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Some Examples of Discrete Groups and Hyperbolic Orbifolds of Infinite Volume

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Abstract. We study the class of two-generator subgroups of $\mathrm{PSL}(2,\mathbf{C})$ with real parameters which was introduced recently by Gehring, Gilman, and Martin. We give criteria for discreteness of non-elementary and non-Fuchsian groups of this class that are generated by two hyperbolic elements. We construct all hyperbolic orbifolds uniformized by the discrete groups of such type. The orbifolds described are of infinite volume.

Introduction

Throughout this paper we will consider $\mathrm{PSL}(2,\mathbf{C})$ which acts as the full group of orientation preserving isometries of hyperbolic space $\mathbf{H}^3$. In the last years its 2-generator subgroups have been the subject of investigation from different points of view, see [1], [3], [4], [8], [9], [12]–[20], [26], [27] (this list is not complete). A series of papers is devoted to the following problem:

Problem. *When is* $\langle f,g\rangle < \mathrm{PSL}(2,\mathbf{C})$ *discrete?*

By now, only necessary or only sufficient conditions for the discreteness are presented in most of the papers. For the former we mention the remarkable Jørgensen inequality, the inequality of Shimizu–Leutbecher and their analogs, see eg. [15]–[16]. For the latter see [9] and references therein.

Given a discrete subgroup Γ of $\mathrm{PSL}(2,\mathbf{C})$, one can consider the orbit space $\mathbf{H}^3/\Gamma$, which is a hyperbolic orbifold. Therefore, we can regard different examples of 2-generator hyperbolic orbifolds as a kind of sufficient conditions for discreteness. Such examples were constructed, for instance, in [12]–[14].

Still there are no criteria (that is, necessary and sufficient conditions) to determine the discreteness of 2-generator groups. It is quite natural to solve this problem first for particular classes of subgroups of $\mathrm{PSL}(2,\mathbf{C})$. What are the classes completely described?

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The description of all elementary discrete groups is well known (see [2], [7], or [25]). If both generators keep invariant a hyperbolic plane in $\mathbf{H}^3$, then the problem was also completely solved: if the generators preserve orientation of that plane, see [7], [10], [11], and [28]; if at least one of the generators reverses orientation of the invariant plane, see [23]. From the geometric point of view, almost all 2-generator discrete groups in this paragraph (except some elementary groups) are, in a sense, 2-dimensional (all of them are isomorphic to discrete subgroups of $\mathbf{E}^2$, $\mathbf{S}^2$, or $\mathbf{H}^2$, in the last case they are not necessarily orientation-preserving).

In [8], Gehring, Gilman, and Martin suggest the investigation of the class of two-generator groups with real parameters (β, β', γ), where $\beta = \mathrm{tr}^2 f - 4$, $\beta' = \mathrm{tr}^2 g - 4$, and $\gamma = \mathrm{tr}[f,g] - 2$. In case when such a group is discrete they obtained necessary conditions on the parameters. In [20], a geometric characterization of real parameters is given, moreover, it is shown that the class contains many "truly spatial" groups (non-elementary groups without invariant plane). We study those spatial groups in series of papers [17]–[22].

The purpose of the present paper is to describe all non-Fuchsian groups with real parameters generated by two hyperbolic elements. Namely, we find necessary and sufficient conditions for their discreteness and construct the hyperbolic orbifolds uniformized by the discrete groups of such type. The orbifolds described are of infinite volume.

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1. Preliminaries

1.1. Hyperbolic geometry.

We recall some facts of hyperbolic geometry.

Two (hyperbolic) planes are either intersecting, or parallel, or disjoint. Two (hyperbolic) lines either lie in a plane (and then they are intersecting, parallel, or disjoint), or they are skew.

In the next section we will need the following three propositions.

Proposition 1.1. *Let ε and ρ be mutually orthogonal planes, and let a be the line by which they intersect. Let b be a straight line which lies in ρ and is orthogonal to a. Then b is orthogonal to ε.*

Proposition 1.2. *If a plane ε is orthogonal to a straight line b lying in another hyperbolic plane σ, then ε and σ are mutually orthogonal.*

Propositions 1.1 and 1.2 are simple and occur not only in hyperbolic geometry, but also in Euclidean and spherical ones.

Proposition 1.3. *Let σ_1 and σ_2 be disjoint planes in $\mathbf{H}^3$, and let ε be a plane which is orthogonal to both of them. Then ε passes through the common perpendicular to σ_1 and σ_2.*

Proof. Indeed, the lines of intersection of σ_1 and σ_2 with ε are disjoint and, consequently, have their common perpendicular (lying in ε). That perpendicular is orthogonal to both σ_1 and σ_2 by Proposition 1.1. The proof is complete. ∎

We recall also that the *angle between two skew lines* in $\mathbf{H}^3$ is defined to be the dihedral angle between two planes which pass through the common perpendicular to these lines and each of them separately. Two lines are mutually orthogonal if and only if there is a plane containing one of the lines and orthogonal to the other.

1.2. PSL$(2,\mathbf{C})$ **as the isometry group of hyperbolic 3-space.**

Consider 3-dimensional Lobachevsky space realized as the upper half-space

$$\mathbf{H}^3 = \{\ (z,t) \mid z \in \mathbf{C},\ t > 0\}$$

with the hyperbolic metric

$$ds^2 = (|dz|^2 + dt^2)/t^2.$$

As is known, PSL$(2,\mathbf{C})$ acts as the full group of orientation preserving isometries of hyperbolic space $\mathbf{H}^3$. A *Kleinian group* is any discrete subgroup of PSL$(2,\mathbf{C})$.

We need a classification of elements of PSL$(2,\mathbf{C})$. For $f \in \mathrm{PSL}(2,\mathbf{C})$, let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbf{C})$ be one of the two matrices which represent f. Then as usual $\mathrm{tr}^2 f \equiv (a+d)^2$. A non-trivial element f is *elliptic* (*parabolic*, *hyperbolic*, or *strictly loxodromic*) if and only if $\mathrm{tr}^2 f \in [0,4)$ ($\mathrm{tr}^2 f = 4$, $\mathrm{tr}^2 f \in (4,+\infty)$, or $\mathrm{tr}^2 f \notin [0,+\infty)$, respectively). Geometrically this is rotation (respectively, limit rotation, translation or rotation followed by translation with the same axis).

An elliptic, parabolic or hyperbolic element, in contrast to a strictly loxodromic one, can be presented as the composition of reflections in two hyperbolic planes. Those planes are either intersecting (the line of the intersection is called the axis of the elliptic element; this is the fixed point set under its action), or parallel (the common point of the planes, which lies on $\overline{\mathbf{C}} = \partial\mathbf{H}^3$, is the unique fixed point of the parabolic element), or disjoint, respectively (in the last case, the common perpendicular to the planes is the unique invariant line under the action of the hyperbolic element and is called the axis of this element).

An elliptic element is determined by its axis, the angle and the direction of rotation about this axis. An elliptic element of finite order is called *primitive* (or *geometrically primitive*), if it is conjugate, on $\overline{\mathbf{C}}$, to a rotation of the form $F(z) = \exp(2\pi i/p)z$, $p \in \mathbf{N}$, and non-primitive otherwise. A non-primitive elliptic element of order p is conjugate to a rotation $F(z) = \exp(2\pi iq/p)z$, where p, $q \in \mathbf{N}$ are relatively prime, $q > 1$ and $q/p < 1/2$.

Further information on Kleinian groups and hyperbolic geometry can be found in [2], [5], [6], [25], and [29].

We remind only the following

Proposition 1.4. *Let f and g be hyperbolic elements. Then $\mathrm{tr}[f,g] \in \mathbf{R}$ if and only if the axes of f and g either lie in one hyperbolic plane or they are mutually orthogonal skew lines.*

This is a special case of Theorem 1 in [20].

2. Main result

Suppose that both elements f and g of $\mathrm{PSL}(2,\mathbf{C})$ are hyperbolic and $\mathrm{tr}[f,g] = \mathrm{tr} fgf^{-1}g^{-1}$ is real.

By Proposition 1.4 the axes of f and g either lie in a hyperbolic plane (and then this plane is invariant under the action of $\Gamma = \langle f, g\rangle$; i.e., Γ is discrete if and only if it is Fuchsian), or those axes are mutually orthogonal skew lines. In the last case, necessary and sufficient conditions for Γ to be discrete are given by the following

Theorem 2.1. *Let $f, g \in \mathrm{PSL}(2,\mathbf{C})$ be hyperbolic elements with mutually orthogonal skew axes. Then*

(1) there exists a unique element $h \in \mathrm{PSL}(2,\mathbf{C})$ such that $h^2 = [f,g]$ and $(hg)^2 = 1$; and

(2) $\Gamma = \langle f, g\rangle$ is discrete if and only if h is hyperbolic, parabolic, or a primitive elliptic element.

Proof. To prove the theorem we will use the methods expounded in papers [17]–[19].

1. The first step is to construct a group of reflections Γ^* so that Γ is its subgroup of finite index. In this case, the question about discreteness of Γ is equivalent to the same question for Γ^*. Simultaneously we prove conclusion *(1)* of the theorem.

We denote an element and its axis by the same letter. So, f and g are mutually orthogonal skew lines. Let e be the half-turn about the common perpendicular to f and g. There exist two other half-turns e_f and e_g such that

$$f = e_f e, \qquad g = e_g e.$$

We have

$$[f,g] = (e_f e)(e_g e)(e e_f)(e e_g) = (f e_g)^2. \tag{1}$$

It is clear that the axes e_f and e lie in one hyperbolic plane, are disjoint, and f is their common perpendicular. An analogous statement is true for the three axes e_g, e, and g.

Let λ, μ, σ, τ be hyperbolic planes defined as follows: λ contains e_f and is orthogonal to f; μ contains e_g and is orthogonal to g; σ contains e_f, e, and f; and finally τ contains e_g, e, and g. Note that the pairs of planes λ and σ, σ and τ, τ and μ intersect at an angle of $\pi/2$, λ and τ are disjoint (f is their common perpendicular), μ and σ are also disjoint (with g as their perpendicular). The planes λ and μ are either disjoint, or parallel, or intersecting.

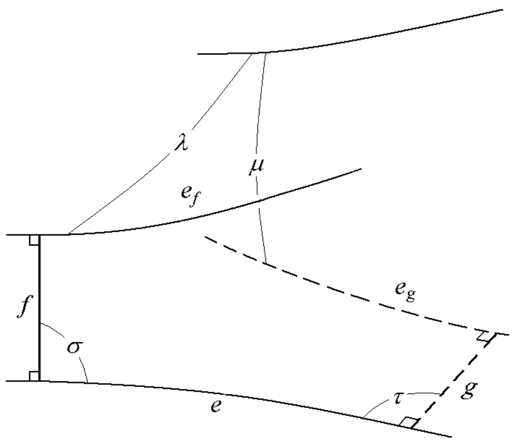

Figure 1:

Let $\mathcal{P}_0$ denote the convex polyhedron (of infinite volume) bounded by λ, μ, σ and τ with edges e_f, e, e_g, and possibly $\lambda \cap \mu$, if this intersection is not empty (see Figure 1[2]). If we denote by R_λ, R_μ, R_σ, R_τ the reflections in λ, μ, σ, τ respectively, then $f = R_\lambda R_\tau$, $e_g = R_\tau R_\mu$ and from (1) it follows that

$$[f, g] = (R_\lambda R_\mu)^2.$$

Denote by h the composition of reflections R_μ and R_λ,

$$h = R_\lambda R_\mu.$$

Then h is the very square root of $[f, g]$ we need. Indeed, the product $hg = (R_\lambda R_\mu)(R_\mu R_\sigma) = R_\lambda R_\sigma = e_f$ is a half-turn. (Note that the other orientation preserving square root h' of non-parabolic commutator $[f, g]$ does not satisfy the condition that $h'g$ is a half-turn. Namely, $h'g$ is hyperbolic in case that $[f, g]$ is elliptic, and $h'g$ is π-loxodromic when $[f, g]$ is hyperbolic.) We have proved conclusion *(1)* of the theorem.

Define

$$\widetilde{\Gamma} = \langle f, g, e \rangle \quad \text{and} \quad \Gamma^* = \langle f, g, e, R_\tau \rangle = \langle R_\lambda, R_\mu, R_\sigma, R_\tau \rangle.$$

Since $efe^{-1} = f^{-1}$ and $ege^{-1} = g^{-1}$, we have $\widetilde{\Gamma} = \Gamma \cup \Gamma e$. It is clear that either $\widetilde{\Gamma} = \Gamma$ (if $e \in \Gamma$), or Γ is an index 2 subgroup of $\widetilde{\Gamma}$. Moreover, $\widetilde{\Gamma}$ is the orientation preserving subgroup of Γ^* of index 2. Thus Γ, $\widetilde{\Gamma}$ and Γ^* are either all discrete or all non-discrete.

2. Assume that $h = R_\lambda R_\mu$ is hyperbolic, parabolic, or a primitive elliptic element of order p (that means that λ and μ are disjoint, parallel, or intersecting at an angle of π/p, respectively). Then $\mathcal{P}_0$, together with the reflections R_λ, R_μ, R_σ, R_τ in its faces, satisfies the hypotheses of Poincaré's polyhedron theorem, and so $\Gamma^* = \langle R_\lambda, R_\mu, R_\sigma, R_\tau \rangle$ is discrete. We have proved the "if" part of *(2)*.

3. Assume that $h = R_\lambda R_\mu$ is neither hyperbolic, nor parabolic, nor a primitive elliptic element. We have two possibilities: either h is an elliptic element of infinite

[2]All figures are done irrespective of any model of Lobachevsky space.

order; or h is a non-primitive elliptic element of finite order. In the first case, both Γ^* and Γ are not discrete. In the second case, we need further investigation. To complete the proof of the theorem it suffices to prove the following

Lemma 2.2. *Let $f, g \in \mathrm{PSL}(2, \mathbf{C})$ be hyperbolic elements with mutually orthogonal skew axes, and let h as above be a non-primitive elliptic element of finite order. Then Γ is not discrete.*

Proof. We will prove it by contradiction. Assume that Γ is discrete, and h is a non-primitive elliptic element, that is, the planes λ and μ intersect at an angle of $q(\pi/p)$, where p is the order of h, $(p,q)=1$, $1<q<p/2$. Then Γ^* (which is also discrete) contains reflections in $q-1$ planes which pass through h and decompose the corresponding dihedral angle of $\mathcal{P}_0$ into q smaller angles of π/p.

Now we need an additional construction.

The lines e_f and h lying in λ are disjoint since they lie in disjoint planes σ and μ. Denote by ε the plane which passes through the common perpendicular to e_f and h and is orthogonal to λ. Then ε is orthogonal to e_f and h (Proposition 1.1), and consequently, to both μ and σ (Proposition 1.2). Therefore ε passes through the common perpendicular to μ and σ, i.e., through g (Proposition 1.3). Analogous argument shows that there exists a plane η which is orthogonal to λ, μ, and τ and passes through f (Figure 2). Denote by $\mathcal{P}_c$ the compact polyhedron bounded by λ, μ, σ, τ, ε, and η.

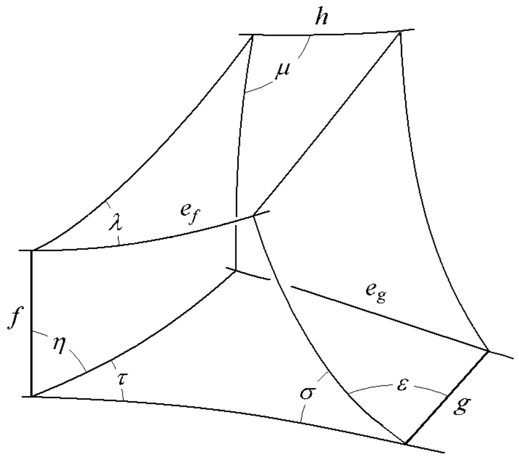

Figure 2:

Recall that Γ^* contains the reflections R_λ, R_μ, R_σ, and R_τ (and does not contain the reflections in ε and η); moreover, Γ^* contains reflections in $q-1$ planes $(q-1 \geq 1)$ passing through h and intersecting the interior of $\mathcal{P}_c$. We call these $q-1$ planes the *additional reflection planes (through h).*

Consider the planes ε and η. Set $\Gamma^*_\varepsilon = \langle R_\lambda, R_\mu, R_\sigma \rangle$, $\Gamma^*_\eta = \langle R_\lambda, R_\mu, R_\tau \rangle$. It is easy to see that Γ^*_ε keeps ε invariant, and Γ^*_η keeps η invariant (since λ, μ, and σ are orthogonal to ε; and λ, μ, and τ are orthogonal to η). The additional reflection planes through h are orthogonal to both ε and η; and the reflections in them belong to both Γ^*_ε and Γ^*_η.

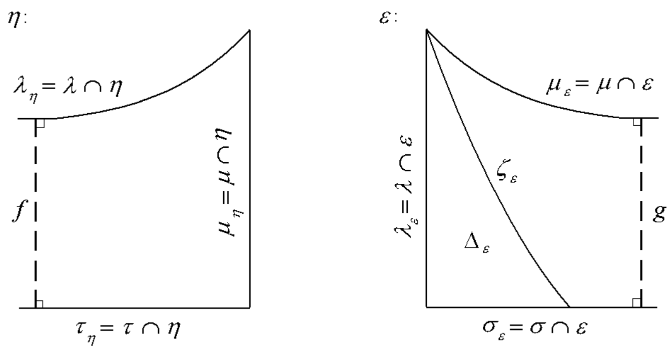

Figure 3:

Both Γ^*_ε and Γ^*_η are subgroups of Γ^* (which is discrete by assumption), hence they are also discrete.

For convenience denote $\lambda_\varepsilon = \lambda \cap \varepsilon$, $\mu_\varepsilon = \mu \cap \varepsilon$, $\sigma_\varepsilon = \sigma \cap \varepsilon$, $\lambda_\eta = \lambda \cap \eta$, $\mu_\eta = \mu \cap \eta$, $\tau_\eta = \tau \cap \eta$ (see Figure 3, where η and ε are shown).

We concentrate on ε, keeping in mind that analogous conclusions are valid also for η. Let ζ be the nearest to λ additional reflection plane through h. Then, ζ and σ are either disjoint, parallel, or intersecting. Denote by $\triangle_\varepsilon$ the triangle (compact or non-compact of finite or even infinite area) formed by λ_ε, σ_ε, and $\zeta_\varepsilon = \zeta \cap \varepsilon$. Using discreteness of Γ^*_ε and Knapp's list [24], we have only those possibilities for $\triangle_\varepsilon$ that are presented in Figure 4 (in case when $\triangle_\varepsilon$ is compact).

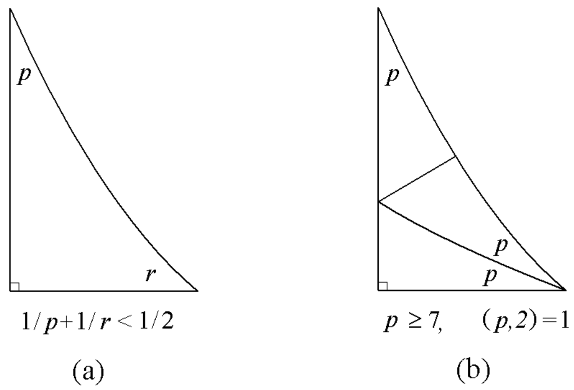

Figure 4:

A priori, for every triangle $\triangle_\varepsilon$ with fixed angles there are as many different decompositions of ε (by the reflection planes that belong to Γ^*_ε) as there are different q's that are relatively prime to p and satisfy the inequality $1 < q < p/2$. (We remind that $q\pi/p$ is the angle between λ and μ.)

We show below that in fact there is at most one decomposition for each admissible triangle $\triangle_\varepsilon$.

First of all, for each $\triangle_\varepsilon$ we select exactly one decomposition D of ε that satisfies the following condition:

(*C1*) Among all decompositions with the same $\triangle_\varepsilon$, D has the minimal number of additional reflection planes through h.

All possible types of decompositions satisfying (*C1*) are represented in Figure 5, where we suppose that case (i) contains subcases of non-compact triangle $\triangle_\varepsilon$ and of disjoint planes ζ and σ. Notice also that the minimal number of additional reflection planes through h in case (ii) equals two, because otherwise μ_ε intersects σ_ε.

We will denote the edges (and their lengths) and faces of the compact polyhedron $\mathcal{P}_c$ by $\overline{g} = g \cap \mathcal{P}_c$, $\overline{\lambda}_\varepsilon = \lambda_\varepsilon \cap \mathcal{P}_c$, $\overline{\tau}_\varepsilon = \tau_\varepsilon \cap \mathcal{P}_c$, ... It is easy to see that for every decomposition shown in Figure 5 we have:

(*C2*) $\overline{\sigma}_\varepsilon$ meets at most one additional reflection plane.

Thus, (*C2*) follows from (*C1*). It is not difficult to see that (*C2*) holds even if (*C1*) does not occur. Indeed, one can check it by adding (to the corresponding decomposition in Figure 5) one or several lines passing through the vertex $h \cap \varepsilon$.

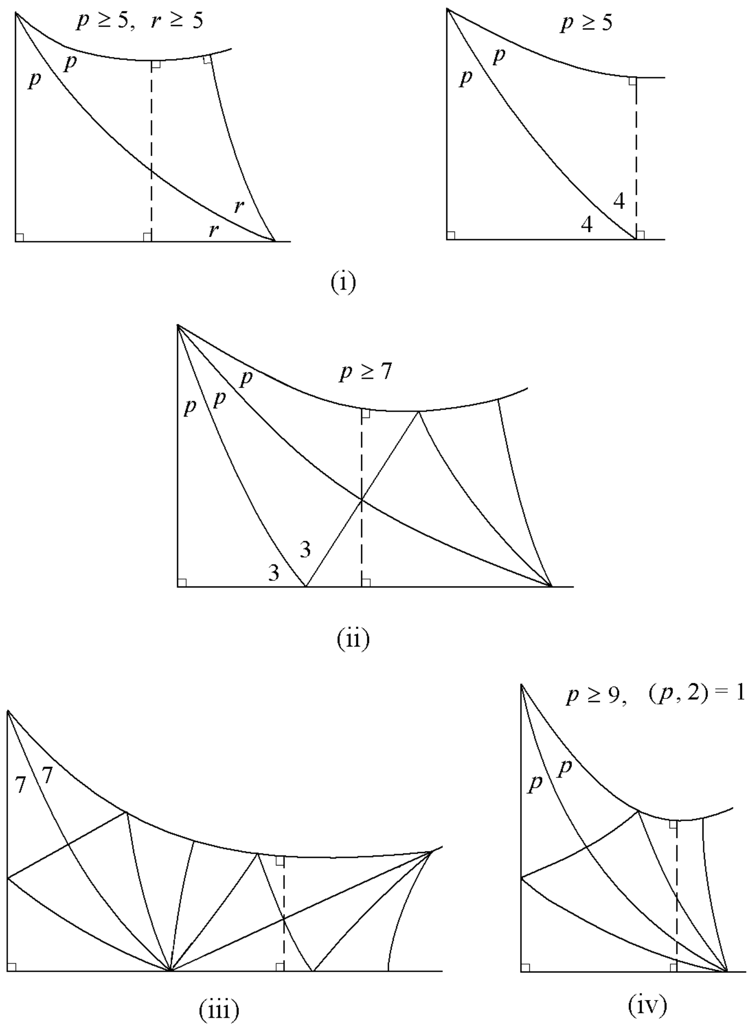

Figure 5:

The similar condition is always realized for η:

(*C2'*) $\overline{\tau}_\eta$ meets at most one additional reflection plane.

So, we have proved that the discreteness of Γ^* implies $(C2)$ and $(C2')$. Finally, we show that decompositions of ε (by the reflections of Γ^*_ε) other than in Figure 5 do not exist. Otherwise, at least two additional reflection planes intersect $\overline{g}$ and, consequently, also intersect $\overline{\tau}_\eta$, which contradicts condition $(C2')$.

It is clear that Figure 5 presents not only all decompositions of ε, but also those of η (by the reflection planes of Γ^*_η).

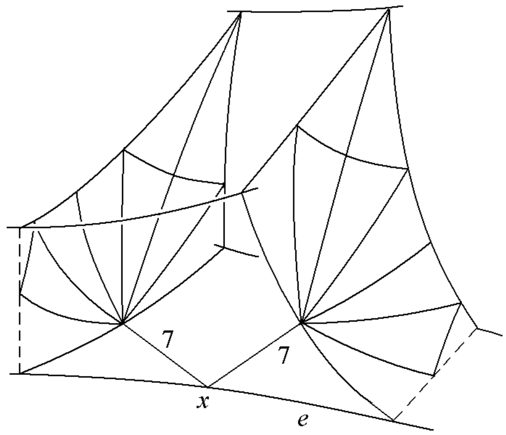

Figure 6:

Since the reflection planes passing through h are the same for ε and η, *a priori* there are the following three possibilities for the pair $\{\varepsilon,\ \eta\}$:

$\langle 1\rangle$ $\{(i),(iii)\}$ or $\{(iii),(i)\}$;

$\langle 2\rangle$ $\{(ii),(ii)\}$;

$\langle 3\rangle$ $\{(iii),(iii)\}$.

Note that (iv) is impossible (it could be combined only with (iii), but there $p=7$).

To complete the proof of Lemma 2.2 it remains to show that the situations $\langle 1\rangle$ – $\langle 3\rangle$ cannot occur.

In $\langle 3\rangle$, the additional reflection plane intersects e at a point, say x (Figure 6). We have two axes of elliptic elements of order 7 passing through x, what is impossible in a discrete group. (It is an easy corollary of the following fact. If a subgroup of a discrete isometry group of $\mathbf{H}^3$ keeps some sphere invariant, then this subgroup is a discrete group of isometries of 2-dimensional sphere. For our subgroup generated by the two elliptic elements of order 7 any sphere centered at x is invariant.)

In hyperbolic geometry a triangle is determined by its angles, therefore, in case $\langle 2\rangle$, $\overline{\lambda}_\varepsilon = \overline{\mu}_\eta$. Hence $\overline{\mu}_\varepsilon = \overline{\lambda}_\eta$ (see Figure 7). However, on the one hand, $\overline{\lambda}_\varepsilon > \overline{\mu}_\varepsilon$ (it is easy to see from Figure 5(ii) for the plane ε). Moreover, λ_ε is the common perpendicular to h and e_f, and, consequently, $\overline{\lambda}_\varepsilon < \overline{\lambda}_\eta$. We have a contradiction.

In case $\langle 1\rangle$ we can restrict ourselves to handling the pair $\{(iii),(i)\}$. Denote by ζ_σ the line of intersection of the additional reflection plane with σ. First of all, ζ_σ cannot intersect e (otherwise, there are two intersecting axes, ζ_σ of order 7 and e of order 2, in the discrete group what is possible only if the angle of their

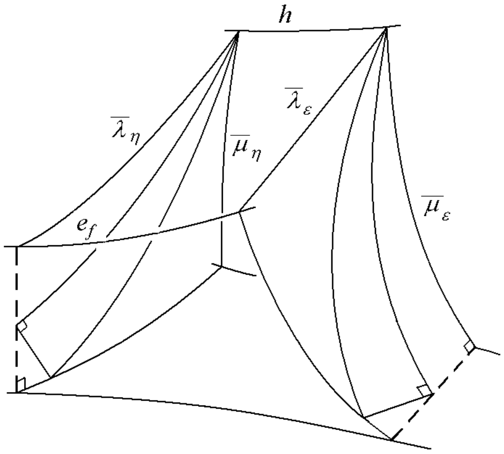

Figure 7:

intersection is equal to $\pi/2$, but then in σ we have a triangle with two right angles). The lines ζ_σ and e cannot be parallel, because none of the discrete groups in Euclidean plane contains elliptic element of order 7. The last possibility is that ζ_σ and e are disjoint. In this case, in the plane orthogonal to both ζ_σ and e we have a triangle with angles of $\pi/2$, $\pi/7$, φ. This triangle is an orthogonal section of an infinite triangle prism formed by reflection planes for which ζ_σ and e are lateral edges. Thus, this triangle have to be smaller than any other section of the prism. On the other hand, from discreteness of Γ^*, we conclude that $\varphi = \pi/n$ $(n \geq 3)$ or $\varphi = 2\pi/7$. However, all such triangles are larger than the one hatched on Figure 8 (the hatched triangle is a part of a triangle with angles $\pi/2$, $\pi/7$, $\pi/3$) and, hence, cannot be the orthogonal section of the prism. We arrived at a contradiction. The proof of Lemma 2.2 (and Theorem 2.1) is complete. ■

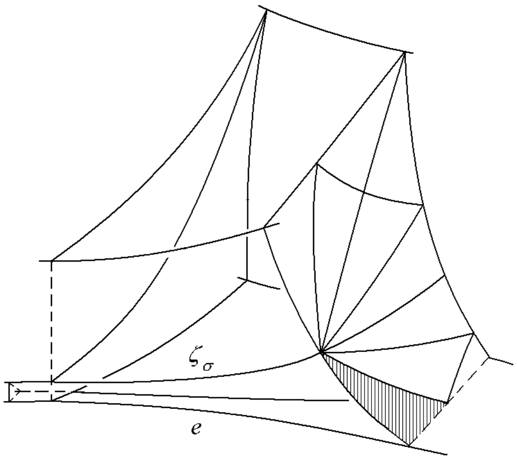

Figure 8:

Remark 2.3. It is easy to reformulate the above result in terms of parameters (β, β', γ), see [20], Table 2.

Remark 2.4. Surprisingly, there are no discrete groups with two hyperbolic generators for which the above mentioned square root h of their commutator is a non-primitive elliptic element (compare this, for example, with the case of two elliptic generators).

3. Two-generator hyperbolic orbifolds of infinite volume

In this section we construct hyperbolic orbifolds that correspond to all the discrete groups described in Theorem 2.1.

Note that if h is a primitive elliptic element of order p, where p is odd, then there is an integer k such that $[f,g]^k = h$. Hence the elements h, $e_f = hg$, and $e = e_f f$ belong to Γ. We have $\widetilde{\Gamma} = \langle f, g, e\rangle = \Gamma$. To construct a fundamental polyhedron for $\widetilde{\Gamma} = \Gamma$ we double $\mathcal{P}_0$ along its face σ and denote by $\mathcal{P}_1$ the doubled polyhedron. It is easy to see that $\mathcal{P}_1$ and the elements e, e_f, and g satisfy the hypotheses of Poincaré's Polyhedron Theorem [5]. We obtain that Γ has the following presentation:

$$\Gamma = \langle e,\ e_f,\ g \mid e^2 = e_f^2 = (e_f g)^p = (ge)^2 = 1\rangle. \qquad (a)$$

To obtain the orbifold with boundary $(\mathbf{H}^3 \cup \Omega(\Gamma))/\Gamma$ (whose interior $\mathbf{H}^3/\Gamma$ is a hyperbolic orbifold and $\Omega(\Gamma)$ is the discontinuity set of Γ), we identify the equivalent faces of $\mathcal{P}_1$. Note that all our orbifolds are embedded into $\mathbf{S}^3$, and we draw them so that ∞ is a regular inner point of the orbifold (see Figure 9 (a)).

In the remaining cases, when h is hyperbolic, parabolic or a primitive elliptic element of even order p, for the construction of the fundamental polyhedron for Γ we double $\mathcal{P}_1$ (the fundamental polyhedron for $\widetilde{\Gamma}$) along τ to obtain the polyhedron $\mathcal{P}_2$ which together with f and g satisfies the hypotheses of Poincaré's Theorem.

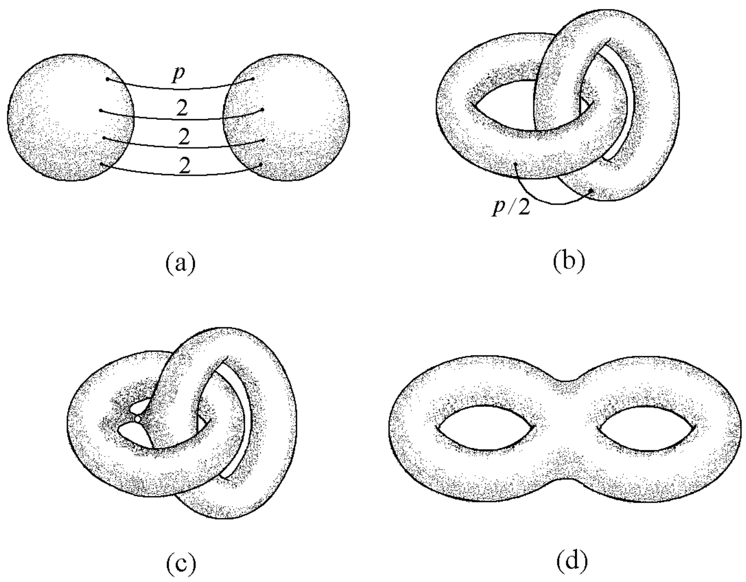

Figure 9:

We see that $\Gamma \neq \widetilde{\Gamma}$ and (as a Kleinian group) has one of the following presentations:

$$\Gamma = \langle f,\ g \ \mid\ (fgf^{-1}g^{-1})^{p/2} = 1\rangle, \tag{b}$$

if $[f,g] = h^2$ is elliptic;

$$\Gamma = \langle f,\ g \ \mid\ (fgf^{-1}g^{-1})^{\infty} = 1\rangle, \tag{c}$$

if $[f,g] = h^2$ is parabolic; and

$$\Gamma = \langle f,\ g \ \mid\ \rangle, \tag{d}$$

if $[f,g] = h^2$ is hyperbolic. For the corresponding orbifolds see Figure 9 (b, c, d, respectively).

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