Journal of Lie Theory
Volume **12** (2002) 289–299


# A Classification of Reductive Linear Groups with Spherical Orbits

**Ivan Arzhantsev**[*]

Communicated by E. B. Vinberg

**Abstract.** We classify finite dimensional $G$-modules $V$ of an algebraic reductive group $G$ such that any $G$-orbit in $V$ is spherical. It is shown that any module with this property can be realized as a spherical module after an extension of the group by a central torus.
*Key words:* reductive groups, spherical modules, algebras of invariants.
*AMS 2000 Subject Classification:* primary 20G05, 17B10; secondary 14M17, 14R20.

## 1. Introduction

Let $G$ be a connected reductive algebraic group over an algebraically closed field $K$ of zero characteristic, and $G^s$ be the maximal connected semisimple subgroup of $G$. Let us recall that an irreducible $G$-variety $Y$ is called *spherical* if the induced action $B : Y$ of a Borel subgroup $B \subseteq G$ has an open orbit. By Rosenlicht's theorem, an action $G : Y$ is spherical iff $K(Y)^B = K$, where $K(Y)^B$ is the field of rational $B$-invariants. An algebraic subgroup $H \subset G$ is said to be *spherical* if the homogeneous space $G/H$ is a spherical $G$-variety.

**Definition 1.1.** Let $X$ be an irreducible algebraic variety. We shall say that an action $G : X$ is *an action with spherical orbits* if there exists an open subset $X_0 \subset X$ such that for any $x \in X_0$ the orbit $Gx$ is spherical.

Below we list some basic facts about actions with spherical orbits.

(1) Any trivial $G$-action is an action with spherical orbits.

(2) Suppose that for an action $G : X$ a generic isotropy subgroup exists, see [9, sec. 7.3]. (This is always the case for linear actions.) Denote this subgroup by $H$. The action $G : X$ is an action with spherical orbits iff $H$ is a spherical subgroup of $G$.

[*] The work was supported by INTAS-OPEN-97-1570, CRDF RM1-2088 and RFBR-98-01-00598.

ISSN 0949–5932 / \$2.50 

(3) Rosenlicht's theorem implies that an action $G : X$ is an action with spherical orbits iff $K(X)^G = K(X)^B$.

(4) It is shown in [1, Corollary 1] that for an action with spherical orbits any $G$-orbit is spherical.

(5) Let $G_1 : X_1$ and $G_2 : X_2$ be actions with spherical orbits. Then the action $(G_1 \times G_2) : (X_1 \times X_2)$ is an action with spherical orbits.

In this paper we classify $G$-modules with spherical orbits. Let $V$ be a finite dimensional $G$-module and $V = V_1 \oplus \ldots \oplus V_k$ be a decomposition into simple $G$-submodules. Denote by $T$ the $k$-dimensional algebraic torus acting by dilatations on any $V_i$. Then $\overline{G} = TG$ is a linear group acting on $V$. It is shown (Proposition 3.3) that the $\overline{G}$-module $V$ is spherical. This result and a classification of spherical modules from [8] allow us to list all "minimal" indecomposable linear actions with spherical orbits (Tables 1-3). In Theorem 6.2 a characterization of arbitrary $G$-modules with spherical orbits is obtained. It is proved that for any $G$-module with spherical orbits the algebras of $U$- and $G^s$-invariants are free. (Here $U$ is a maximal unipotent subgroup of $G$.) Finally we get a classification of $G$-actions with spherical orbits on projective spaces $\mathbb{P}(V)$.

The author is grateful to E. B. Vinberg, A. G. Khovanskii and K. Kaveh for stimulating questions and discussions.

## 2. Indecomposable modules

**Definition 2.1.** A $G$-module $V$ is said to be *a module with spherical orbits* if the linear action $G : V$ is an action with spherical orbits.

**Definition 2.2.** A $G$-module $V$ is *indecomposable* if there exist no proper decompositions $G^s = G_1^s \times G_2^s$ and $V = V_1 \oplus V_2$ such that $(g_1, g_2)(v_1, v_2) = (g_1 v_1, g_2 v_2)$ for any $g = (g_1, g_2) \in G^s$ and any $v = (v_1, v_2) \in V$.

**Definition 2.3.** We say that a $G'$-module $V$ is obtained from a $G$-module $V$ *by a torus extension* if there exists a torus $T$ acting on $V$ such that $T$- and $G$-actions commute and $G' = TG$.

It is clear that any $G$-module $V$ is obtained by a torus extension from the $G^s$-module $V$.

**Lemma 2.4.** *Suppose that $V$ is a $G$-module with spherical orbits and a $G'$-module $V$ is obtained from this module by a torus extension. Then $V$ is a $G'$-module with spherical orbits.*

**Proof.** Let $H$ be the generic isotropy subgroup for the action $G : V$. By assumption, $H$ is spherical in $G$. Then any subgroup of $G'$ containing $H$ is spherical in $G'$. Hence a generic isotropy subgroup for the $G'$-module $V$ is spherical. ∎

**Theorem 2.5.** *All indecomposable $G$-modules with spherical orbits are either indicated in Tables 1-3 or are obtained from the indicated modules by a torus extension.*

**Table 1**

| | $G$ | weights | $\dim V$ | $\mathcal{H}$ | codim |
|---|---|---|---|---|---|
| 0 | $\{e\}$ | 0 | 1 | 0 | 1 |
| 1 | $SL(n)$ | $\phi_1$ | $n$ | $A_{n-2}+R_{n-1}$ | 0 |
| 2 | $\Lambda^2 SL(2n)$ | $\phi_2$ | $2n^2-n$ | $C_n$ | 1 |
| 3 | $\Lambda^2 SL(2n+1)$ | $\phi_2$ | $2n^2+n$ | $C_n+R_{2n}$ | 0 |
| 4 | $S^2 SL(2n)$ | $2\phi_1$ | $2n^2+n$ | $D_n$ | 1 |
| 5 | $S^2 SL(2n+1)$ | $2\phi_1$ | $2n^2+3n+1$ | $B_n$ | 1 |
| 6 | $SO(2n)$ | $\phi_1$ | $2n$ | $B_{n-1}$ | 1 |
| 7 | $SO(2n+1)$ | $\phi_1$ | $2n+1$ | $D_n$ | 1 |
| 8 | $Spin(7)$ | $\phi_3$ | 8 | $G_2$ | 1 |
| 9 | $Spin(9)$ | $\phi_4$ | 16 | $B_3$ | 1 |
| 10 | $Spin(10)$ | $\phi_4$ | 16 | $B_3+R_8$ | 0 |
| 11 | $Sp(2n)$ | $\phi_1$ | $2n$ | $C_{n-1}+R_{2n-1}$ | 0 |
| 12 | $G_2$ | $\phi_1$ | 7 | $A_2$ | 1 |
| 13 | $E_6$ | $\phi_1$ | 27 | $F_4$ | 1 |

**Comments to the Tables.** The column "$G$" contains a reductive group $G$. In Table 1 the linear group $\Lambda^2 SL(n)$ is the image of $SL(n)$ under the action in the second exterior power of the tautological representation, and $S^2 SL(n)$ is the same thing with respect to the second symmetric power.

In the column "weights" the highest weights of the $G$-module are indicated.

**Table 2**

| | $G$ | weights | $\dim V$ | $\mathcal{H}$ | codim |
|---|---|---|---|---|---|
| 14 | $SL(2) \times K^*$ | $\phi_1 \otimes \epsilon + \phi_1 \otimes \epsilon^{-1}$ | 4 | $t_1$ | 1 |
| 15 | $SL(n) \times K^*,\ n > 2$ | $\phi_1 \otimes \epsilon^a + \phi_1 \otimes \epsilon^b,\ a \neq b$ | $2n$ | $A_{n-3} + t_1 + R_{2(n-2)}$ | 0 |
| 16 | $SL(n),\ n > 2$ | $\phi_1 + \phi_{n-1}$ | $2n$ | $A_{n-2}$ | 1 |
| 17 | $SL(2n+1)$ | $\phi_1 + \phi_2$ | $(2n+1)(n+1)$ | $C_n$ | 1 |
| 18 | $SL(2n+1) \times K^*$ | $\phi_1 \otimes \epsilon^a + \phi_{2n-1} \otimes \epsilon^b, a \neq nb$ | $(2n+1)(n+1)$ | $C_{n-1} + t_1 + R_{2(2n-1)}$ | 0 |
| 19 | $SL(2n)$ | $\phi_1 + \phi_2$<br>$\phi_1 + \phi_{2n-2}$ | $n(2n+1)$ | $C_{n-1} + R_{2n-1}$ | 1 |
| 20 | $SO(8)$ | $\phi_1 + \phi_3$ | 16 | $G_2$ | 2 |
| 21 | $Sp(2n) \times K^*$ | $\phi_1 \otimes \epsilon + \phi_1 \otimes \epsilon^{-1}$ | $4n$ | $C_{n-1} + t_1$ | 1 |
| 22 | $SL(n) \times SL(m),\ n > m$ | $\phi_1 \otimes \phi_1$ | $nm$ | $A_{n-m-1} + A_{m-1} + R_{nm-m^2}$ | 0 |
| 23 | $SL(n) \times SL(n)$ | $\phi_1 \otimes \phi_1$ | $n^2$ | $A_{n-1}$ | 1 |
| 24 | $SL(2) \times Sp(2n)$ | $\phi_1 \otimes \phi_1$ | $4n$ | $C_{n-1} + A_1$ | 1 |
| 25 | $SL(3) \times Sp(2n) \times K^*,\ n > 1$ | $\phi_1 \otimes \phi_1 \otimes \epsilon$ | $6n$ | $C_{n-2} + A_1 + t_1 + R_{2n-1}$ | 0 |
| 26 | $SL(4) \times Sp(4)$ | $\phi_1 \otimes \phi_1$ | 16 | $C_2$ | 1 |
| 27 | $SL(n) \times Sp(4),\ n > 4$ | $\phi_1 \otimes \phi_1$ | $4n$ | $A_{n-5} + C_2 + R_{4(n-4)}$ | 0 |

For the group $G_1 \times G_2$ the weight $\phi \otimes \psi$ corresponds to the tensor product of simple $G_1$- and $G_2$-modules with highest weights $\phi$ and $\psi$ respectively. The symbol $+$ denotes a direct sum of modules. If $G^s$ is the product of several simple groups, then their fundamental weights are denoted successively by letters $\phi_i$, $\psi_i$ and $\tau_i$. The fundamental weight of the central torus is denoted by $\epsilon$ (for a two-dimensional torus – by $\epsilon_1$ and $\epsilon_2$).

In the column " $\dim V$" the dimension of the module is shown.

In Tables 1 and 2 the column "$\mathcal{H}$" contains the type of the tangent algebra $\mathcal{H}$ of the generic isotropy subgroup $H$ for our module. Here $t_1$ is the tangent algebra of the one-dimensional central torus in $H$, and $R_k$ is the tangent algebra of the $k$-dimensional unipotent radical of $H$. The information of this column is taken from Elashvili's tables [4], [5].

**Table 3**

| | $G$ | weights | $\dim V$ | codim |
|---|---|---|---|---|
| 28 | $SL(n)\times SL(n)\times K^*$ | $\phi_1\otimes\epsilon+\phi_1\otimes\psi_1$<br>$\phi_1\otimes\epsilon+\phi_{n-1}\otimes\psi_{n-1}$ | $n(n+1)$ | 1 |
| 29 | $SL(n+1)\times SL(n)\times K^*$ | $\phi_1\otimes\epsilon^n+\phi_1\otimes\psi_1\otimes\epsilon^{-1}$ | $(n+1)^2$ | 1 |
| 30 | $SL(n+1)\times SL(n)\times K^*\times K^*,\ n>1$ | $\phi_1\otimes\epsilon_1+\phi_n\otimes\psi_{n-1}\otimes\epsilon_2$ | $(n+1)^2$ | 0 |
| 31 | $SL(n)\times SL(m)\times K^*,\ n>m+1$ | $\phi_1\otimes\epsilon^a+\phi_1\otimes\psi_1\otimes\epsilon^b,\ a\neq b$ | $n(m+1)$ | 0 |
| 32 | $SL(n)\times SL(m)\times K^*,\ n>m+1>2$ | $\phi_1\otimes\epsilon^a+\phi_{n-1}\otimes\psi_{m-1}\otimes\epsilon^b,\ a\neq -b$ | $n(m+1)$ | 0 |
| 33 | $SL(n)\times SL(m)\times K^*,\ n<m$ | $\phi_1\otimes\epsilon^a+\phi_1\otimes\psi_1\otimes\epsilon^b,\ a\neq 0$<br>$\phi_1\otimes\epsilon^a+\phi_{n-1}\otimes\psi_{m-1}\otimes\epsilon^b,\ a\neq 0$ | $n(m+1)$ | 0 |
| 34 | $SL(n)\times SL(2)\times SL(m),\ n>2,\ m>2$ | $\phi_1\otimes\psi_1+\psi_1\otimes\tau_1$ | $2(n+m)$ | 0 |
| 35 | $SL(n)\times SL(2)\times Sp(2m),\ n>2,\ m\geq 1$ | $\phi_1\otimes\psi_1+\psi_1\otimes\tau_1$ | $2(n+2m)$ | 1 |
| 36 | $Sp(2n)\times SL(2)\times Sp(2m),\ n,m\geq 1$ | $\phi_1\otimes\psi_1+\psi_1\otimes\tau_1$ | $4(m+n)$ | 2 |
| 37 | $SL(2)\times Sp(2n)\times K^*$ | $\phi_1\otimes\epsilon+\phi_1\otimes\psi_1$ | $2(2n+1)$ | 1 |

In the last column the codimension of a generic $G$-orbit in $V$ is shown.

## 3. Spherical modules

It is natural to say that a $G$-module $V$ is *spherical* if $V$ is a spherical $G$-variety. (In particular, $G$ acts on $V$ with an open orbit.) Such modules are also known as *multiplicity free modules.* The motivation for this term is the following characterization of spherical $G$-modules: any simple $G$-module appears in the decomposition of the $G$-module $K[V]$ of polynomial functions on $V$ with multiplicity $\leq 1$ [7]. A classification of spherical modules was obtained in [6] (simple modules), and in [2], [3], [8] (non-simple modules).

Let $V = V_1 \oplus \ldots \oplus V_k$ be a decomposition of a $G$-module as a sum of simple submodules, and $T = T_1 \times \ldots \times T_k$ be a torus acting on $V$, where $T_i$ is a one-dimensional torus acting on $V_i$ by dilatations and trivially on the other components. We shall say that $T$ is the *saturating* torus for the $G$-module $V$, and set $\overline{G} = GT$.

**Definition 3.1.** A $G$-module $V$ is said to be *saturated* if $G = \overline{G}$.

We need the following well-known lemma.

**Lemma 3.2.** ( **[6, Lemma 3.11]**) *If $V$ is a simple $G$-module and $H$ is a spherical subgroup of $G$, then for the subspace $V^H$ of $H$-fixed points one has* $\dim V^H \leq 1$.

**Proposition 3.3.** *If $V$ is a $G$-module with spherical orbits, then $V$ is a spherical $\overline{G}$-module.*

**Proof.** Let $H$ be the generic isotropy subgroup for $V$. By assumption, $H$ is a spherical subgroup of $G$. Hence $\dim V_i^H \leq 1$ and $\dim V^H \leq k$. This implies that the torus $T$ acts on $V^H$ with an open orbit, and the $\overline{G}$-orbit of a point of the open $T$-orbit in $V^H$ is open in $V$. On the other hand, this orbit is spherical by Lemma 2.4. ■

## 4. The algebra of $U$-invariants

Let $U$ be a maximal unipotent subgroup of $G$. For any $G$-module $V$ the algebra $K[V]^U$ of polynomial $U$-invariants is the linear span of the highest weight vectors of the $G$-module $K[V]$. The following proposition can be found, for example, in [8, Theorem 1.2].

**Proposition 4.1.** *A $G$-module $V$ is spherical iff $K[V]^U = K[g_1, \ldots, g_m]$, where the $g_i'$s are algebraically independent and the weights of $g_i$ are linearly independent.*

In particular, the algebra $K[V]^U$ is free for any spherical $G$-module $V$.

**Proposition 4.2.** *For any $G$-module $V$ with spherical orbits the algebras $K[V]^U$ and $K[V]^{G^s}$ are free.*

**Proof.** The subgroup $U$ is a maximal unipotent subgroup of the saturated linear group $\overline{G}$. Thus the first statement follows from Proposition 3.3 and Proposition 4.1. For a semisimple group the semigroup of dominant weights generates (over $\mathbb{Q}$) a strictly convex cone. Hence the algebra $K[V]^{G^s}$ is generated by $G^s$-invariant free generators $g_i$ of the algebra $K[V]^U$. ■

Finally we get an analog of Proposition 4.1 for modules with spherical orbits.

**Proposition 4.3.** *Let $G$ be a connected semisimple group. The following conditions are equivalent:*

*(1) $V$ is a $G$-module with spherical orbits;*

*(2) $K[V]^U = K[f_1, \ldots, f_h, g_1, \ldots, g_m]$, where the $f_i'$s are generators of $K[V]^G$, the $g_i'$s are algebraically independent and their weights are linearly independent.*

**Proof.** (1) $\Rightarrow$ (2) We may suppose that $f_i, g_j$ are free generators of $K[V]^U$. Denote by $\omega_i$ the weight of $g_i$. If $\omega_i$ are linearly dependent, then (after reindexing) one has $b_1\omega_1 + \ldots + b_p\omega_p = c_{p+1}\omega_{p+1} + \ldots + c_m\omega_m$, where $b_i, c_j$ are non-negative integers. Hence

$$\frac{g_1^{b_1}\ldots g_p^{b_p}}{g_{p+1}^{c_{p+1}}\ldots g_m^{c_m}} \in K(V)^B = K(V)^G.$$

But for a semisimple group any rational $G$-invariant is the ratio of two polynomial $G$-invariants [9, sec. 3.2]. Therefore

$$g_1^{b_1}\ldots g_p^{b_p}F_1(f_1,\ldots,f_h) = g_{p+1}^{c_{p+1}}\ldots g_m^{c_m}F_2(f_1,\ldots,f_h).$$

This contradiction concludes the proof.

(2) $\Rightarrow$ (1) Any rational $B$-invariant is the ratio of two polynomial $B$-semi-invariants of the same weight [9, sec. 3.2]. Thus any element of $K(V)^B$ is the ratio of $g_1^{d_1}\ldots g_m^{d_m}F_1(f_1,\ldots,f_h)$ and $g_1^{d_1}\ldots g_m^{d_m}F_2(f_1,\ldots,f_h)$. This shows that $K(V)^B = K(V)^G$. ■

## 5. A technical lemma

Let $T$ be an algebraic torus and $X(T)$ be the character group of $T$. Consider a field $K_1 = K(a_1, \ldots, a_n)$ with a $T$-action (by automorphisms) such that $t * a_i = \lambda_i(t)a_i$ for some $\lambda_i \in X(T)$. The subfield of $T$-fixed elements of $K_1$ is denoted by $K_1^T$. By tr.deg $F$ we denote the transcendency degree of a field $F$ over $K$. Consider the subfield $K_2 = K(a_1, \ldots, a_q)$, $q \le n$. Denote by $S_1 =< \lambda_1, \ldots, \lambda_n >$ and $S_2 =< \lambda_1, \ldots, \lambda_q >$ the corresponding sublattices in $X(T)$. For any sublattice $L \subset X(T)$ define $L_{\mathbb{Q}}$ as the vector space $L \otimes_{\mathbb{Z}} \mathbb{Q}$.

A sublattice $L$ determines the subgroup

$$T_L = \{t \in T \mid \lambda(t) = 1 \ \ \forall \lambda \in L\}.$$

**Lemma 5.1.** *Suppose that $K_1^T = K$. Then the following conditions are equivalent:*

*(1)* tr.deg $K_1^{T_L} =$ tr.deg $K_2^{T_L}$*;*

*(2) $L \cap S_1 \subset (S_2)_{\mathbb{Q}}$.*

**Proof.** Set $A_1 = K[a_1, \ldots, a_n]$ and $A_2 = K[a_1, \ldots, a_q]$. Denote by Spec $A_i$ the affine variety corresponding to the algebra $A_i$, $i = 1, 2$. For the natural $T$-action one has tr.deg $K_i^{T_L} = \dim \mathrm{Spec}\ A_i - h_i$, where $h_i$ is the dimension of generic $T$-orbits in Spec $A_i$. The condition $K_1^T = K$ implies $\dim \mathrm{Spec}\ A_1 = \mathrm{rk}\ S_1$ and $\dim \mathrm{Spec}\ A_2 = \mathrm{rk}\ S_2$. Moreover, $h_i = \dim T_L - \dim T_{S_i + L}$. Thus condition (1) is equivalent to $\mathrm{rk}\,(S_1 \cap L) = \mathrm{rk}\,(S_2 \cap L)$. This is a reformulation of (2). ■

## 6. Modules with spherical orbits

Let $V$ be a saturated spherical $G$-module. Any character of the saturating torus $T$ can be considered as a character of $G$ with trivial restriction to $G^s$. For any sublattice $L \subset X(T)$ define

$$G_L = \{g \in G \mid \lambda(g) = 1 \ \ \forall \lambda \in L\}.$$

Let $B^s$ be a Borel subgroup of $G^s$. Then $B = TB^s$ and $B_L = T_L^0 B^s$ are Borel subgroups of $G$ and $G_L$ respectively. Set

$$S(V) = \{\lambda \in X(T) \mid \exists f \in K(V)^{B^s} : \ t * f = \lambda(t) f \ \ \forall t \in T\}$$

and

$$M(V) = \{\lambda \in X(T) \mid \exists f \in K(V)^{G^s} : \ t * f = \lambda(t) f \ \ \forall t \in T\}.$$

**Proposition 6.1.** *(cf. [8, Theorem 2.6]) The $G_L$-module $V$ is spherical iff $L \cap S(V) = \{0\}$.*

**Proof.** Lemma 5.1 with $K_1 = K(V)^{B^s}$ and $K_2 = K$ shows that the condition $L \cap S(V) = \{0\}$ is equivalent to tr.deg $K(V)^{B_L} = 0$, or $K(V)^{B_L} = K$. ■

**Theorem 6.2.** *A $G$-module $V$ is a module with spherical orbits iff the $\overline{G}$-module $V$ is spherical and $G = G_L$ with $L \cap S(V) \subset M(V)_{\mathbb{Q}}$.*

**Proof.** Appling Lemma 5.1 with $K_1 = K(V)^{B^s}$ and $K_2 = K(V)^{G^s}$ one shows that the second condition is equivalent to

$$\text{tr.deg}\,(K(V)^{B^s})^{T_L} = \text{tr.deg}\,(K(V)^{G_s})^{T_L},$$

or $K(V)^{B_L} = K(V)^{G_L}$. ■

In Table 4 we indicate bases of the subspaces $S(V)_{\mathbb{Q}}$ and $M(V)_{\mathbb{Q}}$ for all indecomposable saturated spherical $G$-modules. There is a natural bijection between such modules and items of Tables 1-3, see [8, Theorem 2.5]. Our calculations are based on [8, Table 2].

## 7. Proof of Theorem 2.5

Suppose that $V$ is a spherical indecomposable saturated $\overline{G}$-module. By Proposition 3.3 and Lemma 2.4, we need to find all minimal (with respect to inclusion) subgroups $G$, $G^s \subseteq G \subseteq \overline{G}$, such that $V$ is a $G$-module with spherical orbits. By Theorem 6.2, these subgroups coincide with $G_L$ for a maximal sublattice $L \subset X(T)$ satisfying $L \cap S(V) \subset M(V)_{\mathbb{Q}}$.

*Case 1.* $S(V) = M(V) = 0$. Here $V$ is a spherical $G^s$-module, $G = G^s$, and we obtain items 1, 3, 10, 11, 22, 27, 34.

**Table 4**

| items | $S(V)_{\mathbb{Q}}$ | $M(V)_{\mathbb{Q}}$ |
|---|---|---|
| 1, 3, 10, 11, 22, 27, 34 | $0$ | $0$ |
| 0, 2, 4 – 9, 12, 13, 23, 24, 26 | $<\epsilon_1>$ | $<\epsilon_1>$ |
| 19, 35 | $<\epsilon_2>$ | $<\epsilon_2>$ |
| 25, 33 | $<\epsilon_1>$ | $0$ |
| 20, 36 | $<\epsilon_1,\epsilon_2>$ | $<\epsilon_1,\epsilon_2>$ |
| 15, 31 | $<\epsilon_1-\epsilon_2>$ | $0$ |
| 18 | $<\epsilon_1-n\epsilon_2>$ | $0$ |
| 30 | $<\epsilon_1,\epsilon_2>$ | $0$ |
| 32 | $<\epsilon_1+\epsilon_2>$ | $0$ |
| 16 | $<\epsilon_1+\epsilon_2>$ | $<\epsilon_1+\epsilon_2>$ |
| 14, 21 | $<\epsilon_1,\epsilon_2>$ | $<\epsilon_1+\epsilon_2>$ |
| 17 | $<\epsilon_1+n\epsilon_2>$ | $<\epsilon_1+n\epsilon_2>$ |
| 28, 37 | $<\epsilon_1,\epsilon_2>$ | $<\epsilon_2>$ |
| 29 | $<\epsilon_1,\epsilon_2>$ | $<\epsilon_1+n\epsilon_2>$ |

*Case 2.* rk $S(V)=$ rk $M(V)\neq 0$. Here $V$ is a $G^s$-module with spherical orbits. One gets items 0, 2, 4 – 9, 12, 13, 16, 17, 19, 20, 23, 24, 26, 35, 36.

*Case 3.* rk $S(V)=1$ and $M(V)=0$. In this case $G$ can be obtained as an extension of $G^s$ by any one-dimensional subtorus of $T_L$ with the restriction $L\cap S(V)=\{0\}$. We get items 15, 18, 25, 31, 32, 33.

*Case 4.* rk $S(V) = 2$ and $M(V) = 0$. In this case the $\overline{G}$-module $V$ is the only module with spherical orbits – item 30.

*Case 5.* rk $S(V) = 2$ and rk $M(V) = 1$. In this case $G$ is obtained as an extension of $G^s$ by a one-dimensional subtorus of $T_L$ with $L_{\mathbb{Q}} = M(V)_{\mathbb{Q}}$ (this is the torus fixing generic $G^s$-orbits in $V$). One gets items 14, 21, 28, 29, 37.

This completes the proof of Theorem 2.5.

**Remark 7.1.** Suppose that there is a decomposable $G$-module $V$. We are going to check that all orbits in $V$ are spherical. First we verify that for all indecomposable components $V_i$ the $G$-module $V_i$ is a module with spherical orbits (Theorem 2.5). If this is the case we apply Theorem 6.2 and use Table 4.

## 8. Actions with spherical orbits on projective spaces

Any linear action $G : V$ determines an action $G : \mathbb{P}(V)$ on the projective space $\mathbb{P}(V)$. Let $T'$ be a one-dimensional torus acting on $V$ by dilatations and $G' = T'G$ be the torus extension. It is clear that the action $G : \mathbb{P}(V)$ is an action with spherical orbits iff the action $G' : V$ has this property. Let us recall that $T$ is the saturating torus for the $G$-module $V$. Denote by $F$ the sublattice in $X(T)$ corresponding to $T'$. (In standart coordinates $F$ is given as $x_1 + \ldots + x_k = 0$.) The following results are immediate consequences of Theorem 6.2 and Theorem 2.5.

**Proposition 8.1.** *An action $G : \mathbb{P}(V)$ is an action with spherical orbits iff the $\overline{G}$-module $V$ is spherical and $G = G_L$ with $(L \cap F \cap S(V)) \subset M(V)_{\mathbb{Q}}$.*

**Proposition 8.2.** *Suppose that $G$ is semisimple and a $G$-module $V$ is indecomposable. The action $G : \mathbb{P}(V)$ is an action with spherical orbits iff either the $G$-module $V$ is a module with spherical orbits or $V$ is one of the modules indicated in Table 5.*

**Table 5**

| | $G$ | weights | $\dim V$ |
|---|---|---|---|
| 1 | $SL(3) \times Sp(2n),\ \ n > 1$ | $\phi_1 \otimes \psi_1$ | $6n$ |
| 2 | $SL(2n+1),\ \ n > 1$ | $\phi_1 + \phi_{2n-1}$ | $(2n+1)(n+1)$ |
| 3 | $SL(n) \times SL(m),\ n > m+1 > 2$ | $\phi_1 + \phi_{n-1} \otimes \psi_{m-1}$ | $n(m+1)$ |
| 4 | $SL(n) \times SL(m),\ n < m$ | $\phi_1 + \phi_1 \otimes \psi_1$<br>$\phi_1 + \phi_{n-1} \otimes \psi_{m-1}$ | $n(m+1)$ |

## References


[1] Arzhantsev, I. V., *On $SL_2$-actions of complexity one*, Izvestiya RAN, Ser. Mat.**61(4)** (1997), 3–18 (in Russian); English transl.: Izvestiya Mathematics **61(4)** (1997), 685–698.

[2] Brion, M., *Représentations exceptionnelles des groupes semi-simples*, Ann. Sci. École Norm. Sup. (4), **18** (1985), 345–387.

[3] Benson, C., and G. Ratcliff, *A classification of multiplicity free actions*, Journal of Algebra, **181** (1996), 152–186.

[4] Elashvili, A. G., *Canonical forms and stationary subalgebras of points of general position for simple linear Lie groups*, Funkts. Analiz i ego Prilog. **6(1)** (1972), 51–62 (in Russian); English transl.: Func. Anal. and its Appl. **6(1)** (1972), 44–53.

[5] —, *Stationary subalgebras of points of the common state for irreducible linear Lie groups*, Funkts. Analiz i ego Prilog. **6(2)** (1972), 65–78 (in Russian); English transl.: Func. Anal. and its Appl. **6(2)** (1972), 139–148.

[6] Kac, V. G., *Some remarks on nilpotent orbits*, Journal of Algebra, **64** (1980), 190–213.

[7] Kimel'feld, B. N., and E. B. Vinberg, *Homogeneous domains on flag manifolds and spherical subgroups of semisimple Lie groups*, Funkts. Analiz i ego Prilog. **12(3)** (1978), 12–19 (in Russian); English transl.: Func. Anal. and its Appl. **12(3)** (1978), 168–174.

[8] Leahy, A. S., *A classification of multiplicity free representations*, Journal of Lie Theory, **8** (1998), 367–391.

[9] Popov, V. L., and E. B. Vinberg, *Invariant theory*, Itogi Nauki i Tekhniki, Sovr. Probl. Mat. Fund. Napravl., vol. 55, VINITI, Moscow 1989, pp. 137–309 (in Russian); English transl.: Algebraic Geometry IV, Encyclopaedia of Math. Sciences, vol. 55, Springer-Verlag, Berlin 1994, pp. 123–278.



Ivan Arzhantsev
Chair of Algebra
Moscow State University
Vorobievy Gory,
Moscow, 119899, Russia
arjantse@mccme.ru



Received March 2, 2001
and in final form April 1, 2001