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Characterization of the $\mathbf{L^p}$-Range of the Poisson Transform in Hyperbolic Spaces $\mathbf{B(\mathbb{F}^n)}$

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Abstract. The aim of this paper is to give, in a unified manner, the characterization of the L^p-range $(p \geq 2)$ of the Poisson transform P_λ for the Hyperbolic spaces $B(\mathbb{F}^n)$ over $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ or the quaternions $\mathbb{H}$. Namely, if Δ is the Laplace-Beltrami operator of $B(\mathbb{F}^n)$ and sF a $\mathbb{C}$-valued function on $B(\mathbb{F}^n)$ satisfying $\Delta F = -(\lambda^2 + \sigma^2)F; \lambda \in \mathbb{R}^*$ then we establish:

i) F is the Poisson transform of some $f \in L^2(\partial B(\mathbb{F}^n))$ (ie $P_\lambda f = F$) if and only if it satisfies the growth condition:

$$\sup_{t>0} \frac{1}{t} \int_{B(0,t)} |F(x)|^2 d\mu(x) < +\infty,$$

where $B(0,t)$ is the ball of radius t centered at 0 and $d\mu$ the invariant measure on $B(\mathbb{F}^n)$.

ii) F is the Poisson transform of some $f \in L^p(\partial B(\mathbb{F}^n))$, $p \geq 2$; if and only if it satisfies the following Hardy-type growth condition:

$$\sup_{0 \leq r < 1} (1 - r^2)^{-\frac{\sigma}{2}} \left(\int_{\partial B(\mathbb{F}^n)} |F(r\theta)|^p d\theta \right)^{\frac{1}{p}} < +\infty.$$

Key words: Hyperbolic spaces, Poisson transform, Calderon Zygumund estimates, Jacobi functions.

1. Introduction.

It is known that the Poisson transforms associated to symmetric spaces X of non compact type play an important role in reproducing joint-eigenfunctions of invariant differential operators on X from the boundaries of X (see [6, 8, 9]).

Also, in rank one symmetric spaces of non compact type, the Poisson transform enters in a natural way through the Fourier-Helgason transform in the L^2-Plancherel formula of the Laplace-Beltrami operator on X.

Since then, it becomes natural to characterize the range of the Poisson transform on classical spaces on the Fustenberg boundary B of X such as the

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spaces $\mathcal{C}^\infty(B)$, $L^p(B)$ and the space of distributions $\mathcal{D}^{'}(B)$. (See for instance Strichartz [12] for a large discussion on this subject.) In [2], the first author characterized the L^2-range of the Poisson transform in the complex hyperbolic space.

The aim of this paper is on one hand to extend in a unified manner the result in [2] to the classical hyperbolic spaces $B(\mathbb{F}^n)$ over $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ or the quaternions $\mathbb{H}$, and on the other hand to characterize, for $p \geq 2$, the L^p-range of the Poisson transform in the hyperbolic spaces $B(\mathbb{F}^n)$.

The organization of this paper is as follows. In section 2, we state the main results of this paper. In section 3, we recall some classical results of harmonic analysis on hyperbolic spaces $B(\mathbb{F}^n)$. Section 4 is devoted to the proofs of Theorem A and Theorem B. In the last section, we establish the Key Lemma that gives the uniform L^p-boundedness of the family of operators $Q_r(\lambda)$ associated to superficial Poisson integrals on the boundary $\partial B(\mathbb{F}^n)$ of $B(\mathbb{F}^n)$.

2. Notations and statement of the main results.

Let $\mathbb{F}$ be one of the classical fields (that is $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ or the quaternions $\mathbb{H}$) and let $B(\mathbb{F}^n)$ be the bounded realization of the hyperbolic space $U(n,1;\mathbb{F})/U(n;\mathbb{F}) \times U(1;\mathbb{F})$.

The Poisson transform P_λ associated to $B(\mathbb{F}^n)$ is defined for every $\lambda \in \mathbb{C}$ and every integrable function f on the topological boundary $\partial B(\mathbb{F}^n)$ of $B(\mathbb{F}^n)$ by the following formula:

$$(P_\lambda f)(x) = \int_{\partial B(\mathbb{F}^n)} \left(\frac{1 - |x|^2}{|1 - \langle x, \omega \rangle|^2} \right)^{\frac{i\lambda + \sigma}{2}} f(\omega) d\omega, \tag{1}$$

where $\sigma = \frac{d}{2}(n+1) - 1$ and $d = dim_{\mathbb{R}}\mathbb{F}$

Let $B(0,t)$ be the ball of radius t centered at 0 with respect to the $U(n,1;\mathbb{F})$-invariant metric on $B(\mathbb{F}^n)$, then for every locally integrable function F with respect to the $U(n,1,\mathbb{F})$-invariant measure $d\mu$ on $B(\mathbb{F}^n)$ we set

$$\|F\|_*^2 = \sup_{t>0} \frac{1}{t} \int_{B(0,t)} |F(x)|^2 d\mu(x). \tag{2}$$

Also, for any $\lambda \in \mathbb{R}\backslash 0$, we define the following eigenspace of the Laplace-Beltrami operator Δ of $B(\mathbb{F}^n)$ given by:

$$E_\lambda^*(B(\mathbb{F}^n)) = \{F : B(\mathbb{F}^n) \longrightarrow \mathbb{C}; \Delta F = -(\lambda^2 + \sigma^2)F \text{ and } \|F\|_*^2 < +\infty\}.$$

Finally, by γ we will denote any constant depending only on the dimension of $B(\mathbb{F}^n)$.

Now, with the help of the above notations, the first result of this paper can be stated as follow:

Theorem A. *Let λ be a non zero real number. Then we have:*

i) The Poisson transform P_λ is a topological isomorphism from $L^2(\partial B(\mathbb{F}^n))$ onto the Banach space $E^*_\lambda(B(\mathbb{F}^n))$.

More precisely there exists positive constants γ_1 and γ_2 depending only on n such that for every $f \in L^2(\partial B(\mathbb{F}^n))$ we have the following estimates:

$$\gamma_1|c(\lambda)|\,\|f\|_{L^2(\partial B(\mathbb{F}^n))} \leq \|P_\lambda f\|_* \leq \gamma_2(1+|\lambda|+\frac{1}{|\lambda|})\|f\|_{L^2(\partial B(\mathbb{F}^n))}, \tag{3}$$

where $c(\lambda) = \frac{2^{\sigma-i\lambda}\Gamma(i\lambda)}{\Gamma(\frac{i\lambda+\sigma}{2})\Gamma(\frac{i\lambda+\sigma+2-d}{2})}$ is the Harish-Chandra c-function associated to $B(\mathbb{F}^n)$.

ii) The L^2-boundary value f of the eigenfunction $F \in E^*_\lambda(B(\mathbb{F}^n))$ is given by the following L^2-inversion formula:

$$f(\omega) = |c(\lambda)|^{-2} \lim_{t\mapsto+\infty} \frac{1}{t} \int_{B(0,t)} \left(\frac{1-|x|^2}{|1-\langle x,\omega\rangle|^2}\right)^{\frac{-i\lambda+\sigma}{2}} F(x)d\mu(x), \tag{4}$$

in $L^2(\partial B(\mathbb{F}^n))$.

In the case of the real hyperbolic space $B(\mathbb{R}^n)$, the assumption i) of theorem A was stated in [12, Lemma 4.2]. However the given proof in [12] is not correct and remains related to the conjecture 5.4 on Jacobi functions (See Strichartz corrigendum [13]).

In this paper, we prove the part i) of Theorem A as well as its L^p-counterpart, see Theorem B in below, by discussing the L^p-boundedness of a family of superficial Poisson integrals $(Q_r(\lambda))_{r\in[0,1[}$ on $\partial B(\mathbb{F}^n)$ given by

$$[Q_r(\lambda)f](\theta) = \int_{\partial B(\mathbb{F}^n)} |1-r\langle\theta,\omega\rangle|^{-i\lambda-\sigma} f(w)dw. \tag{5}$$

More precisely, we will establish the following Key lemma of this paper.

Key Lemma. *Let λ be a non zero real number and let $Q_r(\lambda)$ be the operator given by (5). Then we have:*

$$\sup_{0\leq r<1} \|Q_r(\lambda)\|_2 \leq \gamma(1+|\lambda|+\frac{1}{|\lambda|}).$$

Moreover, for every $p \in]1,+\infty[$, there exists a constant $A(\lambda,p)>0$ such that

$$\sup_{0\leq r<1} \|Q_r(\lambda)\|_p \leq A(\lambda,p), \tag{6}$$

where $\|\|_p$ stands for the L^p-operatorial norm.

Let $\Phi_{\lambda,p,q} = P_\lambda \Phi_{p,q}$ be the generalized spherical function associated to $B(\mathbb{F}^n)$ given here in terms of Jacobi functions. Then as an immediate consequence of the Key Lemma we obtain the following uniform pointwise estimate on $\Phi_{\lambda,p,q}$ (See section 3 for precise notations.)

Corollary 2.1. *Let λ be a non zero real number. Then*

$$\sup_{p,q\in\hat{K}_0} |\Phi_{\lambda,p,q}(r)| \leq \gamma(1-r^2)^{\frac{\sigma}{2}}(1+|\lambda|+\frac{1}{|\lambda|}),$$

for some numerical positive constant γ.

Now, we state the second result of this paper.

Theorem B. *Let $p\in[2,+\infty[$, λ be a non zero real number and let F be a $\mathbb{C}$-valued function on $B(\mathbb{F}^n)$ satisfying $\Delta F=-(\lambda^2+\sigma^2)F$. Then we have :*
F is the Poisson transform by P_λ of some $f\in L^p(\partial B(\mathbb{F}^n))$ (i.e $P_\lambda f=F$) if and only if it satisfies the following Hardy-type growth condition:

$$\|F\|_{*,p} = \sup_{0\leq r<1}(1-r^2)^{-\frac{\sigma}{2}}\left(\int_{\partial B(\mathbb{F}^n))}|F(r\theta)|^p d\theta\right)^{\frac{1}{p}} < +\infty.$$

Moreover, there exist a positive constants γ and $A(\lambda,p)$ such that for every $f\in L^p(\partial B(\mathbb{F}^n))$ the following estimates hold:

$$\gamma|c(\lambda)|\,\|f\|_{L^p(\partial B(\mathbb{F}^n))} \leq \|P_\lambda f\|_{*,p} \leq A(\lambda,p)\|f\|_{L^p(\partial B(\mathbb{F}^n))}.$$

Remark 2.2. Notice that for $p=2$, we have $\|F\|_* \leq c\|F\|_{*,2}$. But in general these norms are not comparable.

However, using Theorem A and Theorem B, we can see that they are comparable on the eigenspace

$$E^*_\lambda(B(\mathbb{F}^n)) = \{F : B(\mathbb{F}^n)\longrightarrow\mathbb{C}; \Delta F=-(\lambda^2+\sigma^2)F \text{ and } \|F\|^2_* < +\infty\}.$$

Remark 2.3. While we have considered only the case of $p\geq 2$, we think that the result of Theorem B holds for $1<p\leq 2$. We hope to return to this case in a near future.

3. Background and preliminary results.

In this section, we review some known results of harmonic analysis on the hyperbolic space $B(\mathbb{F}^n)=U(n,1;\mathbb{F})/U(n,\mathbb{F})\times U(1,\mathbb{F})$, referring to [6, 7] for more details on this subject.

Let $\mathbb{F}$ be one of the classical fields (that is $\mathbb{F}=\mathbb{R}$, $\mathbb{C}$ or the quaternions $\mathbb{H}$). On $\mathbb{F}^{n+1}$ considered as a right vector space over $\mathbb{F}$, we consider the following quadratic form

$$J(x_1,\ldots,x_{n+1}) = \sum_{j=1}^n |x_j|^2 - |x_{n+1}|^2,$$

with $|x|^2=x\bar{x}$, where $x\to\bar{x}$ is the standard involution of $\mathbb{F}$.
Let $G=U(n,1;\mathbb{F})$ be the group of all $\mathbb{F}$-linear transformations g on $\mathbb{F}^{n+1}$ keeping the quadratic form J invariant, with the additional property, $detg=1$ if $\mathbb{F}=\mathbb{R}$

or $\mathbb{C}$. Then G is one of the classical split rank one groups, $SO(n,1)$, $SU(n,1)$ or $Sp(n,1)$ accordingly to $\mathbb{F}=\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

The group G acts on the unit ball $B(\mathbb{F}^n)=\{x\in\mathbb{F}^n; |x|<1\}$ by fractional transforms:

$$G\ni g=\begin{pmatrix} A & B\\ C & D\end{pmatrix}: x\longmapsto (Ax+B)(Cx+D)^{-1}.$$

with $A\in\mathbb{F}^{n\times n}$, $B\in\mathbb{F}^{n\times 1}$, $C\in\mathbb{F}^{1\times n}$ and $D\in\mathbb{F}$.
This action of G on $B(\mathbb{F}^n)$ is transitive (See [7]) and as homogeneous space we have the identification $B(\mathbb{F}^n)=G/K$ where K is the stabilizer of $0\in B(\mathbb{F}^n)$. Explicitly, $K=U(n,\mathbb{F})\times U(1,\mathbb{F})$ consisting of pairs (A,D) of unitaries with $det(AD)=1$, when $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$.

The action of G mentioned above extends naturally to $\overline{B(\mathbb{F}^n)}$ and under this action, K is transitive on the topological boundary $\partial B(\mathbb{F}^n)=\{w\in\mathbb{F}^n;\ |w|=1\}$ of $B(\mathbb{F}^n)$.
Let e denote the unit vector of $\mathbb{F}^n$, given by $e=(1,0,\ldots,0)$ and let M be the stabilizer in K of e. Then $\partial B(\mathbb{F}^n)=K/M$. The group M can be identified to $U(n-1,\mathbb{F})\times U(1,\mathbb{F})$.
Let $L^2(\partial B(\mathbb{F}^n))$ be the space of all square integrable $\mathbb{C}$-valued (classes) functions on $\partial B(\mathbb{F}^n)$, with respect to the normalized superficial measure of $\partial B(\mathbb{F}^n)$. Then the group K acts on $L^2(\partial B(\mathbb{F}^n))$ by composition $f\longmapsto f\circ k;\ k\in K$.
It is well known (see [6] or [7] for examples) that under the action of K, the Peter-Weyl decomposition of $L^2(\partial B(\mathbb{F}^n))$ is given by $L^2(\partial B(\mathbb{F}^n))=\bigoplus_{p,q\in\hat{K}_0}V_{p,q}$, where $V_{p,q}$ is the finite linear span $\{\varphi_{pq}\circ k,\ k\in K\}$ and φ_{pq} the zonal spherical function.
The parametrized set $\hat{K}_0$ consist of pairs (p,q) of integers satisfying:

$$\begin{array}{ll} i) & p\equiv q \quad (mod 2),\\ ii) & p\geq 0 \quad \text{and} \quad 0\leq q\leq 1 \quad \text{if} \quad \mathbb{F}=\mathbb{R},\\ & p\geq |q| \quad \text{if} \quad \mathbb{F}=\mathbb{C},\\ & p\geq q\geq 0 \quad \text{if} \quad \mathbb{F}=\mathbb{H}.\end{array}$$

In below, we recall some known results on the Poisson transform which will be useful in the sequel.

Proposition 3.1. [6] *Let λ be a complex number and let f in V_{pq}. Then we have*

$$(P_\lambda f)(x)=\Phi_{\lambda,pq}(|x|)f(\frac{x}{|x|});\quad x\in B(\mathbb{F}^n),$$

where $\Phi_{\lambda,pq}(|x|)$ is the generalized spherical function associated to the hyperbolic space $B(\mathbb{F}^n)$ given by:

$$\Phi_{\lambda,pq}(|x|)=(\frac{i\lambda+\sigma}{2})_{\frac{p+q}{2}}(\frac{i\lambda+\sigma+2-d}{2})_{\frac{p-q}{2}}\{(1)_{p+\frac{dn}{2}}\}^{-1}|x|^p(1-|x|^2)^{\frac{i\lambda+\sigma}{2}}\times$$
$${}_2F_1(\frac{i\lambda+\sigma+p+q}{2},\frac{i\lambda+\sigma+2-d+p-q}{2},p+\frac{dn}{2};|x|^2).$$

Here $(a)_k=a(a+1)\ldots(a+k-1)$ is the Pochammer symbol and ${}_2F_1(a,b,c;x)$ is the classical Gauss hypergeometric function.

Let Δ be the invariant Laplacian of the hyperbolic space $B(\mathbb{F}^n)$, and let $E_\lambda(B(\mathbb{F}^n))$ be the space of all eigenfunctions F of Δ with eigenvalue $-(\lambda^2+\sigma^2)$, λ being a complex number.

Proposition 3.2. [6] *Let λ be a complex number and let F in $E_\lambda(B(\mathbb{F}^n))$. Then there exists a sequence of spherical harmonics $f_{pq} \in V_{pq}$ such that F can be expanded in $\mathcal{C}^\infty(B(\mathbb{F}^n))$ as follows:*

$$F(x) = (1-|x|^2)^{\frac{i\lambda+\sigma}{2}} \cdot$$
$$\sum_{p,q\in\hat{K}_0} |x|^p {}_2F_1\left(\frac{i\lambda+\sigma+p+q}{2}, \frac{i\lambda+\sigma+2-d+p-q}{2}, p+\frac{dn}{2}; |x|^2\right) f_{pq}\left(\frac{x}{|x|}\right).$$

Remark 3.3. The above proposition is proved in [6] in group Theoretical way. But we can also prove it directly as follows: We write the invariant Laplacian Δ into its geodesic polar coordinates decomposition: $(t,b) \in [0,\infty[\times \partial\mathbf{B}(\mathbf{F}^n)$ (as given in Faraut [5])

$$\Delta = \frac{d^2}{dt^2} + ((d-1)tht + (dn-1)\coth t)\frac{d}{dt} - \frac{1}{ch^2 t}\Delta_1 + \frac{1}{sh^2 t}\Delta_2,$$

where Δ_1 and Δ_2 denote (respectively) the restriction to S^{d-1} and S^{dn-1} of the Laplacian of $\mathbf{R}^d$ and $\mathbf{R}^{dn-1}$ respectively (we drop Δ_1 in the case $d=1$). Next, using the fact that the spherical harmonic functions f_{pq} satisfy the following eigenfunction equations:

$$\Delta_1 f_{pq} = -q(q+d-2)f_{pq} \quad \text{and} \quad \Delta_2 f_{pq} = -p(p+dn-2)f_{pq},$$

we can reduce the eigenfunction problem $\Delta F = -(\lambda^2+\sigma^2)F$ to a sequence of second order ordinary differential equations, which can be transformed to the well known hypergeometric differential equation (see also Faraut [5]).

We finish this section by recalling a well known result on the asymptotic behaviour of the generalized spherical function $\Phi_{\lambda,pq}$ (See for instance [12]).

Lemma 3.4. *Let λ be a non zero real number. Then there exists a positive constant $\gamma > 0$ such that we have:*

$$\lim_{t\mapsto+\infty} \frac{1}{t}\int_{B(0,t)} |\Phi_{\lambda,pq}|^2 (1-|x|^2)^{-\sigma-1} dm(x) = \gamma |c(\lambda)|^2,$$

for every $p,q \in \hat{K}_0$.

4. Proof of Theorem A and Theorem B

Proof of Theorem A: The main difficulty in proving Theorem A is to show that the $L^2(\partial B(\mathbb{F}^n))$-range of the Poisson transform P_λ is continuousely embedded in the eigenspace $E^*_\lambda(B(\mathbb{F}^n))$. That is to show that $\|P_\lambda f\|_*$ is finite for every $f \in L^2(\partial\mathbf{B}(\mathbf{F}^n))$. As explained in section 2 this will be derived from the Key

Lemma stated in section 2.
Now let start by proving the necessary condition in i) of Theorem A .
i) Let f in $L^2(\partial\mathbf{B}(\mathbf{F}^n))$ and let $x = r\theta$; $r \in [0,1[$ and $\theta \in \partial B(\mathbb{F}^n)$. Then, we have

$$(P_\lambda f)(r\theta) = (1-r^2)^{\frac{i\lambda+\sigma}{2}}[Q_r(\lambda)f](\theta).$$

Next, from

$$\|P_\lambda f\|_*^2 = \sup_{t>0}\frac{1}{t}\int_0^{tht}\|Q_r(\lambda)f\|^2_{L^2(\partial\mathbf{B}(\mathbb{F}^n))}(1-r^2)^{-1}r^{dn-1}dr,$$

and the uniform L^2-boundedness of the operators $Q_r(\lambda)$ given by the key Lemma we get $\|P_\lambda f\|_* \leq \gamma(1+|\lambda|+\frac{1}{|\lambda|})\|f\|_{L^2(\partial B(\mathbb{F}^n))}$.
Next, we turn to the proof of the sufficiency condition in the i)part of Theorem A. For this, let $F \in E^*_\lambda(B(\mathbb{F}^n))$. Since λ is a non zero real number, the eigenfunction F can be expanded in $\mathcal{C}^\infty([0,1[\times\partial B(\mathbb{F}^n))$ as:

$$F(r\theta) = \sum_{p,q\in\hat{K}_0}\Phi_{\lambda,pq}(r)f_{pq}(\theta),$$

where f_{pq} is some sequence of spherical harmonic functions. By the growth condition on F, that is $\|F\|_* < +\infty$, we get:

$$\sum_{p,q\in\hat{K}_0}\frac{1}{t}\int_0^{tht}|\Phi_{\lambda,pq}(r)|^2\|f_{pq}\|^2_{L^2(\partial B(\mathbb{F}^n))}(1-r^2)^{-\sigma-1}r^{dn-1}dr \leq \|F\|_*^2 < +\infty,$$

for every $t > 0$.
Next, by using Lemma 3.4 on the uniform asymptotic behaviour of the functions $\Phi_{\lambda,pq}$ we obtain :

$$\gamma|c(\lambda)|^2\sum_{p,q\in\hat{K}_0}\|f_{pq}\|^2_{L^2(\partial B(\mathbb{F}^n)})\leq \|F\|_*^2 < +\infty \tag{7}$$

Thus the function $f = \sum_{p,q\in\hat{K}_0} f_{pq}$ is in $L^2(\partial B(\mathbb{F}^n))$, and the representation $F = P_\lambda f$ follows from Proposition 3.1 and the necessary condition of Theorem A established before. Furthermore from (7) we get $\gamma|c(\lambda)|\|f\|_{L^2(\partial B(\mathbb{F}^n)} \leq \|P_\lambda f\|_*$. This finishes the proof of the first part in Theorem A.

ii) Now, we turn to the proof of the L^2-inversion formula. Let F in $E^*_\lambda(B(\mathbb{F}^n))$. By the first part of Theorem A , we know that there exists f in $L^2(\partial B(\mathbb{F}^n))$ such that F can be written as $F = P_\lambda f$. Hence, expanding f into its K-type series,
$f = \sum_{p,q\in\hat{K}_0} f_{pq}$, the proposition 3.1 shows that

$$F(r\theta) = \sum_{p,q\in\hat{K}_0}\Phi_{\lambda,pq}(r)f_{pq}(\theta) \tag{8}$$

in $\mathcal{C}^\infty([0,1[\times\partial B(\mathbb{F}^n)$.
Next, set for each $t > 0$, the following $\mathbb{C}$-valued function on $\partial B(\mathbb{F}^n)$

$$\begin{aligned} g_t(\omega) &= \frac{1}{t}\int_{B(0,t)}F(x)P_{-\lambda}(x,\omega)d\mu(x) \\ &= \frac{1}{t}\int_0^{tht}\left(\int_{\partial B(\mathbb{F}^n)}F(r\theta)P_{-\lambda}(r\theta,\omega)d\theta\right)(1-r^2)^{-\sigma-1}r^{dn-1}dr \end{aligned}$$

Then, replacing F by its above series expansion in (8) and using again Proposition 3.1, the function g_t can be rewritten as:

$$g_t(\omega) = \frac{1}{t} \sum_{p,q\in\hat{K}_0} \left\{ \int_0^{tht} |\Phi_{\lambda,pq}(r)|^2 (1-r^2)^{-\sigma-1} dr \right\} f_{pq}(\omega)$$

Hence the $L^2(\partial B(\mathbb{F}^n))$-norm of the function g_t is given by:

$$\|g_t\|^2_{L^2(\partial B(\mathbb{F}^n))} = \frac{1}{t^2} \sum_{p,q\in\hat{K}_0} \left\{ \int_0^{tht} |\Phi_{\lambda,pq}(r)|^2 (1-r^2)^{-\sigma-1} dr \right\}^2 \|f_{pq}\|^2_{L^2(\partial B(\mathbb{F}^n))}.$$

Now using corollary 2.1 giving the uniform pointwise estimate on the generalized spherical function $\Phi_{\lambda,pq}$ we obtain:

$$\|g_t\|_{L^2(\partial B(\mathbb{F}^n))} \leq \gamma(1+|\lambda|+\frac{1}{|\lambda|})\|f\|_{L^2(\partial B(\mathbb{F}^n))}.$$

This shows that the functions $(g_t)_{t\geq 0}$ are in $L^2(\partial B(\mathbb{F}^n))$, uniformly in $t\in]0,+\infty[$. Also, since

$$\left\| |c(\lambda)|^{-2} g_t - f \right\|^2_{L^2(\partial B(\mathbb{F}^n))}$$

$$= \sum_{p,q\in\hat{K}_0} \left[\left\{ \frac{|c(\lambda)|^{-2}}{t} \int_0^{tht} |\Phi_{\lambda,pq}(r)|^2 (1-r^2)^{-\sigma-1} r^{dn-1} dr \right\}^2 - 1 \right] \times \|f_{pq}\|^2_{L^2(\partial B(\mathbb{F}^n))},$$

we can see, using the uniform pointwise boundedness of $\Phi_{\lambda,pq}$ given by corollary 2.1 as well as their uniform asymptotic behaviour in lemma 3.4, that we have:

$$\lim_{t\to+\infty} \left\| |c(\lambda)|^{-2} g_t - f \right\|^2_{L^2(\partial B(\mathbb{F}^n))} = 0.$$

This finishes the proof of the part ii) in Theorem A.

Remark 4.1. we should mention that the family of functions $(g_t)_{t>0}$ is in fact L^p-uniformly bounded, for every $p>1$, provided that $\|F\|_{*,p} < +\infty$.

Proof of Theorem B: The main tools that will be used in proving Theorem B, are the Key Lemma and the L^2-inversion type formula for the Poisson transform. In fact the Key Lemma shows that, for every $p>1$, the range $P_\lambda(L^p(\partial B(\mathbb{F}^n)))$ is continuously embedded in $E^*_{\lambda,p}(B(\mathbb{F}^n))$.
Now, in order to show that, for $p\geq 2$, the Poisson transform operator P_λ maps $L^p(\partial B(\mathbb{F}^n))$ onto $E^*_{\lambda,p}(B(\mathbb{F}^n))$, we first mention that for, $p\geq 2$, we have the inclusion $E^*_{\lambda,p}(B(\mathbb{F}^n)) \subset E^*_\lambda(B(\mathbb{F}^n))$.
Then, for given $F\in E^*_{\lambda,p}(B(\mathbb{F}^n))$, we know by Theorem A that there exists $f\in L^2(\partial B(\mathbb{F}^n))$ such that $P_\lambda f = F$, with the property that the function f can be recovered from F via the formula $f(\omega) = |c(\lambda)|^{-2} \lim_{t\to+\infty} g_t(\omega)$ in $L^2(\partial B(\mathbb{F}^n))$, where:

$$\begin{aligned} g_t(\omega) &= \frac{1}{t}\int_{B(0,t)} F(x) P_{-\lambda}(x,\omega) d\mu(x) \\ &= \frac{1}{t}\int_0^{tht} \left(\int_{\partial B(\mathbb{F}^n)} F(r\theta) P_{-\lambda}(r\theta,\omega) d\theta \right) (1-r^2)^{-\sigma-1} r^{dn-1} dr. \end{aligned}$$

Now we will show that the above given function is in fact in the space $L^p(\partial B(\mathbb{F}^n)), p \geq 2$.
Let h be a $\mathbb{C}$-valued continuous function on $\partial B(\mathbb{F}^n)$.
We have:

$$|c(\lambda)|^{-2} \lim_{t\to+\infty} \int_{\partial B(\mathbb{F}^n)} g_t(\omega)\overline{h(\omega)}d\omega = \int_{\partial B(\mathbb{F}^n)} f(\omega)\overline{h(\omega)}d\omega$$

But

$$\begin{aligned}
&\int_{\partial B(\mathbb{F}^n)} g_t(\omega)\overline{h(\omega)}d\omega \\
= &\int_{\partial B(\mathbb{F}^n)} \left[\frac{1}{t}\int_{B(0,t)} P_{-\lambda}(x,\omega)F(x)d\mu(x)\right]\overline{h(\omega)}d\omega \\
= &\int_{\partial B(\mathbb{F}^n)} \left[\frac{1}{t}\int_0^{tht}\int_{\partial B(\mathbb{F}^n)} P_{-\lambda}(r\theta,\omega)F(r\theta)(1-r^2)^{-\sigma-1}r^{dn-1}dr\right]\overline{h(\omega)}d\omega
\end{aligned}$$

Since the Poisson kernel $P_\lambda(r\theta,\omega)$ is symmetric in θ and ω, then by using the Fubini Theorem, the last integral can be rewritten as:

$$\frac{1}{t}\int_0^{tht}\left(\int_{\partial B(\mathbb{F}^n)} \overline{(P_\lambda h)(r\theta)}F(r\theta)d\theta\right)(1-r^2)^{-\sigma-1}r^{dn-1}dr$$

Thus by using the Holder inequality in the integral with respect to θ, we obtain

$$\left|\int_{\partial B(\mathbb{F}^n)} \overline{(P_\lambda h)(r\theta)}F(r\theta)d\theta\right| \leq \left[\int_{\partial B(\mathbb{F}^n)} |(P_\lambda h)(r\theta)|^q d\theta\right]^{\frac{1}{q}}\left[\int_{\partial B(\mathbb{F}^n)} |F(r\theta)|^p d\theta\right]^{\frac{1}{p}}$$

where q is such that $\frac{1}{p}+\frac{1}{q}=1$.
Next, the first part of the proof of Theorem B shows that, for every $q>1$, the following estimate holds

$$\left[\int_{\partial B(\mathbb{F}^n)} |(P_\lambda h)(r\theta)|^q d\theta\right]^{\frac{1}{q}} \leq (1-r^2)^{\frac{\sigma}{2}}A(\lambda,q)\|h\|_{L^q(\partial B(\mathbb{F}^n))}.$$

Hence,

$$\begin{aligned}
&\left|\int_{\partial B(\mathbb{F}^n)} g_t(\omega)\overline{h(\omega)}d\omega\right| \\
\leq\ &A(\lambda,q)\|h\|_{L^q(\partial B(\mathbb{F}^n))}\frac{1}{t}\int_0^{tht}(1-r^2)^{-\sigma/2-1}\left(\int_{\partial B(\mathbb{F}^n)}|F(r\theta)|^p d\theta\right)^{\frac{1}{p}} r^{dn-1}dr \\
\leq\ &A(\lambda,q)\|F\|_{*,p}\|h\|_{L^q(\partial B(\mathbb{F}^n))}.
\end{aligned}$$

Thus

$$\left|\int_{\partial B(\mathbb{F}^n)} f(\omega)\overline{h(\omega)}d\omega\right| \leq |c(\lambda)|^{-2}A(\lambda,q)\|F\|_{*,p}\|h\|_{L^q(\partial B(\mathbb{F}^n))}.$$

Taking the supremum over all continuous functions h with $\|h\|_{L^q(\partial B(\mathbb{F}^n))} \leq 1$, we deduce that $f \in L^p(\partial B(\mathbb{F}^n))$ with $|c(\lambda)|\|f\|_{L^p(\partial B(\mathbb{F}^n))} \leq A(\lambda,p)\|F\|_{*,p}$. This finishes the proof of Theorem B.

5. Proof of the Key Lemma

In this section we will establish the L^p-uniform boundedness in $r \in [0,1[$ of the family of superficial Poisson transforms $Q_r(\lambda)$:

$$[Q_r(\lambda)f](\theta) = \int_{\partial B(\mathbb{F}^n)} Q_r(\lambda,\theta,\omega) f(\omega) d\omega,$$

where $Q_r(\lambda,\theta,\omega) = |1 - r\langle\theta,\omega\rangle|^{-i\lambda-\sigma}$.
For this, we endow the homogeneous space $\partial B(\mathbb{F}^n) = \frac{K}{M}$ of the vector space $\mathbb{F}^n$ with the following non-isotropic metric $\rho(\theta,\omega) = |1 - \langle\theta,\omega\rangle|^{\frac{1}{2}}$ so that $(\partial B(\mathbb{F}^n), \rho)$ becomes a space of homogeneous type in the sense of Coifman and Weiss [4]. Hence, for the proof of the Key Lemma, we can use the technics of singular integrals in the setting of spaces of homogeneous type.

For this, we will establish the following two lemmas.

Lemma 5.1. *i) The triangle inequality*

$$\rho(a,c) \le \rho(a,b) + \rho(b,c)$$

holds for all a, b and c in $\overline{B(\mathbb{F}^n)}$.

ii) On $\partial B(\mathbb{F}^n)$, ρ is a metric.

iii) Let $B(\omega,\delta)$ be the ball centered at ω with radius δ with respect to ρ. Then the volume of $B(\omega,\delta)$ with respect to the superficial measure of $\partial B(\mathbb{F}^n)$ behaves as $\delta^{2\sigma}$.

Remark 5.2. Notice that in the case of $\mathbb{F} = \mathbb{R}$, $\rho(a,b) = \frac{\sqrt{2}}{2}|a-b|$. Also in the case of $\mathbb{F} = \mathbb{C}$ the metric $\rho(a,b)$ is the so called nonisotropic metric (see Rudin [12] for more informations). Hence, we will establish the above proposition only in the quaternion case.

Proof. Since ρ is $Sp(n)$-invariant, we may take $b = re\,(0 \le r \le 1)$, and we have to prove that:

$$|1 - \langle a,c\rangle| \le \{|1 - ra_1|^{\frac{1}{2}} + |1 - rc_1|^{\frac{1}{2}}\}^2. \tag{9}$$

Put $a = (a_1, a')$ and $c = (c_1, c')$, the left hand side of (9) is then:

$$\left|1 - a_1\bar{c}_1 - \left\langle a', c'\right\rangle\right| \le |1 - a_1\bar{c}_1| + \left|a'\right|\left|c'\right|,$$

Since

$$|1 - a_1\bar{c}_1| \le |1 - ra_1| + |1 - rc_1|,$$

and

$$\left|a'\right|^2 \le 1 - |a_1|^2 \le 1 - r^2|a_1|^2 \le 2|1 - ra_1|,$$

with similar estimate for $\left|c'\right|^2$, we get (9).

It is obvious that on $\partial B(\mathbb{H}^n)$ we have $\rho(a,b)=0$ if and only if $a=b$.
iii) Since the metric $\rho(\theta,\omega)$ and the superficial measure are $Sp(n)$-invariant we have:

$$\int_{\rho(\theta,\omega)<\delta} d\theta = \int_{\rho(\theta,e)<\delta} d\theta.$$

Now use Lemma 5.4 giving below to get:

$$\int_{\rho(\theta,e)<\delta} d\theta = c\int_{q\in\mathbb{C},|q|<1;\,|1-q|<\delta^2} (1-|q|^2)^{\sigma-d} dm(q).$$

By putting $1-q=t(\cos\theta+\sin\theta y)$ with $\theta\in[0,\pi]$; $t\geq 0$ and $y\in\mathbb{H}$; such that $Re\, y=0$ and $|y|=1$, we obtain:

$$\int_{\rho(\theta,e)<\delta} d\theta \;=\; \int_{\{0<\theta<\pi;0<t<\delta^2 \text{ and } |1-te^{i\theta}|<1\}} (2\cos\theta-t)^{\sigma-d}t^{\sigma-d+3}\sin^2\theta d\theta dt.$$

From which it is easy to see that

$$\int_{\rho(\theta,e)<\delta} d(\theta) \leq \gamma\delta^{2\sigma}.$$

Lemma 5.3. *Let λ be a non-zero real number and let*

$$Q_r(\lambda,\theta,\omega)=|1-r\langle\theta,\omega\rangle|^{-i\lambda-\sigma}.$$

Then there exists a positive constant $\gamma=\gamma(n)$ such that

$$i)\ \sup_{0\leq r<1}|Q_r(\lambda,\theta,\omega)|\leq\gamma[\rho(\theta,\omega)]^{-2\sigma}$$

for every $\theta,\omega\in\partial B(\mathbb{F}^n)$.

$$ii)\ \sup_{0\leq r<1}\Big|Q_r(\lambda,\theta,\omega)-Q_r(\lambda,\theta',\omega)\Big|\leq\gamma(1+|\lambda|)\frac{\rho(\theta,\theta')}{\rho(\theta,\omega)^{2\sigma+1}}$$

for every θ,θ',ω in $\partial B(\mathbb{F}^n)$ such that $\rho(\theta,\omega)\geq 2\rho(\theta,\theta')$

$$iii)\ \sup_{0\leq r<1}\left|\int_{\rho(\omega,\theta)\leq d} Q_r(\lambda,\theta,\omega)d\omega\right|\leq\gamma(1+\frac{1}{|\lambda|})$$

for every $d>0$.

Now, to prove the Key lemma, by classical device (see [Stein, 11]), the uniform estimates given in Lemma 5.3 and the behaviour of the ball (see iii) in Lemma 5.1) are sufficient to get the L^2-uniform boundedness of the operators $Q_r(\lambda)$. Next, to handle the L^p case, we can combine iii) in Lemma 5.1 and ii) in Lemma 5.3 to get the Hörmander condition on the Schwartz kernel $Q_r(\lambda,\theta,\omega)$.

$$\sup_{0\leq r\leq 1}\int_{\rho(\omega,e)\geq\rho(e,\theta')}\Big|Q_r(\lambda,e,\omega)-Q_r(\lambda,\theta',\omega)\Big|d\omega\leq b(\lambda),$$

where $b(\lambda)$ is some positive constant.

Proof of Lemma 5.3: For $r \in [0,1[$ and $d > 0$, we set

$$I(\lambda, r, d) = \int_{\rho(\theta,\omega)<d} |1 - r\langle\theta,\omega\rangle|^{i\lambda-\sigma} d\omega. \tag{10}$$

Then by using the K-invariance of the kernel $|1 - r\langle\theta,\omega\rangle|^{i\lambda-\sigma}$, and the metric $\rho(\theta,\omega)$ as well as the K- invariance of $d\omega$, we can rewrite $I(\lambda, r, d)$ as:

$$I(\lambda, r, d) = \int_{|1-\omega_1|<d^2} |1 - r\omega_1|^{-i\lambda-\sigma}, \tag{11}$$

where ω_1 is the first component of $\omega = (\omega_1, \ldots, \omega_n) \in S^{4n+1}$.

Notice that (11) is uniformly bounded in $r \in [0, \frac{1}{2}[$ and in $d > 0$.

To show its uniform boundedness in $r \in [\frac{1}{2}, 1[$ we will use the following standard calculus lemma.

Lemma 5.4. *Let f be a $\mathbb{C}$-valued function on $\partial B(\mathbb{F}^n)$ with $f(\omega_1, \omega_2, \ldots, \omega_n) = g(\omega_1)$. Then we have:*

$$\int_{\partial B(\mathbb{F}^n)} f(\omega) d\omega = c \int_{\{q\in\mathbb{F},\, |q|<1\}} g(q)(1 - |q|^2)^{\sigma-d} dm(q),$$

for some positive constant c.

Applying the above lemma, we see that the integral $I(\lambda, r, d)$ can be rewritten as:

$$I(\lambda, r, d) = \int_{\{q\in\mathbb{F},\, |q|<1 \quad \text{and} \quad |1-q|<d^2\}} |1 - rq|^{-i\lambda-\sigma} (1 - |q|^2)^{\sigma-d} dm(q).$$

We will establish the uniform boundedness of the previous integral only on the quaternion case.

Therefore $I(\lambda, r, d)$ becomes:

$$I(\lambda, r, d) = \int_{\{q\in\mathbb{H},\, |q|<1 \quad \text{and} \quad |1-q|<d^2\}} |1 - rq|^{-i\lambda-2n-1} (1 - |q|^2)^{2n-3} dm(q).$$

Now using the following change of variables $\omega = 1 - rq$ as well as the polar coordinates on $\mathbb{H}$: $\omega = t(\cos\theta + y\sin\theta)$ with $\theta \in [0,\pi]$; $t \geq 0$ and $y \in \mathbb{H}$, such that $Re\, y = 0$ and $|y| = 1$, the above integral can be written-up to the area of the unit sphere of $\mathbb{R}^4$ as:

$$I(\lambda, r, d) = \int_{\Gamma_{d,r}} t^{-i\lambda-2n-1} \left[\left|r^2 - \left|1 - 2t\cos\theta + t^2\right|\right|\right]^{2n-3} t^3 dt \sin^2\theta d\theta,$$

where $\Gamma_{d,r}$ is the set

$$\Gamma_{d,r} = \{(t,\theta) \in \mathbb{R}^+ \times [0,\pi[; \left|1 - te^{i\theta}\right| < r \quad \text{and} \quad \left|(1-r) - te^{i\theta}\right| < rd^2\}.$$

Thus:

$$|I(\lambda, r, d)| \leq 2^{-(4n-2)} \left| \int_{\Gamma_{d,r}} t^{-i\lambda-(2n-1)+1} \left|r^2 - \left|1 - te^{i\theta}\right|^2\right|^{2n-3} dt \sin^2\theta d\theta \right|.$$

Next, set $N = 2n-1$ and replacing $\sin^2\theta$ by:

$$\sin^2\theta = \frac{1}{4}(e^{2i\theta} + e^{-2i\theta} - 2),$$

we can use the result in [3] to conclude that, for every $r \in [\frac{1}{2}, 1[$ and every $\lambda \in \mathbb{R}\backslash\{0\}$

$$|I(\lambda, r, d)| \leq \gamma(1 + \frac{1}{|\lambda|}),$$

for some positive constant $\gamma = \gamma(n)$. This finishes the proof of iii).

We end this paper by further remarks and comments.

Remark 5.5. We should notice to the reader that the natural method in proving that $\|P_\lambda f\|_*$ is finite for every $f \in L^2(\partial B(\mathbb{F}^n))$ in Theorem A, is to take spherical harmonic expansion of $f \in L^2(\partial B(\mathbb{F}^n))$. But doing so we will be faced on establishing some uniform estimates on Jacobi functions. More precisely we prove estimates of the following type:

$$\sup_{t\geq 0} \frac{1}{t}\int_{B(0,t)} |\Phi_{\lambda,pq}|^2 (1-|x|^2)^{-\sigma-1} dm(x) \leq \left[\gamma(1 + |\lambda| + \frac{1}{|\lambda|})\right]^2,$$

for some constant independently on the pair $p, q \in \hat{K}_0$.
The needed estimate is closely related to Strichartz conjecture for general Jacobi functions (see conjecture 5.4 in [12]), which reads in our case as:

$$\sup_{t\geq 0} \frac{1}{t}\int_{B(0,t)} |\Phi_{\lambda,pq}|^2 (1-|x|^2)^{-\sigma-1} dm(x) \leq \gamma |c(\lambda)|^2,$$

for every $p, q \in \hat{K}_0$.

Here, we have turn around this conjecture by discussing the uniform L^2-boundedness of the family of superficial Poisson integrals $Q_r(\lambda)$ (see the Key Lemma of this paper). But our constant is not too sharp comparatively to the constant $|c(\lambda)|^2$ in Strichartz conjecture.

Remark 5.6. We should mention that along the lines of the above proof and the results given in the appendix of [3] we can establish the following " reduction formula from $\partial B(\mathbb{H}^n)$ to $\partial B(\mathbb{C}^N)$, $N = 2n-1$"

$$\begin{aligned}
&\int_{\partial B(\mathbb{H}^n)} |1 - r\langle\theta,\omega\rangle|^{-i\lambda-2n-1} d\omega \\
= \quad & \frac{C(n)}{r^2} \times \Big\{ \int_{\partial B(\mathbb{C}^N)} |1 - r\langle u,v\rangle|^{-i\lambda-2n+1} dv \\
& - \int_{\partial B(\mathbb{C}^N)} |1 - r\langle u,v\rangle|^{-i\lambda-2n+1} \left(\frac{1 - r\langle u,v\rangle}{\overline{1 - r\langle u,v\rangle}} + \frac{\overline{1 - r\langle u,v\rangle}}{1 - r\langle u,v\rangle} \right) dv \Big\},
\end{aligned}$$

where $\langle\theta,\omega\rangle$ is the quaternion scalar-product, whereas $\langle u,v\rangle$ is the usual Hermitian scalar in $\mathbb{C}^n$.

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References

[1] van den Ban, E. P., and H. Schlichtkrull, *Asymptotic expansions and boundary values of eigenfunctions on Riemannian symmetric spaces*, J. reine angew. Math. **380** (1987), 108–165.

[2] Boussejra, A., and A. Intissar, *Caractérisation des intégrales de Poisson-Szegö $L^2(\partial B(F)^n)$ dans la boule de Bergman B^n $(n \geq 2)$*, C. R. Acad. Sci. Paris I (1992), 309–313.

[3] —, *L^2-Concrete Spectral Analysis of the Invariant Laplacian $\Delta_{\alpha\beta}$ in the Unit Complex Ball* $\mathbf{B^n}$, J. Funct. Anal. **160**, (1998), 115–140.

[4] Coifman, R., and G. Weiss, *Extension of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. **83** (1987), 569–644.

[5] Faraut, J., *Distributions sphériques sur les espaces hyperboliques*, J. Math. Pures. Appl. **58** (1979), 379–444.

[6] Helgason, S., *Eigenspaces of the Laplacian; integral representation and irreducibility*, J. Funct. Anal. **17** (1974), 328–353.

[7] Johnson, D. K., and R. N. Wallach, *Compositon series and interwining operators for the spherical principal series*, I. Trans. Amer. Math. Soc. **229** (1977), 137–173.

[8] Kashiwara, M., A. Kowata, K. Minemura, K. Okamoto, T. Oshima, and M. Tanaka, *Eigenfunctions of invariant differential operators on a symmetric space*, Ann. of Math **107** (1978), 1–39.

[9] Lewis, J., *Eigenfunctions on symmetric spaces with distribution-valued boundary forms*, J. Funct. Anal. **29** (1978), 287–307.

[10] Rudin, W., “Function Theory in the unit complex ball of $\mathbb{C}^n$,” Springer-Verlag, Berlin, New York, 1980.

[11] Stein, E. M., “Harmonic analysis,” Princeton Univ. Press, NJ, 1993.

[12] Strichartz, R., *Harmonic analysis as spectral theory of Laplacians*, J. Funct. Anal. **87** (1989), 51–178.

[13] —, *Corrigendum to Harmonic analysis as spectral theorry of Laplacians*, J. Funct. Anal. **109** (1992), 457–460.

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