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The Abelian Subgroup Conjecture: A Counter Example

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Abstract. If an abelian subgroup A of a locally compact group G has the same weight as G, it is termed large (see [3]). It has been conjectured that every compact group has a large abelian subgroup. In this note we show that no free pro-p group $F(X)$ on set X of cardinality greater than $\aleph_0$ contains a large abelian subgroup.

1. The Background

The weight $w(G)$ of a locally compact topological group is defined as the minimal cardinality of a base of open sets. K. H. Hofmann and S. A. Morris in [3] proposed the following:

Conjecture 1.1. (The Abelian Subgroup Conjecture) Every infinite compact group G has an abelian subgroup A whose weight equals that of G.

For brevity, we adopt the convention in [3], to say G is an LAS-group, if it satisfies the conjecture. As an important consequence of their investigations the authors prove the following extension Theorem (Theorem D):

Theorem 1.2. *Let G be an infinite compact group and N a closed subgroup such that G/N and N are LAS-groups. Then G is an LAS-group.*

In particular, when $N = G_0$ is the connected component of G they show in Corollary E:

Corollary 1.3. *Let G be a compact group and assume that G/G_0 is an LAS-group. Then G is an LAS-group.*

Therefore, if the Abelian Subgroup Conjecture is false, it should fail for some profinite group. According to [2], a compact group G is *strictly reductive*, if it is isomorphic to the direct product of simple compact groups (a compact group is called simple, if it does not contain nontrivial closed normal subgroups). Then in [3] the same authors show in Corollary F:

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Corollary 1.4. *Every strictly reductive compact group is an LAS-group.*

The most important step in further reducing the Abelian Subgroup Conjecture is Theorem 3.4, the *Countable Layer Theorem*, in [2], which, together with Theorem 4.4, the *Topological Decomposition Theorem*, implies the following theorem:

Theorem 1.5. *Every profinite group G has a series $G = G^0 > G^1 > \dots$ of characteristic closed subgroups with each G^i/G^{i-1} strictly reductive and $\bigcap_{i\in\mathbb{N}} G^i = \{1\}$. Moreover $w(G) = \sup_{i\in\mathbb{N}}\{w(G^i/G^{i+1})$.*

Proof. The last statement is a consequence of Theorem 4.14 in [2]. ∎

2. The Counterexample

Profinite groups admit, for every prime p, p-Sylow subgroups, any two of them being conjugate, and every pro-p subgroup of G being contained in a suitable p-Sylow subgroup. The *cofinality* of a cardinal $\aleph$ is the smallest cardinal $\aleph'$ such that the set of predecessors of $\aleph$ contains a subset S of cardinality $\aleph'$ satisfying $\aleph = \sup S$.

Lemma 2.1. *Let $G = \prod_{j\in J} S_j$ with simple compact groups S_j.*

(i) *If $w(G)$ has uncountable cofinality, then G has either a torus subgroup T or a p-Sylow subgroup P for a some prime p such that $w(T) = w(G)$ or $w(P) = w(G)$, respectively.*

(ii) *If the cardinal $\aleph$ has countably infinite cofinality, then a family of groups S_j of prime order can be selected so that G has weight $\aleph$ and all its Sylow-subgroups have a weight that is properly smaller than $\aleph$.*

Proof. In both cases $w(G)$ is infinite. We assume $w(G) \ge \aleph_0$. The identity component G_0 is a product of simple Lie groups. It has a maximal torus T such that $w(T) = w(G_0)$. Let $\mathcal{S}$ be a system of representatives of the finite simple groups; note that $\mathcal{S}$ is countable. Then $G = G_0 \times \prod_{S\in\mathcal{S}} S^{J(S)}$ for a countable family of cardinals $J(S)$. Then $w(G) = \max\bigl\{\aleph_0, w(G_0), \sup\{\operatorname{card} J(S) : S \in \mathcal{S}\}\bigr\}$

Assume now that $w(G)$ has uncountable cofinality. Then, since $\mathcal{S}$ is countable, either $w(G) = w(G_0) = w(T)$ or there is an $S \in \mathcal{S}$ such that $w(G) = \operatorname{card} J(S)$. In the second case let p be any prime such that S has a nontrivial p-Sylow subgroup. Then $S^{J(S)}$ contains a subgroup isomorphic to $\mathbb{Z}(p)^{w(G)}$ and thus G has a p-Sylow subgroup of weight $w(G)$.

Assume next that $\aleph$ is an infinite cardinal of countable cofinality. Let $\{\aleph_n : n \in \mathbb{N}\}$ be a sequence of infinite cardinals $\aleph_n < \aleph$ such that $\aleph = \sup_{n\in\mathbb{N}} \aleph_n$. Let $\{p_n : n \in \mathbb{N}\} = \{2, 3, 5, 7, \dots\}$ be the sequence of primes in ascending order. Set $G = \prod_{n\in\mathbb{N}} \mathbb{Z}(p_n)^{\aleph_n}$. The weight of the p_n-Sylow subgroup $\mathbb{Z}(p_n)^{\aleph_n}$ of G is $\aleph_n < \aleph = \sup_{m\in\mathbb{N}} \aleph_m = w(G)$. ∎

Lemma 2.2. *For any profinite group G whose weight $w(G)$ has uncountable cofinality there exists a p-Sylow subgroup G_p with $w(G_p) = w(G)$.*

Proof. The Countable Layer Theorem 1.5 yields $w(G) = \sup_{i\in N} w(G_i/G_{i+1})$. Since $w(G)$ has infinite cofinality, there is a natural number i such that $w(G) = w(G_i/G_{i+1})$. The hypotheses of 2.1 apply to G_i/G_{i+1} and secure the existence of a prime p such that the p-Sylow subgroup P_i of G_i/G_{i+1} satisfies $w(P_i) = w(G)$. Let P be the p-Sylow subgroup of G. The image PG_{i+1}/G_{i+1} of the p-Sylow subgroup P of G in the quotient G/G_{i+1} is a p-Sylow subgroups. Thus it contains a conjugate of the p-group P_i. Therefore $w(P) \geq w(PG_{i+1}/G_{i+1}) \geq w(P_i) = w(G)$. Trivially, $w(P) \leq w(G)$. Hence $w(P) = w(G)$. This proves the lemma. ∎

Thus, if there is any profinite counterexample to Conjecture 1.1, there should be one which is a pro-p group and we turn to describing one of them.

For this purpose recall the concept of a free pro-p group: Let X be any set, then there is a pro-p group $F_p(X)$, unique up to isomorphism, containing X in such a way that every open normal subgroup contains all but a finite number of elements of X, and having the following universal property: Assume that $f\colon X \to G$ is any function into a profinite group such that for every open normal subgroup N of G, the identity neighborhood $f^{-1}(N)$ contains all but a finite number of the elements of X; then there is a unique morphism of topological groups $F_p(X) \to G$ extending f. The details can be found e.g. in Chapter 3 of [4] and in Chapter 5 [5].

Example 2.3. Fix any prime p and an infinite set X. Then $F_p(X)$ is a compact group of weight $\mathrm{card}(X)$ such that every closed abelian subgroup is isomorphic to the additive group of p-adic integers $\mathbb{Z}_p$, whose weight is $\aleph_0$.

In particular, whenever $\mathrm{card}(X) > \aleph_0$, $F_p(X)$ is a counterexample to Conjecture 1.1.

Proof. Since $|X|$ is infinite, [4], Proposition 2.6.1 (b) together with Proposition 2.6.2, or [5], p. 84, Lemma 5.5.1, yield $w(F_p(X)) = |X|$. Let A be any maximal abelian subgroup of $F_p(X)$. Then, by Corollary 7.7.5 in [4], or by [5], p. 83, Theorem 5.4.6, or by [1], the subgroup A is again a free pro-p group; since A is abelian and nonsingleton, $A \cong \mathbb{Z}_p$. ∎

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References

[1] Cossey, J., G. H. Kegel, and L. G. Kovács, *Maximal Frattini extension*, Arch. Math. **35** (1980), 210–217.

[2] Hofmann, K. H., and S. A. Morris, *A structure theorem on compact groups*, Math. Proc. Cambridge Philos. Soc. **130** (2001), 409–426.

[3] —, *Compact groups with large abelian subgroups*, Math. Proc. Cambridge Philos. Soc. **133** (2002), to appear.

[4] Ribes, L., and P. A. Zalesskii, *Profinite Groups*, Springer-Verlag, Berlin etc., 2000, xiv+435 pp.

[5] Wilson, J. S., "Profinite Groups," Oxford Univ. Press, 1998, xi+284pp.

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