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Classification of two Involutions on Compact Semisimple Lie Groups and Root Systems

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Abstract. Let $\mathfrak{g}$ be a compact semisimple Lie algebra. Then we first classify pairs of involutions (σ,τ) of $\mathfrak{g}$ with respect to the corresponding double coset decompositions $H\backslash G/L$. (Note that we don't assume $\sigma\tau = \tau\sigma$.) In [4], we defined a maximal torus A, a (restricted) root system Σ and a "generalized" Weyl group J and then we proved

$$J\backslash A \cong H\backslash G/L$$

when G is connected. In this paper, we also compute Σ and J for some representatives of all the pairs of involutions when G is simply connected. By these data, we can compute Σ and J for "all" the pairs of involutions.

1. Introduction

Let G be a reductive Lie group with two involutions σ and τ ($\sigma^2 = \tau^2 =$ id.). Let H and L be subgroups of G such that

$$(G^\sigma)_0 \subset H \subset G^\sigma \quad \text{and} \quad (G^\tau)_0 \subset L \subset G^\tau.$$

Here $G^\rho = \{g \in G \mid \rho(g) = g\}$ for an automorphism ρ of G and F_0 denote the connected component of F containing the identity e for a Lie group F. In [4], we gave fundamental theorems on the structure of the double coset decompositions $H\backslash G/L$.

In this paper, we consider the case that G is compact. We also assume that G is connected and semisimple for the sake of simplicity.

Let x be an element of G and ρ an automorphism of G. Then the double coset decompositions $H\backslash G/L$ and $\rho(H)\backslash G/x\rho(L)x^{-1}$ are identified by the map $g \mapsto \rho(g)x^{-1}$ because

$$hg\ell \mapsto \rho(hg\ell)x^{-1} = \rho(h)\rho(g)x^{-1}x\rho(\ell)x^{-1}$$

for $h \in H$ and $\ell \in L$ ([4] Remark 2).

By this remark, we can define an equivalence of pairs of involutions on a Lie algebra $\mathfrak{g}$ as follows.

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Definition 1.1. Let σ, σ', τ and τ' be involutions on $\mathfrak{g}$. Then we write

$$(\sigma, \tau) \sim (\sigma', \tau')$$

if and only if there exist an automorphism ρ and an inner automorphism ρ_0 of $\mathfrak{g}$ such that

$$\sigma' = \rho\sigma\rho^{-1} \qquad \text{and that} \qquad \tau' = \rho_0\rho\tau\rho^{-1}\rho_0^{-1}.$$

The first aim of this paper is to classify pairs of involutions on $\mathfrak{g}$ by this equivalence relation. Precise results are written in Section 2. Typical representatives will also be given for classical case. As a result, we can give the following interesting remark.

Remark 1.2. (i) When $\mathfrak{g}$ is of exceptional type, then we can take representatives (σ, τ) such that $\sigma\tau = \tau\sigma$.

(ii) On the other hand, when $\mathfrak{g}$ is classical, there are three types of equivalence classes such that we can not take commuting pairs of involutions as representatives. (DI-III, AIII-II (p: odd, $p \neq m$) and DI-I' in Section 2. See also Remark 1 in [4].)

(iii) When $\mathfrak{g}$ is not simple, there are many examples as in Section 2 such that we can not take commuting pairs of involutions as representatives.

Let $\mathfrak{g} = \mathfrak{g}^{\sigma} \oplus \mathfrak{g}^{-\sigma} = \mathfrak{g}^{\tau} \oplus \mathfrak{g}^{-\tau}$ be the decompositions of the Lie algebra $\mathfrak{g}$ of G into the $+1$ and -1-eigenspaces for σ and τ, respectively. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{g}^{-\sigma} \cap \mathfrak{g}^{-\tau}$ and put $A = \exp \mathfrak{a}$. Define subgroups

$$\begin{aligned} N_A &= \{(h, \ell) \in H \times L \mid hA\ell^{-1} = A\}, \\ Z_A &= \{(h, \ell) \in H \times L \mid ha\ell^{-1} = a \text{ for all } a \in A\} \end{aligned}$$

of $H \times L$ and put $J = N_A/Z_A$. Then the following theorem is proved in [4].

Theorem (Theorem 1 in [4]). (i) $G = HAL$.

(ii) *By the inclusion map, we have a bijection*

$$J\backslash A \cong H\backslash G/L.$$

For a linear form $\alpha : \mathfrak{a} \to i\mathbb{R}$ (with values in pure imaginary numbers), we define the "root space"

$$\mathfrak{g}_{\mathbb{C}}(\mathfrak{a}, \alpha) = \{X \in \mathfrak{g} \mid [Y, X] = \alpha(Y)X \text{ for all } Y \in \mathfrak{a}\}.$$

Put

$$\Sigma = \Sigma(\mathfrak{g}_{\mathbb{C}}, \mathfrak{a}) = \{\alpha \in i\mathfrak{a}^* - \{0\} \mid \mathfrak{g}_{\mathbb{C}}(\mathfrak{a}, \alpha) \neq \{0\}\}.$$

Then we have the root space decomposition

$$\mathfrak{g}_{\mathbb{C}} = \bigoplus_{\alpha \in \Sigma \cup \{0\}} \mathfrak{g}_{\mathbb{C}}(\mathfrak{a}, \alpha).$$

Since $\mathfrak{a} \subset \mathfrak{g}^{\sigma\tau}$, we also have the following eigenspace decomposition

$$\mathfrak{g}_{\mathbb{C}}(\mathfrak{a}, \alpha) = \bigoplus_{|\lambda|=1} \mathfrak{g}_{\mathbb{C}}(\mathfrak{a}, \alpha, \lambda)$$

where $\mathfrak{g}_{\mathbb{C}}(\mathfrak{a}, \alpha, \lambda) = \{X \in \mathfrak{g}_{\mathbb{C}}(\mathfrak{a}, \alpha) \mid \sigma\tau X = \lambda X\}$. We proved the following in [4].

Proposition (Proposition 1 in [4]). *Σ satisfies the axiom of the root systems.*

Let (σ', τ') be another pair of involutions on $\mathfrak{g}$ such that $\sigma' = \rho\sigma\rho^{-1}$ and that $\tau' = \rho_0\rho\tau\rho^{-1}\rho_0^{-1}$ with some automorphism ρ and inner automorphism ρ_0 of $\mathfrak{g}$. ($(\sigma,\tau) \sim (\sigma',\tau')$ by Definition 1.) Since $\rho^{-1}\rho_0\rho$ is an inner automorphism of $\mathfrak{g}$, we can write $\rho^{-1}\rho_0\rho = \mathrm{Ad}(ha\ell)$ with some $h \in H$, $a \in A$ and $\ell \in L$ by Theorem (i). If we put $\rho' = \rho\mathrm{Ad}(h)$, then we have

$$\sigma' = \rho'\sigma\rho'^{-1} \quad \text{and} \quad \tau' = \rho\mathrm{Ad}(ha\ell)\tau\mathrm{Ad}(ha\ell)^{-1}\rho^{-1} = \rho'\tau_a\rho'^{-1}$$

where $\tau_a = \mathrm{Ad}(a)\tau\mathrm{Ad}(a)^{-1}$. Hence (σ',τ') is $\mathrm{Aut}\,(\mathfrak{g})$-conjugate to (σ,τ_a). Let X be an element of $\mathfrak{g}_{\mathbb{C}}(\mathfrak{a},\alpha,\lambda)$. Then we have

$$\sigma\tau_a X = \sigma\tau\mathrm{Ad}(a)^{-2}X = \lambda a^{-2\alpha}X = \lambda e^{-2\alpha(Y)}X \tag{1.1}$$

if $a = \exp Y$. On the other hand, the map

$$(h,\ell) \mapsto (h, a\ell a^{-1})$$

gives a natural isomorphism of J onto the group for the pair (H, aLa^{-1}). So we have only to compute Σ, $\mathfrak{g}_{\mathbb{C}}(\mathfrak{a},\alpha,\lambda)$ and J for some representatives of the equivalence classes of (σ,τ).

The second aim of this paper is to compute Σ, $\dim \mathfrak{g}_{\mathbb{C}}(\mathfrak{a},\alpha,\lambda)$ and J (when G is simply connected) for representatives (σ,τ) given in Section 2.

In Section 3, we show that the group J is determined by

$$\widetilde{\Sigma} = \{(\alpha,\lambda) \in \Sigma \times U(1) \mid \mathfrak{g}_{\mathbb{C}}(\mathfrak{a},\alpha,\lambda) \neq \{0\}\}$$

when G is simply connected (Proposition 3.1).

In Section 4, we give lists of Σ, dimensions of $\mathfrak{g}_{\mathbb{C}}(\mathfrak{a},\alpha,\lambda)$ and the groups J when G is connected and simply connected. (When $\sigma\tau = \tau\sigma$, the dimensions of $\mathfrak{g}_{\mathbb{C}}(\mathfrak{a},\alpha,\lambda)$ are given in Table V of [5].)

Section 5 is an appendix for Section 3 and Section 4 to prove the following.

Corollary (of Proposition 5.2 and Proposition 5.3). *When G is simply connected, the lattice $\{Y \in \mathfrak{a} \mid \exp Y = e\}$ in $\mathfrak{a}$ is generated by the set*

$$\left\{ Y_\alpha = \frac{4\pi i\alpha}{(\alpha,\alpha)} \;\middle|\; \alpha \in \Sigma(\mathfrak{g}_{\mathbb{C}},\mathfrak{a}) \right\}.$$

Example 1.3. Let $\mathfrak{g} = \mathfrak{u}(n,\mathbb{F})$ where $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Define involutions σ and τ of $\mathfrak{g}$ by

$$\sigma X = I_{p,q}XI_{p,q} \quad \text{and} \quad \tau X = I_{r,s}XI_{r,s}$$

for $X \in \mathfrak{g}$, respectively, where $n = p+q = r+s$, $r \geq p \geq q \geq s$ and

$$I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}.$$

Then we can take a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{g}^{-\sigma} \cap \mathfrak{g}^{-\tau}$ of the form

$$\mathfrak{a} = \left\{ Y(\theta_1,\dots,\theta_s) = \begin{pmatrix} & 0 & & d(\theta_1,\dots,\theta_s) \\ & & 0 & \\ -d(\theta_1,\dots,\theta_s) & & & 0 \end{pmatrix} \;\middle|\; \theta_1,\dots,\theta_s \in \mathbb{R} \right\}$$

where

$$d(\theta_1, \dots, \theta_s) = \begin{pmatrix} \theta_1 & & 0 \\ & \ddots & \\ 0 & & \theta_s \end{pmatrix}.$$

Define $e_j \in i\mathfrak{a}^*$ by

$$e_j : Y(\theta_1, \dots, \theta_s) \mapsto i\theta_j.$$

Then it is wellknown that dimensions of $\mathfrak{g}_{\mathbb{C}}(\mathfrak{a}, \alpha)$ are as follows.

α	$\pm e_j \pm e_k$	$\pm e_j$	$\pm 2e_j$
$\dim \mathfrak{g}_{\mathbb{C}}(\mathfrak{a}, \alpha)$	c	$c(r-s)$	$c-1$

Here $j \neq k$ and

$$c = \begin{cases} 1 & (\mathbb{F} = \mathbb{R}) \\ 2 & (\mathbb{F} = \mathbb{C}) \\ 4 & (\mathbb{F} = \mathbb{H}). \end{cases}$$

Since $(\sigma\tau)^2 = id.$, we have $\lambda = \pm 1$. Since $\mathfrak{g}^{\sigma\tau} \cong \mathfrak{u}(p+s, \mathbb{F}) \oplus \mathfrak{u}(q-s, \mathbb{F})$ and $\mathfrak{g}^\sigma \cap \mathfrak{g}^\tau \cong \mathfrak{u}(p, \mathbb{F}) \oplus \mathfrak{u}(q-s, \mathbb{F}) \oplus \mathfrak{u}(s, \mathbb{F})$, we can easily compute $\dim \mathfrak{g}_{\mathbb{C}}(\mathfrak{a}, \alpha, \lambda)$ as follows.

α	$\pm e_j \pm e_k$	$\pm e_j$	$\pm 2e_j$
$\lambda = 1$	c	$c(p-s)$	$c-1$
$\lambda = -1$	0	$c(q-s)$	0

Let Y be an element of $\mathfrak{a}$ given by

$$Y = Y(\frac{\pi}{2}, \dots, \frac{\pi}{2}, 0, \dots, 0). \qquad \text{(There are } s_1 \ \tfrac{\pi}{2}\text{'s and } s_2 \ 0\text{'s. } s_1 + s_2 = s.\text{)}$$

Put

$$a = \exp Y = \begin{pmatrix} 0 & 0 & I_{s_1} & 0 \\ 0 & I_{r-s_1} & 0 & 0 \\ -I_{s_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{s_2} \end{pmatrix} \in U(n, \mathbb{F}).$$

Then we have

$$\tau_a X = \tau \operatorname{Ad}(a)^{-2} X = I'_{r,s} X I'_{r,s}$$

where

$$I'_{r,s} = I_{r,s} a^{-2} = \begin{pmatrix} -I_{s_1} & & 0 \\ & I_r & \\ 0 & & -I_{s_2} \end{pmatrix}.$$

Since

$$a^{2e_j} = e^{2e_j(Y)} = \begin{cases} -1 & (j \leq s_1) \\ 1 & (j > s_1), \end{cases}$$

we can get by (1.1) the list of $\dim \mathfrak{g}_{\mathbb{C}}(\mathfrak{a}, \alpha, \lambda)$ for the pair (σ, τ_a) of involutions as follows.

α	$\pm e_j \pm e_k$ $(j,k \le s_1)$	$\pm e_j \pm e_k$ $(j,k > s_1)$	$\pm e_j \pm e_k$ $(j \le s_1 < k)$	$\pm e_j$ $(j \le s_1)$	$\pm e_j$ $(j > s_1)$	$\pm 2e_j$
$\lambda = 1$	c	c	0	$c(q-s)$	$c(p-s)$	$c-1$
$\lambda = -1$	0	0	c	$c(p-s)$	$c(q-s)$	0

Remark 1.4. In this example, we choose $a \in A$ so that $\sigma\tau_a = \tau_a\sigma$. For such pairs of involutions, dimensions of $\mathfrak{g}_{\mathbb{C}}(\mathfrak{a}, \alpha, \lambda)$ $(\lambda = \pm 1)$ are computed in [5] Table V. In the same way, we can reproduce all the data on $\dim \mathfrak{g}_{\mathbb{C}}(\mathfrak{a}, \alpha, \lambda)$ given in [5] from the results in Section 4.

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2. Classification of pairs of involutions

Let $\mathfrak{g}$ be a compact semisimple Lie algebra. Let $\widetilde{G} = \operatorname{Aut}(\mathfrak{g})$ be the group of automorphisms of $\mathfrak{g}$. Then $\widetilde{G}_0 = \operatorname{Int}(\mathfrak{g})$ is the subgroup of $\widetilde{G}$ consisting of inner automorphisms of $\mathfrak{g}$. Classification of $\widetilde{G}$-conjugacy classes of involutions on $\mathfrak{g}$ is well-known since it corresponds to the classification of symmetric pairs $(\mathfrak{g}, \mathfrak{h})$. As in Section 1, we define an equivalence of pairs of involutions on $\mathfrak{g}$ as follows.

Definition 1.1. Let σ, σ', τ and τ' be involutions on $\mathfrak{g}$. Then we write

$$(\sigma, \tau) \sim (\sigma', \tau')$$

if and only if there exist a $\rho \in \widetilde{G}$ and $\rho_0 \in \widetilde{G}_0$ such that

$$\sigma' = \rho\sigma\rho^{-1} \qquad \text{and that} \qquad \tau' = \rho_0\rho\tau\rho^{-1}\rho_0^{-1}.$$

Fix a pair of involutions (σ, τ) on $\mathfrak{g}$. We have only to study the equivalence in the set

$$S = \{(\sigma', \tau') \mid \sigma' = \rho\sigma\rho^{-1},\ \tau' = \rho'\tau\rho'^{-1} \text{ for some } \rho, \rho' \in \widetilde{G}\}.$$

Since every equivalence class in S contains an element of

$$S_0 = \{(\sigma, \tau') \mid \tau' = \rho'\tau\rho'^{-1} \text{ for some } \rho' \in \widetilde{G}\},$$

we have only to study the equivalence of elements in S_0. If $(\sigma, \tau') \sim (\sigma, \tau'')$, then

$$\rho\sigma\rho^{-1} = \sigma \quad \text{and} \quad \rho_0\rho\tau'\rho^{-1}\rho_0^{-1} = \tau''$$

for some $\rho \in \widetilde{G}$ and $\rho_0 \in \widetilde{G}_0$. Write $\tau' = \rho'\tau\rho'^{-1}$ with a $\rho' \in \widetilde{G}$. Then

$$\tau'' = (\rho_0\rho\rho')\tau(\rho_0\rho\rho')^{-1}.$$

There exists a natural identification between S_0 and $\widetilde{G}/\widetilde{G}^\tau$ by the map

$$(\sigma, \rho'\tau\rho'^{-1}) \mapsto \rho'\widetilde{G}^\tau.$$

By this identification, (σ, τ'') corresponds to $\rho_0\rho\rho'\widetilde{G}^\tau$. Hence the equivalence class in S_0 containing τ' is identified with the subset

$$\widetilde{G}_0\widetilde{G}^\sigma\rho'\widetilde{G}^\tau$$

of $\widetilde{G}$. Let π be the projection

$$\pi : \widetilde{G} \to \widetilde{G}/\widetilde{G}_0.$$

Then every equivalence class in S_0 corresponds to a double coset

$$\pi(\widetilde{G}^\sigma)\pi(\rho')\pi(\widetilde{G}^\tau)$$

in $\pi(\widetilde{G})$ with some $\rho' \in \widetilde{G}$. Hence we have only to consider the double coset decomposition

$$\pi(\widetilde{G}^\sigma)\backslash\pi(\widetilde{G})/\pi(\widetilde{G}^\tau).$$

Suppose that $\mathfrak{g}$ is simple. Then it is wellknown that

$$\pi(\widetilde{G}) \cong \begin{cases} \mathbb{Z}_2 & \text{if } \mathfrak{g} \cong \mathfrak{su}(n)\ (n \geq 3), \mathfrak{o}(2m)\ (m \geq 5) \text{ or } \mathrm{E}_6 \\ S_3 & \text{if } \mathfrak{g} \cong \mathfrak{o}(8) \\ \{e\} & \text{otherwise.} \end{cases} \tag{2.1}$$

Moreover the group $\pi(\widetilde{G}^\sigma)$ is computed in [3] p.156 for every symmetric pair $(\mathfrak{g}, \mathfrak{g}^\sigma)$. As a result, we have $\pi(\widetilde{G}^\sigma) = \pi(\widetilde{G})$ unless

$$(\mathfrak{g}, \mathfrak{g}^\sigma) \cong (\mathfrak{o}(4m), \mathfrak{u}(2m)) \text{ with } m \geq 3 \quad (\pi(\widetilde{G}) \cong \mathbb{Z}_2,\ \pi(\widetilde{G}^\sigma) = \{e\})$$

or

$$(\mathfrak{g}, \mathfrak{g}^\sigma) \cong (\mathfrak{o}(8), \mathfrak{o}(8-q) \oplus \mathfrak{o}(q)) \text{ with } q = 1,2,3 \quad (\pi(\widetilde{G}) \cong S_3,\ \pi(\widetilde{G}^\sigma) \cong \mathbb{Z}_2).$$

Suppose that $\mathfrak{g}$ is classical. Then it is known that every involution σ of $\mathfrak{g}$ is $\widetilde{G}$-conjugate to one of the following involutions.

type	$\mathfrak{g}$	$\sigma(X)$	$\mathfrak{h} = \mathfrak{g}^\sigma$
AI	$\mathfrak{su}(n)$	$\overline{X}$	$\mathfrak{o}(n)$
AII	$\mathfrak{su}(2m)$	$J_m\overline{X}J_m^{-1}$	$\mathfrak{u}(m, \mathbb{H})$
AIII	$\mathfrak{su}(n)$	$I_{p,q}XI_{p,q}$	$(\mathfrak{u}(p) \oplus \mathfrak{u}(q)) \cap \mathfrak{su}(n)$
BDI	$\mathfrak{o}(n)$	$I_{p,q}XI_{p,q}$	$\mathfrak{o}(p) \oplus \mathfrak{o}(q)$
DIII	$\mathfrak{o}(2m)$	$J_mXJ_m^{-1}$	$\mathfrak{u}(m)$
CI	$\mathfrak{u}(n, \mathbb{H})$	$(i_1I_n)X(i_1I_n)^{-1}$	$\mathfrak{u}(n)$
CII	$\mathfrak{u}(n, \mathbb{H})$	$I_{p,q}XI_{p,q}$	$\mathfrak{u}(p, \mathbb{H}) \oplus \mathfrak{u}(q, \mathbb{H})$

Here $p + q = n$,

$$I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}, \qquad J_m = \begin{pmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_1 \end{pmatrix} \quad \text{where} \quad J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and the field $\mathbb{H}$ of quarternions is defined as

$$\mathbb{H} = \{a_0 + a_1i_1 + a_2i_2 + a_3i_3 \mid a_0, \ldots, a_3 \in \mathbb{R}\}$$

where $i_1^2 = i_2^2 = i_3^2 = -1$, $i_1i_2 = -i_2i_1 = i_3$, $i_2i_3 = -i_3i_2 = i_1$ and $i_3i_1 = -i_1i_3 = i_2$.

Proposition 2.1. *Let (σ,τ) be a pair of involutions on a classical simple compact Lie algebra $\mathfrak{g}$. Then (σ,τ) or (τ,σ) is equivalent to one of the following pairs if σ and τ are not $\widetilde{G}_0$-conjugate.*

type	$\mathfrak{g}$	$\begin{pmatrix}\sigma(X)\\ \tau(X)\end{pmatrix}$	$\begin{pmatrix}\mathfrak{h}\\ \mathfrak{l}\end{pmatrix}$
BDI − I	$\mathfrak{o}(n)$	$\begin{pmatrix}I_{p,q}XI_{p,q}\\ I_{r,s}XI_{r,s}\end{pmatrix}$	$\begin{pmatrix}\mathfrak{o}(p)\oplus\mathfrak{o}(q)\\ \mathfrak{o}(r)\oplus\mathfrak{o}(s)\end{pmatrix}$
AIII − III	$\mathfrak{su}(n)$	$\begin{pmatrix}I_{p,q}XI_{p,q}\\ I_{r,s}XI_{r,s}\end{pmatrix}$	$\begin{pmatrix}(\mathfrak{u}(p)\oplus\mathfrak{u}(q))\cap\mathfrak{su}(n)\\ (\mathfrak{u}(r)\oplus\mathfrak{u}(s))\cap\mathfrak{su}(n)\end{pmatrix}$
CII − II	$\mathfrak{u}(n,\mathbb{H})$	$\begin{pmatrix}I_{p,q}XI_{p,q}\\ I_{r,s}XI_{r,s}\end{pmatrix}$	$\begin{pmatrix}\mathfrak{u}(p,\mathbb{H})\oplus\mathfrak{u}(q,\mathbb{H})\\ \mathfrak{u}(r,\mathbb{H})\oplus\mathfrak{u}(s,\mathbb{H})\end{pmatrix}$
AI − III	$\mathfrak{su}(n)$	$\begin{pmatrix}\overline{X}\\ I_{p,q}XI_{p,q}\end{pmatrix}$	$\begin{pmatrix}\mathfrak{o}(n)\\ (\mathfrak{u}(p)\oplus\mathfrak{u}(q))\cap\mathfrak{su}(n)\end{pmatrix}$
CI − II	$\mathfrak{u}(n,\mathbb{H})$	$\begin{pmatrix}(i_1I_n)X(i_1I_n)^{-1}\\ I_{p,q}XI_{p,q}\end{pmatrix}$	$\begin{pmatrix}\mathfrak{u}(n)\\ \mathfrak{u}(p,\mathbb{H})\oplus\mathfrak{u}(q,\mathbb{H})\end{pmatrix}$
DI − III	$\mathfrak{o}(2m)$	$\begin{pmatrix}I_{p,q}XI_{p,q}\\ J_mXJ_m^{-1}\end{pmatrix}$	$\begin{pmatrix}\mathfrak{o}(p)\oplus\mathfrak{o}(q)\\ \mathfrak{u}(m)\end{pmatrix}$
AIII − II	$\mathfrak{su}(2m)$	$\begin{pmatrix}I_{p,q}XI_{p,q}\\ J_m\overline{X}J_m^{-1}\end{pmatrix}$	$\begin{pmatrix}(\mathfrak{u}(p)\oplus\mathfrak{u}(q))\cap\mathfrak{su}(n)\\ \mathfrak{u}(m,\mathbb{H})\end{pmatrix}$
AI − II	$\mathfrak{su}(2m)$	$\begin{pmatrix}\overline{X}\\ J_m\overline{X}J_m^{-1}\end{pmatrix}$	$\begin{pmatrix}\mathfrak{o}(2m)\\ \mathfrak{u}(m,\mathbb{H})\end{pmatrix}$
DIII − III′	$\mathfrak{o}(4m)$	$\begin{pmatrix}J_{2m}XJ_{2m}^{-1}\\ J'_{2m}XJ'^{-1}_{2m}\end{pmatrix}$	$\begin{pmatrix}\mathfrak{u}(2m)\\ I_{4m-1,1}\mathfrak{u}(2m)I_{4m-1,1}\end{pmatrix}$
DI − I′ ($q,s=1$ *or* 3)	$\mathfrak{o}(8)$	$\begin{pmatrix}I_{p,q}XI_{p,q}\\ \kappa^{-1}(I_{r,s}\kappa(X)I_{r,s})\end{pmatrix}$	$\begin{pmatrix}\mathfrak{o}(p)\oplus\mathfrak{o}(q)\\ \kappa(\mathfrak{o}(r)\oplus\mathfrak{o}(s))\end{pmatrix}$

Here $n=p+q=r+s,\ r\geq p\geq q\geq s$,

$$J'_{2m}=\begin{pmatrix}J_{2m-1} & 0\\ 0 & -J_1\end{pmatrix}$$

and κ *is an outer automorphism of* $\mathfrak{o}(8)$ *of order* 3. (*In* BDI-I, AIII-III *and* CII-II *type, we have* $q>s$ *since we assume that* σ *and* τ *are not* G_0*-conjugate.*)

Proof. The assertion is clear from the preceding argument except when $\mathfrak{g}=\mathfrak{o}(8)$. Suppose that $\mathfrak{g}=\mathfrak{o}(8)$. Then σ and τ are $\widetilde{G}$-conjugate to $\mathrm{Ad}(I_{p,q})$ and $\mathrm{Ad}(I_{r,s})$, respectively, with some $q,s=1,2,3,4$. If $q=4$ or $s=4$, then $\pi(\widetilde{G}^\sigma)$ or $\pi(\widetilde{G}^\tau)$ equals $\pi(\widetilde{G})$ and therefore we have $(\sigma,\tau)\sim(\mathrm{Ad}(I_{p,q}),\mathrm{Ad}(I_{r,s}))$ (BDI-I type). So we may assume that $q,s=1,2,3$. Note that we have a set of representatives $\{e,\pi(\kappa)\}$ (κ is an outer automorphism of $\mathfrak{g}$ of order 3) of the double coset decomposition

$$\pi(\widetilde{G}^\sigma)\backslash\pi(\widetilde{G})/\pi(\widetilde{G}^\tau)$$

since $\pi(\widetilde{G})\cong S_3$ and $\pi(\widetilde{G}^\sigma)\cong\pi(\widetilde{G}^\tau)\cong\mathbb{Z}_2$. If $q=2$ or $s=2$, then (σ,τ) or (τ,σ) is equivalent to BDI-I type or DI-III type because $\mathrm{Ad}(I_{6,2})$ and $\mathrm{Ad}(J_4)$ are conjugate by an outer automorphism of $\mathfrak{g}$ of order 3. If $q, s = 1$ or 3, then (σ, τ) is equivalent to BDI-I type or DI-I' type. ■

Next suppose that $\mathfrak{g}$ is exceptional. Since $\pi(\widetilde{G}^\sigma) = \pi(\widetilde{G})$ for every $(\mathfrak{g}, \sigma)$, we have only to consider all the combinations of two $\widetilde{G}$-conjugacy classes of involutions. In Table V of [5] (c.f. Tableau II of [1]), commuting pairs of involutions are classified. But we can see that there appear "all" the combinations of involutions in that table. Thus we can take representatives (σ, τ) of pairs of involutions so that $\sigma\tau = \tau\sigma$ for exceptional case. (See the table in Section 4, B.)

Finally, suppose that $\mathfrak{g}$ is not simple and that $\mathfrak{g}$ is irreducible under the action of the pair (σ, τ), which means that there is no non-trivial ideal $\mathfrak{g}_0$ of $\mathfrak{g}$ such that $\sigma\mathfrak{g}_0 = \tau\mathfrak{g}_0 = \mathfrak{g}_0$. Write

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$$

with simple Lie algebras $\mathfrak{g}_j$, $j = 1, \ldots, k$. By the (σ, τ)-irreducibility of $\mathfrak{g}$, $\sigma\mathfrak{g}_1$ or $\tau\mathfrak{g}_1$ equals $\mathfrak{g}_j$ with some $j \neq 1$. Exchanging the roles of σ and τ, if necessary, and transposing $\mathfrak{g}_j$ with $\mathfrak{g}_2$ if $j \neq 2$, we may assume that

$$\tau\mathfrak{g}_1 = \mathfrak{g}_2.$$

If $k = 2$, then $\sigma\mathfrak{g}_2 = \mathfrak{g}_1$ or $\mathfrak{g}_2$. If $k \geq 3$, then we have

$$\sigma\mathfrak{g}_2 = \mathfrak{g}_j$$

with some $j \geq 3$ by the (σ, τ)-irreducibility of $\mathfrak{g}$. Transposing $\mathfrak{g}_j$ with $\mathfrak{g}_3$ if $j \neq 3$, we may assume that

$$\sigma\mathfrak{g}_2 = \mathfrak{g}_3.$$

Repeating this argument, we may assume that

$$\sigma\mathfrak{g}_{2j} = \mathfrak{g}_{2j+1} \qquad \text{if} \quad 2j + 1 \leq k$$

and that

$$\tau\mathfrak{g}_{2j-1} = \mathfrak{g}_{2j} \qquad \text{if} \quad 2j \leq k.$$

Suppose that k is even. Then $\sigma\mathfrak{g}_k$ must equal $\mathfrak{g}_1$ or $\mathfrak{g}_k$. If $\sigma\mathfrak{g}_k = \mathfrak{g}_k$, then we have clearly $\sigma\mathfrak{g}_1 = \mathfrak{g}_1$. On the other hand, if k is odd, then we have $\tau\mathfrak{g}_k = \mathfrak{g}_k$ and $\sigma\mathfrak{g}_1 = \mathfrak{g}_1$. Thus we have obtained the following classification of $(\mathfrak{g}, \sigma, \tau)$ when $\mathfrak{g}$ is not simple.

Proposition 2.2. *Suppose that $\mathfrak{g}$ is compact semisimple and that $\mathfrak{g}$ is irreducible under the action of the pair (σ, τ). Then $\mathfrak{g}$ is the direct sum of k-copies of a simple Lie algebra $\mathfrak{g}_1$ with some $k \geq 1$ and the pair (σ, τ) (or (τ, σ)) is $Aut(\mathfrak{g})$-conjugate to one of the following three types.*

(I) *k is even and*

$$\begin{aligned} \sigma(X_1, X_2, \ldots, X_{k-1}, X_k) &= (\rho_1(X_k), X_3, X_2, \ldots, X_{k-1}, X_{k-2}, \rho_1^{-1}(X_1)) \\ \tau(X_1, X_2, \ldots, X_{k-1}, X_k) &= (X_2, X_1, \ldots, X_k, X_{k-1}) \end{aligned}$$

with some automorphism ρ_1 of $\mathfrak{g}_1$.

(II) *k is even and*

$$\begin{aligned}\sigma(X_1, X_2, \ldots, X_{k-1}, X_k) &= (\sigma_1(X_1), X_3, X_2, \ldots, X_{k-1}, X_{k-2}, \tau_1(X_k)) \\ \tau(X_1, X_2, \ldots, X_{k-1}, X_k) &= (X_2, X_1, \ldots, X_k, X_{k-1})\end{aligned}$$

with some involutions σ_1 and τ_1 of $\mathfrak{g}_1$.

(III) *k is odd and*

$$\begin{aligned}\sigma(X_1, X_2, \ldots, X_{k-1}, X_k) &= (\sigma_1(X_1), X_3, X_2, \ldots, X_k, X_{k-1}) \\ \tau(X_1, X_2, \ldots, X_{k-1}, X_k) &= (X_2, X_1, \ldots, X_{k-1}, X_{k-2}, \tau_1(X_k))\end{aligned}$$

with some involutions σ_1 and τ_1 of $\mathfrak{g}_1$.

Remark 2.3. Suppose that $(\mathfrak{g}, \sigma, \tau)$ is type (I).

(i) Let ρ_0 be an inner automorphism of $\mathfrak{g}_1$ and put

$$\widetilde{\rho_0} = (\rho_0, \mathrm{id.}, \ldots, \mathrm{id.}) \in \mathrm{Int}\,(\mathfrak{g}).$$

Then we have

$$\begin{aligned}\widetilde{\rho_0}\sigma\widetilde{\rho_0}^{-1}(X_1, \ldots, X_k) &= \widetilde{\rho_0}\sigma(\rho_0^{-1}(X_1), X_2, \ldots, X_k) \\ &= \widetilde{\rho_0}(\rho_1(X_k), X_3, X_2, \ldots, X_{k-1}, X_{k-2}, \rho_1^{-1}\rho_0^{-1}(X_1)) \\ &= (\rho_0\rho_1(X_k), X_3, X_2, \ldots, X_{k-1}, X_{k-2}, (\rho_0\rho_1)^{-1}(X_1)).\end{aligned}$$

So we may replace $\rho_1 \in \mathrm{Aut}\,(\mathfrak{g}_1)$ by any element $\rho_0\rho_1$ of $\mathrm{Int}\,(\mathfrak{g}_1)\rho_1$ since $(\sigma, \tau) \sim (\widetilde{\rho_0}\sigma\widetilde{\rho_0}^{-1}, \tau)$ by the definition of equivalence.

(ii) The space $\mathfrak{g}^{-\sigma} \cap \mathfrak{g}^{-\tau}$ is written as

$$\mathfrak{g}^{-\sigma} \cap \mathfrak{g}^{-\tau} = \{X^{(k)} \mid X \in \mathfrak{g}_1^{\rho_1}\}$$

where $X^{(k)} = (X, -X, \ldots, X, -X)$. Hence a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{g}^{-\sigma} \cap \mathfrak{g}^{-\tau}$ is of the form

$$\{X^{(k)} \mid X \in \mathfrak{u}_1\}$$

where $\mathfrak{u}_1$ is a maximal abelian subalgebra of $\mathfrak{g}_1^{\rho_1}$.

(iii) Let G_1 be a connected Lie group with Lie algebra $\mathfrak{g}_1$ and G the direct product of k-copies of G_1. Suppose that ρ_1 lifts to an automorphism of G_1. Then σ and τ lift to automorphisms of G. Suppose that $H = G^\sigma$ and $L = G^\tau$. Let $(x_1, \ldots, x_k)$ be an element of G. Then we have

$$\begin{aligned}H(x_1, \ldots, x_k)L &= H(x_1, \ldots, x_k)(e, \ldots, e, x_k^{-1}, x_k^{-1})L \\ &= H(x_1, \ldots, x_{k-2}, x_{k-1}x_k^{-1}, e)L \\ &= H(e, \ldots, e, x_kx_{k-1}^{-1}, x_kx_{k-1}^{-1}, e)(x_1, \ldots, x_{k-2}, x_{k-1}x_k^{-1}, e)L \\ &= H(x_1, \ldots, x_{k-3}, x_kx_{k-1}^{-1}x_{k-2}, e, e)L \\ &= \cdots \\ &= H(x_1x_2^{-1}\cdots x_{k-1}x_k^{-1}, e, \ldots, e)L.\end{aligned}$$

Hence it is clear that the double coset decomposition $H\backslash G/L$ can be identified with the set

$$\{\{\rho_1(x)yx^{-1} \mid x \in G_1\} \mid y \in G_1\}$$

of ρ_1-twisted conjugacy classes in G_1 by the map

$$G \ni (x_1, \dots, x_k) \mapsto x_1 x_2^{-1} \cdots x_{k-1} x_k^{-1} \in G_1.$$

By Theorem 1 of [4], we have

$$G = HAL$$

where $A = \exp \mathfrak{a}$. Hence we have

$$G_1 = \bigcup_{x \in G_1} \rho_1(x) U_1 x^{-1}$$

where $U_1 = \exp \mathfrak{u}_1$. Let $\widetilde{G}_1$ be a Lie group such that $(\widetilde{G}_1)_0 = G_1$ and that

$$\rho_1(g) = y^{-1} g y \quad (g \in G_1)$$

with some $y \in \widetilde{G}_1$. Then we have

$$yG_1 = \bigcup_{x \in G_1} y\rho_1(x) U_1 x^{-1} = \bigcup_{x \in G_1} xyU_1 x^{-1}$$

(c.f. [7] Chap.II Theorem). Thus we can say that Theorem 1 of [4] includes [7] Chap.II Theorem.

Remark 2.4. Suppose that $(\mathfrak{g}, \sigma, \tau)$ is type (II) or (III). Then the space $\mathfrak{g}^{-\sigma} \cap \mathfrak{g}^{-\tau}$ is written as

$$\mathfrak{g}^{-\sigma} \cap \mathfrak{g}^{-\tau} = \{X^{(k)} \mid X \in \mathfrak{g}_1^{-\sigma_1} \cap \mathfrak{g}_1^{-\tau_1}\}$$

where

$$X^{(k)} = \begin{cases} (X, -X, \dots, X, -X) & \text{if } k \text{ is even} \\ (X, -X, \dots, -X, X) & \text{if } k \text{ is odd.} \end{cases}$$

Hence a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{g}^{-\sigma} \cap \mathfrak{g}^{-\tau}$ is of the form

$$\{X^{(k)} \mid X \in \mathfrak{a}_1\}$$

where $\mathfrak{a}_1$ is a maximal abelian subspace of $\mathfrak{g}_1^{-\sigma_1} \cap \mathfrak{g}_1^{-\tau_1}$.

Let G_1 be a connected Lie group with Lie algebra $\mathfrak{g}_1$ and G the direct product of k-copies of G_1. Suppose that σ_1 and τ_1 lift to automorphisms of G_1. Then σ and τ lift to G. By the same argument as in Remark 2.3 (iii), we have

$$G^\sigma \backslash G / G^\tau \cong G_1^{\sigma_1} \backslash G_1 / G_1^{\tau_1}$$

by the map

$$(x_1, \dots, x_k) \mapsto \begin{cases} x_1 x_2^{-1} \cdots x_{k-1} x_k^{-1} & \text{if } k \text{ is even} \\ x_1 x_2^{-1} \cdots x_{k-1}^{-1} x_k & \text{if } k \text{ is odd.} \end{cases}$$

3. Structure of J when G is simply connected

In this section, we assume that G is simply connected. Then it is known that

$$G^\rho \text{ is connected}$$

for any automorphism ρ of G ([2] 3.4.Theorem). As in Section 5, we also have

$$\{Y \in \mathfrak{a} \mid \exp Y = e\} = \Gamma_{\mathfrak{a}}$$

where $\Gamma_{\mathfrak{a}}$ is the lattice in $\mathfrak{a}$ generated by the set

$$\left\{ Y_\alpha = \frac{4\pi i\alpha}{(\alpha,\alpha)} \;\middle|\; \alpha \in \Sigma(\mathfrak{g}_{\mathbb{C}}, \mathfrak{a}) \right\}.$$

As in [4] Remark 3, the group J is identified with the subgroup of $W_H(A) \ltimes A$ (semidirect product of $W_H(A)$ and A) given by

$$\{(w,a) \in W_H(A) \ltimes A \mid w = \mathrm{Ad}(h)|_{\mathfrak{a}} \text{ and } a = h\ell^{-1} \text{ for some } h \in N_H(A) \text{ and } \ell \in L\}$$

where $W_S(A) = N_S(A)/Z_S(A)$ for a subgroup S of G.

Let $\widetilde{J}$ be the group of affine transformations on $\mathfrak{a}$ defined by

$$\widetilde{J} = \{(w,Y) \mid w = \mathrm{Ad}(h)|_{\mathfrak{a}} \text{ and } \exp Y = h\ell^{-1} \quad \text{for some } h \in N_H(\mathfrak{a}) \text{ and } \ell \in L\}$$

where $(w,Y)X = wX + Y$ for $X \in \mathfrak{a}$. Then it is clear that

$$J \cong \widetilde{J}/\Gamma_{\mathfrak{a}}$$

by the exponential map.

Let $(\alpha, e^{2\mu})$ $(\mu \in i\mathbb{R})$ be an element of

$$\widetilde{\Sigma} = \{(\alpha,\lambda) \in \Sigma(\mathfrak{g}_{\mathbb{C}}, \mathfrak{a}) \times U(1) \mid \mathfrak{g}_{\mathbb{C}}(\mathfrak{a}, \alpha, \lambda) \neq \{0\}\}.$$

Let X_α be a nonzero element of $\mathfrak{g}_{\mathbb{C}}(\mathfrak{a}, \alpha, e^{2\mu})$ such that $\sigma X_\alpha = \overline{X_\alpha}$ (Lemma 2 of [4]) and Y an element of $\mathfrak{a}$ such that $\alpha(Y) = \mu$. Put $a = \exp Y$ and $\tau_a = \mathrm{Ad}(a)\tau\mathrm{Ad}(a)^{-1} = \tau\mathrm{Ad}(a)^{-2}$. Then we have

$$\sigma\tau_a X_\alpha = \sigma\tau\mathrm{Ad}(a)^{-2}X_\alpha = X_\alpha.$$

Hence

$$m_\alpha = \exp k(X_\alpha + \sigma X_\alpha) \in H \cap aLa^{-1}$$

defines the reflection $w_\alpha = \mathrm{Ad}(m_\alpha)|_{\mathfrak{a}}$ for some $k \in \mathbb{R}$. Put $h = m_\alpha \in H$ and $\ell = a^{-1}m_\alpha a \in L$. Then

$$h\ell^{-1} = m_\alpha a^{-1} m_\alpha^{-1} a = \exp(Y - w_\alpha Y) = \exp(\alpha(Y)\alpha^\vee) = \exp \mu\alpha^\vee$$

where $\alpha^\vee = 2\alpha/(\alpha,\alpha) \in i\mathfrak{a}$ is the co-root of α. Hence we have

$$(w_\alpha, \mu\alpha^\vee) \in \widetilde{J}.$$

Since

$$(w_\alpha, \mu\alpha^\vee)X = w_\alpha X + \mu\alpha^\vee = X - \alpha(X)\alpha^\vee + \mu\alpha^\vee$$

for $X \in \mathfrak{a}$, $(w_\alpha, \mu\alpha^\vee)$ is the reflection in $\mathfrak{a}$ with respect to the hyperplane $\alpha(X) = \mu$.

Proposition 3.1. (i) $\widetilde{J}$ *is generated by*

$$\{(w_\alpha, \mu\alpha^\vee) \mid (\alpha, e^{2\mu}) \in \widetilde{\Sigma}\}.$$

(ii) J *is generated by*

$$\{(w_\alpha, \exp\mu\alpha^\vee) \mid (\alpha, e^{2\mu}) \in \widetilde{\Sigma}\}.$$

(iii) *If* $W_{H\cap L}(\mathfrak{a}) = W(\Sigma)$ $(\Sigma = \Sigma(\mathfrak{g}_\mathbb{C}, \mathfrak{a}))$, *then*

$$J = W(\Sigma) \ltimes J_0$$

where J_0 *is the subgroup of* A *generated by*

$$\{\exp\mu\alpha^\vee \mid (\alpha, e^{2\mu}) \in \widetilde{\Sigma}\}.$$

Proof. (i) Let $G_{\mathrm{reg}} = \{x \in G \mid \mathfrak{g}^{-\sigma} \cap \mathfrak{g}^{-\tau_x} \text{ is abelian}\}$ be the set of regular elements in G defined in [4]. Then it is clear that

$$hG_{\mathrm{reg}}\ell^{-1} = G_{\mathrm{reg}}$$

for any $h \in H$ and $\ell \in L$. Hence we have

$$hA_{\mathrm{reg}}\ell^{-1} = A_{\mathrm{reg}}$$

for any $(h, \ell) \in N_A$ if we put

$$A_{\mathrm{reg}} = A \cap G_{\mathrm{reg}} = \{a \in A \mid a^{2\alpha} \neq \lambda \text{ for all } (\alpha, \lambda) \in \widetilde{\Sigma}\}.$$

Consider

$$\mathfrak{a}_{\mathrm{reg}} = \{Y \in \mathfrak{a} \mid \exp Y \in A_{\mathrm{reg}}\} = \{Y \in \mathfrak{a} \mid e^{2\alpha(Y)} \neq \lambda \text{ for all } (\alpha, \lambda) \in \widetilde{\Sigma}\}.$$

Then we have

$$j\,\mathfrak{a}_{\mathrm{reg}} = \mathfrak{a}_{\mathrm{reg}} \tag{3.1}$$

for any $j \in \widetilde{J}$.

Let j be an element of $\widetilde{J}$ and Δ a connected component of $\mathfrak{a}_{\mathrm{reg}}$. Then by (3.1), $j\Delta$ is a connected component of $\mathfrak{a}_{\mathrm{reg}}$. Let $\widetilde{J}_r$ denote the subgroup of $\widetilde{J}$ generated by

$$\{(w_\alpha, \mu\alpha^\vee) \mid (\alpha, e^{2\mu}) \in \widetilde{\Sigma}\}.$$

Since $(w_\alpha, \mu\alpha^\vee)$ is the reflection with respect to the hyperplane $\{Y \in \mathfrak{a} \mid \alpha(Y) = \mu\}$ and since

$$\mathfrak{a} - \mathfrak{a}_{\mathrm{reg}} = \bigcup_{(\alpha, e^{2\mu}) \in \widetilde{\Sigma}} \{Y \in \mathfrak{a} \mid \alpha(Y) = \mu\},$$

it is clear that there exists a $j_r \in \widetilde{J}_r$ such that

$$j_r j\Delta = \Delta.$$

Put $j_0 = j_r j = (w_0, Z)$. Then we have only to show that $j_0 = \text{id}$. Let N be a positive integer such that $w_0^N = \text{id}$. Since

$$j_0^N = (w_0^N, Z') = (\text{id.}, Z')$$

with some $Z' \in \mathfrak{a}$ and since Δ is bounded, we have $Z' = 0$ and therefore $j_0^N = \text{id}$. Let X be an element of Δ and put

$$X_0 = \frac{1}{N}(X + j_0(X) + \cdots + j_0^{N-1}(X)).$$

Then $j_0(X_0) = X_0$ and $X_0 \in \Delta$ because Δ is convex. Put $x_0 = \exp X_0$.

Let (h, ℓ) be an element of N_A such that $\mathrm{Ad}(h)|_{\mathfrak{a}} = w_0$ and that $h\ell^{-1} = \exp Z$. Then we have

$$h x_0 \ell^{-1} = x_0.$$

Hence

$$h = x_0 \ell x_0^{-1} \in G^{\sigma} \cap G^{\tau_{x_0}} \subset G^{\sigma\tau_{x_0}}.$$

By [2] 3.4.Theorem, $G^{\sigma\tau_{x_0}}$ is connected. Since x_0 is regular, we have

$$\mathfrak{g}^{\sigma\tau_{x_0}} \subset \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}).$$

Hence we have $h \in G^{\sigma\tau_{x_0}} \subset Z_G(\mathfrak{a})$ and therefore

$$w_0 = \mathrm{Ad}(h)|_{\mathfrak{a}} = \text{id}.$$

which implies $N = 1$ and $j_0 = \text{id}$.

The assertions (ii) and (iii) are clear from (i). ■

4. Lists of $\Sigma(\mathfrak{g}_{\mathbb{C}}, \mathfrak{a})$, $\dim \mathfrak{g}_{\mathbb{C}}(\mathfrak{a}, \alpha, \lambda)$ and J when G is simply connected

A. Classical case

We can give as follows the list of $\Sigma = \Sigma(\mathfrak{g}_{\mathbb{C}}, \mathfrak{a})$, $\dim \mathfrak{g}_{\mathbb{C}}(\mathfrak{a}, \alpha, \lambda)$ and J for all $(\mathfrak{g}, \sigma, \tau)$ given in Proposition 2.1.

<table>
<tr><th>type
$\mathfrak{g}$</th><th>$\begin{pmatrix}\sigma(X)\\ \tau(X)\end{pmatrix}$</th><th>$\Sigma$</th><th>$\dim \mathfrak{g}_{\mathbb{C}}(\mathfrak{a}, \alpha, \lambda)$</th><th>$c$</th><th>$a \in J_0$
$\Longleftrightarrow$</th></tr>
<tr><td>BDI – I
$\mathfrak{o}(n)$</td><td rowspan="3">$\begin{pmatrix}I_{p,q} X I_{p,q}\\ I_{r,s} X I_{r,s}\end{pmatrix}$</td><td>$\mathrm{B}_s$</td><td rowspan="3">$\begin{pmatrix}c & c(p-s) & c-1\\ 0 & c(q-s) & 0\end{pmatrix}$</td><td>1</td><td>(4.2)</td></tr>
<tr><td>AIII – III
$\mathfrak{su}(n)$</td><td>BC_s</td><td>2</td><td>$a^2 = e$</td></tr>
<tr><td>CII – II
$\mathfrak{u}(n, \mathbb{H})$</td><td>$\mathrm{BC}_s$</td><td>4</td><td>$a^2 = e$</td></tr>
<tr><td>AI – III
$\mathfrak{su}(n)$</td><td>$\begin{pmatrix}\overline{X}\\ I_{p,q} X I_{p,q}\end{pmatrix}$</td><td rowspan="2">$\mathrm{BC}_q$</td><td rowspan="2">$\begin{pmatrix}c & c(p-q) & c-1\\ c & c(p-q) & c\end{pmatrix}$</td><td>1</td><td>$a^4 = e$ $(p > q)$
(4.3) $(p = q)$</td></tr>
<tr><td>CI – II
$\mathfrak{u}(n, \mathbb{H})$</td><td>$\begin{pmatrix}(i_1 I_n) X (i_1 I_n)^{-1}\\ I_{p,q} X I_{p,q}\end{pmatrix}$</td><td>2</td><td>$a^4 = e$</td></tr>
</table>

DI – III $\mathfrak{o}(2m)$ AIII – II $\mathfrak{su}(2m)$	$\begin{pmatrix} I_{p,q}XI_{p,q} \\ J_mXJ_m^{-1} \end{pmatrix}$ $\begin{pmatrix} I_{p,q}XI_{p,q} \\ J_m\overline{X}J_m^{-1} \end{pmatrix}$	$\mathrm{BC}_{q'}$	$\begin{pmatrix} 2c & c(p-q) & 2c-1 \\ 2c & c(p-q) & c-1 \\ 0 & 2c\varepsilon_q & 0 \end{pmatrix}$ $(q'=[q/2],\ \varepsilon_q = q-2q')$	1 2	$a^4=e$ $(\varepsilon_q=1)$ (4.4) $(\varepsilon_q=0)$ $a^4=e$
AI – II $\mathfrak{su}(2m)$	$\begin{pmatrix} \overline{X} \\ J_m\overline{X}J_m^{-1} \end{pmatrix}$	A_{m-1}	$\begin{pmatrix} 2 \\ 2 \end{pmatrix}$		$a^4=e$
DIII – III′ $\mathfrak{o}(4m)$	$\begin{pmatrix} J_{2m}XJ_{2m}^{-1} \\ J'_{2m}XJ'^{-1}_{2m} \end{pmatrix}$	BC_{m-1}	$\begin{pmatrix} 4 & 4 & 1 \\ 0 & 4 & 0 \end{pmatrix}$		$a^2=e$
DI – I′ $\mathfrak{o}(8)$	$\begin{pmatrix} I_{p,q}XI_{p,q} \\ \kappa^{-1}(I_{r,s}\kappa(X)I_{r,s}) \end{pmatrix}$ $(q,s=1$ or $3)$	(4.1)	$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$		(4.5)

Here the notations and assumptions for σ and τ are as in Section 2. But we choose an outer automorphism κ of $\mathfrak{o}(8)$ of order 3 so that $\kappa\mathrm{Ad}(I_{4,4}) = \mathrm{Ad}(I_{4,4})\kappa$ and that $\mathrm{Ad}(I'_{7,1})\kappa\mathrm{Ad}(I'_{7,1}) = \kappa^{-1}$ where $I'_{7,1} = I_{4,4}I_{5,3}$ (c.f. [3] p.106, p.155). The root system Σ for DI-I' type is

$$\Sigma = \begin{cases} \mathrm{G}_2 & \text{if } q=s=3 \\ \phi & \text{otherwise.} \end{cases} \tag{4.1}$$

Matrices for $\dim \mathfrak{g}_{\mathbb{C}}(\mathfrak{a},\alpha,\lambda)$ imply the following for each type of Σ. Here we write $d(\alpha,\lambda) = \dim \mathfrak{g}_{\mathbb{C}}(\mathfrak{a},\alpha,\lambda)$ and take a standard orthogonal basis $\{e_1,\dots,e_*\}$ of $i\mathfrak{a}^*$ if Σ is BC_* or B_*-type.

BC_* or B_*	$\begin{pmatrix} d(\pm e_j \pm e_k, 1) & d(\pm e_j, 1) & d(\pm 2e_j, 1) \\ d(\pm e_j \pm e_k, -1) & d(\pm e_j, -1) & d(\pm 2e_j, -1) \\ (d(\pm e_j \pm e_k, \pm i) & d(\pm e_j, \pm i) & d(\pm 2e_j, \pm i)) \end{pmatrix}$
A_{m-1}	$\begin{pmatrix} d(\alpha,1) \\ d(\alpha,-1) \end{pmatrix}$ for all $\alpha \in \Sigma$
G_2	$\begin{pmatrix} d(\alpha,1) & d(\beta,1) \\ d(\alpha,\omega) & d(\beta,\omega) \end{pmatrix}$ $\begin{pmatrix} \alpha: \text{ short roots} \\ \beta: \text{ long roots} \\ \omega = (-1 \pm \sqrt{3}i)/2 \end{pmatrix}$

Since G is simply connected, we can compute the group J by Proposition 3.1 if we know $\widetilde{\Sigma} = \{(\alpha,\lambda) \in \Sigma \times U(1) \mid \dim \mathfrak{g}_{\mathbb{C}}(\mathfrak{a},\alpha,\lambda) > 0\}$. We have

$$J = W(\Sigma) \ltimes J_0$$

except for AI-III type $(p=q)$. The group J_0 (or J) is as follows when it is not of the form $\{a \in A \mid a^k = e\}$ with $k = 2$ or 4.

The group J_0 for BDI-I type is given by

$$\begin{aligned} J_0 &= \left\{ \exp \frac{\pi i}{2} \sum_{j=1}^{s} n_j e_j^\vee \;\middle|\; n_j \in \mathbb{Z} \right\} \\ &= \left\{ \exp \frac{1}{4} \sum_{j=1}^{s} n_j Y_{e_j} \;\middle|\; n_j \in \mathbb{Z} \right\} \\ &= \{ a \in A \mid a^2 \in Z \} \end{aligned} \tag{4.2}$$

where Z is the center of $G = Spin(n)$. (Note that $|A \cap Z| = 2$.)

The group J for AI-III type $(p = q)$ is given by

$$J = J^{(0)} \sqcup J^{(1)} \tag{4.3}$$

where $J^{(\varepsilon)} = \{(w, a) \mid w \in W^{(\varepsilon)},\ a \in J_0^{(\varepsilon)}\}$, $W(\Sigma) = W^{(0)} \sqcup W^{(1)}$ ($W^{(0)}$ is D_q-type) and

$$J_0^{(\varepsilon)} = \left\{ \exp \frac{1}{4} \sum_{j=1}^{q} n_j Y_{2e_j} \;\middle|\; n_j \in \mathbb{Z},\ \sum_{j=1}^{q} n_j \in 2\mathbb{Z} + \varepsilon \right\}.$$

The group J_0 for DI-III type $(\varepsilon_q = 0)$ is given by

$$J_0 = \{ \exp \frac{1}{4} \sum_{j=1}^{q'} n_j Y_{2e_j} \mid n_j \in \mathbb{Z},\ \sum_{j=1}^{q'} n_j \in 2\mathbb{Z} \}. \tag{4.4}$$

The group J_0 for DI-I' type $(q = s = 3)$ is generated by

$$\{ \exp \frac{1}{6} Y_\alpha \mid \alpha : \text{short roots} \}. \tag{4.5}$$

(Note that $J_0 / \{a \in A \mid a^2 = e\} \cong \mathbb{Z}_3 = \mathbb{Z}/3\mathbb{Z}$.)

For each type of $(\mathfrak{g}, \sigma, \tau)$, we compute $\mathfrak{g}^{\sigma\tau}$, choose a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{g}^{-\sigma} \cap \mathfrak{g}^{-\tau}$ and then we find Σ and $\dim \mathfrak{g}_{\mathbb{C}}(\mathfrak{a}, \alpha, \lambda)$ as follows. (The group J is determined by $\dim \mathfrak{g}_{\mathbb{C}}(\mathfrak{a}, \alpha, \lambda)$ as in Section 3 since G is simply connected.) If $\mathfrak{g} = \mathfrak{su}(n)$, then we may consider $\mathfrak{g} = \mathfrak{u}(n)$ because the center has no effect on the problems for Lie algebras.

Remark 4.1. (i) If $(\mathfrak{g}, \sigma, \tau)$ is not of type DI-III, AIII-II (q: odd) or DI-I', then we have chosen σ and τ in Proposition 1 so that

$$\sigma\tau = \tau\sigma.$$

Hence we have $(\sigma\tau)^2 = \text{id.}$ and therefore $\lambda = \pm 1$ for such cases. Although dimensions of $\mathfrak{g}_{\mathbb{C}}(\mathfrak{a}, \alpha, \lambda)$ are given in Table V of [5] in these cases, we will give an explicit computation for the sake of convenience.

(ii) As is remarked in Section 1, we can easily get all the data on root multiplicities given in Table V of [5] from our two lists for classical case and exceptional case.

(iii) Type $(\mathfrak{k}_\varepsilon)$ in [5] corresponds to the case when σ and τ are G_0-conjugate.

BDI-I, AIII-III and CII-II type: For these cases, dimensions of $\mathfrak{g}_{\mathbb{C}}(\mathfrak{a}, \alpha, \lambda)$ are already computed in Example 1.3. We can compute J by Proposition 3.1 if we note that

$$Y_{e_j} = \frac{4\pi i e_j}{(e_j, e_j)} = Y(0, \dots, 4\pi, \dots, 0)$$

where 4π is in the j-th component.

AI-III and CI-II type: Put

$$(\mathbb{F}, \mathbb{F}', \eta, c) = \begin{cases} (\mathbb{R}, \mathbb{C}, i, 1) & \text{(AI-III type)} \\ (\mathbb{C}, \mathbb{H}, i_2, 2) & \text{(CI-II type).} \end{cases}$$

We have $\mathfrak{g}^{\sigma} \cap \mathfrak{g}^{\tau} = \mathfrak{u}(p, \mathbb{F}) \oplus \mathfrak{u}(q, \mathbb{F})$ and $\mathfrak{g}^{\sigma\tau} \cong \mathfrak{u}(n, \mathbb{F})$ because $\sigma\tau$ is $\operatorname{Aut}(\mathfrak{g})$-conjugate to

$$\begin{aligned}
& \operatorname{Ad}\begin{pmatrix} I_p & 0 \\ 0 & \eta I_q \end{pmatrix} \sigma\tau \operatorname{Ad}\begin{pmatrix} I_p & 0 \\ 0 & \eta I_q \end{pmatrix}^{-1} \\
= \; & \operatorname{Ad}\begin{pmatrix} I_p & 0 \\ 0 & \eta I_q \end{pmatrix} \sigma \operatorname{Ad}\begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \operatorname{Ad}\begin{pmatrix} I_p & 0 \\ 0 & \eta I_q \end{pmatrix}^{-1} \\
= \; & \operatorname{Ad}\begin{pmatrix} I_p & 0 \\ 0 & \eta I_q \end{pmatrix} \sigma \operatorname{Ad}\begin{pmatrix} I_p & 0 \\ 0 & \eta I_q \end{pmatrix} \\
= \; & \sigma \operatorname{Ad}\begin{pmatrix} I_p & 0 \\ 0 & -\eta I_q \end{pmatrix} \operatorname{Ad}\begin{pmatrix} I_p & 0 \\ 0 & \eta I_q \end{pmatrix} \\
= \; & \sigma.
\end{aligned}$$

(Note that σ is defined by an automorphism of $\mathbb{F}'$ which maps η to $-\eta$.) We can take $\mathfrak{a}$ of the form

$$\mathfrak{a} = \left\{ Y(\theta_1, \dots, \theta_q) = \begin{pmatrix} 0 & & \eta d(\theta_1, \dots, \theta_q) \\ & 0 & \\ \eta d(\theta_1, \dots, \theta_q) & & 0 \end{pmatrix} \;\middle|\; \theta_1, \dots, \theta_q \in \mathbb{R} \right\}.$$

Let e_j be an element of $i\mathfrak{a}^*$ defined by

$$e_j : Y(\theta_1, \dots, \theta_q) \mapsto i\theta_j.$$

Then it is clear that

$$\dim \mathfrak{g}_{\mathbb{C}}(\mathfrak{a}, \pm e_j \pm e_k) = 2c, \quad \dim \mathfrak{g}_{\mathbb{C}}(\mathfrak{a}, \pm e_j) = 2c(p-q), \quad \dim \mathfrak{g}_{\mathbb{C}}(\mathfrak{a}, \pm 2e_j) = 2c - 1$$

and that $\mathfrak{g}_{\mathbb{C}}(\mathfrak{a}, \alpha) = \{0\}$ for other $\alpha \in i\mathfrak{a}^* - \{0\}$. On the other hand, we have

$$\dim \mathfrak{g}_{\mathbb{C}}(\mathfrak{a}, \pm e_j \pm e_k, 1) = c, \ \dim \mathfrak{g}_{\mathbb{C}}(\mathfrak{a}, \pm e_j, 1) = c(p-q) \text{ and } \dim \mathfrak{g}_{\mathbb{C}}(\mathfrak{a}, \pm 2e_j, 1) = c - 1$$

since $\mathfrak{g}^{\sigma\tau}/\mathfrak{g}^{\sigma} \cap \mathfrak{g}^{\tau} \cong \mathfrak{u}(n, \mathbb{F})/\mathfrak{u}(p, \mathbb{F}) \oplus \mathfrak{u}(q, \mathbb{F})$.

DI-III and AIII-II type: Put

$$(\mathbb{F}, \mathbb{F}', c) = \begin{cases} (\mathbb{R}, \mathbb{C}, 1) & \text{(DI-III type)} \\ (\mathbb{C}, \mathbb{H}, 2) & \text{(AIII-II type).} \end{cases}$$

First suppose that q is even. Then $\mathfrak{g}^\sigma \cap \mathfrak{g}^\tau \cong \mathfrak{u}(p', \mathbb{F}) \oplus \mathfrak{u}(q', \mathbb{F})$ ($p' = p/2$, $q' = q/2$) and $\mathfrak{g}^{\sigma\tau} \cong \mathfrak{g}^\tau \cong \mathfrak{u}(m, \mathbb{F})$ because $I_{p,q}J_m$ is $O(2m)$-conjugate to J_m. We can take a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{g}^{-\sigma} \cap \mathfrak{g}^{-\tau}$ of the form $\mathfrak{a} =$

$$\left\{ Y(\theta_1, \dots, \theta_{q'}) = \begin{pmatrix} 0 & & d(\theta_1 I_2', \dots, \theta_{q'} I_2') \\ & 0 & \\ -d(\theta_1 I_2', \dots, \theta_{q'} I_2') & & 0 \end{pmatrix} \,\middle|\, \theta_1, \dots, \theta_{q'} \in \mathbb{R} \right\}$$

(4.6), where

$$I_2' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad d(A_1, \dots, A_{q'}) = \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_{q'} \end{pmatrix}$$

for matrices $A_1, \dots, A_{q'}$. Let e_j be an element of $i\mathfrak{a}^*$ defined by

$$e_j : Y(\theta_1, \dots, \theta_{q'}) \mapsto i\theta_j.$$

Take an abelian subspace

$$\mathfrak{b} = \left\{ Z(\varphi_1, \dots, \varphi_q) = \begin{pmatrix} 0 & & d'(\varphi_1, \dots, \varphi_q) \\ & 0 & \\ -{}^t d'(\varphi_1, \dots, \varphi_q) & & 0 \end{pmatrix} \,\middle|\, \varphi_1, \dots, \varphi_q \in \mathbb{R} \right\}$$

of $\mathfrak{g}$ containing $\mathfrak{a}$, where $d'(\varphi_1, \dots, \varphi_q) = d(\varphi_1, \dots, \varphi_q) d(I_2', \dots, I_2')$. Define $f_j \in i\mathfrak{b}^*$ by

$$f_j : Z(\varphi_1, \dots, \varphi_q) \mapsto i\varphi_j.$$

Then we have

$$f_{2j-1}|_{\mathfrak{a}} = f_{2j}|_{\mathfrak{a}} = e_j$$

for $j = 1, \dots, q'$. So we have

$$\begin{aligned}
\dim \mathfrak{g}_{\mathbb{C}}(\mathfrak{a}, \pm e_j \pm e_k) &= \dim \mathfrak{g}_{\mathbb{C}}(\mathfrak{b}, \pm f_{2j-1} \pm f_{2k-1}) + \dim \mathfrak{g}_{\mathbb{C}}(\mathfrak{b}, \pm f_{2j-1} \pm f_{2k}) \\
&\quad + \dim \mathfrak{g}_{\mathbb{C}}(\mathfrak{b}, \pm f_{2j} \pm f_{2k-1}) + \dim \mathfrak{g}_{\mathbb{C}}(\mathfrak{b}, \pm f_{2j} \pm f_{2k}) \\
&= 4c, \\
\dim \mathfrak{g}_{\mathbb{C}}(\mathfrak{a}, \pm e_j) &= \dim \mathfrak{g}_{\mathbb{C}}(\mathfrak{b}, \pm f_{2j-1}) + \dim \mathfrak{g}_{\mathbb{C}}(\mathfrak{b}, \pm f_{2j}) \\
&= 2c(p-q), \\
\dim \mathfrak{g}_{\mathbb{C}}(\mathfrak{a}, \pm 2e_j) &= \dim \mathfrak{g}_{\mathbb{C}}(\mathfrak{b}, \pm 2f_{2j-1}) + \dim \mathfrak{g}_{\mathbb{C}}(\mathfrak{b}, \pm 2f_{2j}) \\
&\quad + \dim \mathfrak{g}_{\mathbb{C}}(\mathfrak{b}, \pm(f_{2j-1} + f_{2j})) \\
&= 2(c-1) + c \\
&= 3c - 2.
\end{aligned}$$

On the other hand, we have

$$\dim \mathfrak{g}_{\mathbb{C}}(\mathfrak{a}, \pm e_j \pm e_k, 1) = 2c, \ \dim \mathfrak{g}_{\mathbb{C}}(\mathfrak{a}, \pm e_j, 1) = c(p-q) \text{ and } \dim \mathfrak{g}_{\mathbb{C}}(\mathfrak{a}, \pm 2e_j, 1) = 2c - 1$$

since $\mathfrak{g}^{\sigma\tau}/\mathfrak{g}^\sigma \cap \mathfrak{g}^\tau \cong \mathfrak{u}(m, \mathbb{F})/\mathfrak{u}(p', \mathbb{F}) \oplus \mathfrak{u}(q', \mathbb{F})$.

Next suppose that q is odd. Then we have

$$\mathfrak{g}^{(\sigma\tau)^2} \cong \mathfrak{u}(2m-2,\mathbb{F}) \oplus \mathfrak{u}(2,\mathbb{F})$$

because $(\sigma\tau)^2(X) = (I_{p,q}J_m)^2 X (I_{p,q}J_m)^{-2}$ and

$$(I_{p,q}J_m)^2 = \begin{pmatrix} J_{p'} & & 0 \\ & I_2' & \\ 0 & & -J_{q'} \end{pmatrix}^2 = \begin{pmatrix} -I_{2p'} & & 0 \\ & I_2 & \\ 0 & & -I_{2q'} \end{pmatrix}$$

where $p' = [p/2]$. Moreover we have

$$\mathfrak{g}^{(\sigma\tau)^2} \cap \mathfrak{g}^{\sigma} \cong \mathfrak{u}(2p',\mathbb{F}) \oplus \mathfrak{u}(2q',\mathbb{F}) \oplus \mathfrak{u}(1,\mathbb{F}) \oplus \mathfrak{u}(1,\mathbb{F})$$

and

$$\mathfrak{g}^{(\sigma\tau)^2} \cap \mathfrak{g}^{\tau} \cong \mathfrak{u}(m-1,\mathbb{F}') \oplus \mathfrak{u}(1,\mathbb{F}').$$

Since $\mathfrak{a} \subset \mathfrak{g}^{-\sigma} \cap \mathfrak{g}^{-\tau} \subset \mathfrak{g}^{\sigma\tau} \subset \mathfrak{g}^{(\sigma\tau)^2}$ and since $\mathfrak{u}(2,\mathbb{F}) = (\mathfrak{u}(1,\mathbb{F}) \oplus \mathfrak{u}(1,\mathbb{F})) + \mathfrak{u}(1,\mathbb{F}')$, we can choose $\mathfrak{a}$ of the same form as (4.6). So we can compute $\dim \mathfrak{g}_{\mathbb{C}}(\mathfrak{a},\alpha)$ and $\dim \mathfrak{g}_{\mathbb{C}}(\mathfrak{a},\alpha,\pm 1)$ for all $\alpha \in \Sigma$ by the results when q is even. Especially we have

$$\begin{aligned} \dim \mathfrak{g}_{\mathbb{C}}^{(\sigma\tau)^2}(\mathfrak{a},\pm e_j) &= 2c(2p'-2q'), \\ \dim \mathfrak{g}_{\mathbb{C}}(\mathfrak{a},\pm e_j) &= 2c(2m-4q') = 2c(2p'-2q'+2) \end{aligned}$$

and $\dim \mathfrak{g}_{\mathbb{C}}^{(\sigma\tau)^2}(\mathfrak{a},\alpha) = \dim \mathfrak{g}_{\mathbb{C}}(\mathfrak{a},\alpha)$ if $\alpha \neq \pm e_j$. Hence we have

$$\dim \mathfrak{g}_{\mathbb{C}}(\mathfrak{a},\alpha,\pm i) = \begin{cases} 2c & \text{if } \alpha = \pm e_j \\ 0 & \text{otherwise.} \end{cases}$$

AI-II type: Since $\sigma\tau(X) = J_m X J_m^{-1}$ and since J_m is $U(2m)$-conjugate to

$$\begin{pmatrix} iI_m & 0 \\ 0 & -iI_m \end{pmatrix},$$

we have $\mathfrak{g}^{\sigma\tau} \cong \mathfrak{u}(m) \oplus \mathfrak{u}(m)$. On the other hand, we have $\mathfrak{g}^{\sigma} \cap \mathfrak{g}^{\tau} = \{X \in \mathfrak{o}(2m) \mid J_m X J_m^{-1} = X\} \cong \mathfrak{u}(m)$. We can take a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{g}^{-\sigma} \cap \mathfrak{g}^{-\tau}$ of the form

$$\mathfrak{a} = \{Y(\theta_1,\ldots,\theta_m) = d(i\theta_1 I_2,\ldots,i\theta_m I_2) \mid \theta_1,\ldots,\theta_m \in \mathbb{R}\}.$$

Let e_j be an element of $i\mathfrak{a}^*$ defined by

$$e_j : Y(\theta_1,\ldots,\theta_m) \mapsto i\theta_j.$$

Take an abelian subspace

$$\mathfrak{b} = \{Z(\varphi_1,\ldots,\varphi_{2m}) = d(i\varphi_1,\ldots,i\varphi_{2m}) \mid \varphi_1,\ldots,\varphi_{2m} \in \mathbb{R}\}$$

of $\mathfrak{g}$ containing $\mathfrak{a}$ and define $f_j \in i\mathfrak{b}^*$ by

$$f_j : Z(\varphi_1,\ldots,\varphi_q) \mapsto i\varphi_j.$$

Then we have

$$f_{2j-1}|_{\mathfrak{a}} = f_{2j}|_{\mathfrak{a}} = e_j$$

for $j = 1, \ldots, m$. So we have

$$\begin{aligned}
\dim \mathfrak{g}_{\mathbb{C}}(\mathfrak{a}, e_j - e_k) &= \dim \mathfrak{g}_{\mathbb{C}}(\mathfrak{b}, f_{2j-1} - f_{2k-1}) + \dim \mathfrak{g}_{\mathbb{C}}(\mathfrak{b}, f_{2j-1} - f_{2k}) \\
&\quad + \dim \mathfrak{g}_{\mathbb{C}}(\mathfrak{b}, f_{2j} - f_{2k-1}) + \dim \mathfrak{g}_{\mathbb{C}}(\mathfrak{b}, f_{2j} - f_{2k}) \\
&= 4
\end{aligned}$$

and $\mathfrak{g}_{\mathbb{C}}(\mathfrak{a}, \alpha) = \{0\}$ for other $\alpha \in i\mathfrak{a}^* - \{0\}$. On the other hand, we have $\dim \mathfrak{g}_{\mathbb{C}}(\mathfrak{a}, e_j - e_k, 1) = 2$ since $\mathfrak{g}^{\sigma\tau}/\mathfrak{g}^{\sigma} \cap \mathfrak{g}^{\tau} \cong \mathfrak{u}(m) \oplus \mathfrak{u}(m)/\{(X, X) \mid X \in \mathfrak{u}(m)\}$.

DIII-III' type: We have $\mathfrak{g}^{\sigma\tau} \cong \mathfrak{o}(4m-2) \oplus \mathfrak{o}(2)$ and $\mathfrak{g}^{\sigma} \cap \mathfrak{g}^{\tau} \cong \mathfrak{u}(2m-1) \oplus \mathfrak{u}(1)$. We can take a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{g}^{-\sigma} \cap \mathfrak{g}^{-\tau}$ consisting of the elements

$$Y(\theta_1, \ldots, \theta_{m-1}) = \begin{pmatrix} 0 & d(\theta_1 I_2', \ldots, \theta_{m-1} I_2') & 0 \\ -d(\theta_1 I_2', \ldots, \theta_{m-1} I_2') & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $\theta_1, \ldots, \theta_{m-1} \in \mathbb{R}$. Let e_j be an element of $i\mathfrak{a}^*$ defined by

$$e_j : Y(\theta_1, \ldots, \theta_{m-1}) \mapsto i\theta_j.$$

Since the symmetric pair $(\mathfrak{g}^{\sigma\tau}, \mathfrak{g}^{\sigma} \cap \mathfrak{g}^{\tau})$ is DIII-type, we have $\dim \mathfrak{g}_{\mathbb{C}}(\mathfrak{a}, \pm e_j, 1) = \dim \mathfrak{g}_{\mathbb{C}}(\mathfrak{a}, \pm e_j \pm e_k, 1) = 4$ and $\dim \mathfrak{g}_{\mathbb{C}}(\mathfrak{a}, \pm 2e_j, 1) = 1$. We also have easily that

$$\dim \mathfrak{g}_{\mathbb{C}}(\mathfrak{a}, \alpha, -1) = \begin{cases} 4 & \text{if } \alpha = \pm e_j \\ 0 & \text{otherwise.} \end{cases}$$

(We may take an abelian subspace $\mathfrak{b}$ of $\mathfrak{g}$ as in DI-III and AIII-II type.)

DI-I' type: Remark that $G = HL$ if and only if $G = H'L'$ with some $\mathrm{Int}(\mathfrak{g})$-conjugates H' and L' of H and L, respectively.

First we consider the case $q = s = 1$ and show that $G = HL$. Put $\sigma' = \mathrm{Ad}(I_{7,1}')$ and $\tau' = \kappa \mathrm{Ad}(I_{7,1}')\kappa^{-1}$. Then we have only to show that $G = G^{\sigma'} G^{\tau'}$ by the above remark. We have

$$\sigma'\tau' = \sigma'\kappa^{-1}\sigma'\kappa = \kappa^2 = \kappa^{-1}$$

and therefore $\mathfrak{g}^{\sigma'\tau'} = \mathfrak{g}^{\kappa}$ is G_2-type. Since κ and σ' generate a subgroup of $\mathrm{Aut}(\mathfrak{g})$ isomorphic to S_3, it is known that

$$\mathfrak{g}^{\kappa} \subset \mathfrak{g}^{\sigma'}.$$

Hence we have $\mathfrak{g}^{\sigma'\tau'} \subset \mathfrak{g}^{\sigma'}$ and therefore

$$\mathfrak{g}^{-\sigma'} \cap \mathfrak{g}^{-\tau'} = \mathfrak{g}^{\sigma'\tau'} \cap \mathfrak{g}^{-\sigma'} = \{0\},$$

which implies $G = G^{\sigma'} G^{\tau'}$.

Next we consider the case $(q, s) = (3, 1)$ and show that $G = HL$. Put $\tau' = \kappa^{-1}\mathrm{Ad}(I_{7,1}')\kappa$. Then we have only to show that

$$\mathfrak{g}^{-\sigma} \cap \mathfrak{g}^{-\tau'} = \{0\}$$

as in the case of $q=s=1$. Since $I_{5,3}=I_{4,4}I'_{7,1}$, we have

$$\sigma\tau' = \mathrm{Ad}(I_{4,4})\mathrm{Ad}(I'_{7,1})\kappa^{-1}\mathrm{Ad}(I'_{7,1})\kappa = \mathrm{Ad}(I_{4,4})\kappa^{-1}$$

and $(\sigma\tau')^2 = \kappa^{-2} = \kappa$ because $\kappa\mathrm{Ad}(I_{4,4}) = \mathrm{Ad}(I_{4,4})\kappa$. Hence $\mathfrak{g}^{\sigma\tau'} \subset \mathfrak{g}^{(\sigma\tau')^2} = \mathfrak{g}^{\kappa} \subset \mathfrak{g}^{\tau'}$ and therefore

$$\mathfrak{g}^{-\sigma} \cap \mathfrak{g}^{-\tau'} = \mathfrak{g}^{\sigma\tau'} \cap \mathfrak{g}^{-\tau'} = \{0\}.$$

Finally, consider the case $q=s=3$. We have

$$\begin{aligned}\sigma\tau &= \mathrm{Ad}(I_{5,3})\kappa^{-1}\mathrm{Ad}(I_{5,3})\kappa \\ &= \mathrm{Ad}(I_{4,4})\mathrm{Ad}(I'_{7,1})\kappa^{-1}\mathrm{Ad}(I_{4,4})\mathrm{Ad}(I'_{7,1})\kappa \\ &= \mathrm{Ad}(I'_{7,1})\kappa^{-1}\mathrm{Ad}(I'_{7,1})\kappa \\ &= \kappa^{-1}\end{aligned}$$

since $\kappa\mathrm{Ad}(I_{4,4}) = \mathrm{Ad}(I_{4,4})\kappa$. Hence $\mathfrak{g}^{\sigma\tau} = \mathfrak{g}^{\kappa}$ is G_2-type and $\mathfrak{g}^{\sigma}\cap\mathfrak{g}^{\tau} = \mathfrak{g}^{\sigma}\cap\mathfrak{g}^{\sigma\tau} \cong \mathfrak{su}(2)\oplus\mathfrak{su}(2)$ since $(\mathfrak{g}^{\sigma\tau}, \mathfrak{g}^{\sigma}\cap\mathfrak{g}^{\tau})$ is a nontrivial symmetric pair. So we can choose a two-dimensional maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{g}^{-\sigma}\cap\mathfrak{g}^{-\tau}$. It is clear that dimensions of $\mathfrak{g}_{\mathbb{C}}(\mathfrak{a},\alpha,\lambda)$ are as in the list.

B. Exceptional case

When $\mathfrak{g}$ is exceptional, we can give as follows the list of Σ, $\dim\mathfrak{g}_{\mathbb{C}}(\mathfrak{a},\alpha,\lambda)$ and J for representatives of all the triples $(\mathfrak{g},\sigma,\tau)$ such that σ and τ are not G_0-conjugate. We may assume that $\sigma\tau=\tau\sigma$ and that $(\mathfrak{g}^{\sigma\tau}, \mathfrak{h}\cap\mathfrak{l})$ are as follows (c.f. Table V in [5]). Dimensions of $\mathfrak{g}_{\mathbb{C}}(\mathfrak{a},\alpha,\lambda)$ are given in Table V in [5]. (We can easily compute them by the same arguments as in the classical cases.) By Proposition 3.1, the group J is determined by $\widetilde{\Sigma}$.

type $\mathfrak{g}$	$\mathfrak{h}$ $\mathfrak{l}$	$\mathfrak{g}^{\sigma\tau}$ $\mathfrak{h}\cap\mathfrak{l}$	Σ	$\dim\mathfrak{g}_{\mathbb{C}}(\mathfrak{a},\alpha,\lambda)$	$a\in J_0$ $\Longleftrightarrow$
EI − II E_6	$\mathfrak{sp}(4)$ $\mathfrak{su}(6)\oplus\mathfrak{su}(2)$	F_4 $\mathfrak{sp}(3)\oplus\mathfrak{sp}(1)$	F_4	$\begin{pmatrix}1&1\\1&0\end{pmatrix}$	(4.7)
EV − VI E_7	$\mathfrak{su}(8)$ $\mathfrak{o}(12)\oplus\mathfrak{su}(2)$	$\mathrm{E}_6\oplus\mathbb{R}$ $\mathfrak{su}(6)\oplus\mathfrak{su}(2)\oplus\mathbb{R}$		$\begin{pmatrix}2&1\\2&0\end{pmatrix}$	
EVIII − IX E_8	$\mathfrak{o}(16)$ $\mathrm{E}_7\oplus\mathfrak{su}(2)$	$\mathrm{E}_7\oplus\mathfrak{su}(2)$ $\mathfrak{o}(12)\oplus\mathfrak{o}(4)$		$\begin{pmatrix}4&1\\4&0\end{pmatrix}$	
EV − VII E_7	$\mathfrak{su}(8)$ $\mathrm{E}_6\oplus\mathbb{R}$	$\mathfrak{o}(12)\oplus\mathfrak{su}(2)$ $\mathfrak{su}(6)\oplus\mathfrak{su}(2)\oplus\mathbb{R}$	C_3	$\begin{pmatrix}4&0&1\\4&0&0\end{pmatrix}$	(4.4)
EII − III E_6	$\mathfrak{su}(6)\oplus\mathfrak{su}(2)$ $\mathfrak{o}(10)\oplus\mathbb{R}$	$\mathfrak{o}(10)\oplus\mathbb{R}$ $\mathfrak{su}(5)\oplus\mathbb{R}^2$	BC_2	$\begin{pmatrix}4&4&1\\2&4&0\end{pmatrix}$	(4.4)
EVI − VII E_7	$\mathfrak{o}(12)\oplus\mathfrak{su}(2)$ $\mathrm{E}_6\oplus\mathbb{R}$	$\mathrm{E}_6\oplus\mathbb{R}$ $\mathfrak{o}(10)\oplus\mathbb{R}^2$		$\begin{pmatrix}6&8&1\\2&8&0\end{pmatrix}$	

EI – III E_6	$\mathfrak{sp}(4)$ $\mathfrak{o}(10) \oplus \mathbb{R}$	$\mathfrak{sp}(4)$ $\mathfrak{sp}(2) \oplus \mathfrak{sp}(2)$	BC_2	$\begin{pmatrix} 3 & 4 & 0 \\ 3 & 4 & 1 \end{pmatrix}$	$a^4 = e$
EI – IV E_6	$\mathfrak{sp}(4)$ F_4	$\mathfrak{su}(6) \oplus \mathfrak{su}(2)$ $\mathfrak{sp}(3) \oplus \mathfrak{sp}(1)$	A_2	$\begin{pmatrix} 4 \\ 4 \end{pmatrix}$	$a^4 = e$
EII – IV E_6	$\mathfrak{su}(6) \oplus \mathfrak{su}(2)$ F_4	$\mathfrak{sp}(4)$ $\mathfrak{sp}(3) \oplus \mathfrak{sp}(1)$	BC_1	$\begin{pmatrix} 8 & 3 \\ 8 & 5 \end{pmatrix}$	$a^4 = e$
EIII – IV E_6	$\mathfrak{o}(10) \oplus \mathbb{R}$ F_4	F_4 $\mathfrak{o}(9)$	BC_1	$\begin{pmatrix} 8 & 7 \\ 8 & 1 \end{pmatrix}$	$a^4 = e$
FI – II F_4	$\mathfrak{sp}(3) \oplus \mathfrak{sp}(1)$ $\mathfrak{o}(9)$	$\mathfrak{sp}(3) \oplus \mathfrak{sp}(1)$ $\mathfrak{o}(5) \oplus \mathfrak{o}(4)$	BC_1	$\begin{pmatrix} 4 & 3 \\ 4 & 4 \end{pmatrix}$	$a^4 = e$

Here the matrices for $\dim \mathfrak{g}_{\mathbb{C}}(\mathfrak{a}, \alpha, \lambda)$ are the same as in the classical case if Σ is A_2, BC_2 or C_3-type. If Σ is F_4-type or BC_1-type, then the matrix implies

$$\begin{pmatrix} \dim \mathfrak{g}_{\mathbb{C}}(\mathfrak{a}, \alpha, 1) & \dim \mathfrak{g}_{\mathbb{C}}(\mathfrak{a}, \beta, 1) \\ \dim \mathfrak{g}_{\mathbb{C}}(\mathfrak{a}, \alpha, -1) & \dim \mathfrak{g}_{\mathbb{C}}(\mathfrak{a}, \beta, -1) \end{pmatrix}$$

for short roots α and long roots β. If Σ is F_4-type, then J_0 is generated by

$$\{\exp \frac{1}{4} Y_\alpha \mid \alpha : \text{short roots}\}. \tag{4.7}$$

(Note that $J_0/\{a \in A \mid a^2 = e\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.)

C. The case $\mathfrak{g}$ is not simple

Type I: Let $(\mathfrak{g}, \sigma, \tau)$ be as in Proposition 2.2 (I) and take a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{g}^{-\sigma} \cap \mathfrak{g}^{-\tau}$ as in Remark 2.3 (ii). Then the space $\mathfrak{g}_{\mathbb{C}}(\mathfrak{a}, \alpha, \lambda)$ consists of the elements

$$(X_1, X_2, \lambda^{-1} X_1, \lambda X_2, \ldots, \lambda^{-m+1} X_1, \lambda^{m-1} X_2)$$

with $X_1 \in (\mathfrak{g}_1)_{\mathbb{C}}(\mathfrak{u}_1, \alpha_1, \lambda^m)$ and $X_2 \in (\mathfrak{g}_1)_{\mathbb{C}}(\mathfrak{u}_1, -\alpha_1, \lambda^{-m})$. Here $m = k/2$, $\alpha_1(Y) = \alpha(Y^{(k)})$ for $Y \in \mathfrak{u}_1$ and $(\mathfrak{g}_1)_{\mathbb{C}}(\mathfrak{u}_1, \beta, \mu) = \{X \in (\mathfrak{g}_1)_{\mathbb{C}}(\mathfrak{u}_1, \beta) \mid \rho_1 X = \mu X\}$ for $(\beta, \mu) \in i\mathfrak{u}_1^* \times U(1)$.

Let G_1, G, H, L, A and U_1 be as in Remark 2.3 (iii). Let (h, ℓ) be an element of N_A. Write $h = (h_1, \ldots, h_k)$ and $\ell = (\ell_1, \ldots, \ell_k)$ where $h_1, \ldots, h_k, \ell_1, \ldots, \ell_k \in G_1$. Since $h\ell^{-1} \in A$, we can write

$$(h_1 \ell_1^{-1}, \ldots, h_k \ell_k^{-1}) = h\ell^{-1} = a^{(k)} = (a, a^{-1}, \ldots, a, a^{-1})$$

with some $a \in U_1$. Hence we have

$$\begin{aligned} \ell_{k-1} &= \ell_k = a h_k \\ h_{k-2} &= h_{k-1} = a^2 h_k \\ \ell_{k-3} &= \ell_{k-2} = a^3 h_k \\ &\cdots \\ \rho_1(h_k) &= h_1 = a^k h_k \end{aligned}$$

by the definition of H and L. Thus we can identify the group J with the group

$$\{(\mathrm{Ad}(g)|_{U_1}, b) \in W_{G_1}(U_1) \ltimes U_1 \mid \rho_1(g)g^{-1} = b^k\}$$

by the map

$$(h, \ell) \mapsto (\mathrm{Ad}(h_k)|_{U_1}, a).$$

(Note that A is identified with U_1 by $a^{(k)} \mapsto a$.)

Define subgroups

$$N_1 = \{g \in G_1 \mid \rho_1(g)U_1 g^{-1} = U_1\}, \quad Z_1 = \{g \in G_1 \mid \rho(g)ag^{-1} = a \text{ for all } a \in U_1\}$$

of G_1 and put $J_1 = N_1/Z_1$. Then by the same argument as in [4] Remark 3, we can identify J_1 with the group

$$\{(\mathrm{Ad}(g)|_{U_1}, b) \in W_{G_1}(U_1) \ltimes U_1 \mid \rho_1(g)g^{-1} = b\}.$$

Comparing J and J_1, we can see that the map $A \ni a^{(k)} \mapsto a^k \in U_1$ induces a bijection

$$J\backslash A \cong J_1\backslash U_1.$$

Summarizing the above arguments, we have the following natural commuting diagram of bijections.

$$\begin{array}{ccc} J\backslash A & \longrightarrow & J_1\backslash U_1 \\ \Big\downarrow & & \Big\downarrow \\ H\backslash G/L & \longrightarrow & G_1/\rho_1\text{-twisted conj.} \end{array}$$

Here the vertical arrows are given by the inclusions and the horizontal ones are given by the map

$$(x_1, \ldots, x_k) \mapsto x_1 x_2^{-1} \cdots x_{k-1} x_k^{-1}.$$

Note that the restriction of this map to A is $a^{(k)} \mapsto a^k$.

If ρ_1 is an inner automorphism, then we may assume that $\rho_1 = \mathrm{id}$. by Remark 2.3 (i). If G_1 is simply connected, then it is known that $J_1 \cong W_{G_1}(U_1)$. Hence

$$J \cong W_{G_1}(U_1) \ltimes \{a \in U_1 \mid a^k = e\}.$$

Of course, this follows also from Proposition 3.1.

On the other hand, if ρ_1 is an outer automorphism, then we may assume that $(\mathfrak{g}_1, \mathfrak{g}_1^{\rho_1})$ is one of the following five types by Remark 2.3 (i) and (2.1).

$\mathfrak{g}_1$	$\mathfrak{g}_1^{\rho_1}$	$\Sigma((\mathfrak{g}_1)_{\mathbb{C}}, \mathfrak{u}_1)$	$\dim(\mathfrak{g}_1)_{\mathbb{C}}(\mathfrak{u}_1, \alpha, \lambda)$	J_0 (or J) when $k=2$
$\mathfrak{su}(2m)$	$\mathfrak{sp}(m)$	C_m	$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	(4.4)
$\mathfrak{su}(2m+1)$	$\mathfrak{o}(2m+1)$	BC_m	$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$	(4.3)
$\mathfrak{o}(2m)$	$\mathfrak{o}(2m-1)$	B_{m-1}	$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	(4.2)
E_6	F_4	F_4	$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$	(4.7)
$\mathfrak{o}(8)$	G_2	G_2	$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$	(4.5)

Here the matrices for $\dim(\mathfrak{g}_1)_{\mathbb{C}}(\mathfrak{u}_1, \alpha, \lambda)$ are as in the simple cases. If $k=2$, then we can see that every $\widetilde{\Sigma} = \{(\alpha, \lambda) \in i\mathfrak{a}^* \times U(1) \mid \mathfrak{g}_{\mathbb{C}}(\mathfrak{a}, \alpha, \lambda) \neq \{0\}\}$ is equal to one of the classical cases. So the group J is the same by Proposition 3.1.

Type (II) and (III): Let $(\mathfrak{g}, \sigma, \tau)$ be as in Proposition 2.2 (II) and (III). Take a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{g}^{-\sigma} \cap \mathfrak{g}^{-\tau}$ as in Remark 2.4. If $k=2m$ is even, then the space $\mathfrak{g}_{\mathbb{C}}(\mathfrak{a}, \alpha, \lambda)$ consists of the elements

$$(X, \lambda^{-2m+1}\tau_1 X, \lambda^{-1}X, \lambda^{-2m+2}\tau_1 X, \ldots, \lambda^{-m+2}X, \lambda^{-m-1}\tau_1 X, \lambda^{-m+1}X, \lambda^{-m}\tau_1 X)$$

with $X \in (\mathfrak{g}_1)_{\mathbb{C}}(\mathfrak{a}_1, \alpha_1, \lambda^k)$. Here $\alpha_1(Y) = \alpha(Y^{(k)})$ for $Y \in \mathfrak{a}_1$ and $(\mathfrak{g}_1)_{\mathbb{C}}(\mathfrak{a}_1, \beta, \mu) = \{(X \in (\mathfrak{g}_1)_{\mathbb{C}}(\mathfrak{a}_1, \beta) \mid \sigma_1\tau_1 X = \mu X\}$ for $(\beta, \mu) \in i\mathfrak{a}_1^* \times U(1)$. Similarly, if $k = 2m+1$ is odd, then the space $\mathfrak{g}_{\mathbb{C}}(\mathfrak{a}, \alpha, \lambda)$ consists of the elements

$$(X, \lambda^{-2m}\tau_1 X, \lambda^{-1}X, \lambda^{-2m+1}\tau_1 X, \ldots, \lambda^{-m-2}\tau_1 X, \lambda^{-m+1}X, \lambda^{-m-1}\tau_1 X, \lambda^{-m}X)$$

with $X \in (\mathfrak{g}_1)_{\mathbb{C}}(\mathfrak{a}_1, \alpha_1, \lambda^k)$.

Let G_1, G, A and A_1 be as in Remark 2.4. Let J_1 be the group for $(G_1, G_1^{\sigma_1}, G_1^{\tau_1})$. Then by a similar argument as in type I case, we have the following natural commuting diagram of bijections.

$$\begin{array}{ccc} J\backslash A & \longrightarrow & J_1\backslash A_1 \\ \downarrow & & \downarrow \\ G^\sigma\backslash G/G^\tau & \longrightarrow & G_1^{\sigma_1}\backslash G_1/G_1^{\tau_1} \end{array}$$

Here the vertical arrows are given by the inclusions and the horizontal ones are given by the map

$$(x_1, \ldots, x_k) \mapsto \begin{cases} x_1 x_2^{-1} \cdots x_{k-1} x_k^{-1} & \text{if } k \text{ is even} \\ x_1 x_2^{-1} \cdots x_{k-1}^{-1} x_k & \text{if } k \text{ is odd.} \end{cases}$$

Note that the restriction of this map to A is $a^{(k)} \mapsto a^k$.

5. Appendix

Let ρ be an automorphism of $\mathfrak{g}$. Take a maximal abelian subalgebra $\mathfrak{u}$ of $\mathfrak{g}^\rho$ and let $\mathfrak{t}$ denote the centralizer of $\mathfrak{u}$ in $\mathfrak{g}$. Then the following lemma seems to be known. For the sake of completeness, we will give a proof by the same argument as in the proof of [2] 3.4.Theorem.

Lemma 5.1. *$\mathfrak{t}$ is abelian.*

Proof. Let $\mathfrak{z}$ be the center of $\mathfrak{t}$ and $\mathfrak{s} = [\mathfrak{t}, \mathfrak{t}]$ the semisimple part of $\mathfrak{t}$. Then

$$\mathfrak{t} = \mathfrak{z} \oplus \mathfrak{s}.$$

Suppose that $\mathfrak{s} \neq \{0\}$. Then we will get a contradiction.

Since $\mathfrak{u}$ is ρ-stable, $\mathfrak{s}$ is also ρ-stable. By [7] Chap.II Section 2, we have $\mathfrak{s}^\rho \neq \{0\}$ and therefore $\mathfrak{s}^\rho$ contains a nontrivial abelian subalgebra $\mathfrak{u}_\mathfrak{s}$. Thus we get an abelian subalgebra $\mathfrak{u} \oplus \mathfrak{u}_\mathfrak{s}$ of $\mathfrak{g}^\rho$. But this contradicts to the assumption that $\mathfrak{u}$ is maximal abelian in $\mathfrak{g}^\rho$. ∎

Let Γ_* ($* = \mathfrak{t}$ or $\mathfrak{u}$) denote the lattices in $*$ generated by the set

$$\left\{ Y_\alpha = \frac{4\pi i \alpha}{(\alpha, \alpha)} \;\middle|\; \alpha \in \Sigma(\mathfrak{g}_\mathbb{C}, *) \right\}. \tag{5.1}$$

Let G be the connected simply connected Lie group with Lie algebra $\mathfrak{g}$. Then it is known that

$$\{Y \in \mathfrak{t} \mid \exp Y = e\} = \Gamma_\mathfrak{t}.$$

So the following proposition implies that

$$\{Y \in \mathfrak{u} \mid \exp Y = e\} = \Gamma_\mathfrak{u}.$$

Proposition 5.2. $\Gamma_\mathfrak{u} = \Gamma_\mathfrak{t} \cap \mathfrak{u}$.

Proof. Since $\mathfrak{g}$ is semisimple, $\mathfrak{g}$ is written as a direct sum

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$$

of simple Lie algebras $\mathfrak{g}_1, \ldots, \mathfrak{g}_k$. Considering each irreducible component of the ρ-action on $\mathfrak{g}$, we may assume that $\mathfrak{g}_1 = \mathfrak{g}_2 = \cdots = \mathfrak{g}_k$ and that

$$\rho(X_1, \ldots, X_k) = (\rho_1(X_k), X_1, \ldots, X_{k-1})$$

with some automorphism ρ_1 of $\mathfrak{g}_1$. Let $X = (X_1, \ldots, X_k)$ be an element of $\mathfrak{g}^\rho$. Then we have

$$(\rho_1(X_k), X_1, \ldots, X_{k-1}) = (X_1, \ldots, X_k)$$

and therefore $\rho_1(X_k) = X_1 = X_2 = \cdots = X_k \in \mathfrak{g}_1^{\rho_1}$. Hence we have shown that

$$\mathfrak{g}^\rho = \{(X, \ldots, X) \in \mathfrak{g} \mid X \in \mathfrak{g}_1^{\rho_1}\} \cong \mathfrak{g}_1^{\rho_1}.$$

Thus the space $\mathfrak{u}$ is of the form

$$\mathfrak{u} = \{(X, \dots, X) \in \mathfrak{g} \mid X \in \mathfrak{u}_1\}$$

where $\mathfrak{u}_1$ is a maximal abelian subalgebra of $\mathfrak{g}_1^{\rho_1}$. So we have only to consider the case that $\mathfrak{g}$ is simple in the following.

Let $\widetilde{W}(*) = W(*) \ltimes \Gamma_*$ be the group of affine transformations on the spaces $* = \mathfrak{t}$ or $\mathfrak{u}$ generated by reflections

$$w_{\alpha,n} : Y \mapsto w_\alpha(Y) + nY_\alpha = Y - \alpha(Y)\frac{2\alpha}{(\alpha,\alpha)} + nY_\alpha$$

with respect to the hyperplanes $\{Y \in * \mid \alpha(Y) = 2\pi in\}$ for $\alpha \in \Sigma(*)$ and $n \in \mathbb{Z}$. We can define the set $C(*)$ of "Weyl chambers" in $* = \mathfrak{t}, \mathfrak{u}$ consisting of connected components of the set

$$\{Y \in * \mid \alpha(Y) \neq 2\pi in \text{ for all } \alpha \in \Sigma(*) \text{ and } n \in \mathbb{Z}\}.$$

Then it is known that $\widetilde{W}(*)$ acts simply transitively on $C(*)$. Note that every $\Delta_\mathfrak{u} \in C(\mathfrak{u})$ can be written as

$$\Delta_\mathfrak{u} = \Delta_\mathfrak{t} \cap \mathfrak{u}$$

with a unique $\Delta_\mathfrak{t} \in C(\mathfrak{t})$ because there is no root $\alpha \in \Sigma(\mathfrak{t})$ such that $\alpha|_\mathfrak{u} = 0$ by Lemma 5.1.

Put $\widetilde{W}(\mathfrak{t})_\mathfrak{u} = \{w \in \widetilde{W}(\mathfrak{t}) \mid w(\mathfrak{u}) = \mathfrak{u}\}|_\mathfrak{u}$. Then we have only to show that

$$\widetilde{W}(\mathfrak{t})_\mathfrak{u} = \widetilde{W}(\mathfrak{u}) \tag{5.2}$$

because the subgroups consisting of parallel translations in $\widetilde{W}(\mathfrak{t})_\mathfrak{u}$ and $\widetilde{W}(\mathfrak{u})$ are

$$\{e\} \ltimes (\mathfrak{u} \cap \Gamma_\mathfrak{t}) \quad \text{and} \quad \{e\} \ltimes \Gamma_\mathfrak{u},$$

respectively.

We will prove (5.2) by applying the wellknown argument due to I.Satake ([6] Appendix). If $\mathfrak{u} = \mathfrak{t}$, then the assertion is clear. So we may assume that ρ is an outer automorphism of $\mathfrak{g}$. First we will show that $\widetilde{W}(\mathfrak{u}) \subset \widetilde{W}(\mathfrak{t})_\mathfrak{u}$. We have only to show that $w_{\alpha,n} \in \widetilde{W}(\mathfrak{t})_\mathfrak{u}$ for every $\alpha \in \Sigma(\mathfrak{u})$ and n by case-by-case checking. Let β be a root in $\Sigma(\mathfrak{t})$ such that $\alpha = \beta|_\mathfrak{u}$. Note that $(\beta, \rho(\beta)) \leq 0$ if $\rho(\beta) \neq \beta$ because $\beta - \rho(\beta) \notin \Sigma(\mathfrak{t})$ by Lemma 5.1. Let c be the least positive integer such that

$$\rho^c \in \operatorname{Int}(\mathfrak{g}).$$

Then $c = 2$ or 3 since $\mathfrak{g}$ is simple (c.f. (2.1)). Extend $\mathfrak{u}$ to a maximal abelian subalgebra $\mathfrak{t}'$ of $\mathfrak{g}^{\rho^c}$. Then $\mathfrak{t}'$ is maximal abelian in $\mathfrak{g}$ since ρ^c is an inner automorphism. Hence $\mathfrak{t}' = \mathfrak{t}$. Thus we have proved that

$$\rho^c|_\mathfrak{t} = \text{id}.$$

First suppose that $c = 2$. Then we have $|\beta|/|\alpha| = 1$, $\sqrt{2}$ or 2.

If $|\beta|/|\alpha| = 1$, then $w_{\alpha,n} = w_{\beta,n}|_\mathfrak{u} \in \widetilde{W}(\mathfrak{t})_\mathfrak{u}$.

If $|\beta|/|\alpha| = \sqrt{2}$, then $w_{\alpha,n} = w_{\beta,n} w_{\rho(\beta),n}|_\mathfrak{u} \in \widetilde{W}(\mathfrak{t})_\mathfrak{u}$ since $(\beta, \rho(\beta)) = 0$.

If $|\beta|/|\alpha| = 2$, then $\gamma = \beta + \rho(\beta) \in \Sigma(\mathfrak{t})$ and therefore $w_{\alpha,n} = w_{\gamma,2n}|_{\mathfrak{u}} \in \widetilde{W}(\mathfrak{t})_{\mathfrak{u}}$.

Next consider the case that $c = 3$. If $\rho(\beta) = \beta$, then $w_{\alpha,n} = w_{\beta,n}|_{\mathfrak{u}} \in \widetilde{W}(\mathfrak{t})_{\mathfrak{u}}$. On the other hand, if $\rho(\beta) \neq \beta$, then $\angle\beta O\rho(\beta) = \pi/2,\ 2\pi/3$ or π since $(\beta, \rho(\beta)) \leq 0$.

If $\angle\beta O\rho(\beta) = \pi$, then $\rho(\beta) = -\beta$. Since $\rho^3|_{\mathfrak{t}} = \mathrm{id.}$, we have $\beta = \rho^3(\beta) = -\beta$, a contradiction.

If $\angle\beta O\rho(\beta) = 2\pi/3$, then we have

$$\angle\rho(\beta)O\rho^2(\beta) = \angle\rho^2(\beta)O\beta = \frac{2\pi}{3}.$$

Hence we have

$$\beta + \rho(\beta) + \rho^2(\beta) = 0$$

which implies $\alpha = \beta|_{\mathfrak{u}} = (1/3)(\beta + \rho(\beta) + \rho^2(\beta))|_{\mathfrak{u}} = 0$, a contradiction.

So we have $\angle\beta O\rho(\beta) = \angle\rho(\beta)O\rho^2(\beta) = \angle\rho^2(\beta)O\beta = \pi/2$. Then it is clear that

$$w_{\alpha,n} = w_{\beta,n}w_{\rho(\beta),n}w_{\rho^2(\beta),n}|_{\mathfrak{u}} \in \widetilde{W}(\mathfrak{t})_{\mathfrak{u}}.$$

Conversely, we will show that $\widetilde{W}(\mathfrak{t})_{\mathfrak{u}} \subset \widetilde{W}(\mathfrak{u})$. Let w be an element of $\widetilde{W}(\mathfrak{t})^{\mathfrak{u}} = \{w \in \widetilde{W}(\mathfrak{t}) \mid w(\mathfrak{u}) = \mathfrak{u}\}$ and $\Delta_{\mathfrak{u}} = \Delta_{\mathfrak{t}} \cap \mathfrak{u}$ an element of $C(\mathfrak{u})$ ($\Delta_{\mathfrak{t}} \in C(\mathfrak{t})$). Then $w\Delta_{\mathfrak{u}} = w\Delta_{\mathfrak{t}} \cap \mathfrak{u} \in C(\mathfrak{u})$ and there exists a $w_0 \in \widetilde{W}(\mathfrak{u})$ such that $w_0^{-1}w\Delta_{\mathfrak{u}} = \Delta_{\mathfrak{u}}$. Since $\widetilde{W}(\mathfrak{u}) \subset \widetilde{W}(\mathfrak{t})_{\mathfrak{u}}$, we can choose $\widetilde{w}_0 \in \widetilde{W}(\mathfrak{t})^{\mathfrak{u}}$ such that $\widetilde{w}_0|_{\mathfrak{u}} = w_0$. Then we have $\widetilde{w}_0^{-1}w\Delta_{\mathfrak{u}} = \Delta_{\mathfrak{u}}$ and therefore $\widetilde{w}_0^{-1}w\Delta_{\mathfrak{t}} = \Delta_{\mathfrak{t}}$. This implies $w = \widetilde{w}_0$. Hence

$$w|_{\mathfrak{u}} = \widetilde{w}_0|_{\mathfrak{u}} = w_0 \in \widetilde{W}(\mathfrak{u}).$$ ∎

Let $\mathfrak{g}, \sigma, \tau$ and $\mathfrak{a}$ be as in Section 1. Let $\mathfrak{u}$ be a maximal abelian subalgebra of $\mathfrak{g}^{\sigma\tau}$ containing $\mathfrak{a}$. Then $\Gamma_{\mathfrak{a}}$ is also defined by (5.1) for $* = \mathfrak{a}$.

Proposition 5.3. $\Gamma_{\mathfrak{a}} = \Gamma_{\mathfrak{u}} \cap \mathfrak{a}$.

Proof. As in the proof of Proposition 5.2, we can also define

$$\widetilde{W}(\mathfrak{a}) = W(\mathfrak{a}) \ltimes \Gamma_{\mathfrak{a}}$$

and $C(\mathfrak{a})$ in the same way. Note that for every $\Delta_{\mathfrak{a}} \in C(\mathfrak{a})$, we can write

$$\overline{\Delta_{\mathfrak{a}}} = \overline{\Delta_{\mathfrak{u}}} \cap \mathfrak{a}$$

with some $\Delta_{\mathfrak{u}} \in C(\mathfrak{u})$.

Put $\widetilde{W}(\mathfrak{u})_{\mathfrak{a}} = \{w \in \widetilde{W}(\mathfrak{u}) \mid w(\mathfrak{a}) = \mathfrak{a}\}|_{\mathfrak{a}}$. Then we have only to show that

$$\widetilde{W}(\mathfrak{u})_{\mathfrak{a}} = \widetilde{W}(\mathfrak{a})$$

as in the proof of Proposition 5.2.

The following lemma will be proved later.

Lemma 5.4. *Let α be a root in $\Sigma(\mathfrak{u})$ such that $\sigma(\alpha) \neq -\alpha$. Then $(\alpha, \sigma(\alpha)) \geq 0$.*

Let $\alpha \in \Sigma(\mathfrak{a})$ and let β be a root in $\Sigma(\mathfrak{u})$ such that $\beta|_{\mathfrak{a}} = \alpha$. Then by Lemma 5.4, we have $|\beta|/|\alpha| = 1$, $\sqrt{2}$ or 2. Hence we can prove $\widetilde{W}(\mathfrak{a}) \subset \widetilde{W}(\mathfrak{u})_{\mathfrak{a}}$ by the same argument as in the proof of (5.2).

Next we will show that $\widetilde{W}(\mathfrak{u})_{\mathfrak{a}} \subset \widetilde{W}(\mathfrak{a})$. Let w be an element of $\widetilde{W}(\mathfrak{u})^{\mathfrak{a}} = \{w \in \widetilde{W}(\mathfrak{u}) \mid w(\mathfrak{a}) = \mathfrak{a}\}$. Let $\Delta_{\mathfrak{a}}$ be an element of $C(\mathfrak{a})$ and $\Delta_{\mathfrak{u}}$ an element of $C(\mathfrak{u})$ such that

$$\overline{\Delta_{\mathfrak{a}}} = \overline{\Delta_{\mathfrak{u}}} \cap \mathfrak{a}.$$

Then we have $w\overline{\Delta_{\mathfrak{a}}} = w\overline{\Delta_{\mathfrak{u}}} \cap \mathfrak{a}$ and there exists a $w_0 \in \widetilde{W}(\mathfrak{a})$ such that $w_0^{-1} w \Delta_{\mathfrak{a}} = \Delta_{\mathfrak{a}}$. Since $\widetilde{W}(\mathfrak{a}) \subset \widetilde{W}(\mathfrak{u})_{\mathfrak{a}}$, we can choose $\widetilde{w}_0 \in \widetilde{W}(\mathfrak{u})^{\mathfrak{a}}$ such that $\widetilde{w}_0|_{\mathfrak{a}} = w_0$. Let Y be an element of $\Delta_{\mathfrak{a}}$. Then we have

$$\widetilde{w}_0^{-1} w(Y) = Y$$

since $\overline{\Delta_{\mathfrak{u}}}$ is a complete set of representatives of $\widetilde{W}(\mathfrak{u})$-orbits on $\mathfrak{u}$. Hence we have $\widetilde{w}_0^{-1} w|_{\mathfrak{a}} = \mathrm{id}.$ and therefore

$$w|_{\mathfrak{a}} = \widetilde{w}_0|_{\mathfrak{a}} = w_0 \in \widetilde{W}(\mathfrak{a}). \qquad \blacksquare$$

Proof of Lemma 5.4. Let α be a root in $\Sigma(\mathfrak{u})$ such that $\sigma(\alpha) \neq -\alpha$ and X_α a nonzero element of $\mathfrak{g}_{\mathbb{C}}(\mathfrak{u}, \alpha)$ such that

$$\sigma\tau X_\alpha = \lambda X_\alpha$$

with some $\lambda \in U(1)$. Suppose that $(\alpha, \sigma(\alpha)) < 0$. Then

$$X = [X_\alpha, \sigma X_\alpha]$$

is a nonzero element of $\mathfrak{g}_{\mathbb{C}}(\mathfrak{u}, \beta)$ where $\beta = \alpha + \sigma(\alpha) \neq 0$. We have

$$\sigma X = [\sigma X_\alpha, X_\alpha] = -[X_\alpha, \sigma X_\alpha] = -X$$

and

$$\sigma\tau(\sigma X_\alpha) = \sigma(\sigma\tau)^{-1} X_\alpha = \lambda^{-1} \sigma X_\alpha.$$

Hence we have

$$X \in \mathfrak{g}_{\mathbb{C}}(\mathfrak{u}, \beta) \cap \mathfrak{g}_{\mathbb{C}}^{-\sigma} \cap \mathfrak{g}_{\mathbb{C}}^{\sigma\tau} = \mathfrak{g}_{\mathbb{C}}(\mathfrak{u}, \beta) \cap \mathfrak{g}_{\mathbb{C}}^{-\sigma} \cap \mathfrak{g}_{\mathbb{C}}^{-\tau}.$$

On the other hand, we have $[X, \mathfrak{a}] = \{0\}$ since $\beta|_{\mathfrak{a}} = 0$. Thus we have a contradiction to the assumption that $\mathfrak{a}$ is maximal abelian in $\mathfrak{g}^{-\sigma} \cap \mathfrak{g}^{-\tau}$. $\blacksquare$

As a corollary of Proposition 5.2 and Proposition 5.3, we have:

Corollary . *If G is simply connected, then*

$$\{Y \in \mathfrak{a} \mid \exp Y = e\} = \Gamma_{\mathfrak{a}}.$$

References

[1] Berger, M., *Les espace symétriques non compacts*, Ann. Sci. École Norm. Sup. **74** (1957), 85–177.

[2] Borel, A., *Sous-groupes commutatifs et torsion des groupes de Lie compacts connexes*, Tôhoku Math. J. **13** (1961), 216–240.

[3] Loos, O., Symmetric spaces II, Benjamin, New York, 1969.

[4] Matsuki, T., *Double coset decompositions of reductive Lie groups arising from two involutions.*, Journal of Algebra **197** (1997), 49–91.

[5] Oshima, T., and J. Sekiguchi, *The restricted root system of a semisimple symmetric pair*, Advanced Studies in Pure Math. **4** (1984), 433–497.

[6] Satake, I., *On representations and compactifications of a symmetric Riemannian spaces*, Ann. of Math. **71** (1960), 77–110.

[7] de Siebenthal, J., *Sur les groupes de Lie compacts non connexes*, Comm. Math. Helv **31** (1956), 41–89.

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