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Determinantally Homogeneous Polynomials on Representations of Euclidean Jordan Algebras

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Abstract. The classical notion of determinantally homogeneous polynomial is presented in the context of representations of Euclidean Jordan algebras. When the Jordan algebra is of classical type, the study of the algebra of determinantally homogeneous polynomials is strongly related to classical invariant theory and a fairly complete description is obtained. For the Euclidean Jordan algebra of Lorentzian type, the representations are related to Clifford modules. In this case, only partial results are obtained, including complete answers for pinor spaces associated to Clifford algebras of low dimension.

0. Introduction

Representations of Euclidean Jordan algebras have been recognized in the last ten years as a firm ground for many questions classically related to algebra or analysis on matrix spaces. Among other themes, *zeta* integrals were studied by several authors ([FK] ch. XVI, [A1],[A2],[C2]). The notion of *determinantally homogeneous* polynomials (called Φ-homogeneous in [C2]) emerges as an important notion. The present paper is devoted to a systematic study of these polynomials.

Section 1 introduces notation for the rest of the paper. Section 2 is concerned with the "classical" Euclidean Jordan algebras and makes connexion with classical invariant theory. The four last sections are devoted to the *Lorentzian* case, which turns out to be strongly connected with the theory of Clifford algebras and pinor spaces. Complete answers are only obtained in low dimensional cases (Section 6).

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1. Determinantally homogeneous polynomials

Let V be a real Euclidean Jordan algebra, which we assume for simplicity to be simple. A general reference for notation and results is [FK]. $V^\times$ denotes the set of invertible elements in V, and Ω is the open component of $V^\times$ containing the neutral element e. Its closure is the set of squares in V. Let n be the dimension of V, r its rank. Denote (*) by tr (resp. det) the trace and the generic norm (or determinant) of V, and let $(x,y)=\operatorname{tr} xy$ be the symmetric inner product on V.

Recall that a *representation* Φ of V in a Euclidean space $(E,\langle .,.\rangle)$ is a Jordan algebra homomorphism $\Phi : V \longrightarrow \operatorname{Sym}(E)$, or more explicitly a linear mapping $\Phi : V \longrightarrow \operatorname{End}(E)$ which satisfies

$$\Phi(xy) = \frac{1}{2}(\Phi(x)\Phi(y)+\Phi(y)\Phi(x)) \tag{1}$$

$$\Phi(e) = \mathrm{Id} \tag{2}$$

$$\langle \Phi(x)\xi\,,\,\eta\rangle = \langle \xi\,,\,\Phi(x)\eta\rangle \tag{3}$$

for all $x,y\in V$ and $\xi,\eta\in E$. Let $Q : E \longrightarrow V$ be the associated quadratic map defined by the formula

$$(Q(\xi),x) = \langle \Phi(x)\xi,\xi\rangle,\ \forall x\in V. \tag{4}$$

Notice that the image of Q is contained in $\overline{\Omega}$. The representation is said to be *regular*, if the image of Q contains e (hence all the elements of Ω, see [C1]). An equivalent characterization of a regular representation is the condition

$$\det Q(\xi) \not\equiv 0 \text{ on } E.$$

Also recall the formulæ

$$\det Q(\Phi(x)\xi) = (\det x)^2 \det Q(\xi), \qquad \operatorname{Det}\Phi(x) = (\det x)^{\frac{N}{r}} \tag{5}$$

for $x\in V$ and $\xi\in E$, where $N=\dim E$ ($\frac{N}{r}$ is always an integer).

A polynomial p defined on E is said to be *determinantally homogeneous* of degree l (l a nonnegative integer) if

$$p(\Phi(x)\xi) = (\det x)^l p(\xi) \tag{6}$$

for all $x\in V$ and $\xi\in E$. Denote by $\mathcal{P}$ the vector space of polynomials with complex coefficients on E, and for $m\in\mathbb{N}$ let $\mathcal{P}^m$ be the subspace of polynomials which are homogeneous of degree m (in the ordinary sense). For $l\in\mathbb{N}$, denote by $\mathcal{P}^{\det,l}$ the space of polynomials which are determinantally homogeneous of degree l, and let $\mathcal{P}^{\det}=\bigoplus_{l\in\mathbb{N}}\mathcal{P}^{\det,l}$. Clearly $\mathcal{P}^{\det}$ is an algebra.

Notice that a polynomial which is determinantally homogeneous of degree l is homogeneous (in the ordinary sense) of degree lr.

(*) Det and Tr are used for the determinant and the trace of an endomorphism.

A first exemple of a non trivial determinantally homogeneous polynomial is the polynomial $\det Q(\xi)$ which by (5) is determinantally homogeneous of degree 2, and non trivial if and only if the representation is regular.

When $x \in V^\times$, then $\Phi(x)$ has $\Phi(x^{-1})$ as its inverse, and so belongs to $GL(E)$. Moreover from (5), we see that $\operatorname{Det}\Phi(x) = 1$ if $\det x = 1$. Let Γ be the closed subgroup of $SL(E)$ generated by the operators $\{\Phi(x) \mid \det x = 1\}$. Observe that Γ is a reductive subgroup, as it is generated by symmetric operators. The following result makes connexion with invariant theory.

Theorem 1.1. *A polynomial p belongs to $\mathcal{P}^{\det,l}$ if and only if*

$$p(\gamma\xi) = p(\xi), \quad \forall \gamma \in \Gamma$$

$$p(\lambda\xi) = \lambda^{lr} p(\xi), \quad \forall \lambda \in \mathbb{R}\ .$$

Proof. The conditions are obviously necessary. Conversely, let p be a polynomial which satisfies both conditions. Let $x \in \Omega$, so that in particular $\det x > 0$, and set $\lambda = (\det x)^{\frac{1}{r}}$. Then $x = \lambda x_1$ with $\det x_1 = 1$. From the two conditions we get

$$p(\Phi(x)\xi) = p(\Phi(\lambda x_1)\xi) = \lambda^{lr} p(\xi) = (\det x)^l p(\xi).$$

But now for any ξ fixed in E, the polynomial functions $x \mapsto p(\Phi(x)\xi)$ and $x \mapsto (\det x)^l p(\xi)$ agree on Ω hence everywhere. As ξ is arbitrary this shows that p is determinantally homogeneous of degree l. ∎

In order to determine the group Γ, the following result will be helpful.

Lemma 1.2. *Let G be a connected semi-simple real Lie group of the non-compact type with finite center. Let $\mathfrak{g} = Lie(G)$ be its Lie algebra, let θ be a Cartan involution of G, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the associated Cartan decomposition of $\mathfrak{g}$. Then the set of all finite products $\{\exp X_1 \exp X_2 \dots \exp X_n,\ X_j \in \mathfrak{p}\}$ is dense in G.*

Proof. Let Γ be the set of all such products. As $(\exp X)^{-1} = \exp(-X)$, the set Γ is a subgroup of G. Its closure $\overline{\Gamma}$ is a closed, hence a Lie subgroup of G, whose Lie algebra contains $\mathfrak{p}$, hence also $[\mathfrak{p}, \mathfrak{p}]$. As G has no compact factor, $[\mathfrak{p},\, \mathfrak{p}] \oplus \mathfrak{p} = \mathfrak{g}$. Thus $\overline{\Gamma}$ has Lie algebra $\mathfrak{g}$ and, as G is assumed to be connected, this implies $\overline{\Gamma} = G$. ∎

2. The classical cases

In this section we determine the determinantally homogeneous polynomials in the classical cases. So $\mathbb{K}$ will denote either $\mathbb{R}, \mathbb{C}$ or the field of quaternions $\mathbb{H}$, and $V = \operatorname{Herm}(r, \mathbb{K})$ is the space of $(r \times r)$ $\mathbb{K}$-Hermitian matrices, with the Jordan product $x.y = \frac{1}{2}(xy + yx)$. The representation space is $E = \operatorname{End}(r, k, \mathbb{K})$, and the action of V on E is by left multiplication

$$\Phi(x)\xi = x\xi.$$

The quadratic map is given by $Q(\xi)=\xi\xi^*$, and the representation is regular if and only if $k\geq r$ (see [C1]).

The next result essentially amounts to the determination of the group previously denoted by Γ, but in a different presentation.

Theorem 2.1. *Let $\mathbb{K}=\mathbb{R},\mathbb{C}$ or $\mathbb{H}$, and let $V=\mathrm{Herm}(r,\mathbb{K})$. The closed subgroup Γ of $GL(r,\mathbb{K})$ generated by the elements $\{x\in V \mid \det x=1\}$ is $\Gamma=SL(r,\mathbb{K})$.*

Proof. For $\mathbb{K}=\mathbb{R}$, first observe that $\det x=\mathrm{Det}\, x$ for any element x of $Sym(r,\mathbb{R})$, so that the subgroup generated is contained in $SL(r,\mathbb{R})$. For the standard Cartan involution of $SL(r,\mathbb{R})$

$$\mathfrak{p}=\{X\in \mathrm{Sym}(r,\mathbb{R}) \mid \mathrm{Tr}(x)=0\} \quad ,$$

and for $X\in\mathfrak{p}$, $\exp X$ is belongs to V and has determinant 1. By Lemma 1.2. the closed subgroup generated by such elements is equal to $SL(r,\mathbb{R})$. So, *a fortiori* $\Gamma=SL(r,\mathbb{R})$.

Next, assume $\mathbb{K}=\mathbb{C}$, and $V=\mathrm{Herm}(r,\mathbb{C})$. In this case $\det x=\mathrm{Det}_{\mathbb{C}}\, x$, so that clearly Γ is a subgroup of $SL(r,\mathbb{C})$. For the standard Cartan involution of $SL(r,\mathbb{C})$,

$$\mathfrak{p}=\{X\in \mathrm{Herm}(r,\mathbb{C}) \mid \mathrm{Tr}\, X=0\}.$$

Now if $X\in\mathfrak{p}$, then $\exp X$ belongs to V and has determinant 1. So, again the closed subgroup generated by these elements is $SL(r,\mathbb{C})$. Hence $\Gamma=SL(r,\mathbb{C})$.

Now assume $\mathbb{K}=\mathbb{H}$, and realize the quaternions as usual, using $(1,\mathbf{i},\mathbf{j},\mathbf{k})$ as a basis of $\mathbb{H}$ over $\mathbb{R}$. Consider the space $\mathbb{C}^{2r}$, and let

$$J=J_r=\begin{pmatrix} 0 & \mathrm{Id}_r \\ -\mathrm{Id}_r & 0 \end{pmatrix}.$$

Then $\mathbb{C}^{2r}$ can be viewed as a right $\mathbb{H}$ vector space by letting

$$z\,\mathbf{i}=iz,\ \ z\,\mathbf{j}=J\overline{z},\ \ z\,\mathbf{k}=iJ\overline{z}.$$

Now

$$\mathrm{End}(r,\mathbb{H})=\{x\in \mathrm{End}(2r,\mathbb{C}) \mid xJ=J\overline{x}\}.$$

The $\mathbb{H}$-Hermitian matrices are realized as

$$V=\mathrm{Herm}(r,\mathbb{H})=\{x\in\mathrm{End}(2r,\mathbb{C}) \mid xJ=J\overline{x},\ x^*=x\}.$$

Notice that the two conditions $xJ=J\overline{x}$ and $x^*=x$ imply that $(xJ)^t=-xJ$, that is xJ is skew-symmetric. Moreover, $J(xJ)J^t=Jx=\overline{x}J=\overline{xJ}$.

For any skew-symmetric matrix, denote by $\mathrm{Pf}\,(A)$ its *Pfaffian*, with the convention that $\mathrm{Pf}\,(J)=1$ (see e.g. [GW] Appendix B.2.6). Recall the following properties of the Pfaffian : for any skew-symmetric matrix A and any element $g\in GL(2r,\mathbb{C})$,

$$\mathrm{Pf}(\overline{A})=\overline{\mathrm{Pf}(A)},\quad \mathrm{Det}_{\mathbb{C}}A=\mathrm{Pf}(A)^2,\quad \mathrm{Pf}(gAg^t)=\mathrm{Det}_{\mathbb{C}}(g)\ \mathrm{Pf}(A).$$

For $x \in V$, we infer from the last remarks that $\mathrm{Pf}(xJ)$ is real, and in fact $\det x = \mathrm{Pf}(xJ)$.

Let $SL(r,\mathbb{H}) = SL(2r,\mathbb{C}) \cap \mathrm{End}(r,\mathbb{H})$. If $x \in V$ with $\det x = 1$, then

$$\mathrm{Det}_{\mathbb{C}}(x) = \mathrm{Det}_{\mathbb{C}}(xJ) = \mathrm{Pf}(xJ)^2 = (\det x)^2 = 1 ,$$

so that the subgroup Γ is certainly contained in $SL(r,\mathbb{H})$.

For the standard Cartan involution on $SL(2r,\mathbb{C})$,

$$\mathfrak{p} = \{x \in \mathrm{End}(2r,\mathbb{C}) \mid Jx = \overline{x}J, x = x^*, \mathrm{Tr}(x) = 0\}.$$

For $X \in \mathfrak{p}$, $\exp X$ belongs to V and

$$1 = \mathrm{Det}_{\mathbb{C}}\,(\exp X) = \mathrm{Det}_{\mathbb{C}}\,(J \exp X) = \mathrm{Pf}(J \exp X)^2$$

so that $\mathrm{Pf}(J\exp X) = \pm 1$, but by continuity of $X \mapsto \mathrm{Pf}(J\exp X)$ one gets $\mathrm{Pf}(J\exp X) = 1$, hence $\det \exp X = 1$. By Lemma 1.2, the closed subgroup generated by such elements is equal to $SL(r,\mathbb{H})$. Hence the result. ∎

After this preparation we are ready for the determination of $\mathcal{P}^{\det}$. Let us consider the real case. So let $V = \mathrm{Sym}(r,\mathbb{R})$, and $E = \mathrm{End}(r,k,\mathbb{R})$.

Theorem 2.2. *There exist non trivial determinantally homogeneous polynomials on E if and only if $k : geqr$. Assume $k \geq r$. Then for $\eta \in E$ define*

$$p_\eta(\xi) = \mathrm{Det}(\xi\,\eta^t).$$

The polynomials p_η belong to $\mathcal{P}^{\det,1}$, and as η runs through E, they generate $\mathcal{P}^{\det}$ as an algebra.

Remark. A finite set of generators can be obtained as follows. For $1 \leq j \leq r$, the j-th column of the matrix $\xi\eta^t$ is a linear combination of the k columns of the matrix ξ, the coefficients of the linear combination being the elements of the j-th row of η. By using the properties of the determinant, it is clear that $\mathrm{Det}(\xi\eta^t)$ can be written as a linear combination of determinants of $r \times r$ matrices, whose columns are (a certain selection of) columns of the matrix ξ. In other words, $\mathcal{P}^{\det,1}$ is generated (as a vector space) by the rank r minors of the matrix ξ. Hence $\mathcal{P}^{\det}$ is generated (as an algebra) by the same elements.

Proof. The result, in this last version is a consequence of the characterization of $\mathcal{P}^{\det}$ obtained in Theorem (1.1) and a classical result in invariant theory for $SL(r,\mathbb{R})$ acting on k copies of the natural representation on $\mathbb{R}^r$ (see [W] p. 54). ∎

Let us consider the case where $\mathbb{K} = \mathbb{C}$. Then $V = \mathrm{Herm}(r,\mathbb{C})$ and $E = \mathrm{End}(r,k,\mathbb{C})$. It is important to recall that we are looking to E as a real vector space, and in particular polynomials on E need not be holomorphic.

Theorem 2.3. *There exist non trivial determinantally homogeneous polynomials on E if and only if $k \geq r$. Assume $k \geq r$. Then for $\eta \in E$ define*

$$p_\eta(\xi) = \mathrm{Det}(\xi\,\eta^t),\ \ q_\eta(\xi) = \mathrm{Det}(\overline{\xi}\,\eta^t).$$

The polynomials p_η *and* q_η *belong to* $\mathcal{P}^{\det,1}$, *and as* η *runs through* E, *they generate* $\mathcal{P}^{\det}$ *as an algebra.*

Proof. Let m be an integer, and for $s,t\in\mathbb{N}$, with $s+t=m$, let

$$\mathcal{P}^{s,t}=\{p\in\mathcal{P}\mid p(\lambda\xi)=\lambda^s\overline{\lambda}^t p(\xi),\ \forall\lambda\in\mathbb{C}^*\}.$$

By looking at the natural action of $\mathbb{C}^*$ on $\mathcal{P}^m$,

$$\mathcal{P}^m=\bigoplus_{s+t=m}\mathcal{P}^{s,t}.$$

There is a similar decomposition for $\mathcal{P}^{\det}$. First observe that

$$SL(r,\mathbb{C})\cap\mathbb{C}^*\,\mathrm{Id}=\{\lambda\,\mathrm{Id},\lambda^r=1\}\simeq\mathbb{U}_r$$

($\mathbb{U}_r$ is the multiplicative group of r-th roots of unity). So if p is invariant by $SL(r,\mathbb{C})$, and satisfies $p(\lambda\xi)=\lambda^s\overline{\lambda}^t p(\xi)$ with $s+t=lr$, then by comparing these two properties on $SL(r,\mathbb{C})\cap\mathbb{C}^*\,\mathrm{Id}$, it is necessary that $s-t\equiv 0 \mod r$, and hence $s\equiv 0\mod r, t\equiv 0\mod r$. So, set

$$P^{\det,s,t}=\{p\in\mathcal{P}\mid p(z\xi)=(\operatorname{Det} z)^s(\operatorname{Det}\overline{z})^t p(\xi),\ \forall z\in\operatorname{End}(r,\mathbb{C})\}. \tag{7}$$

Then clearly

$$\mathcal{P}^{\det,l}=\bigoplus_{s+t=l}\mathcal{P}^{\det,s,t}.$$

Let us now complexify the whole situation. Realize the complexification of $E=\operatorname{End}(r,\mathbb{C})$ as $E^{\mathbb{C}}=\operatorname{End}(r,\mathbb{C})\times\operatorname{End}(r,\mathbb{C})$, with the conjugation

$$(z_1,z_2)\longmapsto(\overline{z}_2,\overline{z}_1)$$

through the embedding $(z\longmapsto(z,\overline{z}))$ and in a similar fashion, realize the complexification of $\operatorname{End}(r,k,\mathbb{C})$ as $\operatorname{End}(r,k,\mathbb{C})\times\operatorname{End}(r,k,\mathbb{C})$ with the conjugation $(\xi_1,\xi_2)\longmapsto(\overline{\xi}_2,\overline{\xi}_1)$ through the embedding $(\xi\longmapsto(\xi,\overline{\xi}))$. The natural action of $\operatorname{End}(r,\mathbb{C})\times\operatorname{End}(r,\mathbb{C})$ on $\operatorname{End}(r,k,\mathbb{C})\times\operatorname{End}(r,k,\mathbb{C})$ given by

$$(z_1,z_2)(\xi_1,\xi_2)\longmapsto(z_1\xi_1,z_2\xi_2)$$

is a complexification of the action of $\operatorname{End}(r,\mathbb{C})$ on $\operatorname{End}(r,k,\mathbb{C})$. The polynomial $(\operatorname{Det} z)^s(\operatorname{Det}\overline{z})^t$ on $\operatorname{End}(r,\mathbb{C})$ is the restriction of the holomorphic polynomial $(\operatorname{Det} z_1)^s(\operatorname{Det} z_2)^t$ on $\operatorname{End}(r,\mathbb{C})\times\operatorname{End}(r,\mathbb{C})$. A polynomial $p\in\mathcal{P}^{\det,s,t}$ extends in a unique way to a holomorphic polynomial $\widetilde{p}$ on $E^{\mathbb{C}}=\operatorname{End}(r,k,\mathbb{C})\times\operatorname{End}(r,k,\mathbb{C})$ which satisfies

$$\widetilde{p}\,(z_1\xi_1,z_2\xi_2)=(\operatorname{Det} z_1)^s(\operatorname{Det} z_2)^t\,\widetilde{p}(\xi_1,\xi_2) \tag{8}$$

for any $z_1,z_2\in\operatorname{End}(r,\mathbb{C})$. Denote by $\widetilde{\mathcal{P}}^{s,t}$ the space of polynomials which satisfy (8).

The determination of $\widetilde{\mathcal{P}}^{s,t}$ is now easy using the isomorphism

$$\mathcal{P}(\mathrm{End}(r,k,\mathbb{C}) \times \mathrm{End}(r,k,\mathbb{C})) \simeq \mathcal{P}(\mathrm{End}(r,k,\mathbb{C})) \otimes \mathcal{P}(\mathrm{End}(r,k,\mathbb{C})).$$

Clearly

$$\widetilde{\mathcal{P}}^{s,t} \simeq \mathcal{P}^{\mathrm{Det},s} \otimes \mathcal{P}^{\mathrm{Det},t}.$$

But now a classical result in invariant theory (in fact the complex version of the theorem we used in the real case) shows that $\mathcal{P}^{\mathrm{Det},s}$ (resp. $\mathcal{P}^{\mathrm{Det},t}$)is generated by (the products of s (resp. t)) polynomials of the form $\mathrm{Det}(\xi_1\eta^t)$ (resp. $\mathrm{Det}(\xi_2\eta^t)$), where η is an arbitrary element of $\mathrm{End}(r,k,\mathbb{C})$. Theorem 4 follows by taking the restriction of these polynomials to the real form $E = \{(\xi,\overline{\xi}) \mid \xi \in \mathrm{End}(r,k,\mathbb{C})\}$ of $\mathrm{End}(r,k,\mathbb{C}) \times \mathrm{End}(r,k,\mathbb{C})$. ■

Let us now assume $\mathbb{K} = \mathbb{H}$, and use the same realization of $\mathrm{Herm}(r,\mathbb{H})$ as above. Similarly,

$$\mathrm{End}(r,k,\mathbb{H}) \simeq \{\xi \in \mathrm{End}(2r,2k,\mathbb{C}) \mid \xi J_k = J_r\overline{\xi}\}.$$

Notice that these realizations are compatible in the sense that the action of $\mathrm{End}(r,\mathbb{H})$ on $\mathrm{End}(r,k,\mathbb{H})$ is the restriction of the natural action of $\mathrm{End}(2r,\mathbb{C})$ on $\mathrm{End}(2r,2k,\mathbb{C})$ given by $(x,\xi) \longmapsto x\xi$.

The conjugation $z \mapsto -J\overline{z}J$ realizes $\mathrm{End}(r,\mathbb{H})$ as a real form of $\mathrm{End}(2r,\mathbb{C})$, and similarly, $\xi \mapsto -J_r\overline{\xi}J_k$ realizes $\mathrm{End}(r,k,\mathbb{H})$ as a real form of $\mathrm{End}(2r,2k,\mathbb{C})$. The group $SL(r,\mathbb{H})$ is realized as a real form of $SL(2r,\mathbb{C})$.

Now a polynomial p on $E = \mathrm{End}(r,k,\mathbb{H})$ can be extended in a unique way as a holomorphic polynomial $\widetilde{p}$ on $\mathrm{End}(2r,2k,\mathbb{C})$. Assume p is determinantally homogeneous of degree l. Theorems 1.1 and 2.1 imply $\widetilde{p}$ satisfies

$$\forall g \in SL(2r,\mathbb{C}), \qquad \widetilde{p}(g\xi) = \widetilde{p}(\xi)\ .$$

By the same classical result as before, the algebra of $SL(2r,\mathbb{C})$-invariants polynomials is generated by the polynomials $\widetilde{p}_\eta = \mathrm{Det}(\xi\eta^t)$, where η is an arbitrary element in $\mathrm{End}(2r,2k,\mathbb{C})$. Observe that $\widetilde{p}_\eta$ is homogeneous of degree $2r$. But $\widetilde{p}$ has the same degree of homogeneity as p, namely rl. So if l is odd $\mathcal{P}^{\mathrm{det},l} = \{0\}$. Here is the final statement for this case.

Theorem 2.4. *Let $V = \mathrm{Herm}(r,\mathbb{H})$ acting on $E = \mathrm{End}(r,k,\mathbb{H})$ by left multiplication. There exist non trivial determinantally homogeneous polynomials on E if and only if $k \geq r$. Assume $k \geq r$, and let $E = \mathrm{End}(r,k,\mathbb{H})$ realized in $\mathrm{End}(2r,2k,\mathbb{C})$ as above. Then $\mathcal{P}^{\mathrm{det},l} \neq \{0\}$ if and only if l is even. For $\eta \in \mathrm{End}(2r,2k,\mathbb{C})$, define*

$$p_\eta(\xi) = \mathrm{Det}\,(\xi\,\eta^t).$$

Then p_η belongs to $\mathcal{P}^{\mathrm{det},2}$ and the polynomials p_η generate $\mathcal{P}^{\mathrm{det}}$ when η runs through $\mathrm{End}(2r,2k,\mathbb{C})$. ■

3. Representations of the Jordan algebra of Lorentzian type and Clifford modules

Let $W, (.|.)$ be a Euclidean vector space of dimension q. Define on $V = \mathbb{R} \oplus W$ the *Jordan product*

$$(\lambda, w)(\lambda', w') = (\lambda\lambda' + (w|w'), \lambda w' + \lambda' w) \quad . \tag{9}$$

This turns V into a Jordan algebra with unit element $e = (1, 0)$. For the inner product, we use

$$\big((\lambda, w)\, ,\, (\lambda', w')\big) = \lambda\lambda' + (w \mid w') \tag{10}$$

which is self-adjoint for the Jordan multiplication, so that V has a structure of Euclidean Jordan algebra. The corresponding cone is

$$\Omega = \{(\lambda, w) \mid \lambda^2 - (w \mid w) > 0, \lambda > 0\}$$

and this is the reason to say that V is of *Lorentzian type*. The case $q = 1$ is peculiar, because V decomposes as a sum of two ideals, so that we may assume that $q \geq 2$. All Euclidean Jordan algebras of rank 2 are obtained in this manner (see [FK] ch.V).

The *determinant* function for V is given by

$$\det(\lambda, w) = \lambda^2 - (w \mid w). \tag{11}$$

The element $(\lambda, w) \in V$ is invertible if and only if $\det(\lambda, w) \neq 0$, and its inverse is given by

$$(\lambda, w)^{-1} = (\lambda^2 - (w \mid w))^{-1}(\lambda, -w) \ .$$

There is an other approach to this family of Jordan algebras, via the *Clifford algebra* $Cl(W)$. Recall that it is the associative algebra (the product is denoted by $a.b$) generated by W with the relations $w\,.w = \|w\|^2$ (for most of the references, this is the Clifford algebra for the *negative*-definite form $-\|w\|^2$). There exists a canonical injection from V into $Cl(W)$, which maps $(\lambda, w) \in V$ to $\lambda + w$ in $Cl(W)$. Moreover, $Cl(W)$, which is an associative algebra can be turned into a Jordan algebra by using the symmetrized product

$$\frac{1}{2}(a.b + b.a).$$

Then $\mathbb{R} \oplus W$ is a Jordan subalgebra of $Cl(W)$ and the induced Jordan product coincides with the one we introduced *supra*. So we indifferently denote elements of V either as (λ, w) or as $\lambda + w$, where $\lambda \in \mathbb{R}$ and $w \in W$.

On $Cl(W)$, there exists a unique involutive automorphism, denoted by $a \mapsto \tilde{a}$ such that $\tilde{w} = -w$ for all $w \in W$. There exists also a unique involutive anti-automorphism denoted by $\check{}$ such that $\check{w} = w$ for $w \in W$, and by composing the two, there is a second involutive anti-automorphism, denoted by $\hat{}$ such that $\hat{w} = -w$ for $w \in W$. With these notation, one has for any $v \in V^\times$,

$$v^{-1} = (\det v)^{-1}\hat{v} \ .$$

Let Φ be a representation of the Jordan algebra V on an Euclidean vector space $(E, \langle .\,,.\rangle)$. Let $w \in W$. Then $(0,w)^2 = (\|w\|^2, 0) = \|w\|^2 e$. Hence $\Phi(w)^2 = \|w\|^2\,\mathrm{Id}$. The universal property of the Clifford algebra implies that there exists a (unique) homomorphism of associative algebras

$$\widetilde{\Phi} : Cl(W) \longrightarrow \mathrm{End}(E)$$

extending Φ. Hence E is a *Clifford module* for $Cl(W)$.

Conversely, let E be a Clifford module for $Cl(W)$, that is a real vector E of finite dimension, with an action of $Cl(W)$. Fix an orthonormal basis $(e_1, e_2, \dots, e_q)$ of W. For $I = \{i_1 < i_2 < \dots < i_{|I|}\}$ any ordered subset of $\{1, 2, \dots, q\}$, let $e_I = e_{i_1}e_{i_2}\dots e_{i_{|I|}}$. Then the set $F = \{\pm e_I\}$, where I runs through all subsets of $\{1, 2, \dots, q\}$, is a multiplicative group for the Clifford product, of cardinality 2^{q+1}. Fix an inner product $\langle . \mid . \rangle_0$ on E and let

$$\langle \xi, \eta\rangle \ = \ 2^{-(q+1)} \sum_{a\in F} \langle a.\xi, a.\eta\rangle_0.$$

This defines a new inner product on E. Let $b \in F$. As F is a group,

$$\langle b.\xi, b.\eta\rangle \ = \ \langle \xi, \eta\rangle$$

for all $\xi, \eta \in E$. In particular, for $i, 1 \le i \le q$, $\langle e_i.\xi, e_i.\eta\rangle = \langle \xi, \eta\rangle$. Let $x = \sum_{i=1}^q x_i e_i$ an arbitrary element of W. Then

$$\langle x.\xi, \eta\rangle \ = \sum_{i=1}^q x_i\langle e_i.\xi, \eta\rangle \ = \sum_{i=1}^q x_i\langle e_i.(e_i.\xi), e_i.\eta\rangle$$

$$= \sum_{i=1}^q x_i\langle \xi, e_i.\eta\rangle \ = \ \langle \xi, x.\eta\rangle.$$

Define $\Phi : V \longrightarrow \mathrm{End}(E)$ by the formula

$$\Phi(v)\xi = v.\xi \quad ,$$

where we use the identification $V \simeq \mathbb{R}\oplus W \subset Cl(W)$. Then Φ is a Jordan algebra homomorphism from V into $\mathrm{Sym}(E)$, where $\mathrm{Sym}(E)$ is defined with respect to the new inner product $\langle .\,,.\rangle$. This shows the equivalence between representations of V and $Cl(W)$-modules. This correspondance had been observed before (see e.g. [Do])

Let Φ be a representation of $V = \mathbb{R}\oplus W$ on a Euclidean vector space E and let $\widetilde{\Phi}$ be the extension of Φ to $Cl(W)$. As the algebra $\{\widetilde{\Phi}(a), a \in Cl(W)\}$

is generated by $\{\Phi(w), w \in W\}$, the notions of equivalent or irreducible representations are the same for V or $Cl(W)$. The classification of the irreducible Clifford modules is well known, and in fact it is essentially equivalent to the structure theorems for Clifford algebras. Up to equivalence, there is a unique irreducible representation (when q is even) or two (when q is odd) ($\sharp$). The space for an irreducible representation is called a *pinor space*. The reference [H] contains an exhaustive description of the pinor spaces (see also [D]). The following table indicates the main facts. For each value of q, we list the dimension p of the corresponding pinor space (in the odd case, both pinor spaces have same dimension) and the commutant $\mathbb{K}$ of the representation (which is either $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$).

q	$8k$	$8k+1$	$8k+2$	$8k+3$	$8k+4$	$8k+5$	$8k+6$	$8k+7$
p	16^k	16^k	2.16^k	4.16^k	8.16^k	8.16^k	16.16^k	16.16^k
$\mathbb{K}$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}$	$\mathbb{H}$	$\mathbb{C}$

Table 1

Let E be a pinor space. Then there exists on E a unique (up to a positive real number) Euclidean inner product $\langle . , . \rangle$ such that

$$(*) \qquad \langle \Phi(x)\xi \, , \, \eta\rangle \; = \; \langle \xi \, , \, \Phi(x)\eta\rangle$$

In fact it is enough to verify the uniqueness, because the existence is true for any Clifford module. But if $\langle . , . \rangle'$ is another inner product with the property $(*)$, then there exists a linear operator A such that $\langle \xi \, , \, \eta\rangle' \; = \; \langle A\xi \, , \, \eta\rangle$. The symmetry of $\langle . , . \rangle$ shows that A has to be symmetric. Then, property $(*)$ is satisfied if and only if A commutes with $\Phi(x)$, for all $x \in V$. But as A is symmetric, and the representation is irreducible, A can have only one eigenvalue, hence is a multiple of the identity.

Let us express Q using the orthonormal basis $\{e_1, e_2, \dots, e_q\}$ of W already introduced. Let $v = (\lambda, \sum_{i=1}^{q} x_i e_i) \in V$. Then

$$\big(v, Q(\xi)\big) = \langle \Phi(x)\xi, \xi\rangle = \lambda\langle \xi, \xi\rangle + \sum_{i=1}^{q} x_i \langle e_i.\xi, \xi\rangle$$

so that

$$(12) \qquad Q(\xi) = (\|\xi\|^2, \sum_{i=1}^{q} \langle e_i.\xi, \xi\rangle e_i).$$

For the group Γ, we use a slightly different approach as before, by realizing Γ in the Clifford algebra (in Section 2, the group Γ was constructed

($\sharp$) I wish to take the opportunity to indicate that Théorème 2 in [C1] is partially incorrect, as there are *two* non equivalent irreducible representations in the "complex case" ($q \equiv 3, 7$ mod. 8) through an irreducible representation). Our approach is parallel to the classical approach of the *Clifford* group or *Pin* group (see [GM]).

Let $Cl(W)^{\times}$ be the (multiplicative) group of invertible elements of $Cl(W)$ and let Γ be the closed subgroup of $Cl(W)^{\times}$ generated by the elements $v \in \mathbb{R} \oplus W$ with $\det(v) = 1$. Observe that $\det(\lambda, w) = \lambda^2 - \|w\|^2$ is a quadratic form on V of signature $(1,q)$. Let $SO_0(V, \det) \simeq SO_0(1,q)$ be the connected component of the identity in $O(V, \det)$.

Lemma 3.1. *Let $v \in V$ with $\det v = 1$. Then*

$$\forall z \in V, \quad v.z.v \in V$$

and the mapping $\gamma_v : z \longmapsto v.z.v$ belongs to the group $SO_0(V, \det)$.

Proof. Let b be the bilinear form associated to the quadratic form $\det$. Then, for any v and $z \in V$, the following identity holds

$$v.z.v = 2b(v, \hat{z})v - (\det v)\,\hat{z}\ ,$$

so that $v.z.v$ belongs to V. Moreover, if $\det v = 1$ then clearly γ_v belongs to $O(V, \det)$. As $\gamma_{-v} = \gamma_v$ we may assume that v belongs to the set

$$\{u = (\lambda, w) \in V \mid \lambda^2 - \|w\|^2 > 0, \lambda > 0\}.$$

The connectedness of this set implies the fact that $\gamma_v \in SO_0(V, \det)$.

Now, by composition and continuity, it is possible to extend Lemma 3.1 to all of Γ. ∎

Theorem 3.2. *For any element a in Γ, and $z \in V$, $az\check{a}$ belongs to V, and the corresponding mapping $\gamma_a : z \longmapsto a.z.\check{a}$ is in $SO_0(V, \det)$. The mapping $\gamma : \Gamma \longrightarrow SO_0(V, \det)$ is a surjective homomorphism, with kernel $\{\pm \mathrm{Id}\}$.*

Proof. We may apply Lemma 1.2 to $SO_0(V, \det)$. With the standard Cartan involution, the $\mathfrak{p}$-part of the Lie algebra $\mathfrak{so}(V, \det) \simeq \mathfrak{so}(1,q)$ is given by

$$\left\{ X_w = \begin{pmatrix} 0 & w^t \\ w & 0 \end{pmatrix}, w \in W \right\}.$$

If $w = te_1$, then

$$\exp X_{te_1} = \begin{pmatrix} \cosh t & \sinh t & 0 \\ \sinh t & \cosh t & 0 \\ 0 & 0 & \mathrm{Id}_{q-2} \end{pmatrix}$$

belongs to the image of Γ, as can be seen from (13). This argument clearly applies to any element $w \in W$. By Lemma 1.2, this implies the surjectivity of γ. If $a \in \Gamma$ is an element in the kernel of γ, then $a.w = w.(\check{a})^{-1}$ for any $w \in W$. But as $a\hat{a} = 1$ for any element of Γ, we get $(\check{a})^{-1} = \widetilde{a}$, so that $a.w = w.\widetilde{a}$ for any $w \in W$. This implies that $a \in \mathbb{R}$ (see [GM] ch. 1, Lemma (5.25)). But as $a.\hat{a} = 1$, necessarily $a = \pm 1$. Conversely, it is clear that -1 belongs to the kernel of γ. ∎

For further use, notice that the subgroup in $Cl(W)^{\times}$ generated by the elements of the form $w_1.w_2.\dots.w_{2p}$, where $w_j \in W$ and $\|w_j\| = 1$ for each j is a subgroup of Γ, called the spin group and denoted by $Spin(W)$. $Spin(W)$ is mapped by γ onto $SO(W)$ viewed as a subgroup of $SO_0(\mathbb{R} \oplus W, \det)$.

4. Euclidean algebras and the non-regular cases

The regularity of the pinor representation was discussed in [C1], and it was shown that the pinor representation is regular if and only if $q \neq 2, 3, 5, 9$. We digress from our general programm to give a short proof of the non-regularity when $q = 2, 3, 5, 9$. It also serves as a presentation of some material to be used thereafter. We closely follow [H] (see also [FK] ch. V).

By a *Euclidean algebra* (or Euclidean Hurwitz algebra) we mean a (not necessarily associative, nor commutative) finite dimensional algebra $\mathbb{A}$ over $\mathbb{R}$ with unit element 1, equipped with a Euclidean inner product $\langle \, , \rangle$ whose associated square norm $\| \, \|^2$ satisfies

$$\|ab\|^2 = \|a\|^2 \|b\|^2 \quad \text{for } a, b \in \mathbb{A}.$$

Let $\operatorname{Im} \mathbb{A}$ be the orthogonal complement of $\mathbb{R}1$. Each element $a \in \mathbb{A}$ has a unique decomposition as $a = a_1 + a'$, where $a_1 \in \mathbb{R}1, a' \in \operatorname{Im} \mathbb{A}$. For convenience, set $\operatorname{Re} a = a_1, \operatorname{Im} a = a'$. *Conjugation* is defined by

$$\overline{a} = a_1 - a' = \operatorname{Re} a - \operatorname{Im} a.$$

Lemma 4.1. *For all* $a, b \in \mathbb{A}$

(i) $\quad \overline{\overline{a}} = a$

(ii) $\quad \overline{ab} = \overline{b}\,\overline{a}$

(iii) $\quad a\overline{a} = \overline{a}a = \|a\|^2$

(iv) $\quad \langle a, b\rangle = \operatorname{Re} a\overline{b} = \operatorname{Re} \overline{a}b.$

For $a \in \mathbb{A}$, denote by R_a (resp. L_a) the right (resp. left) multiplication by a, viewed as a $\mathbb{R}$-linear operator on $\mathbb{A}$.

Lemma 4.2. *Let* $a \in \mathbb{A}$. *Then*

$$R_a^* = R_{\overline{a}} \ , \ L_a^* = L_{\overline{a}} \tag{14}$$

$$R_{\overline{a}} \circ R_a = R_a \circ R_{\overline{a}} = \|a\|^2 \operatorname{Id} \tag{15}$$

$$\textit{for } a, b \in \mathbb{A}, \text{ with } a \perp b, \quad R_{\overline{a}} \circ R_b = -R_{\overline{b}} \circ R_a \tag{16}$$

Lemma 4.3. *The only Euclidean algebras are* $\mathbb{R}, \mathbb{C}, \mathbb{H}$ *(quaternions) and* $\mathbb{O}$ *(octonians or Cayley numbers), of respective dimensions* $1, 2, 4, 8$.

For a proof of these three lemmas, see [H], ch. 6.

Let $\mathbb{A}$ be a Euclidean algebra. Forgetting the extra structures on $\mathbb{A}$, let $W = \mathbb{R} \oplus \mathbb{A}$ viewed as a Euclidean vector space with the inner product defined by

$$\|(x_1, a)\|^2 = x_1^2 + \|a\|^2.$$

Observe that W is of dimension respectively $2, 3, 5$ or 9. To each element $(x_1, a) \in W$ associate the ($\mathbb{R}$ linear) operator $\Phi(x_1, a)$ on $E = \mathbb{A} \oplus \mathbb{A}$ described by the following matrix

$$\Phi(x_1, a) = \begin{pmatrix} x_1 \operatorname{Id} & R_a \\ R_{\overline{a}} & -x_1 \operatorname{Id} \end{pmatrix}.$$

Now

$$\Phi(x_1, a)\Phi(x_1, a) = (x_1^2 + \|a\|^2) \operatorname{Id}$$

so that Φ can be extended to an algebra homomorphism $\widetilde{\Phi}$ of the Clifford algebra $Cliff(W)$ in $\operatorname{End}(E)$. As the dimension of the pinor space(s) is known to be respectively $2, 4, 8, 16$ (see Table 1) which is exactly the dimension (over $\mathbb{R}$) of E, it is clear that this representation is nothing else but the pinor representation of $Cl(W)$. As explained before, this can also be viewed as a representation of the Lorentzian Jordan algebra $V = \mathbb{R} \oplus W$,

$$(x_0, x_1, a) \longmapsto \Phi(x_0, x_1, a) = \begin{pmatrix} (x_0 + x_1) \operatorname{Id} & R_a \\ R_{\overline{a}} & (x_0 - x_1) \operatorname{Id} \end{pmatrix}. \tag{17}$$

As inner product on E we use

$$\langle \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \rangle = \langle \xi_1, \eta_1 \rangle + \langle \xi_2, \eta_2 \rangle.$$

In this formalism, the quadratic map Q is easily determined, as

$$\langle \Phi(x_0, x_1, a) \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \rangle$$

$$\begin{aligned} &= (x_0 + x_1)\|\xi_1\|^2 + (x_0 - x_1)\|\xi_2\|^2 + \langle R_a\, \xi_2, \xi_1 \rangle + \langle R_{\overline{a}} \xi_1, \xi_2 \rangle \\ &= x_0(\|\xi_1\|^2 + \|\xi_2\|^2) + x_1(\|\xi_1\|^2 - \|\xi_2\|^2) + 2\langle a, \overline{\xi_2} \xi_1 \rangle \end{aligned}$$

by using (14) and (15). Hence

$$Q(\xi_1, \xi_2) = \Big(\|\xi_1\|^2 + \|\xi_2\|^2, \|\xi_1\|^2 - \|\xi_2\|^2, \overline{\xi_2}\, \xi_1 \Big). \tag{18}$$

Now

$$\det Q(\xi_1, \xi_2) = (\|\xi_1\|^2 + \|\xi_2\|^2)^2 - (\|\xi_1\|^2 - \|\xi_2\|^2)^2 - 4\|\overline{\xi_2}\, \xi_1\|^2$$

which equals 0 because of the multiplicative property of the square norm. We can now reformulate the result in the language of Clifford algebras and pinor spaces.

Theorem 4.4. *Let $q = 2, 3, 5$ or 9. Let E be the pinor space associated to the Clifford algebra $Cl(W)$. For any orthonormal basis $(e_1, e_2, \dots e_q)$ of W and for any $\xi \in E$*

$$\|\xi\|^4 = \sum_{i=1}^{q} |\langle e_i.\xi, \xi\rangle|^2. \tag{19}$$

Remark. Formula (19) implies several identities for the matrices representing the operators $E_j = \Phi(e_j), 1 \leq j \leq q$. In fact, fix an orthonormal basis $\{\epsilon_1, \epsilon_2, \dots, \epsilon_N\}$ of E and let $\xi = \sum_{\alpha=1}^{N} \xi_\alpha \epsilon_\alpha$. Notice that $N = 2(q-1) = 2, 4, 8$ or 16. Then the identity is a polynomial identity in the variables (ξ_α), hence the coefficients of the monomials on both sides must be the same. For $j = 1, 2 \dots, q$, denote by $E^j_{\alpha,\beta}$ the coefficients of the matrix representing the operator $E_j = \Phi(e_j)$ in the basis $\{\epsilon_\alpha\}$. Then (19) implies

$$\sum_{j=1}^{q} E^j_{\alpha,\alpha} E^j_{\alpha,\beta} = \delta_{\alpha,\beta}. \tag{20}$$

This is a special case of the Fierz identities (see [B-R-R]).

5. Determinantally homogeneous polynomials of degree 1 on pinor spaces

Let E be a pinor space, associated to the Jordan algebra of Lorentzian type $\mathbb{R} \oplus W$. The discussion of the space $\mathcal{P}^{\det,1}$ is strongly related to the nature of the commutant $\mathbb{K}$ of the representation, which is a real field, isomorphic to either $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. The real vector space E can be considered as a (right) $\mathbb{K}$-vector space. If $q \in \mathbb{K}$, the corresponding action on E will be denoted by R_q. We will make use of the fact that $R_q^t = R_{\overline{q}}$ for any $q \in \mathbb{K}$.

Proposition 5.1. *A real-valued polynomial p on E belongs to $\mathcal{P}^{\det,1}$ if and only if $p(\xi) = \langle A\xi, \xi\rangle$ for some $A \in \mathrm{End}(E)$ satisfying*

$$\begin{aligned} &\text{i)} & A^t &= A \\ &\text{ii)} & A^2 &= \lambda\,\mathrm{Id},\ \ \lambda \in [0, \infty) \\ &\text{iii)} & A\Phi(w) &= -\,\Phi(w)A, \quad \forall w \in W \end{aligned} \tag{21}$$

Proof. First let $p \in \mathcal{P}^{\det,1}$. The polynomial p is homogeneous of degree 2. Hence, there exists a symmetric bilinear form b on E such that $p(\xi) = b(\xi, \xi)$. If $w \in W$, then $p(w.\xi) = -\|w\|^2 p(\xi)$ for all $\xi \in E$. By polarization, b has to satisfy

$$b(w.\xi, w.\eta) = -\|w\|^2 b(\xi, \eta)$$

for all $w \in W$. Apply this equality to $\eta = w.\zeta$ to get

$$b(w.\xi, \zeta) = -b(\xi, w.\zeta)$$

for all $w \in W, \xi, \zeta \in E$. The operator $A \in \mathrm{End}(E)$ defined by the relation $b(\xi,\eta) = \langle A\xi\,,\,\eta\rangle$ must satisfy $A^t = A$ and $A \circ \Phi(w) = -\Phi(w) \circ A$ for all $w \in W$. The operator $A^2 = AA^t$ is positive self-adjoint, and must commute with $\Phi(w)$ for all $w \in W$. As the representation is assumed to be irreducible, A^2 must be a multiple of the identity, say $A^2 = \lambda\,\mathrm{Id}$, and λ has to be nonnegative. Hence A satisfies (21).

Conversely, let A satisfy the conditions (21). For $w \in W$ and $\xi \in E$, we have

$$\langle A(w.\xi), \xi\rangle = \langle w.\xi, A\xi\rangle = \langle \xi, w.(A\xi)\rangle = -\langle A(w.\xi), \xi\rangle\ ,$$

hence $\langle A(w.\xi), \xi\rangle = 0$. Let $p(\xi) = \langle A\xi, \xi\rangle$. For $(\mu, w) \in V$, we have

$$p(\Phi(\mu,w)\xi) = \langle A(\mu\xi + w.\xi), \mu\xi + w.\xi) = \mu^2\langle A\xi,\xi\rangle + \langle A(w.\xi), w.\xi\rangle\ .$$

But $\langle A(w.\xi), w.\xi\rangle = -\langle w.A\xi, w\xi\rangle = -\|w\|^2\langle A\xi,\xi\rangle$, and eventually we get

$$p(\Phi(\mu,w)\xi) = (\mu^2 - \|w\|^2)p(\xi)\quad ,$$

which is exactly the property that characterizes elements of $\mathcal{P}^{\det,1}$. ■

The operators which satisfy (21) are related to the *spin structure operators* and our results are related to those in [H] (which cover the case of a quadratic form of arbitrary signature).

Once a solution is known, it is easy to determine all of them. If A and B are two operators which satisfy (21), then AB commutes with the representation, hence there exists $s \in \mathbb{K}$ such that $AB = R_s$. As A^{-1} is a multiple of A, we have $B = A \circ R_q$, for some $q \in \mathbb{K}$. The conditions $(21i), (21ii)$ for B imply that $A \circ R_q = R_{\overline{q}} \circ A$. Conversely, if A satisfies (21), and if $q \in \mathbb{K}$ is such that $A \circ R_q = R_{\overline{q}} \circ A$, then $B = A \circ R_q$ also satisfies (21).

Theorem 5.1. *Let W be a Euclidean vector space of dimension $q \geq 2$, and let E be a pinor space for the Clifford algebra $Cl(W)$. Then the dimension p of $\mathcal{P}^{\det,1}$ is given by the following table*

$q \bmod 8$	0	1	2	3	4	5	6	7
p	1	0	0	0	1	0	3	2.

Proof. Let first assume q is even. Fix an orthonormal basis $e_1, e_2, \dots e_q$ of W. The element $f = e_1e_2\dots e_q$ of $Cl(W)$ verifies $fx = -xf$ for all $x \in W$. Moreover, $f\check{f} = \check{f}f = 1$, and $\check{f} = f$ when $q \equiv 0, 4 \mod 8$, $\check{f} = -f$ when $q \equiv 2, 6 \mod 8$. Let $F = \widetilde{\Phi}(f)$. Then F anti-commutes with the representation, $FF^t = \mathrm{Id}$ and $F^t = F$ when $q \equiv 0, 4 \mod 8$, $F^t = -F$ when $q \equiv 2, 6 \mod 8$. If A is an operator which satisfies the conditions (21), then A and F commute. As A and F anti-commute with the representation, $AF = FA$ commutes with the representation. Hence AF belongs to $\mathbb{K}$. But $(AF)^2 = FAAF = \lambda F^2$, and hence $(AF)^2 = \lambda\,\mathrm{Id}$ when $q \equiv 0, 4 \mod 8$, $(AF)^2 = -\lambda\,\mathrm{Id}$ when $q \equiv 2, 6 \mod 8$. Finally, $(AF)^t = F^tA^t$ and hence AF is symmetric when $q \equiv 0, 4 \mod 8$, skew-symmetric when $q \equiv 2, 6 \mod 8$.

Case $q \equiv 0, 4 \mod 8$

Then F satisfies conditions (21), and if A also satisfies these conditions, AF is a self-adjoint element of $\mathbb{K}$, hence is real. Hence, there exists a unique (up to a scalar) polynomial in $\mathcal{P}^{\det,1}$, and it is given by

$$p(\xi) = \langle F\xi, \xi\rangle \ .$$

Case $q \equiv 2 \mod 8$

In this case, $\mathbb{K} = \mathbb{R}$, and AF is a real multiple of the identity. As $(AF)^2 = -\lambda\,\mathrm{Id}$, with $\lambda \geq 0$, there is no solution except 0, and $\mathcal{P}^{\det,1}$ is trivial.

Case $q \equiv 6 \mod 8$

In this case, $\mathbb{K} = \mathbb{H}$. Let q be a pure imaginary quaternion so that $q^2 = q(-\overline{q}) = -\|q\|^2$. Set $A = F \circ R_q$. Then $A^t = R_q^t \circ F^t = R_{\overline{q}} \circ (-F) = R_q \circ F = F \circ R_q = A$. Moreover, $A^2 = F \circ R_q \circ R_q \circ F = -F \circ F = +\mathrm{Id}$, and finally A anti-commutes with the representation, as F anti-commutes with the representation and R_q commutes with the representation. Hence A satisfies (21), and it is easily verifies that any solution is of that type. So, in this case, $\mathcal{P}^{\det,1}$ is of dimension 3.

Let us now consider the case when q is odd. The element $f = e_1 e_2 \dots e_q$ belongs to the center of the Clifford algebra. Moreover, $f\check{f} = \check{f}f = 1$, with $\check{f} = f$ when $q \equiv 1,5 \mod 8$, $\check{f} = -f$ when $q \equiv 3,7 \mod 8$. Let $F = \widetilde{\Phi}(f)$. Then $F \in \mathbb{K}$, $F^t = F$ when $q \equiv 1,5 \mod 8$ and $F^t = -F$ when $q \equiv 3,7 \mod 8$. If A satisfies (21), then $A \circ F = -F \circ A$.

Case $q \equiv 1,5 \mod 8$

In this case, F is an element of $\mathbb{K}$ which is moreover symmetric, hence a real multiple of Id. If A is an operator which satisfies (21), then $AF = -FA$, which is impossible except if $A = 0$. Hence $\mathcal{P}^{\det,1} = \{0\}$ in this case.

It remains to study the cases when $q \equiv 3,7 \mod 8$. In this cases, $f^2 = -1$. Let $\mathcal{A}$ the subalgebra of even elements in $Cl(W)$. As $Cl(W) = \mathcal{A} \oplus f\mathcal{A}$, one has $Cl(W) \simeq \mathcal{A} \otimes_{\mathbb{R}} \mathbb{C} = \mathcal{A}_{\mathbb{C}}$. Using previous notation, $F^2 = -\mathrm{Id}$ and hence the pinor space has a complex structure given by F and the pinor space can be regarded as a complex irreducible representation of the algebra $\mathcal{A}_{\mathbb{C}}$.

Case $q \equiv 3 \mod 8$

In this case, the algebra $\mathcal{A}$ is isomorphic to $\mathrm{End}_{\mathbb{H}}(E)$ for a $\mathbb{H}$-structure on E. If A is an operator which satisfies (21), then A^2 commutes with the representation, hence belongs to $\mathbb{K}$. The conditions (21) implies that A is a real multiple of the identity, which contradicts (23), except if $A = 0$. Hence again $\mathcal{P}^{\det,1} = \{0\}$.

Case $q \equiv 7 \mod 8$

In this cases, the algebra $\mathcal{A}$ is isomorphic to $\mathrm{End}_{\mathbb{R}}(S)$ where S is a real form of E. The conjugation θ with respect to S commutes with the action of $\mathcal{A}$, and anti-commutes with the action of f, hence anticommutes with the action of odd elements, and in particular $\theta \circ \Phi(w) = -\Phi(w) \circ \theta$ for any $w \in W$. Moreover, the inner product is associated to a Hermitian form on E for which θ is Hermitian. So θ satisfies (21). For any $z \in \mathbb{C}$, the condition $\theta \circ R_z = R_{\overline{z}} \circ \theta$ is satisfied. Hence, in this case the dimension of $\mathcal{P}^{\det,1}$ is 2. ∎

6. The low dimensional cases

In this last section we will determine the space $\mathcal{P}^{\det}$ for the pinor space associated to low dimensional Jordan algebras of Lorentzian type. We keep notation from previous sections.

The cases where $q = 2, 3, 5$ have already been treated. In fact, for these values of q, the Jordan algebra $V = \mathbb{R} \oplus W_q$ is isomorphic to $\mathrm{Herm}(2, \mathbb{K})$, where $\mathbb{K} = \mathbb{R}$ when $q = 2$, $\mathbb{K} = \mathbb{C}$ when $q = 3$ and $\mathbb{K} = \mathbb{H}$ when $q = 5$, the pinor representation being just the fondamental representation on $\mathbb{K}^2$. Hence the results from section 2 show that $\mathcal{P}^{\det}$ only contains the constant polynomials.

Case $q = 4$

We use a specific realization of the Jordan algebra (and of the corresponding Clifford algebra). The pinor space E has dimension 8 and admits a quaternionic structure. So it is isomorphic to $\mathbb{H}^2$ and can be realized as $\mathbb{C}^4$, with the quaternionic structure (essentially) given by the 4×4 matrix

$$J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}.$$

The Lorentzian Jordan algebra $V_4 = \mathbb{R} \oplus \mathbb{R}^4$ is realized as the set of all matrices

$$\Phi(x) = \Phi(x_0, u, v) = \begin{pmatrix} x_0 & u & 0 & v \\ \overline{u} & x_0 & -v & 0 \\ 0 & -\overline{v} & x_0 & \overline{u} \\ \overline{v} & 0 & u & x_0 \end{pmatrix}$$

where $x_0 \in \mathbb{R}$ and $u, v \in \mathbb{C}$. In fact one can verify that the symmetrized Jordan product gives

$$\frac{1}{2}\big(\Phi(x_0, u, v)\Phi(x_0', u', v') + \Phi(x_0', u', v')\Phi(x_0, u, v)\big)$$
$$= \Phi\big(x_0 x_0' + \mathrm{Re}\langle \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u' \\ v' \end{pmatrix} \rangle \,,\, x_0 u' + x_0' u \,,\, x_0 v' + x_0' v\big) \;.$$

where $\langle .\,, .\rangle$ is the standard Hermitian product on $\mathbb{C}^2$.

Consider the Hermitian form on $\mathbb{C}^4$ given by

$$h(\xi_1, \xi_2, \xi_3, \xi_4) = |\xi_1|^2 - |\xi_2|^2 + |\xi_3|^2 - |\xi_4|^2.$$

An elementary computation shows that

$$h(\Phi(x_0, u, v)(\xi_1, \xi_2, \xi_3, \xi_4)) = (x_0^2 - \|u\|^2 - \|v\|^2)\; h(\xi_1, \xi_2, \xi_3, \xi_4)$$

so that h belongs to $\mathcal{P}^{\det,1}$.

Let us introduce the matrices

$$J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad H = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \Omega = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Observe that $\Omega = HJ = JH$ and that Ω is skew-symmetric and defines a complex symplectic form on $\mathbb{C}^4$. Recall the following result.

Proposition 6.1. *Let $g \in GL(4,\mathbb{C})$. Then any two of the following properties imply the third one :*

$$\begin{aligned} &\text{(i)} & Jg =&\overline{g}J \\ &\text{(ii)} & g^*Hg =&H \\ &\text{(iii)} & g^t\Omega g =&\Omega \end{aligned}$$

The elements which satisfy these conditions form a subgroup of $GL(4,\mathbb{C})$ (even of $SL(4,\mathbb{C})$, denoted by $HU(1,1)$ (the notation is borrowed from [H]).

As in the general case, let Γ be the closed subgroup of $GL(4,\mathbb{C})$ generated by the elements $\Phi(x_0,u,v)$ with $x_0^2-|u|^2-|v|^2=1$.

Theorem 6.2. *The group Γ coincides with $HU(1,1)$.*

Proof. As it is clear that $\Gamma \subset HU(1,1)$, we need only to prove the converse statement. For the usual Cartan involution on $SL(4,\mathbb{C})$, the Hermitian elements of $HU(1,1)$ (i.e elements of the form $\exp X$ with X in the $\mathfrak{p}$ part of the Lie algebra of $HU(1,1)$) are described as the elements $g \in GL(\mathbb{C}^4)$ which satisfy $Jg=\overline{g}J$, $g=g^*$ and $g^*Hg=H$.

The first condition guarantees that g is of the form $g=\begin{pmatrix} a & b \\ -\overline{b} & \overline{a}\end{pmatrix}$. The second condition forces $a=a^*$ and $b'=-b$. Hence g is of the form

$$g=\begin{pmatrix} x_0 & u & 0 & v \\ \overline{u} & x_1 & -v & 0 \\ 0 & -\overline{v} & x_0 & \overline{u} \\ \overline{v} & 0 & u & x_1 \end{pmatrix}$$

with $x_0,x_1\in\mathbb{R}, u,v\in\mathbb{C}$. One verifies that the last condition forces $x_1=x_0$ and $x_0^2-|u|^2-|v|^2=1$. Hence, there exists $x=(x_0,u,v)\in V$ with $x_0^2-|u|^2-|v|^2=1$, such that $g=\Phi(x)$. Lemma 1.2 then implies that $\Gamma=HU(1,1)$. ■

Next we complexify the situation. Let us consider on $GL(4,\mathbb{C})$ the conjugation σ defined by

$$\sigma(g)=-J\overline{g}J.$$

The corresponding real form is just $GL(n,\mathbb{H})$. Consider now the subgroup $Sp_2(\mathbb{C})=Sp(4,\mathbb{C})$ given by

$$Sp(4,\mathbb{C})=\{g\in GL(4,\mathbb{C}) \mid g^t\Omega g=\Omega\}.$$

Then σ preserves $Sp(4,\mathbb{C})$ and the fixed points set of σ in $Sp(4,\mathbb{C})$ is just $HU(1,1)$.

On the space $\mathbb{C}^4\times\mathbb{C}^4$, consider the conjugation $(\xi,\eta)\longmapsto(iJ\overline{\eta},-iJ\overline{\xi})$. Then the imbedding $\mathbb{C}^4\longrightarrow\mathbb{C}^4\times\mathbb{C}^4$ given by

$$\xi\longmapsto(\xi,-iJ\overline{\xi})$$

realizes $\mathbb{C}^4\times\mathbb{C}^4$ as a complexification of $\mathbb{C}^4$. The group $Sp(4,\mathbb{C})$ acts on $\mathbb{C}^4\times\mathbb{C}^4$ by $g.(\xi,\eta)=(g\xi,g\eta)$. If $g\in HU(1,1)$, then $g.(\xi,-iJ\overline{\xi})=(g\xi,-iJ(\overline{g\xi})$, so that the diagonal action of $Sp(4,\mathbb{C})$ on $\mathbb{C}^4\times\mathbb{C}^4$ is just the complexification of the action of $HU(1,1)$ on $\mathbb{C}^4$.

Theorem 6.3. $\mathcal{P}^{\det}$ *is generated by the polynomial* h.

Proof. Let $p \in \mathcal{P}^{\det,m}$ for some m. Then there exists a unique (holomorphic) polynomial $\widetilde{p}$ on $\mathbb{C}^4 \times \mathbb{C}^4$ such that $p(\xi) = \widetilde{p}(\xi, -iJ\overline{\xi})$. The fact that $p \in \mathcal{P}^{\det}$ implies that $p(g\xi) = p(\xi)$ for any $g \in HU(1,1)$. By holomorphicity of $\widetilde{p}$, this implies $\widetilde{p}(g\xi, g\eta) = p(\xi, \eta)$ for any $g \in Sp(4, \mathbb{C})$. As p is homogeneous (in the ordinary sense) of degree $2m$, the same is true for $\widetilde{p}$. But now we can apply the First Fondamental Theorem of invariant theory for the symplectic group (cf [GW] Theorem 4.4.2.). It shows that $\widetilde{p}$ is proportional to the m-th power of the symplectic form $\xi^t \Omega \eta$. By restriction to the real form, we see that p is proportional to the m-th power of the Hermitian form h. Conversely, it is clear that h is determinantally homogeneous of degree 1 and so any polynomial in h belongs to $\mathcal{P}^{\det}$. ∎

A corollary of this result is the following identity.

Corollary 6.4. *Let* $(\xi_1, \xi_2, \xi_3, \xi_4)$ *be four complex numbers. Then*

$$(|\xi_1|^2 + |\xi_2|^2 + |\xi_3|^2 + |\xi_4|^2)^2 - 4|\xi_1\overline{\xi}_2 + \overline{\xi}_3\xi_4|^2 - 4|\overline{\xi}_1\xi_4 - \overline{\xi}_2\xi_3|^2$$

$$= (|\xi_1|^2 - |\xi_2|^2 + |\xi_3|^2 - |\xi_4|^2)^2.$$

Proof. The Euclidean product on $\mathbb{C}^4$ is

$$\langle \xi, \eta \rangle = \mathrm{Re}\ (\xi_1\overline{\eta}_1 + \xi_2\overline{\eta}_2 + \xi_3\overline{\eta}_3 + \xi_4\overline{\eta}_4)\ .$$

The map $Q : E \longrightarrow V$ is then given by

$$Q(\xi_1, \xi_2, \xi_3, \xi_4) = \Big(|\xi_1|^2 + |\xi_2|^2 + |\xi_3|^2 + |\xi_4|^2,\ 2(\xi_1\overline{\xi}_2 + \xi_4\overline{\xi}_3),\ 2(\xi_1\overline{\xi}_4 - \xi_2\overline{\xi}_3)\Big).$$

Now $\det Q(\xi_1, \xi_2, \xi_3, \xi_4)$ is equal to the left hand side of the identity and it is determinantally homogeneous of degree 2, hence it must coincide (up to a normalization factor) with $h(\xi_1, \xi_2, \xi_3, \xi_4)^2$. The remaining constant is easily seen to be 1. Needless to say, the identity can be easily verified by straightforward computation. ∎

Case $q = 6, 7, 8, 9$

The pinor spaces associated to the cases where $q = 6, 7, 8, 9$ are all of the same dimension, namely 16, and in fact they admit a (very useful) common realization on the space $E = \mathbb{O} \oplus \mathbb{O}$, equipped with the standard inner product

$$\langle (\xi, \eta), (\xi', \eta') \rangle \ = \ \langle \xi, \xi' \rangle_{\mathbb{O}} + \langle \eta, \eta' \rangle_{\mathbb{O}}$$

Case $q = 9$

The realization for $q = 9$ has already be described in section 4. Let $W_9 = \mathbb{R} \oplus \mathbb{O}$ considered as a 9-dimensional Euclidean space with the inner product given by

$$\langle (x_1, q), (x_1', q') \rangle = x_1 x_1' + \langle q, q' \rangle_{\mathbb{O}}.$$

For $x_0 \in \mathbb{R}, (x_1, q) \in W_9$ let

$$\Phi(x_0, x_1, q) = \begin{pmatrix} (x_0 + x_1)\,\mathrm{Id}_8 & R_q \\ R_{\overline{q}} & (x_0 - x_1)\,\mathrm{Id}_8 \end{pmatrix}.$$

This formula defines a representation of $V_9 = \mathbb{R} \oplus W_9$ on $\mathbb{O} \oplus \mathbb{O}$. The spin group $Spin(9)$ in its 16-dimensional spin representation is known to be transitive on the unit-sphere S^{15} (see [H]), and hence any determinantally homogeneous polynomial being invariant by $Spin(9)$ is a radial polynomial, that is a polynomial in $\|\xi\|^2 + \|\eta\|^2$. But, for any $t \in \mathbb{R}$, the action of the matrix

$$\Phi(\cosh t, \sinh t, 0) = \begin{pmatrix} e^t\,\mathrm{Id}_8 & 0 \\ 0 & e^{-t}\,\mathrm{Id}_8 \end{pmatrix}$$

should also preserve the polynomial, which is impossible, except for the constant polynomials. Hence there are no non trivial determinantally homogeneous polynomials in that case.

Theorem 6.5. *For the representation of the Euclidean algebra of Lorentzian type of dimension $1 + 9$ on the pinor space $\mathbb{O}^2$, $\mathcal{P}^{\det}$ is reduced to the constant polynomials.*

Case $q = 8$

Now let $W_8 = \mathbb{O}, V_8 = \mathbb{R} \oplus \mathbb{O}$ the $1 + 8$-dimensional Euclidean Jordan algebra of Lorentzian type, and consider the representation on $E = \mathbb{O} \oplus \mathbb{O}$ given by

$$\Phi(x_0, q) = \begin{pmatrix} x_0\,\mathrm{Id}_8 & R_q \\ R_{\overline{q}} & x_0\,\mathrm{Id}_8 \end{pmatrix}.$$

The orbits of the corresponding spin group $Spin(8)$ in E are known. In fact (see [H] Theorem 14.69) the orbit of the point (ξ^0, η^0) is given by

$$\mathcal{O}_{\xi,\eta} = \{(\xi, \eta) \mid \|\xi\| = \|\xi^0\|, \|\eta\| = \|\eta^0\|\}$$

If a polynomial p on E is invariant by $Spin(8)$, it is constant on the orbits of $Spin(8)$, and hence there exists a function f defined on $[0, +\infty) \times [0, +\infty)$ such that $p(\xi, \eta) = f(\|\xi\|, \|\eta\|)$. Fix $v_0, w_0 \in \mathbb{O}$ with norm 1. Then $p(\lambda v_0, \mu w_0) = f(\lambda, \mu)$ for all $\lambda, \mu \geq 0$. The left hand side of this equality is a polynomial in λ, μ which is even with respect to both λ and μ, hence can be written as a polynomial of λ^2 and μ^2. So there exists a polynomial q in two variables such that $p(\xi, \eta) = q(\|\xi\|^2, \|\eta\|^2)$. This may be rewritten as

$$p(\xi, \eta) = r(\|\xi\|^2 + \|\eta\|^2, \|\xi\|^2 - \|\eta\|^2)\ ,$$

with r a polynomial in two variables. Now assume that the polynomial p is also invariant under the mappings

$$\Phi(\cosh t, \sinh t) = \begin{pmatrix} \cosh t\ \mathrm{Id}_8 & \sinh t\ \mathrm{Id}_8 \\ \sinh t\ \mathrm{Id}_8 & \cosh t\ \mathrm{Id}_8 \end{pmatrix}$$

for $t \in \mathbb{R}$. As $\|\xi\|^2 - \|\eta\|^2$ is invariant under this transformation, whereas $\|\xi\|^2 + \|\eta\|^2$ is not invariant, it is clear that the invariance of p under these transformations is equivalent to the fact that r does not depend on the first variable. Hence any determinantally homogeneous polynomial can be written as a polynomial in $\|\xi\|^2 - \|\eta\|^2$. But the converse statement is easily seen to be true.

Theorem 6.6. *For the representation of the Euclidean Lorentzian algebra of dimension* $1+8$ *on the pinor space* $\mathbb{O}^2$, $\mathcal{P}^{\det}$ *is generated (as an algebra) by the quadratic invariant* $\|\xi\|^2 - \|\eta\|^2$.

Remark. The quadratic map Q in this realization of the representation is easily seen to be given by $Q(\xi,\eta) = (\|\xi\|^2 + \|\eta\|^2, \overline{\eta}\,\xi)$. The corresponding polynomial $\det Q(\xi,\eta)$ is determinantally homogenous of degree 2 and in fact is easily seen to be equal to $(\|\xi\|^2 - \|\eta\|^2)^2$.

Case $q = 7$

Let $W_7 = \mathrm{Im}\ \mathbb{O} = (\mathbb{R}.1)^\perp$ be the space of pure imaginary elements in $\mathbb{O}$, and let $V_7 = \mathbb{R} \oplus W_7$. We get a representation of V_7 on $E = \mathbb{O} \oplus \mathbb{O}$ by

$$\Phi(x_0, q) = \begin{pmatrix} x_0\,\mathrm{Id}_8 & R_q \\ -R_q & x_0\,\mathrm{Id}_8 \end{pmatrix}, \quad x_0 \in \mathbb{R}, q \in Im(\mathbb{O}).$$

As $\Phi(0,q)\Phi(0,q') = \begin{pmatrix} -R_qR_{q'} & 0 \\ 0 & -R_qR_{q'} \end{pmatrix}$, the action of $Spin(7)$ is just two copies of the spin representation of $Spin(7)$ of dimension 8. This is the situation where the *triality* phenomenon occurs. Recall the following results (see [H] p. 285).

Lemma 6.7. *An element* $g \in O(\mathbb{O})$ *belongs to* $Spin(7)$ *if and only if*

$$g(qq') = g(q)g(g^{-1}(1)q')$$

for all $q, q' \in \mathbb{O}$.

Moreover, the spin representation of $Spin(7)$ on the unit sphere in $\mathbb{O}$ is transitive. The stabilizer of the element 1 is the automorphism group $Aut(\mathbb{O})$ of the algebra $\mathbb{O}$ (which incidentally, turns out to be isomorphic to the special simple group G_2, but we won't use this fact). Moreover, the group $Aut(\mathbb{O})$ acts *transitively* on the unit sphere of $\mathrm{Im}\,\mathbb{O}$.

As a consequence of this, it is possible to give a description of the orbits of $Spin(7)$ in $\mathbb{O} \oplus \mathbb{O}$.

Lemma 6.8. *The orbit of* (ξ^0, η^0) *under* $Spin(7)$ *is given by*

$$\mathcal{O}_{\xi^0,\eta^0} = \{(\xi,\eta) \in \mathbb{O} \oplus \mathbb{O} \mid \|\xi\| = \|\xi^0\|, \|\eta\| = \|\eta^0\|, \langle \xi^0, \eta^0 \rangle = \langle \xi, \eta \rangle\}$$

Proof. First observe that the spin representation being unitary, $\mathcal{O}_{\xi^0,\eta^0}$ contains the orbit of (ξ^0,η^0). Conversely, assume (ξ,η) and (ξ^0,η^0) are two couples which satisfy the conditions

$$\|\xi\| = \|\xi^0\|, \|\eta\| = \|\eta^0\|, \langle \xi^0, \eta^0 \rangle = \langle \xi, \eta \rangle.$$

As $Spin(7)$ is transitive on the unit sphere S_7 (in the spin representation), it is enough to prove the conjugation result when $\xi = \xi^0 = 1$. Now let η be arbitrary and decompose η as $\eta = a + q$ where a is real and $q \in \mathrm{Im}\ \mathbb{O}$. As $a = \langle 1, \eta \rangle = \langle 1, \eta^0 \rangle$, the corresponding decomposition for η^0 is $\eta^0 = a + q^0$ and observe that $\|q\| = \|q^0\|$. But we know that there exists an element $g \in Aut(\mathbb{O})$ such that $g(q) = q^0$. This finishes the proof. ∎

Let $p \in \mathcal{P}^{\det}$. Then p is invariant under the action of $Spin(7)$ and hence only depends on $\|\xi\|, \|\eta\|$ and $\langle \xi, \eta\rangle$, or on $\|\xi\|^2 + \|\eta\|^2$, $\|\xi\|^2 - \|\eta\|^2$ and $\langle \xi, \eta\rangle$. Moreover p is also invariant by the transformations $\Phi(\cosh t, \sinh t\ q)$, where q is a pure imaginary quaternion of norm 1. It is easily seen that $\|\xi\|^2 - \|\eta\|^2$ and $\langle \xi, \eta\rangle$ are invariant under these transforms, but this is *not* the case for $\|\xi\|^2 + \|\eta\|^2$. This implies that $p(\xi, \eta)$ only depends on $\|\xi\|^2 - \|\eta\|^2$ and $\langle \xi, \eta\rangle$. In other words, there is a function f defined on $\mathbb{R}^2$ such that $p(\xi, \eta) = f(\|\xi\|^2 - \|\eta\|^2, \langle \xi, \eta\rangle)$. Now let i, j, k, e be four imaginary quaternions of length 1 and two by two orthogonal. Then for λ, μ any real numbers, one verifies that the following identity holds:

$$f(\lambda, \mu) = p(1 + (\frac{\lambda}{2} + 1)i + \mu j, \mu + \frac{\lambda}{2}k + e)\ .$$

As the right hand side is a polynomial in (λ, μ) it shows that f is a polynomial function. Hence the following result:

Theorem 6.9. *For the representation of the Euclidean Lorentzian algebra of dimension $1+7$ on the pinor space $\mathbb{O}^2$, $\mathcal{P}^{\det}$ is generated (as an algebra) by the quadratic polynomials $\|\xi\|^2 - \|\eta\|^2$ and $\langle \xi, \eta\rangle$.* ■

Case $q = 6$

Fix a imaginary octonion of norm 1, say i, and let $W_6 = (\mathbb{R}1 \oplus \mathbb{R}i)^{\perp}$. We realize the representation of the corresponding Jordan algebra $V_6 = \mathbb{R} \oplus W_6$ on E by

$$\Phi(x_0, q) = \begin{pmatrix} x_0 \,\mathrm{Id} & R_q \\ -R_q & x_0 \,\mathrm{Id} \end{pmatrix}, \quad x_0 \in \mathbb{R}, q \in W_6.$$

The commutant of the representation is isomorphic to $\mathbb{H}$, with generators given by

$$\mathrm{Id}, \begin{pmatrix} R_i & 0 \\ 0 & -R_i \end{pmatrix}, \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}, \begin{pmatrix} 0 & R_i \\ R_i & 0 \end{pmatrix}.$$

The commutation relations are due to the relation $R_q \circ R_i = -R_i \circ R_q$ valid for all $q \in W_6$ (cf property (16)).

It is easy to verify that the three real-valued quadratic polynomials

$$\|\xi\|^2 - \|\eta\|^2,\ \langle \xi, \eta\rangle,\ \langle \xi, R_i\eta\rangle$$

are determinantally homogeneous of degree 1 under the action of the representation.

Through the specified element i, $\mathbb{O}$ is given a complex structure, which allows to identify $\mathbb{O}$ with $\mathbb{C}^4$. The spin group $Spin(6)$ acts diagonally on $\mathbb{O}\oplus\mathbb{O}$, and on each factor, this action commutes with R_i, hence is complex-linear. Moreover, the Hermitian inner product on $\mathbb{C}^4$ given by

$$(\xi, \xi') = \langle \xi, \xi'\rangle + \sqrt{-1}\langle \xi, R_i\xi'\rangle$$

is preserved by the action of $Spin(6)$. More precisely, one can show that $Spin(6) \simeq SU(4)$, the latter acting in the standard way on $\mathbb{O} \simeq \mathbb{C}^4$ (see [H]). Recall now the following classical result of invariant theory.

Lemma 6.10. *Consider the action of $SU(n)$ on $\mathbb{C}^{n\times k} = \underbrace{\mathbb{C}^n \oplus \mathbb{C}^n \oplus \ldots \oplus \mathbb{C}^n}_{k\ times}$, and assume $k < n$. Then the algebra of polynomials on $\mathbb{C}^{n\times k}$ invariant under the action of $SU(n)$ is generated by the quadratic polynomials*

$$(\xi_l\,,\,\xi_m), \quad 1 \le l, m \le\le k\ .$$

Proof. This result is easily obtained by complexification from the corresponding classical result for $SL(n,\mathbb{C})$ acting covariantly on some copies of $\mathbb{C}^n$, and contravariantly on some other copies of $\mathbb{C}^n$ (see [W]). If we want to stick to the real Euclidean inner product $\langle \xi, \xi' \rangle = \mathrm{Re}\,(\xi, \xi')$, we may replace the complex valued invariants by the family of real valued invariants $\langle \xi_l, \xi_m \rangle$ and $\langle \xi_l\,, R_i\,\xi_m \rangle$, with $1 \le l \le m \le k$. ■

Assume p is a polynomial on E which is invariant under the action of $Spin(6)$. So p can be regarded as a polynomial on $\mathbb{C}^4 \oplus \mathbb{C}^4$ invariant under $SU(4)$. By Lemma 6.10 , p can be written as a polynomial in the following quadratic polynomials

$$\|\xi\|^2, \|\eta\|^2, \langle \xi, \eta \rangle, \langle \xi\,, R_i\,\eta \rangle$$

which we prefer to write as

$$p(\xi,\eta) = r(\|\xi\|^2 + \|\eta\|^2,\ \|\xi\|^2 - \|\eta\|^2,\ \langle \xi, \eta \rangle,\ \langle \xi, R_i\,\eta \rangle)$$

where r is some polynomial depending on four variables. Now assume further p is determinantally homogeneous (say of degree l). Let j be any norm 1 imaginary octonion orthogonal to i. Then p has to be invariant under the mapping $\Phi(\cosh t, j \sinh t)$, for $t \in \mathbb{R}$. As we have already seen, the last three expressions $\|\xi\|^2 - \|\eta\|^2$, $\langle \xi, \eta \rangle$, $\langle \xi, R_i\,\eta \rangle$ are invariant under these transformations, whereas it is easily seen that this is *not* the case for the first one $\|\xi\|^2 + \|\eta\|^2$. It implies that r does not depend on the first variable.

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