Journal of Lie Theory
Volume **12** (2002) 205–216

The Approximative Centre of a Lie Algebra

Grant Cairns

Communicated by M. Cowling

Abstract. This paper examines the approximative centre of a Lie algebra; this is the set of elements which are not sent uniformly to infinity by the adjoint action of the underlying Lie group.

The centre $Z(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is the set of elements that are fixed by the adjoint action of the underlying Lie group. A natural generalization would be to consider the set of elements with *bounded* orbit under the adjoint action. It turns out that another useful notion is the set of elements which are not sent uniformly to infinity. More generally, one has:

Definition 1. If $\mathfrak{g}$ is a finite dimensional real Lie algebra, and $\mu \in \mathbb{R}$, let $\mathfrak{C}_\mu(\mathfrak{g})$, or simply $\mathfrak{C}_\mu$, denote the complement of the set of elements $X \in \mathfrak{g}$ for which there exists a neighbourhood $U \ni X$ and an element $Y \in \mathfrak{g}$ such that the map

$$(\exp t\mu.\operatorname{tr}(\operatorname{ad} Y)) \exp \operatorname{ad}(t.Y)|_U : U \to \mathfrak{g}$$

tends uniformly to infinity as t goes to positive infinity.

When $\mathfrak{g}$ is unimodular, $\mathfrak{C}_\mu$ is independent of μ and is called the *approximative centre* of $\mathfrak{g}$; this notion is due to Etienne Ghys and was introduced in [4]. Some of the ideas presented below were developed in [3], but were not published. A further application for $\mu = 1$ is given in [5]. The present paper is devoted to the following:

Theorem. *Let $\mathfrak{g}$ be a finite dimensional real Lie algebra. Then:*

(a) $\mathfrak{C}_\mu = \mathfrak{s} \oplus \mathfrak{n}$, *where $\mathfrak{s}$ is a compact semisimple direct summand of $\mathfrak{g}$, and $\mathfrak{n}$ is a nilpotent characteristic ideal of $\mathfrak{g}$. Moreover:*
  - (i) $\mathfrak{n}$ *is metabelian,*
  - (ii) $Z(\mathfrak{n})(= Z(\mathfrak{C}_\mu))$ *is an ideal of $\mathfrak{g}$ and* $\mathfrak{n} = Z(\mathfrak{n}) + [\mathfrak{n}, \mathfrak{g}]$,
  - (iii) $\mathfrak{C}_\mu \subset Z(\mathfrak{g}) + [\mathfrak{g}, \mathfrak{g}]$.

(b) *If $\mathfrak{g}$ is unimodular or $\mu = 0$, then $\mathfrak{s}$ is the maximal compact semisimple direct summand of $\mathfrak{g}$, and* $Z(\mathfrak{g}) \subset \mathfrak{n}$.

(c) *If $\mathfrak{g}$ is not unimodular and $\mu \neq 0$, then $\mathfrak{s} = 0$ and $\mathfrak{n}$ is abelian.*

ISSN 0949–5932 / \$2.50

Corollary. *With the above notation, one has:*

(a) $\mathfrak{C}_\mu$ *is unimodular,*

(b) *If* $\mathfrak{g}$ *is nilpotent,* $\mathfrak{C}_\mu$ *is abelian,*

(c) *If* $\mathfrak{g}$ *is solvable,* $\mathfrak{C}_\mu$ *is metabelian,*

(d) $\mathfrak{C}_\mu = \mathfrak{g}$ *if and only if* $\mathfrak{g}$ *is the Lie algebra of a compact Lie group.*

Following the proofs of the theorem and its corollary, the paper concludes with 3 examples.

Notation. In the following, if $\mathfrak{k}$ is a subalgebra of $\mathfrak{g}$, then

(a) $R(\mathfrak{k})$ denotes the radical of $\mathfrak{k}$, and we write $R(\mathfrak{g}) = \mathfrak{r}$,

(b) $Z(\mathfrak{k})$ is the centre of $\mathfrak{k}$, and $Z_{\mathfrak{g}}(\mathfrak{k})$ is the centralizer of $\mathfrak{k}$ in $\mathfrak{g}$.

Proof of the theorem. For convenience, throughout this proof, we will use the l_1 norm $\|(z_1, \dots, z_n)\| = \sum_{i=1}^n |z_i|$ on $\mathbb{C}^n$, rather than the Euclidean norm. We begin by studying the automorphism $\exp(t.C)$ of $\mathbb{C}^n$, where C is the following Jordan form:

$$C = \begin{pmatrix} \lambda & & & & 0 \\ 1 & \lambda & & & \\ & \ddots & \ddots & & \\ & & 1 & \lambda & \\ 0 & & & 1 & \lambda \end{pmatrix}.$$

First, note that

$$(*) \qquad \exp t.C = \exp t.\lambda \begin{pmatrix} 1 & & & & 0 \\ t & 1 & & & \\ t^2/2! & t & 1 & & \\ \vdots & \ddots & \ddots & \ddots & \\ t^{n-1}/(n-1)! & \dots & t^2/2! & t & 1 \end{pmatrix}.$$

Lemma 1. *Let* $T \in \mathbb{R}$.

(a) *If the real part of* $\lambda + T$ *is positive (or negative), then for all* $x \in \mathbb{C}^n \setminus \{0\}$, *there exists a neighbourhood* U *of* x *in* $\mathbb{C}^n$ *such that*

$$(\exp t.T) \exp t.C|_U : U \to \mathbb{C}^n$$

tends uniformly to infinity as $t \to +\infty$ *(or* $t \to -\infty$*).*

(b) *If* $\lambda + T$ *is imaginary, then for all* $x = (x_1, \dots, x_n) \in \mathbb{C}^n$, *the following conditions are equivalent:*

(i) *there exists a neighbourhood* U *of* x *in* $\mathbb{C}^n$ *such that*

$$(\exp t.T) \exp t.C|_U : U \to \mathbb{C}^n$$

tends uniformly to infinity when t *tends to infinity,*

(ii) *there exists* $1 \le l < \frac{n+1}{2}$ *such that* $x_l \neq 0$.

Proof. (a) Suppose $\lambda + T > 0$. Setting $A_t = (\exp t.T)\exp t.C$, one has from (*):

$$(A_t)^{-1} = (\exp -t.(\lambda + T)) \begin{pmatrix} 1 & & & & 0 \\ -t & 1 & & & \\ t^2/2! & -t & 1 & & \\ \vdots & \ddots & \ddots & \ddots & \\ (-t)^{n-1}/(n-1)! & \dots & t^2/2! & -t & 1 \end{pmatrix}.$$

Let $x \in \mathbb{C}^n \backslash \{0\}$, and $0 < K < 1$. Let U be a relatively compact neighbourhood of x in $\mathbb{C}^n$ which doesn't contain zero in its closure.

Let $B > 0$. Clearly, there exists $m \in \mathbb{N}$ such that $\|z\|/K^m > B$ for all $z \in U$. One has

$$\|z\| = \|A_t^{-1} \circ A_t(z)\| \leq \|A_t^{-1}\|.\|A_t(z)\|$$

and so $\|A_t(z)\| \geq \|z\|/\|A_t^{-1}\|$. There exists $t_0 > 0$ such that for all $t \geq t_0$ the operator norm $\|A_t^{-1}\|$ of the automorphism A_t^{-1} is less than K. Hence, for all $t > t_0$,

$$\|A_{mt}(z)\| \geq \|z\|/\|A_{mt}^{-1}\| = \|z\|/\|A_t^{-m}\| \geq \|z\|/\|A_t^{-1}\|^m > \|z\|/K^m > B.$$

Thus, $\|A_s(z)\| > B$ for all $z \in U$ and $s > m.t_0$. The case where $\mathrm{Re}(\lambda + T) < 0$ is analogous.

(b) First suppose that $x_l = 0$ for all $1 \leq l < (n+1)/2$. We will show that there does not exist any neighbourhood U of x for which the map $(\exp t.T)\exp t.C|_U : U \to \mathbb{C}^n$ tends uniformly to infinity when t tends to infinity. Let U be a relatively compact neighbourhood of x. It suffices to show that there exists $N > 0$ such that, for all sufficiently large $t > 0$, there exists $z_t \in U$ such that $\|(\exp t.T)\exp t.C(z_t)\| < N$.

Let k be the smallest natural number such that $x_k \neq 0$. By hypothesis, $2k \geq n+1$. For all sufficiently large $t > 0$ we will construct an element z_t of U of the form $z_t = (\epsilon_1(t), \dots, \epsilon_{k-1}(t), x_k, x_{k+1}, \dots, x_n)$. First note that if $z = (\epsilon_1, \dots, \epsilon_{k-1}, x_k, \dots, x_n) \in \mathbb{C}^n$, then, by $(*)$, one has

$$\begin{aligned} \|(\exp t.T)\exp tC(z)\| &= |\epsilon_1| + |\epsilon_1 t + \epsilon_2| + \dots + |\epsilon_1 \frac{t^{k-1}}{(k-1)!} + \dots + \epsilon_{k-1} t + x_k| \\ &+ |\epsilon_1 \frac{t^k}{k!} + \dots + \epsilon_{k-1}\frac{t^2}{2} + x_k t + x_{k+1}| + \dots + |\epsilon_1 \frac{t^{n-1}}{(n-1)!} + \dots + x_n|. \end{aligned}$$

We choose the $(k-1)$ numbers ϵ_j such that the last $(n-k)$ terms of the preceding expression are zero. To do this, consider the following equation:

$$(1) \qquad \begin{pmatrix} \frac{t^k}{k!} & & \frac{t^2}{2} \\ & \ddots & \\ \frac{t^{n-1}}{(n-1)!} & & \frac{t^{n+1-k}}{(n+1-k)!} \end{pmatrix} \begin{pmatrix} \epsilon_1(t) \\ \vdots \\ \epsilon_{k-1}(t) \end{pmatrix} = \begin{pmatrix} -x_k t - x_{k+1} \\ \vdots \\ -x_k \frac{t^{n-k}}{(n-k)!} - \dots - x_n \end{pmatrix}$$

We have $(k-1)$ variables (the $\epsilon_j(t)$) and $(n-k)$ equations. Because $2k \geq n+1$, one has $n-k \leq k-1$. In order to solve equation (1), set $\epsilon_j(t) = 0$ for all $n-k < j \leq k-1$, and solve the following equation:

$$\begin{pmatrix} \frac{t^k}{k!} & & \frac{t^{2k-n+1}}{(2k-n+1)!} \\ & \ddots & \\ \frac{t^{n-1}}{(n-1)!} & & \frac{t^k}{k!} \end{pmatrix} \begin{pmatrix} \epsilon_1(t) \\ \vdots \\ \epsilon_{n-k}(t) \end{pmatrix} = \begin{pmatrix} -x_k t - x_{k+1} \\ \vdots \\ -x_k \frac{t^{n-k}}{(n-k)!} - \cdots - x_n \end{pmatrix} \tag{2}$$

This equation possesses a unique solution $(\epsilon_1(t), \ldots, \epsilon_{n-k}(t))$ since the coefficient matrix is invertible for all $1 \leq 2k-n+1 \leq k \leq n$ and $t > 0$. Let F denote the coefficient matrix of the left hand side of (2), and let F_j denote the matrix obtained by replacing the j^{th} column of F by the right hand side of (2). By Cramer's method, $\epsilon_j(t) = \det F_j / \det F$, for all $1 \leq j \leq n-k$. Notice that $\det F = at^{k(n-k)}$ for some non-zero real a, while $\det F_j$ is a polynomial in t of degree $k(n-k)-k+j$ which is divisible by $t^{k(n-k)-n+j}$. Hence the solution $(\epsilon_1(t), \ldots, \epsilon_{n-k}(t))$ of equation (2) has the following form; for all $1 \leq j \leq k-1$

$$\epsilon_j(t) = \frac{a_{j1}}{t^{k-j}} + \frac{a_{j2}}{t^{k-j+1}} + \cdots + \frac{a_{j(n-k+1)}}{t^{n-j}} \tag{3}$$

where the coefficients a_{jl} are constants depending only on $x_k, \ldots, x_n$. Note that as $2k \geq n+1$, one has $k-j \geq 1$. Let

$$z_t = (\epsilon_1(t), \ldots, \epsilon_{k-1}(t), x_k, \ldots, x_n) \in \mathbb{C}^n. \tag{4}$$

By (3), it is clear that $z_t \in U$ for all sufficiently large t. By construction, one has

$$\begin{aligned} \|(\exp t.T) \exp tC(z_t)\| &= |\epsilon_1(t)| + |t\epsilon_1(t) + \epsilon_2| \\ &\quad + \cdots + |\epsilon_1(t) \frac{t^{k-1}}{(k-1)!} + \cdots + \epsilon_{n-k}(t) t^{2k-n} + x_k|. \end{aligned}$$

But, by (3), each of the terms of the preceding expression is bounded as $t \to \infty$. In other words, there exists $N > 0$ such that $\|(\exp t.T) \exp tC(z_t)\| < N$ for all sufficiently large t.

Conversely, we will show that if there exists $1 \leq l < (n+1)/2$ such that $x_l \neq 0$, then there exists a neighbourhood U of x such that the map $(\exp t.T) \exp t.C|_U \colon U \to \mathbb{C}^n$ tends uniformly to infinity as $t \to \infty$. Let k again denote the smallest natural number such that $x_k \neq 0$. By hypothesis, $2k < n+1$. Let U be a relatively compact neighbourhood of x in $\mathbb{C}^n$ such that for all $z = (z_1, \ldots, z_n) \in U$ one has $z_k \neq 0$. One has, for all $t > 0$,

$$\|(\exp t.T) \exp t.C(z)\| = \|z_1\| + \|z_1 t + z_2\| + \cdots + \|\frac{t^{n-1}}{(n-1)!} z_1 + \cdots + z_n\|.$$

Our strategy is consider only the k last terms, and to eliminate the terms z_j for $j < k$. Set $T_j(z_1, \ldots, z_j) = \|\frac{t^{j-1}}{(j-1)!} z_1 + \frac{t^{j-2}}{(j-2)!} z_2 + \cdots + z_j\|$. Note that

$$T_j(z_1, \ldots, z_j) + T_{j+1}(z_1, \ldots, z_{j+1}) = \frac{t^{j-1}}{(j-1)!} \left(\|z_1 + a\| + \frac{t}{j} \|z_1 + b\| \right),$$

where

$$a = \frac{(j-1)!}{t^{j-1}}\left(\frac{t^{j-2}}{(j-2)!}z_2 + \cdots + z_j\right),$$
$$b = \frac{j!}{t^j}\left(\frac{t^{j-1}}{(j-1)!}z_2 + \cdots + z_{j+1}\right).$$

For $t > j$ one has $\|z_1 + a\| + \frac{t}{j}\|z_1 + b\| \geq \|b - a\|$ and so

$$T_j(z_1, \ldots, z_j) + T_{j+1}(z_1, \ldots, z_{j+1}) \geq \frac{t^{j-1}}{(j-1)!}\|b-a\|$$
$$= \|\frac{t^{j-2}}{(j-1)!}z_2 + \frac{t^{j-3}}{(j-2)!}2z_3 + \cdots + \frac{t^{j-k}}{(j-k+1)!}(k-1)z_k + \cdots + \frac{j}{t}z_{j+1}\|.$$

Thus,

$$T_j(z_1, \ldots, z_j) + T_{j+1}(z_1, \ldots, z_{j+1}) \geq \frac{1}{t}T_j(z_2, 2z_3, \ldots, jz_{j+1}). \tag{5}$$

From above,

$$\|(\exp t.T)\exp t.C(z)\| = \sum_{j=1}^{n} T_j(z_1, \ldots, z_j) \geq \sum_{j=n-k+1}^{n} T_j(z_1, \ldots, z_j).$$

Hence, by (5), for all $t > n$,

$$\|(\exp t.T)\exp t.C(z)\| \geq \frac{1}{2t}\sum_{j=n-k+1}^{n-1} T_j(z_2, 2z_3, \ldots, jz_{j+1})$$
$$\geq \frac{1}{2^2t^2}\sum_{j=n-k+1}^{n-2} T_j(2z_3, 3!z_4, \ldots, j(j+1)z_{j+2})$$
$$\vdots$$
$$\geq \frac{1}{2^{k-1}t^{k-1}}T_{n-k+1}((k-1)!z_k, \frac{k!}{1!}z_{k+1}, \ldots, \frac{(n-1)!}{(n-k)!}z_n)$$
$$= \frac{1}{2^{k-1}}\left\|\frac{t^{n-2k+1}(k-1)!}{(n-k)!}z_k + \frac{t^{n-2k}k!}{(n-k-1)!1!}z_{k+1} + \cdots + \frac{(n-1)!}{(n-k)!t^{k-1}}z_n\right\|.$$

So, as $n - 2k + 1 \geq 1$ and $z_k \neq 0$ for all $z \in U$, it is clear that for all $M > 0$ there exists $N > 0$ such that $\|(\exp t.T)\exp t.C(z)\| > M$ for all $t > N$ and $z \in U$. This establishes Lemma 1. ■

Let y be an element of $\mathfrak{g}$. Let M_y denote the Jordan form of the linear endomorphism of $\mathbb{C}^n$ determined by $\mathrm{ad}(y)$, and denote the Jordan blocks of M_y by $C_1(y), \ldots, C_r(y)$, and the corresponding eigenvalues $\lambda_1, \ldots, \lambda_r$. If $x \in \mathfrak{g}$, we write $x_y = x^1 \oplus \cdots \oplus x^r \in \mathbb{C}^n$ where $x^i = (x_1^i, \ldots, x_{n_i}^i)$ are the components of x with respect to the decomposition of $\mathbb{C}^n$ determined by the Jordan form M_y; one has $M_y(x_y) = C_1(y)(x^1) \oplus \cdots \oplus C_r(y)(x^r)$.

Lemma 2. *Let x, $y \in \mathfrak{g}$. The following two conditions are equivalent:*

(a) *there exists a neighbourhood U of x in $\mathfrak{g}$ such that*

$$(\exp t\mu.\operatorname{tr}(\operatorname{ad} y)) \exp \operatorname{ad}(t.y)|_U \colon U \to \mathfrak{g}$$

tends uniformly to infinity as $t \to \infty$,

(b) *there exists $1 \le i \le r$ and a neighbourhood U_i of x^i in $\mathbb{C}^{n_i}$ such that*

$$(\exp t\mu.\operatorname{tr}(\operatorname{ad} y)) \exp(t.C_i(y))|_{U_i} \colon U_i \to \mathbb{C}^{n_i}$$

tends uniformly to infinity as $t \to \infty$.

Proof. (a) follows immediately from (b) because

$$\| \exp \operatorname{ad}(t.y)(z) \| = \sum_{i=1}^{r} \| \exp \operatorname{ad}(t.C_i(y))(z^i) \|$$

for all $z \in \mathbb{C}^n$. To see the converse, notice that if

$$(\exp t\mu.\operatorname{tr}(\operatorname{ad} y)) \exp(t.C_i(y))|_{U_i} \colon U_i \to \mathbb{C}^{n_i}$$

does not tend uniformly to infinity as $t \to \infty$, then the point $z_t \in \mathbb{C}^{n_i}$ given by (4), in the proof of Lemma 1, has real coordinates; the required result follows easily. ∎

Set

$$\begin{aligned}\mathfrak{C}_\mu(y,i) = \{x^i \in \mathbb{C}^{n_i} \mid x^i \equiv 0, \text{ if } \operatorname{Re}(\mu \operatorname{tr}(\operatorname{ad} y) + \lambda_i) \neq 0, \text{ and} \\ x^i_l = 0 \text{ for all } 1 \le l < (n_i+1)/2, \text{ if } \operatorname{Re}(\mu \operatorname{tr}(\operatorname{ad} y) + \lambda_i) = 0\}.\end{aligned}$$

From Lemmas 1 and 2, one has:

Lemma 3. *With the notation introduced above, the approximative centre of $\mathfrak{g}$ is*

$$\mathfrak{C}_\mu = \bigcap_{y \in \mathfrak{g}} \{x \in \mathfrak{g} \mid x_y \in \bigoplus_{i=1}^{r} \mathfrak{C}_\mu(y,i)\}.$$

In particular, $\mathfrak{C}_\mu$ is a vector space. In fact, we have:

Lemma 4. *$\mathfrak{C}_\mu$ is a characteristic ideal of $\mathfrak{g}$.*

Proof. Let $\phi \colon \mathfrak{g} \to \mathfrak{g}$ be a derivation and consider the automorphism $\Phi_s = e^{s\phi}$ of $\mathfrak{g}$. Note that $\mathfrak{C}_\mu$ is Φ_s-invariant. Indeed, if x and y belong to $\mathfrak{g}$, then

$$\Phi_s(\exp(t.\operatorname{ad} y)(x)) = \exp(t.\operatorname{ad}(\Phi_s y))(\Phi_s x)$$

and hence

$$\Phi_s(\exp \mu \operatorname{tr}(\operatorname{ad}(t.y)) \exp(t.\operatorname{ad} y)(x)) = (\exp \mu \operatorname{tr}(\operatorname{ad}(t.\Phi_s y)) \exp(t.\operatorname{ad}(\Phi_s y))(\Phi_s x).$$

It follows that if x is an element of $\mathfrak{C}_\mu$, then $\Phi_s x$ is too, for all $s \in \mathbb{R}$. Differentiating with respect to s, one has that $\mathfrak{C}_\mu$ is invariant under ϕ. ∎

Remark 1. Let $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ denote the complexification of $\mathfrak{g}$. One can define the *approximative centre* $\mathfrak{C}_\mu(\mathfrak{g}_{\mathbb{C}})$ of $\mathfrak{g}_{\mathbb{C}}$ is an analogous manner to that of $\mathfrak{g}$. It is clear from the above that $\mathfrak{C}_\mu(\mathfrak{g}_{\mathbb{C}})$ is an ideal of $\mathfrak{g}_{\mathbb{C}}$. However, it is not true in general that $\mathfrak{C}_\mu(\mathfrak{g}_{\mathbb{C}}) = \mathfrak{C}_\mu \otimes_{\mathbb{R}} \mathbb{C}$. Nevertheless, $\mathfrak{C}_\mu \otimes_{\mathbb{R}} \mathbb{C}$ is clearly an ideal of $\mathfrak{g}_{\mathbb{C}}$. Thus, if $y \in \mathfrak{C}_\mu$ and $z \in \mathfrak{g}_{\mathbb{C}}$, then $\mathrm{ad}(y)(z) \in \mathfrak{C}_\mu \otimes_{\mathbb{R}} \mathbb{C}$ and in particular, $C_i(y)(z) \in \mathfrak{C}_\mu(y,i)$, for all i.

If $y \in \mathfrak{C}_\mu$, let $\mathrm{ad}_{\mathfrak{C}_\mu}(y)\colon \mathfrak{C}_\mu \to \mathfrak{C}_\mu$ denote the restriction of $\mathrm{ad}(y)$ to $\mathfrak{C}_\mu$.

Lemma 5. *If $y \in \mathfrak{C}_\mu$, then:*

(a) *the eigenvalues $\lambda_1, \dots, \lambda_r$ of $\mathrm{ad}(y)\colon \mathfrak{g} \to \mathfrak{g}$ are all imaginary, whence $\mathrm{tr}(\mathrm{ad}\, y) = 0$,*

(b) *for all i, one has:*

  (i) *if $\lambda_i = 0$, then the corresponding Jordan block has size at most 3; that is, $n_i \le 3$,*

  (ii) *if $\lambda_i \ne 0$, then the corresponding Jordan block has size 1; that is, $n_i = 1$,*

(c) $\mathfrak{C}_\mu = \ker(\mathrm{ad}^2_{\mathfrak{C}_\mu}(y)) \oplus \mathrm{im}(\mathrm{ad}^2_{\mathfrak{C}_\mu}(y))$,

Proof. First note that y is an eigenvector of $\mathrm{ad}(y)$ with eigenvalue 0, and so $\mathrm{Re}(\mu\,\mathrm{tr}(\mathrm{ad}\, y)) = 0$, by Lemma 3. Hence $\mu\,\mathrm{tr}(\mathrm{ad}\, y) = 0$. Fix i and let $\{Z^i_j \mid 1 \le j \le n_i\}$ be the canonical basis of $\mathbb{C}^{n_i}$. Using the notation from above, one has

$$C_i(y)(Z^i_j) = \begin{cases} \lambda_i Z^i_j + Z^i_{j+1} & ; \text{ if } j < n_i, \\ \lambda_i Z^i_j & ; \text{ if } j = n_i. \end{cases} \tag{6}$$

(a). If $\mathrm{Re}(\lambda_i) \ne 0$, then $\mathrm{Re}(\lambda_i + \mu\,\mathrm{tr}(\mathrm{ad}\, y)) \ne 0$, since $\mu\,\mathrm{tr}(\mathrm{ad}\, y) = 0$. By Lemmas 1 and 2, one has $Z^i_{n_i} \notin \mathfrak{C}_\mu(y,i)$, but by (6) and the previous remark, one has $Z^i_{n_i} = \frac{1}{\lambda_i} C_i(y)(Z^i_{n_i}) \in \mathfrak{C}_\mu(y,i)$, which is a contradiction. Thus $\mathrm{Re}(\lambda_i) = 0$, for all i.

(b)(i). If $\lambda_i = 0$, then by the previous remark, $Z^i_2 = C_i(y)(Z^i_1) \in \mathfrak{C}_\mu(y,i)$. Hence, by Lemma 1(b), $2 \ge \frac{n_i+1}{2}$; that is, $n_i \le 3$.

(b)(ii). If $\lambda_i \ne 0$, then by the previous remark,

$$Z^i_{n_i} = \frac{1}{\lambda_i} C_i(y)(Z^i_{n_i}) \in \mathfrak{C}_\mu(y,i)$$

and $Z^i_j = \frac{1}{\lambda_i}(C_i(y)(Z^i_j) - Z^i_{j+1})$ for all $1 \le j \le n_i - 1$. So by induction, $Z^i_j \in \mathfrak{C}_\mu(y,i)$ for all $1 \le j \le n_i$. In particular, $Z^i_1 \in \mathfrak{C}_\mu(y,i)$, and hence, by Lemma 1(b), $1 \ge \frac{n_i+1}{2}$; that is, $n_i = 1$.

(c). Now consider $\mathrm{ad}^2(y)$. Note that:

(i) if $n_i = 1$, $\mathfrak{C}_\mu(y,i) = \langle Z^i_1 \rangle$, and $C^2_i(y)(Z^i_1) = 0$ if $\lambda_i = 0$, and $Z^i_1 = \frac{1}{\lambda_i^2} C^2_i(Z^i_1)$ if $\lambda_i \ne 0$.

(ii) if $n_i = 2$, $\mathfrak{C}_\mu(y,i) = \langle Z^i_2 \rangle$, and $C^2_i(y)(Z^i_2) = 0$.

(iii) if $n_i = 3$, $\mathfrak{C}_\mu(y,i) = \langle Z^i_2, Z^i_3 \rangle$, $C^2_i(y)(Z^i_2) = 0$ and $C^2_i(y)(Z^i_3) = 0$.

Hence each of the basis elements of $\mathfrak{C}_\mu(y,i)$ belongs to either the kernel or the image of $C^2_i(y)$. Since $\mathfrak{C}_\mu \subset \{x \in \mathfrak{g} \mid x_y \in \bigoplus_{i=1}^r \mathfrak{C}_\mu(y,i)\}$, it follows that $\mathfrak{C}_\mu$ is the vector subspace sum $\ker(\mathrm{ad}^2_{\mathfrak{C}_\mu}(y)) + \mathrm{im}(\mathrm{ad}^2_{\mathfrak{C}_\mu}(y))$. So for dimension reasons, $\mathfrak{C}_\mu = \ker(\mathrm{ad}^2_{\mathfrak{C}_\mu}(y)) \oplus \mathrm{im}(\mathrm{ad}^2_{\mathfrak{C}_\mu}(y))$. ∎

Lemma 6.
(a) *If* $y \in \mathfrak{g}$, *one has* $\mathfrak{C}_\mu \subset \ker(\mathrm{ad}(y)) + \mathrm{im}(\mathrm{ad}(y))$,
(b) $\mathfrak{C}_\mu \subset Z(\mathfrak{g}) + [\mathfrak{g}, \mathfrak{g}]$.

Proof. (a). Suppose that $y \in \mathfrak{g}$. With the above notation,

$$\mathfrak{C}_\mu(y,i) = \begin{cases} 0 & ; \text{ if } \mathrm{Re}(\mu\,\mathrm{tr}(\mathrm{ad}\,y) + \lambda_i) \neq 0, \\ \langle Z_l^i \mid l \geq (n_i+1)/2 \rangle & ; \text{ if } \mathrm{Re}(\mu\,\mathrm{tr}(\mathrm{ad}\,y) + \lambda_i) = 0. \end{cases}$$

Suppose that $\mathrm{Re}(\mu\,\mathrm{tr}(\mathrm{ad}\,y) + \lambda_i) = 0$. If $\lambda_i = 0$, then:
(i) if $n_i = 1$, then $C_i(y)(Z_1^i) = 0$.
(ii) if $n_i \geq 2$, then $Z_l^i = C_i(y)(Z_{l-1}^i)$, for all $l \geq (n_i+1)/2$.

If $\lambda_i \neq 0$, then $Z_{n_i}^i = C_i(y)(\frac{1}{\lambda_i} Z_{n_i}^i)$ and $Z_l^i = C_i(y)(\frac{1}{\lambda_i} Z_l^i) - \frac{1}{\lambda_i} Z_{l+1}^i$, for all $1 \leq l \leq n_i - 1$. In particular, $Z_l^i \in \mathrm{im}\, C_i(y)$, for all $l \geq (n_i+1)/2$.

Hence each of the basis elements of $\mathfrak{C}_\mu(y,i)$ belongs to either the kernel or the image of $C_i(y)$. Since $\mathfrak{C}_\mu \subset \{x \in \mathfrak{g} \mid x_y \in \bigoplus_{i=1}^r \mathfrak{C}_\mu(y,i)\}$, it follows that $\mathfrak{C}_\mu$ is a subset of the vector subspace sum $\ker(\mathrm{ad}(y)) + \mathrm{im}(\mathrm{ad}(y))$.

(b). From (a), one has:

$$\mathfrak{C}_\mu \subset \bigcap_{y \in \mathfrak{g}} \ker(\mathrm{ad}(y)) + \bigoplus_{y \in \mathfrak{g}} \mathrm{im}(\mathrm{ad}(y)).$$

That is, $\mathfrak{C}_\mu \subset Z(\mathfrak{g}) + [\mathfrak{g}, \mathfrak{g}]$. ∎

Let K denote the Killing–Cartan form of $\mathfrak{C}_\mu$; this is the map $K \colon \mathfrak{C}_\mu \times \mathfrak{C}_\mu \to \mathbb{R}$ defined by $K(x,y) = \mathrm{tr}(\mathrm{ad}_{\mathfrak{C}_\mu}(x) \circ \mathrm{ad}_{\mathfrak{C}_\mu}(y) \colon \mathfrak{C}_\mu \to \mathfrak{C}_\mu)$.

Remark 2. If $y \in \mathfrak{C}_\mu$ and $\{\lambda_i \mid i = 1, \ldots, r\}$ are the eigenvalues of $\mathrm{ad}_{\mathfrak{C}_\mu}(y)$, one has $K(y,y) = \sum_{i=1}^r n_i \lambda_i^2$. By Lemma 5, one has $K(y,y) \leq 0$ and $K(y,y) = 0$ if and only if the map $\mathrm{ad}^2_{\mathfrak{C}_\mu}(y)$ is identically zero.

Let $\mathfrak{n}$ denote the maximal nilpotent ideal of $\mathfrak{C}_\mu$. Consider the radical $R(\mathfrak{C}_\mu)$ of $\mathfrak{C}_\mu$. Recall that $R(\mathfrak{C}_\mu)$ is the K-orthogonal complement of the derived algebra $[\mathfrak{C}_\mu, \mathfrak{C}_\mu]$. Let ${\mathfrak{C}_\mu}^\perp$ denote the K-orthogonal complement of $\mathfrak{C}_\mu$. Recall that

$$R(\mathfrak{C}_\mu) \supset {\mathfrak{C}_\mu}^\perp \supset \mathfrak{n}. \tag{7}$$

Definition 2. For convenience, we introduce four sets:
(a) $A = \{x \in \mathfrak{C}_\mu \mid \mathrm{ad}^2_{\mathfrak{C}_\mu}(x) \equiv 0\}$,
(b) $B = \{x \in \mathfrak{C}_\mu \mid K(x,x) = 0\}$.
(c) $D = \oplus_{x \in \mathfrak{C}_\mu} \mathrm{im}\,\mathrm{ad}^2_{\mathfrak{C}_\mu}(x)$,
(d) $E = \cap_{x \in \mathfrak{C}_\mu} \ker \mathrm{ad}^2_{\mathfrak{C}_\mu}(x)$,

Lemma 7. *$R(\mathfrak{C}_\mu) = \mathfrak{n}$, and $\mathfrak{n}$ is metabelian.*

Proof. First notice that ${\mathfrak{C}_\mu}^\perp \subset B \subset A \subset \mathfrak{n} \subset {\mathfrak{C}_\mu}^\perp$. Indeed,

$${\mathfrak{C}_\mu}^\perp = \{x \in \mathfrak{C}_\mu \mid K(x,y) = 0 \text{ for all } y \in \mathfrak{C}_\mu\}$$

and so $\mathfrak{C}_\mu{}^\perp \subset B$. Remark 2 gives $B \subset A$. Recall that $\mathfrak{n}$ is the set of elements x for which $\mathrm{ad}_{\mathfrak{C}_\mu}(x)\colon \mathfrak{C}_\mu \to \mathfrak{C}_\mu$ is nilpotent [1]. So $A \subset \mathfrak{n}$. Equation (7) gives $\mathfrak{n} \subset \mathfrak{C}_\mu{}^\perp$. So $B = A = \mathfrak{n} = \mathfrak{C}_\mu{}^\perp$. In particular, $\mathfrak{n} = A$ and so $\mathfrak{n}$ is metabelian.

Now note that $E \subset A$. Indeed, if $x \in E$, then for all $y \in \mathfrak{C}_\mu$, $x \in \ker \mathrm{ad}^2_{\mathfrak{C}_\mu}(y)$ and $x \in \ker \mathrm{ad}^2_{\mathfrak{C}_\mu}(x+y)$. So

$$0 = [x+y,[x+y,x]] = [x+y,[y,x]] = [x,[y,x]] + [y,[y,x]] = [x,[y,x]].$$

Thus $\mathrm{ad}^2_{\mathfrak{C}_\mu}(x)(y) = 0$ for all $y \in \mathfrak{C}_\mu$. So $x \in A$.

From Lemma 5, for all $x \in \mathfrak{C}_\mu$, $\ker \mathrm{ad}^2(x) \cap \mathrm{im}\, \mathrm{ad}^2(x) = \{0\}$. Hence $\mathfrak{C}_\mu = D \oplus E$. As $E \subset A = \mathfrak{n}$, we have $E \subset R(\mathfrak{C}_\mu)$, from (7). Thus

$$R(\mathfrak{C}_\mu) = (R(\mathfrak{C}_\mu) \cap D) \oplus E. \tag{8}$$

Note also that $R(\mathfrak{C}_\mu) \cap D \subset A$. Indeed, it suffices to show that if $x, y \in \mathfrak{C}_\mu$, and $z = \mathrm{ad}^2_{\mathfrak{C}_\mu}(x)(y) \in R(\mathfrak{C}_\mu)$, then $z \in B$. But

$$K(z,z) = K([x,[x,y]],z) = K([x,y],[x,z]).$$

Now $[x,z] \in [\mathfrak{C}_\mu, R(\mathfrak{C}_\mu)] \subset \mathfrak{C}_\mu{}^\perp$. So $K(z,z) = 0$.

Since $E \subset A$ and $R(\mathfrak{C}_\mu) \cap D \subset A$, (8) gives $R(\mathfrak{C}_\mu) \subset A = \mathfrak{n}$, and hence by (7), $R(\mathfrak{C}_\mu) = \mathfrak{n}$. ∎

Let $\mathfrak{s}$ be a Levi subalgebra of $\mathfrak{C}_\mu$.

Lemma 8. *$\mathfrak{n}$ is a characteristic ideal of $\mathfrak{g}$ and $\mathfrak{s}$ is a direct summand of $\mathfrak{g}$.*

Proof. As $\mathfrak{r}$ is a characteristic ideal [1], and by Lemma 4, $\mathfrak{C}_\mu$ is a characteristic ideal, so $\mathfrak{r} \cap \mathfrak{C}_\mu$ is one too. But $\mathfrak{n} = \mathfrak{r} \cap \mathfrak{C}_\mu$ since by [6, Theorem 3.8.1], $\mathfrak{r} \cap \mathfrak{C}_\mu = R(\mathfrak{C}_\mu)$, and by the previous Lemma $R(\mathfrak{C}_\mu) = \mathfrak{n}$.

For all $y \in \mathfrak{r}$, Lemma 6 and the fact that $\mathfrak{r}$ is an ideal gives

$$\mathfrak{s} \subset \mathfrak{C}_\mu \subset \ker \mathrm{ad}(y) + \mathrm{im}\, \mathrm{ad}(y) \subset \ker \mathrm{ad}(y) + \mathfrak{r}.$$

Hence $\mathfrak{s} \subset Z_{\mathfrak{g}}(\mathfrak{r}) + \mathfrak{r}$. Thus, as $\mathfrak{s}$ is semisimple,

$$\mathfrak{s} = [\mathfrak{s},\mathfrak{s}] \subset [Z_{\mathfrak{g}}(\mathfrak{r}) + \mathfrak{r}, Z_{\mathfrak{g}}(\mathfrak{r}) + \mathfrak{r}] \subset [Z_{\mathfrak{g}}(\mathfrak{r}), Z_{\mathfrak{g}}(\mathfrak{r})] + [\mathfrak{r},\mathfrak{r}] \subset Z_{\mathfrak{g}}(\mathfrak{r}) + [\mathfrak{r},\mathfrak{r}].$$

Taking repeated brackets of $\mathfrak{s}$ with itself, and using the fact that $\mathfrak{r}$ is solvable, one obtains $\mathfrak{s} \subset Z_{\mathfrak{g}}(\mathfrak{r})$.

By the Malcev–Harish-Chandra Theorem, $\mathfrak{g}$ has a Levi subalgebra $\mathfrak{S}$ such that $\mathfrak{s} \subset \mathfrak{S}$ (See [6, Cor. 3.14.3]). Note that $\mathfrak{s} = \mathfrak{S} \cap \mathfrak{C}_\mu$. Indeed, clearly $\mathfrak{s} \subset \mathfrak{S} \cap \mathfrak{C}_\mu$. On the other hand, since $\mathfrak{S}$ is a subalgebra and $\mathfrak{C}_\mu$ is an ideal of $\mathfrak{g}$, $\mathfrak{S} \cap \mathfrak{C}_\mu$ is an ideal of $\mathfrak{S}$. So $\mathfrak{S} \cap \mathfrak{C}_\mu$ is a semisimple subalgebra. Thus, as $\mathfrak{s}$ is a maximal semisimple subalgebra of $\mathfrak{C}_\mu$, $\mathfrak{s} = \mathfrak{S} \cap \mathfrak{C}_\mu$. In particular, $\mathfrak{s}$ is an ideal of $\mathfrak{S}$. As $\mathfrak{S}$ is semisimple, $\mathfrak{s}$ is a direct summand of $\mathfrak{S}$; that is, there is an ideal $\mathfrak{s}'$ of $\mathfrak{S}$ such that $\mathfrak{S}$ is an internal direct sum of ideals $\mathfrak{S} = \mathfrak{s} \oplus \mathfrak{s}'$. Consider the vector space direct sum $\mathfrak{t} = \mathfrak{s}' \oplus \mathfrak{r}$. By construction $\mathfrak{t}$ is an ideal of $\mathfrak{g}$. From above, $\mathfrak{s} \subset Z_{\mathfrak{g}}(\mathfrak{r})$. So we have an internal direct sum of ideals: $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{t}$. ∎

Remark 3. By Remark 2, $K(x,x) \leq 0$ for all $x \in \mathfrak{s}$, where K is the Killing–Cartan form of $\mathfrak{C}_\mu$. So, by the previous Lemma, the Killing–Cartan form of $\mathfrak{s}$ is negative semi-definite; that is, $\mathfrak{s}$ is compact.

Returning to the statement of the theorem, note that with the exception of (a)(ii), part (a) follows from Lemma 8, Remark 3, and Lemmas 7 and 6(b). From Lemma 6(a), one has:

$$\mathfrak{C}_\mu \subset \bigcap_{z \in \mathfrak{n}} \ker(\mathrm{ad}(z)) + \bigoplus_{z \in \mathfrak{n}} \mathrm{im}(\mathrm{ad}(z)).$$

That is, $\mathfrak{C}_\mu \subset Z_g(\mathfrak{n}) + [\mathfrak{n}, \mathfrak{g}]$. Hence $\mathfrak{n} \subset Z_g(\mathfrak{n}) + [\mathfrak{n}, \mathfrak{g}]$. It follows that as $\mathfrak{n}$ is an ideal, by Lemma 8,

$$\mathfrak{n} \subset \mathfrak{n} \cap Z_g(\mathfrak{n}) + [\mathfrak{n}, \mathfrak{g}] = Z(\mathfrak{n}) + [\mathfrak{n}, \mathfrak{g}].$$

Clearly $Z(\mathfrak{n}) + [\mathfrak{n}, \mathfrak{g}] \subset \mathfrak{n}$. To see that $Z(\mathfrak{n})$ is an ideal of $\mathfrak{g}$, notice that as $\mathfrak{n}$ is an ideal of $\mathfrak{g}$, the Jacobi identity gives

$$[\mathfrak{n}, [Z(\mathfrak{n}), \mathfrak{g}]] = [\mathfrak{g}, [Z(\mathfrak{n}), \mathfrak{n}]] + [Z(\mathfrak{n}), [\mathfrak{n}, \mathfrak{g}]] = 0,$$

and so $[Z(\mathfrak{n}), \mathfrak{g}] \subset Z(\mathfrak{n})$. So we have established (a)(ii).

Suppose that $\mathfrak{g}$ is unimodular or that $\mu = 0$. One sees directly from Lemma 3 that $Z(\mathfrak{g}) \subset \mathfrak{C}_\mu$, and hence $Z(\mathfrak{g}) \subset \mathfrak{n}$. Notice that if $\mathfrak{g}_1$ is a compact semisimple Lie algebra, then $\mathfrak{g}_1$ is the Lie algebra of a compact Lie group and so the orbits of the adjoint action are bounded. Hence $\mathfrak{C}_\mu(\mathfrak{g}_1) = \mathfrak{g}_1$. As $\mathfrak{g}$ is unimodular or $\mu = 0$, Lemma 3 implies that if $\mathfrak{g}$ is a direct sum of ideals, $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, then $\mathfrak{C}_\mu = \mathfrak{C}_\mu(\mathfrak{g}_1) \oplus \mathfrak{C}_\mu(\mathfrak{g}_2)$. In particular, if $\mathfrak{g}_1$ is compact semisimple, $\mathfrak{g}_1 \subset \mathfrak{C}_\mu$, from which it follows that $\mathfrak{g}_1 \subset \mathfrak{s}$. Thus $\mathfrak{s}$ is the maximal compact semisimple direct summand of $\mathfrak{g}$.

Finally, suppose that $\mathfrak{g}$ is not unimodular and $\mu \neq 0$. Let $x \in \mathfrak{g}$ with $\tau := -\operatorname{tr}(\mathrm{ad}(x)) \neq 0$. Consider the Jordan form of the induced derivation $\mathrm{ad}_{\mathfrak{C}_\mu}(x) \colon \mathfrak{C}_\mu \to \mathfrak{C}_\mu$. Suppose that λ_1, λ_2 are (not necessarily distinct) eigenvalues of $\mathrm{ad}_{\mathfrak{C}_\mu}(x)$ and for each $i = 1, 2$ let $\{Z^i_j \mid 1 \leq j \leq n_i\}$ be linearly independent vectors in $\mathfrak{C}_\mu \otimes_{\mathbb{R}} \mathbb{C}$ with $\mathrm{ad}(x)(Z^i_j) = \lambda_i Z^i_j + Z^i_{j-1}$ for all j, where by definition $Z^i_0 = 0$. We will show by induction on $p = j + k$ that $[Z^1_j, Z^2_k] = 0$ for all $j \leq n_1, k \leq n_2$. The claim is obviously true for $p = 0$. Suppose that it holds for $p = l$. Then for $p = l + 1$, the inductive hypothesis gives:

$$\begin{aligned} \mathrm{ad}(x)[Z^1_j, Z^2_k] &= [\mathrm{ad}(x) Z^1_j, Z^2_k] + [Z^1_j, \mathrm{ad}(x) Z^2_k] \\ &= [\lambda_1 Z^1_j, Z^2_k] + [Z^1_{j-1}, Z^2_k] + [Z^1_j, \lambda_2 Z^2_k] + [Z^1_j, Z^2_{k-1}] \\ &= (\lambda_1 + \lambda_2)[Z^1_j, Z^2_k]. \end{aligned} \tag{9}$$

As $Z^1_j, Z^2_k \in \mathfrak{C}_\mu \otimes_{\mathbb{R}} \mathbb{C}$, one has $[Z^1_j, Z^2_k] \in \mathfrak{C}_\mu \otimes_{\mathbb{R}} \mathbb{C}$. By Lemma 3, the eigenvalues of $\mathrm{ad}_{\mathfrak{C}_\mu}(x)$ all have real part equal to $\mu\tau$. So $\mathrm{Re}(\lambda_1) = \mathrm{Re}(\lambda_2) = \mu\tau$ and $\mathrm{Re}(\lambda_1 + \lambda_2) = 2\mu\tau \neq \mu\tau$. So $\lambda_1 + \lambda_2$ is not an eigenvalue of $\mathrm{ad}_{\mathfrak{C}_\mu}(x)$, and thus (9) gives $[Z^1_j, Z^2_k] = 0$. This completes the induction. Thus $\mathfrak{C}_\mu$ is abelian. So $\mathfrak{s} = 0$ and $\mathfrak{n}$ is abelian, as required. This completes the proof of the theorem. ∎

Proof of the corollary. Part (a) follows immediately from Lemma 5(a), while (c) follows immediately from the theorem.

(b) If $\mathfrak{g}$ is nilpotent, then $\mathfrak{C}_\mu = \mathfrak{n}$ and by part (a)(ii) of the theorem,

$$\begin{aligned} \mathfrak{C}_\mu = Z(\mathfrak{C}_\mu) + [\mathfrak{C}_\mu, \mathfrak{g}] &= Z(\mathfrak{C}_\mu) + [Z(\mathfrak{C}_\mu) + [\mathfrak{C}_\mu, \mathfrak{g}], \mathfrak{g}] \\ &= Z(\mathfrak{C}_\mu) + [Z(\mathfrak{C}_\mu), \mathfrak{g}] + [[\mathfrak{C}_\mu, \mathfrak{g}], \mathfrak{g}]. \end{aligned} \tag{10}$$

Now by part (a)(ii) of the theorem, $Z(\mathfrak{C}_\mu)$ is an ideal of $\mathfrak{g}$, and so $[Z(\mathfrak{C}_\mu), \mathfrak{g}] \subset Z(\mathfrak{C}_\mu)$ and (10) gives $\mathfrak{C}_\mu = Z(\mathfrak{C}_\mu) + [[\mathfrak{C}_\mu, \mathfrak{g}], \mathfrak{g}]$. Repeating this argument, one has $\mathfrak{C}_\mu = Z(\mathfrak{C}_\mu) + \mathfrak{g}^k(\mathfrak{C}_\mu)$ for all $k \geq 1$, where $\mathfrak{g}^i(\mathfrak{C}_\mu) = [\mathfrak{g}, \mathfrak{g}^{i-1}(\mathfrak{C}_\mu)]$ and $\mathfrak{g}^1(\mathfrak{C}_\mu) = [\mathfrak{g}, \mathfrak{C}_\mu]$. Thus, if $\mathfrak{g}$ is nilpotent, $\mathfrak{C}_\mu = Z(\mathfrak{C}_\mu)$; that is, $\mathfrak{C}_\mu$ is abelian.

(d) If $\mathfrak{g}$ is the Lie algebra of a compact Lie group, the orbits of the adjoint action of $\mathfrak{g}$ are bounded and thus $\mathfrak{C}_\mu = \mathfrak{g}$. Conversely, if $\mathfrak{C}_\mu = \mathfrak{g}$, then by the theorem, $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{n}$, where $\mathfrak{s}$ is compact semisimple and $\mathfrak{n}$ is metabelian. In fact, by part (a)(ii) of the theorem, $\mathfrak{n} = Z(\mathfrak{n}) + [\mathfrak{n}, \mathfrak{g}] = Z(\mathfrak{n})$, and so $\mathfrak{n}$ is abelian. Hence $\mathfrak{g}$ is the Lie algebra of a compact Lie group. ■

Example 1. The approximative centre of the following (solvable unimodular) Lie algebra is nilpotent non-abelian:

$$\mathfrak{g} = \langle x, y, z, w \mid [x, y] = z,\ [x, z] = -y,\ [y, z] = w \rangle.$$

Indeed, it is easy to see from Lemma 3 that $\mathfrak{C}_\mu = \langle y, z, w \rangle$.

Example 2. Consider the standard filiform nilpotent Lie algebra (see [2]):

$$\mathfrak{g} = \langle x, y_1, \ldots, y_n \mid [x, y_i] = y_{i+1}, \forall i < n \rangle.$$

The approximative centre of $\mathfrak{g}$ is abelian and strictly greater than the centre; indeed, it is easy to see from Lemma 3 that $\mathfrak{C}_\mu = \langle y_i \mid i \geq (n+1)/2 \rangle$.

Example 3. Consider the Lie algebra $\mathfrak{g} = \mathbb{R}^3 \rtimes \mathfrak{so}(3, \mathbb{R})$, where the action of $\mathfrak{so}(3, \mathbb{R})$ on $\mathbb{R}^3$ is the standard linear one. Here $\mathfrak{g}$ has a Levi subalgebra which is compact and simple, but the approximative centre has no simple factor ($\mathfrak{C}_\mu = \mathbb{R}^3$).

Remark 4. If a Lie algebra $\mathfrak{g}$ has an ideal $\mathfrak{a}$ which is the Lie algebra of a compact Lie group, it doesn't necessarily follow that $\mathfrak{a} \subset \mathfrak{C}_\mu$, or that $\mathfrak{C}_\mu \subset \mathfrak{a}$. In Example 2, the ideal $\langle y_1, \ldots, y_n \rangle$ is abelian but it is not contained in $\mathfrak{C}_\mu$. In Example 1, $\mathfrak{C}_\mu$ is not contained in the (maximal) abelian ideal $\langle z, w \rangle$.

Thanks. The author is very grateful to the referee, whose careful reading and sensible suggestions improved the presentation of this paper.

References

[1] N. Bourbaki, "Lie groups and Lie algebras. Chapters 1–3," Springer-Verlag, Berlin, 1998.

[2] D. Burde, *Affine cohomology classes for filiform Lie algebras*, in "Crystallographic groups and their generalizations," Amer. Math. Soc., Providence, RI, Contemp. Math. **262** (2000), 159–170.

[3] G. Cairns, "Feuilletages géodésibles," Thesis, Montpellier, 1987.

[4] G. Cairns and E. Ghys, *Totally geodesic foliations on 4-manifolds*, J. Diff. Geom. **23** (1986), 241–254.

[5] G. Cairns and P. Molino, *Weakly involutive totally geodesic distributions of constant rank*, preprint.

[6] V.S. Varadarajan, "Lie groups, Lie Algebras, and their Representations," Springer-Verlag, 1984.

G. Cairns
Department of Mathematics
La Trobe University
Melbourne, Australia 3083
G.Cairns@latrobe.edu.au

Received November 17, 2000
and in final form May 15, 2001