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A Note on Observable Subgroups of Linear Algebraic Groups and a Theorem of Chevalley

Nazih Nahlus

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Abstract. Let H be an algebraic subgroup of a linear algebraic group G over an algebraically closed field K. We show that H is observable in G if and only if there exists a finite-dimensional rational G-module V and an element v of V such that H is the isotropy subgroup of v as well as the isotropy subgroup of the line Kv.

Moreover, we give a similar result in the case where H contains a normal algebraic subgroup A which is observable in G. In this case, we deduce that H is observable in G whenever H/A has non non-trivial rational characters. We also give an example from complex analytic groups.
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Let K be a fixed algebraically closed field of arbitrary characteristic. Let H be an algebraic subgroup of a linear algebraic group G over K. Then H is called *observable* in G if every finite-dimensional rational H-module is a sub H-module of some finite-dimensional rational G-module. There are many characterizations for the observability of H in G. For example, H is observable in G if and only if H is the isotropy subgroup in G of an element in some finite-dimensional rational G-module. Moreover, H is observable in G if and only if G/H is a quasi-affine variety [2], [5, Thm. 2.1].

Now suppose that H is observable in G. Then, on one hand, H is the isotropy subgroup of an element v in some finite-dimensional rational G-module V. On the other hand, by a theorem of Chevalley , H, like every algebraic subgroup of G, is the isotropy subgroup of a line in some finite-dimensional rational G-module V'. So it would be of interest to find a finite-dimensional rational G-module V and an element v of V such that H is the isotropy subgroup of v as well as the isotropy subgroup of the line Kv. This property is contained in Theorem 1 below. Moreover, Theorem 1 provides a generalization of this property to the case where H contains a normal algebraic subgroup A which is observable in G (for example, one may take A to be $nil(H)$ or $H \cap radG$). In this case, Corollary 2(2) shows that H is observable in G whenever $X(H/A) = 1$ where X stands for "the group of rational characters".

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Theorem 1. *Let H be an algebraic subgroup of a linear algebraic group G over K. Then H is observable in G (if) and only if there exists a finite-dimensional rational G-module V and an element v of V such that H is the isotropy subgroup of v and the line Kv. More generally, let A be a normal algebraic subgroup of H such that A is observable in G (for example, A may be taken to be $nil(H)$ or $H \cap radG$). Then there exists a finite-dimensional rational G-module V and an element v of V such that H is the isotropy subgroup of the line Kv and A fixes v.*

Proof. By Chevalley's theorem [1, 5.1], [3, 11.2], there is a finite-dimensional rational G-module V and an element v of V such that H is the stabilizer in G of Kv. Hence there exists $f \in X(H)$ such that $g.v = f(g)v$ for every element $g \in H$. Since A is observable in G, its dual module on Kv can be imbedded as a sub A-module of a finite-dimensional rational G-module W. So there is a non-zero element $w_0 \in W$ such that $a.w_0 = f(a^{-1})w_0$ for all $a \in A$. Let $W_0 = \{w \in W : a.w = f(a^{-1})w\}$ for all a in A. Then W_0 is H-invariant because, if $w \in W_0$, then $a.(h.w) = h.(h^{-1}ah).w = h.f(ha^{-1}h^{-1})w = f(a^{-1})h.w$ since A is normal in H and $f \in X(H)$. Let $m = dim(W_0)$ and let $V^+ = V \otimes .... \otimes V$ (m-times) $\otimes \bigwedge^m(W)$ which is naturally a G-module. Let $w_1, ..., w_m$ be a basis of W_0, and let $v^+ = v \otimes .... \otimes v \otimes (w_1 \wedge .. \wedge w_m)$. Now we show that the pair (V^+, v^+) has the desired properties. Since W_0 is H-invariant, it follows that $h.(w_1 \wedge .. \wedge w_m) \in K(w_1 \wedge .. \wedge w_m)$, so $h.v^+ \in V^+$. It follows that there exists $k \in X(H)$ such that $h.v^+ = k(h)v^+$ for every element h of H, and that H is the isotropy subgroup of the line Kv^+. Moreover, for every $a \in A$, $a.v^+ = f(a)^m f(a^{-1})^m.v^+ = v^+$, so A fixes v^+.

Finally, we note that $H \cap radG$ is observable in G by transitivity since every algebraic subgroup of a solvable algebraic group X (say) is observable in X [5, Cor. 2.5] and since $rad(G)$ is observable in G for being normal in G. We also note that $nil(H)$ is observable in G since every nilpotent algebraic subgroup of G is observable in G [2, Cor. 2], (see also Corollary 2(4) below). This proves Theorem 1. ∎

Corollary 2. *Let G, H and A be as in Theorem 1. Then*

(1) (cf. [4, Thm. 4]) *there is a rational character on H whose kernel is observable in G and contains A.*
(2) *If $X(H/A) = 1$, then H is observable in G.*
(3) [5, Thm. 2.7] *If $rad(H)$ is observable in G, then so is H.*
(4) [5, Cor. 2.9] *If $rad(H)$ is nilpotent, then H is observable in G.*

Proof. (1) and (2) are evident. To see (3) and (4), we may assume that G and H are connected [5, Cor. 2.2]. If rad(H) is observable in G, then H is observable in G by part (2) since $H/radH$ is semisimple. To see (4), we may assume that H is solvable by part (3) and thus H is nilpotent. Hence $H = U \times T$ where U is the unipotent radical of H and T is the maximal torus of H. But every torus of G is observable in G by [5, Cor. 2.4] or by the proof of [2, Thm. 2(1)]. Hence H is observable in G by part(2).

Remark 3. Let H be an algebraic subgroup of G over K. If $H = A \times T$ where A is an observable subgroup of G and T is a central torus in G, then H may fail to be observable in G. In particular, if A and B are normal algebraic subgroups of H such that A and B are observable in G, then there may not exist a finite-dimensional rational G-module V and an elemrent v of V such that H is the isotropy subgroup of the line Kv and AB fixes v.

To see this, consider $G = GL(n,K)$ and let m be an odd integer such that $m < n$. Write each matrix X in $G = GL(n,K)$ as $X = \begin{pmatrix} X_{(11)} & X_{(12)} \\ X_{(21)} & X_{(22)} \end{pmatrix}$ which is a 2×2 matrix of block matrices such that $X_{(11)}$ is an $m \times m$ matrix. Now consider the parabolic subgroup $H_m = \{X \in G,\ X_{(21)} = 0\}$, let

$$A_m = \{X \in G,\ X_{(11)} \in SL(m,K) \text{ and } X_{(21)} = 0\},$$

and let T be the subgroup of non-zero multiples of the identity $n \times n$ matrix. Then $H_m = A_m \times T$ and H_m is not observable in G since G/H_m is a complete variety. But T is observable in G for being a torus and A_m is observable in G since A_m is the isotropy subgroup of $e_1 \wedge .... \wedge e_m$ in $\wedge^m(V)$ where $V = K^n$ is the natural module for $G = GL(n,K)$ and $\{e_1, ..., e_n\}$ is the standard basis for $V = K^n$.

Remark 4. In $GL(n,C)$ where C is the field of complex numbers, the complex analytic subgroups H_m, A_m, and T defined in the above example, are universally algebraic in the sense that all their finite-dimensional analytic representations are rational. Moreover, $H_m = A_m \times T$ and A_m is observable in G.

To see this, we recall the known fact that a analytic group X is universally algebraic if and only if X is generated by $[X,X]$ and all its reductive subgroups [6, p. 623]. But this is true for every parabolic subgroup P of a reductive algebraic group G (even over K) since $[B,B] = B_u$ for every Borel subgroup B of G [3, Ex. 13, p. 162] (where B_u is the unipotent radical of B), $P_u \subset B_u$ [3, Ex. 3, p. 146] if P contains the Borel subgroup B, and P has a Levi decomposition [3, Thm. 30.2], [1]. Hence H_m and A_m are universally algebraic since $H_m = A_m \times T$. Moreover, A_m is observable in $GL(n,C)$ as shown in the above example.

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Nazih Nahlus
Department of Mathematics
American University of Beirut
c/o New York Office
850 Third Ave. 18th floor
New York, New York 10022-6297
nahlus@aub.edu.lb

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