - 2k = 0 \wedge s - 1 - 2l \ge 0) \\ k + l \equiv \gamma \pmod 2 \end{array} \right\}.$$

Proof. To show this lemma, we notice the following fact:

Fact 7.1. *For any $x \in U(\hat{\mathcal{L}}^{<}_{1:\frac{1}{2}})$, $x.\varphi^{\sigma}_n$ can be written as follows:*

$$x.\varphi^{\sigma}_n = \sum_{y,\alpha,m} c^{\sigma}_{(y,\alpha,m),(x,n)} \; \{y \; (G_{-\frac{1}{2}})^{\alpha}\}.\varphi^{\sigma}_m,$$

where $y \in U(\hat{\mathcal{L}}^{\geq}_{1:\frac{1}{2}})$, $\alpha \in \{0, 1\}$, $m \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ and $c^{\sigma}_{(y,\alpha,m),(x,n)} \in \mathbb{C}$.

Using this fact and the embedding diagrams of Verma modules, we can prove this lemma by the same argument as in the proof of Lemma 5.1. ■

Remark 7.5. For the number $c = c(\frac{p}{q})$ which satisfies the condition in Case 2', $X^{(0)}_{q-1,p-1}|_{t=\frac{p}{q}} \neq 0$ and $Y^{(1)}_{q-1,p-1}|_{t=\frac{p}{q}} = 0$ if and only if $h \in \Lambda^{c}_{0}$.

We return to the proof of Theorem 7.2. Combining Proposition 7.1, Lemma 7.4 and Remark 7.5, we obtain (31). Thus we complete the proof of Theorem 7.2. ■

At the end of this paper, we comment on the following proposition related with the rationality of the minimal models for the $N = 1$ super Virasoro algebras.

Proposition 7.6. *Let* $\epsilon = \frac{1}{2}, 0$. *Suppose that* $c = c(\frac{p}{q})$ *for some* $p, q \in \mathbb{Z}_{>0}$ *such that* $p, q \geq 2$, $p - q \in 2\mathbb{Z}$ *and* $(\frac{1}{2}(p-q), q) = 1$. *For* $h, h' \in \Lambda_\epsilon^c$ *we have*

$$\mathrm{Ext}^j_{(Vir_\epsilon, Vir^0_\epsilon)}(L_\epsilon(c,h), L_\epsilon(c,h')) = \begin{cases} \mathbb{C} & \text{if } j = 0 \text{ and } h = h', \\ \mathbb{C}^2 & \text{if } j \in 2\mathbb{Z}_{>0} \text{ and } h = h', \\ 0 & \text{otherwise} . \end{cases}$$

In particular, $\mathrm{Ext}^1_{(Vir_\epsilon, Vir^0_\epsilon)}(L_\epsilon(c,h), L_\epsilon(c,h')) = 0$ *for any* $h, h' \in \Lambda_\epsilon^c$.

Proof. Combining Theorem 4.5 in [4] and Theorem 2 in [20], we obtain this theorem by a standard argument, since we already have BGG type resolutions for the minimal series representations over the $N = 1$ super Virasoro algebras. ■

Finally, let us remark on a relation between our fusion algebras and the super vertex operator algebra associated with $L_{\frac{1}{2}}(c(\frac{p}{q}), 0)$.

Remark 7.7. 1. By [16], $L_{\frac{1}{2}}(c(\frac{p}{q}), 0)$ becomes a vertex operator superalgebra and the set of the irreducible representations of the SVOA $L_{\frac{1}{2}}(c(\frac{p}{q}), 0)$ is given by

$$\left\{ L_{\frac{1}{2}}(c(\frac{p}{q}), h) \,\middle|\, h \in \Lambda_{\frac{1}{2}}^{c(\frac{p}{q})} \right\}.$$

The fusion rule of this SVOA was computed in [1], and it gives the same result as in Theorem 3.5.

2. The rationality of the SVOA $L_{\frac{1}{2}}(c(\frac{p}{q}), 0)$ was conjectured in [16] and proved in [1]. From Theorem 3.3 and Proposition 7.6 for $\epsilon = \frac{1}{2}$, we obtain another proof of the rationality of the SVOA $L_{\frac{1}{2}}(c(\frac{p}{q}), 0)$.

References

[1] Adamović D., *Rationality of Neveu-Schwarz vertex operator superalgebras.*, Int. Math. Res. Not. **17** (1997), 865–874.

[2] Astashkevich A., *On the structure of Verma modules over Virasoro and Neveu-Schwarz algebras*, Comm. Math. Phys. **186** (1997), 531–562.

[3] Awata H., and Y. Yamada, *Fusion Rules for the fractional level* $\widehat{\mathfrak{sl}(2)}$ *algebra*, Mod. Phys. Lett. A **7** (1992), 1185–1195.

[4] Deodhar V. V., D. Gabber, and V. Kac, *Structure of some categories of representations of infinite dimensional Lie algebras*, Adv. Math. **45** (1982), 92–116.

[5] Eholzer W., and R. Hübel, *Fusion Algebras of Fermionic Rational Conformal Field Theories via a Generalized Verlinde Formula*, Nucl. Phys. B **414** (1994), 348-378.

[6] Feigin B. L., and D. B. Fuchs, *Representations of the Virasoro algebra*, Gordon and Breach Science Publ. New York, Adv. Stud. Contemp. Math. **7** (1990), 465–554.

[7] —, *Cohomology of some nilpotent subalgebras of the Virasoro and Kac-Moody Lie algebras*, J. Geom. Phys. **5** (1988), 209–235.

[8] Feigin, B. L., and E. Frenkel, *Quantization of the Drinfel'd-Sokolov reduction*, Phys. Lett. B **246** (1990), 75–81.

[9] Feigin B., and F. Malikov, *Fusion algebra at a rational level and cohomology of nilpotent subalgebras of* $\hat{sl}_2$, Lett. Math. Phys. **31** (1994), 315–325.

[10] —, *Modular functor and representation theory of* $\hat{sl}_2$ *at a rational level, Operads: Proceedings of Renaissance Conference,* 357–405, Contemp. Math. **202**, Amer. Math. Soc. Providence, RI, (1997).

[11] Frenkel E., V. Kac, and M. Wakimoto, *Characters and fusion rules for* W*-algebras via quantized Drinfel'd-Sokolov reduction*, Comm. Math. Phys. **147** (1992), 295–328.

[12] Iohara K. and Y. Koga, *Résolutions de type Bernstein-Gel'fand-Gel'fand pour les super-algèbres de Virasoro N=1*, C. R. Acad. Sci. Paris, t. **328** Série I (1999), 381–384.

[13] —, *Fusion algebras for N=1 superconformal field theories through coinvariants* I*:*$\hat{osp}(1|2)$*-symmetry*, J. Reine Angew. Math., to appear.

[14] Kimura K., *Hamiltonian reduction of super* $OSP(1,2)$ *and* $SL(2,1)$ *Kac-Moody algebras*, Internat. J. Mod. Phys. A, **7**, Suppl. 1B (1992), 533–544.

[15] Kac V. G., and M. Wakimoto, *Modular invariant representations of infinite dimensional Lie algebras and superalgebras*, Proc. Natl. Acad. Soc., **35** (1988), 4956–4960.

[16] Kac V. G., and W. Wang, *Vertex operator superalgebras and their representations*, *Mathematical aspects of conformal and topological field theories and quantum groups*, 161–191, Contemp. Math. **175**, Amer. Math. Soc. Providence, RI, (1994).

[17] Matsuo Y., and S. Yahikozawa, *Superconformal field theory with modular invariance on a torus*, Phys. Lett. B **178** (1986), 211-218.

[18] Meurman A., and A. Rocha-Caridi, *Highest weight representation of the Neveu-Schwarz and Ramond algebras*, Comm. Math. Phys. **107** (1986), 263–294.

[19] Minces P., M. A. Namazie, and C. Núñez, *Fusion rules in* $N=1$ *superconformal minimal models*, Phys. Lett. B **422** (1998), 117-125.

[20] Rocha-Caridi A., and N. R. Wallach, *Projective modules over graded Lie algebras.* I, Math. Z. **180** (1982), 151–177.

[21] Verlinde E., *Fusion rules and modular transformations in* $2D$ *conformal field theory*, Nucl. Phys. B **300** (1988), 360-376.

Kenji Iohara
Department of Mathematics
Faculty of Science
Kobe University
Kobe 657-8501
Japan

Yoshiyuki Koga
Department of Mathematics
Faculty of Science
Osaka University
Osaka 560-0043
Japan

Received April 27, 1999
and in final form October 24, 2000