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Poisson Kernels and Pluriharmonic $\mathbf{H}^2$-Functions on Homogeneous Siegel Domains

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Abstract. In the paper we prove that a real function F defined on a homogeneous not necessarily symmetric Siegel domain satisfying an $\mathcal{H}^2$ condition is pluriharmonic if and only if $\mathbf{H}F = 0$, $\mathcal{L}F = 0$, $LF = 0$, where $\mathbf{H}$, $\mathcal{L}$, L are second order differential operators. This generalizes the result of [3] where symmetric domains were considered. Our approach to study non-symmetric case is based on T-algebras introduced by Vinberg in [11].

1. Introduction

This paper treats pluriharmonic functions on homogeneous Siegel domains. These are the functions locally characterized by the equations

$$\partial_{z_j}\partial_{\overline{z}_k}F = 0 \quad \text{for} \quad j,k = 1,\dots,n$$

n being the dimension of the underlying complex space. There are other local characterizations, one of them being Forelli's theorem [5].

Here we are mostly interested in a global question i.e., we impose a growth condition, and we look for a characterization of pluriharmonic functions among the ones satisfying it. Similar problems have already been studied by various authors (for recent results see e.g. [1], [3], [2], [7], [8]), all of them being interested in symmetric domains while here we do not need symmetry at all.

Let Ω be an irreducible homogeneous cone, and let $\mathcal{D}$ be a corresponding homogeneous Siegel domain. We identify $\mathcal{D}$ with a solvable Lie group S that acts simply transitively on $\mathcal{D}$ as a group of biholomorphisms. We study S–invariant real elliptic degenerate second order operators on $\mathcal{D}$, which annihilate holomorphic functions and, consequently, their real and imaginary part. Such operators will be called *admissible*. The particular interest in restricting our attention to second order degenerate elliptic operators is caused by the fact that for such operators there is a very well understood potential theory. Theory of

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bounded functions harmonic with respect to an S–invariant operator L satisfying Hörmander condition was studied in [2], [4] and was based on an earlier work of H. Furstenberg, Y. Guivarc'h and A. Raugi. The basic result of this theory we use here is a description of bounded L–harmonic functions as Poisson integrals on a nilpotent subgroup $N(L)$ of S. For an admissible L on a Siegel domain the boundary $N(L)$ always contains a group $N(\Phi)$ which acts transitively on the Shilov boundary. In our case L is an arbitrary elliptic admissible operator and then we choose two other operators: $\mathbf{H}$ and $\mathcal{L}$ in the way that $N(L+\mathbf{H}) = N(\Phi)$. Thus

$$F(s) = \int_{N(\Phi)} f(s \circ v) P(v) dv$$

where P is the Poisson kernel corresponding to $\mathbf{L} = L + \mathbf{H}$, and $v \mapsto s \circ v$ is the action of S on the Bergman–Shilov boundary $N(\Phi)$. $\mathcal{L}$ is closely related to the tangential holomorphic structure of the Siegel domain of type two and it does not appear in the tube case. *We show that three operators L, $\mathbf{H}$ and $\mathcal{L}$ are sufficient to characterize pluriharmonic functions F with ($\mathcal{H}^2$) growth condition* (Theorem 5.1)

$$\sup_{z \in \mathcal{D}} \int_{N(\Phi)} |F(w \cdot z)|^2 dw < \infty. \tag{$\mathcal{H}^2$}$$

For symmetric domains, this theorem was proved in [3]. Our strategy is to prove that the support of the integrated representation U_f^λ is included in $\overline{\Omega} \cup -\overline{\Omega}$. For this we use the operator $\mathbf{H}$ which is basically the Laplace–Beltrami operator on a product of upper-planes. The proof exploits both the algebra of the underlying cone, and the Fourier analysis on $N(\Phi)$. The latter is pretty much the same as in [3]. Our main contribution here is in the algebraic part. (Section 2.)

Let $V = \bigoplus_{1 \le i \le j r} \mathcal{X}_{ij}$ be the normal decomposition of the clan V, and $c_1, \ldots, c_r$ the corresponding system of simple idempotents. The authors of [3] used heavily the fact that for a symmetric cone $\mathcal{X}_{ij}$'s do not vanish. In this paper we have been able to overcome this difficulty by showing in fact that when Ω is irreducible there are enough of non-vanishing $\mathcal{X}_{ij}$'s.

Let S_0 be a triangular group acting simply transitively on Ω. To push the argument through we study carefully the action of the S_0 on V. Modulo a set of Lebesgue measure 0, V is the sum of the open orbits $\mathcal{O}_\eta$ of S_0 where

$$\eta = \sum_{j=1}^{r} \eta_j c_j$$

for $\eta \in \{-1, 1\}^r$. The action of S_0 on any of them is simply transitive and identifies $\mathcal{O}_\eta$ with S_0 via the diffeomorphism (Theorem 2.9)

$$s \mapsto s \circ \sum_{j=1}^{r} \eta_j c_j.$$

The diagonalization of any non-degenerate element of V (Proposition 2.7) not only allows us to prove the main theorem, but also that the Fourier transform of P is smooth on the open orbits $\mathcal{O}_\eta$ (Theorem 4.5).

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2. Homogeneous cones

Let Ω be a homogeneous cone in a vector space V. We are going to describe an algebraic structure of V. First, we state some definitions and facts.

Definition 2.1. A matrix algebra of rank r is an algebra $\mathcal{U}$ bigraded by subspaces $\mathcal{U}_{ij}$, $i,j=1,\dots,r$, such that $\mathcal{U}_{ij}\mathcal{U}_{jk}\subset\mathcal{U}_{ik}$, and for $j\neq l$, $\mathcal{U}_{ij}\mathcal{U}_{lk}=0$.

Definition 2.2. An involution of a matrix algebra $\mathcal{U}$ is a linear mapping * of $\mathcal{U}$ onto itself that satisfies the following conditions:

i. $x^{\star\star}=x$;

ii. $(xy)^\star=y^\star x^\star$;

iii. $\mathcal{U}_{ij}^\star=\mathcal{U}_{ji}$;

for all $x,y\in\mathcal{U}$.

Let $\mathcal{U}$ be an algebra with involution *. We define the subspace of Hermitian matrices in $\mathcal{U}$

$$\mathcal{X}=\{x\in\mathcal{U}|\ x^\star=x\}$$

and

$$\mathcal{T}=\bigoplus_{1\le i\le j\le r}\mathcal{U}_{ij},$$

the subalgebra of $\mathcal{U}$ consisting in upper triangular matrices.

Let $\mathcal{U}_{ii}=\mathbb{R}c_i$. We denote by ρ the unique isomorphism of $\mathcal{U}_{ii}$ onto the algebra of real numbers $\mathbb{R}$. For a matrix $x\in\mathcal{U}$,

$$x=\sum_{i=1}^r x_{ii}+\sum_{i\neq j}x_{ij},$$

we define its trace tr as follows

$$\operatorname{tr}x=\sum_{i=1}^r n_i\rho(x_{ii}),\tag{1}$$

$$n_i=1+\frac{1}{2}\sum_{\substack{j=1\\ j\neq i}}^r\dim\ \mathcal{U}_{ij}.\tag{2}$$

We need the following notation

$$[xy]=xy-yx,$$
$$[xyz]=x(yz)-(xy)z.$$

Definition 2.3. A matrix algebra $\mathcal{U}$ with an involution $\star$ is called a T–algebra if the following conditions are satisfied:

i. for $i = 1, \ldots, r$, $\mathcal{U}_{ii} = \mathbb{R}c_i$;

ii. for $x_{ij} \in \mathcal{U}_{ij}$, $c_i x_{ij} = x_{ij} c_j = x_{ij}$;

iii. $\operatorname{tr}[xy] = 0$;

iv. $\operatorname{tr}[xyz] = 0$;

v. if $x \neq 0$, then $\operatorname{tr} xx^\star > 0$;

vi. for all $t, u, w \in \mathcal{T}$, $[tuw] = 0$;

vii. for all $t, u \in \mathcal{T}$, $[tuu^\star] = 0$.

For each matrix $x \in \mathcal{U}$, we put

$$\overline{x} = \frac{1}{2}\sum_{i=1}^{r} x_{ii} + \sum_{i<j}^{r} x_{ij},$$
$$\underline{x} = \frac{1}{2}\sum_{i=1}^{r} x_{ii} + \sum_{j<i}^{r} x_{ij},$$

and define a bilinear operator $\triangle$ by the formula

$$x \triangle y = \overline{x} y + y \underline{x}. \tag{3}$$

Let

$$S_0 = \{t \in \mathcal{T} |\ t_{ii} > 0,\ i = 1, \ldots, r\}.$$

The product in S_0 is associative by property vi. Thus S_0 is open in $\mathcal{T}$ and it is a connected Lie group. Its Lie algebra $\mathcal{S}_0$ can be identified with $\mathcal{T}$ with the bracket

$$[X, Y] = [XY].$$

Then we have $\mathcal{S}_0 = \mathcal{N}_0 \oplus \mathcal{A}$ where $\mathcal{N}_0 = \bigoplus_{1 \leq i < j \leq r} \mathcal{U}_{ij}$ and $\mathcal{A} = \bigoplus_{i=1}^{r} \mathcal{U}_{ii}$. Let $N_0 = \exp \mathcal{N}_0$, $A = \exp \mathcal{A}$.

Definition 2.4. An algebra $\mathcal{L}$ with linear form s and multiplication $\triangle$ is called a clan if the following conditions hold:

i. the operator $L(x)$ defined by $L(x)y = x \triangle y$ has only real eigenvalues;

ii. $[L(x), L(y)] = L(x \triangle y - y \triangle x)$;

iii. $s(x \triangle y) = s(y \triangle x)$;

iv. if $x \neq 0$, then $s(x \triangle x) > 0$;

for all $x, y \in \mathcal{L}$.

For every clan $\mathcal{L}$ with a unit element e there exists a normal decomposition (see [11]). This means that $\mathcal{L}$ has a direct sum decomposition

$$\mathcal{L} = \bigoplus_{1\leq i\leq j\leq r} \mathcal{L}_{ij}$$

such that the subspaces $\mathcal{L}_{ij}$ are mutually orthogonal with respect to the scalar product

$$(x|y) = s(x\triangle y) \quad \text{for} \quad x, y \in \mathcal{L},$$

and the following properties hold:

i. for each $1 \leq i \leq r$ there exists an idempotent c_i such that $\mathcal{L}_{ii} = \mathbb{R}c_i$;

ii. for $1 \leq i \leq j \leq k$,

$$\mathcal{L}_{ij}\triangle\mathcal{L}_{jk} \subset \mathcal{L}_{ik}$$

and

$$\mathcal{L}_{jk}\triangle\mathcal{L}_{ik} + \mathcal{L}_{ik}\triangle\mathcal{L}_{jk} \subset \mathcal{L}_{ij};$$

iii. for $i \leq j$, $k \leq l$, if $j \neq k$ and $j \neq l$, then

$$\mathcal{L}_{ij}\triangle\mathcal{L}_{kl} = 0;$$

iv. for $i < j$ and $x_{ij} \in \mathcal{L}_{ij}$,

$$\begin{aligned} c_i\triangle x_{ij} &= \frac{1}{2}x_{ij}, \\ x_{ij}\triangle c_i &= 0, \\ c_j\triangle x_{ij} &= \frac{1}{2}x_{ij}, \\ x_{ij}\triangle c_j &= x_{ij}. \end{aligned}$$

The number r is invariant under isomorphism and is called the rank of the clan $\mathcal{L}$ or the rank of the associated cone $\Omega(\mathcal{L})$.

Let us define the subspaces $\mathcal{X}_{ij}$ for $1 \leq i < j \leq r$ by

$$\begin{aligned} \mathcal{X}_{ij} &= \mathcal{X} \cap (\mathcal{U}_{ij} + \mathcal{U}_{ji}), \\ \mathcal{X}_{ii} &= \mathcal{U}_{ii}. \end{aligned}$$

Then we can state the following

Theorem 2.5. ([11]) *The subspace of Hermitian matrices $\mathcal{X}$ in a T-algebra $\mathcal{U}$ with the multiplication given by the formula (3) and a linear form* tr *defined by (1) is a clan with a unit element. Moreover, the decomposition $\mathcal{X} = \bigoplus_{1\leq i\leq j\leq r} \mathcal{X}_{ij}$ is a normal decomposition. Conversely, let $\mathcal{L}$ be a clan with a unit element. Then there is a unique T-algebra $\mathcal{U}$ such that $\mathcal{L}$ is isomorphic to the clan $\mathcal{X}$.*

Let $\Omega(\mathcal{X})$ be the homogeneous cone associated with the clan $\mathcal{X}$. Then

$$\Omega(\mathcal{X}) = \{ss^\star |\ s \in S_0\}$$

and the mapping $s \mapsto ss^\star$ is one-to-one. The group S_0 acts simply transitively on $\Omega(\mathcal{X})$ by

$$t \circ (ss^\star) = (ts)(s^\star t^\star). \tag{4}$$

The transformation (4) corresponding to t is the restriction to $\Omega(\mathcal{X})$ of a linear transformation $\pi(t) \in \mathrm{GL}(\mathcal{X})$. For $t = \exp y$, $y \in \mathcal{T}$, the differential $d\pi(y)$ of $\pi(t)$ is given by

$$d\pi(y)x = yx + xy^\star = L(y + y^\star)x$$

for $x \in \mathcal{X}$.

Let G be the identity component of the group $G(\Omega)$ of all transformations in $\mathrm{GL}(\mathcal{X})$ which leave $\Omega(\mathcal{X})$ invariant. Then S_0 is a maximal triangular subgroup of G.

Let $\Omega'(\mathcal{X})$ denote the open dual homogeneous cone defined by

$$\Omega'(\mathcal{X}) = \{x' \in \mathcal{X} |\ (x|x') > 0,\ \forall x \in \overline{\Omega(\mathcal{X})} - \{0\}\}.$$

It was proved in [11] that

$$\Omega'(\mathcal{X}) = \Omega^\star(\mathcal{X})$$

and the group $S_0^\star$ acts simply transitively on $\Omega'(\mathcal{X})$. Thus

$$S_0' = S_0^\star.$$

Here $\Omega^\star(\mathcal{X})$ and $S_0^\star$ denote the images under the involution * of $\Omega(\mathcal{X})$ and S_0, respectively. We shall denote the open dual cone to Ω by $\Omega^\star$.

In the T–algebra $\mathcal{U}$ we consider the subspaces

$$\mathcal{X}^k = \mathcal{X} \cap \bigoplus_{i,j=1}^{k} \mathcal{U}_{ij}$$

for $k = 1, \ldots, r$. With every element $x \in \mathcal{X}$,

$$x = \sum_{i=1}^{r} x_{ii} + \sum_{i<j}^{r} x_{ij} + \sum_{i<j}^{r} x_{ij}^\star,$$

we associate a sequence of matrices $x^k \in \mathcal{X}^k$, $k = 1, \ldots, r$, as follows

$$x^r = x, \tag{5}$$

$$x^{k-1} = \sum_{i,j=1}^{k-1} \left(\rho(x_{kk}^k)x_{ij}^k - x_{ik}^k x_{kj}^k\right). \tag{6}$$

We put

$$p_k(x) = \rho(x_{kk}^k) \quad \text{for} \quad k = 1, \ldots, r.$$

We define

$$J = \{x \in \mathcal{X} |\ p_k(x) \neq 0,\ k = 1, \ldots, r\}.$$

Since p_k are non-zero polynomials, the set J^c is closed in $\mathcal{X}$ and has measure 0 in $\mathcal{X}$.

The following Lemma 2.6 and Proposition 2.7 are based on [11] Lemma 3 and Proposition 2.

Lemma 2.6. *For $a_1, \dots, a_r \in \mathbb{R}$, we put $\mathbf{a} = \sum_{i=1}^r a_i c_i$. Let*

$$x = t\left(\sum_{i=1}^r a_i c_i\right) t^\star$$

for $t \in S_0$. Then

$$x_{ij}^k = \left(\prod_{s=k+1}^r p_s(x)\right) \sum_{p=1}^k a_p t_{ip} t_{jp}^\star \tag{7}$$

for $i, j = 1, \dots, k$ and $k = 1, \dots, r$.

Proof. Since for every $k = 1, \dots, r$ and $1 \le i, j \le k$

$$(a_k t_{ik} c_k) t_{jk}^\star = t_{ik}(a_k c_k t_{jk}^\star),$$

we have $[t\mathbf{a}t^\star] = 0$.

The proof of the formula (7) is inductive. For $k = r$ we simply get

$$x_{ij}^r = \sum_{p=1}^k t_{ip} a_p t_{jp}^\star.$$

Let us assume that (7) holds for some k, $1 \le k \le r$. Then

$$x_{ik}^k = \left(\prod_{s=k+1}^r p_s(x)\right) \sum_{p=1}^k a_p t_{ip} t_{kp}^\star = \left(\prod_{s=k+1}^r p_s(x)\right) a_k \rho(t_{kk}) t_{ik},$$

$$x_{kj}^k = \left(\prod_{s=k+1}^r p_s(x)\right) \sum_{p=1}^k a_p t_{kp} t_{jp}^\star = \left(\prod_{s=k+1}^r p_s(x)\right) a_k \rho(t_{kk}) t_{jk}^\star,$$

and

$$\rho(x_{kk}^k) = \left(\prod_{s=k+1}^r p_s(x)\right) \rho\left(\sum_{p=1}^k a_p t_{kp} t_{kp}^\star\right) = \left(\prod_{s=k+1}^r p_s(x)\right) \rho(t_{kk})^2 a_k.$$

For $i, j = 1, \dots, k-1$, we have

$$\begin{aligned} x_{ij}^{k-1} &= \rho(x_{kk}^k) x_{ij}^k - x_{ik}^k x_{kj}^k \\ &= \left(\prod_{s=k}^r p_s(x)\right) \sum_{p=1}^k a_p t_{ip} t_{jp}^\star - \left(\prod_{s=k+1}^r p_s(x)\right)^2 \rho(t_{kk})^2 a_k^2 t_{ik} t_{jk}^\star \\ &= \left(\prod_{s=k}^r p_s(x)\right) \left(\sum_{p=1}^k a_p t_{ip} t_{jp}^\star - a_k t_{ik} t_{jk}^\star\right) = \left(\prod_{s=k}^r p_s(x)\right) \sum_{p=1}^{k-1} a_p t_{ip} t_{jp}^\star. \end{aligned}$$

which finishes the proof. ∎

Proposition 2.7. *If $x \in \mathcal{X}$, then the element x belongs to J if and only if there are $t \in S_0$ and $\eta \in \{-1,1\}^r$ such that*

$$x = t\left(\sum_{i=1}^{r} \eta_i c_i\right) t^\star. \tag{8}$$

Moreover, the operator t and the sequence η are unique.

Proof. First, we show that every x of the form (8) belongs to J. Let

$$x = t\left(\sum_{i=1}^{r} \eta_i c_i\right) t^\star$$

for $t \in S_0$ and $\eta \in \{-1,1\}^r$. By Lemma 2.6,

$$p_i(x) = \left(\prod_{s=i+1}^{r} p_s(x)\right) \rho(t_{ii})^2 \eta_i \neq 0 \quad \text{for} \quad i = 1, \dots, r, \tag{9}$$

and so $x \in J$. Moreover, for $i = 1, \dots, r$

$$\eta_i = \operatorname{sign} \prod_{s=i}^{r} p_s(x), \tag{10}$$

$$t_{ii} = \sqrt{\left|\frac{p_i(x)}{\prod_{s=i+1}^{r} p_s(x)}\right|} c_i. \tag{11}$$

Further, by Lemma 2.6, we have for $1 \leq i < j \leq r$

$$x_{ij}^j = \left(\prod_{s=j+1}^{r} p_s(x)\right) \eta_j \rho(t_{jj}) t_{ij}.$$

By (9)–(11), we get

$$t_{ij} = \frac{\eta_j \eta_{j+1} x_{ij}^j}{\sqrt{\left|\prod_{s=j}^{r} p_s(x)\right|}} \quad \text{for} \quad 1 \leq i < j < r, \tag{12}$$

and

$$t_{jr} = \frac{\eta_r x_{jr}}{\sqrt{|p_r(x)|}} \quad \text{for} \quad j = 1, \dots, r. \tag{13}$$

Thus t and η are determined by x.

Conversely, if $x \in J$, then we define $t \in S_0$ and $\eta \in \{-1,1\}^r$ by formulas (10)–(13). Let

$$y = t\left(\sum_{i=1}^{r} \eta_i c_i\right) t^\star.$$

Then we have for $1 \le i \le j \le r$

$$y_{ij} = \sum_{p=j}^{r} \eta_p t_{ip} t_{jp}^{\star} = \eta_j \rho(t_{jj}) t_{ij} + \sum_{p=j+1}^{r} \eta_p t_{ip} t_{jp}^{\star}.$$

By (10)–(12) and (5), we may write

$$\begin{aligned} y_{ij} =& \frac{x_{ij}^{j}}{\prod_{s=j+1}^{r} p_s(x)} + \eta_{j+1} t_{i,j+1} t_{j,j+1}^{\star} + \sum_{p=j+2}^{r} \eta_p t_{ip} t_{jp}^{\star} \\ =& \frac{x_{ij}^{j}}{\prod_{s=j+1}^{r} p_s(x)} + \frac{x_{i,j+1}^{j+1} \left(x_{j,j+1}^{j+1}\right)^{\star}}{\prod_{s=j+1}^{r} p_s(x)} + \sum_{p=j+2}^{r} t_{ip} \eta_p t_{jp}^{\star} \\ =& \frac{x_{ij}^{j+1}}{\prod_{s=j+2}^{r} p_s(x)} + \sum_{p=j+2}^{r} t_{ip} \eta_p t_{jp}^{\star}. \end{aligned}$$

This argument allows us to raise by one the upper index in the first term. We may repeat the argument till we reach r and obtain the equation

$$y_{ij} = x_{ij}^{r} = x_{ij}.$$

In the same way we can prove

$$y_{ii} = \sum_{p=i}^{r} \eta_p t_{ip} t_{ip}^{\star} = x_{ii}. \qquad \blacksquare$$

Let x be any element of J. Using Proposition 2.7, we can write

$$x = t \left(\sum_{i=1}^{r} \eta_i c_i \right) t^{\star}.$$

Since the action by t on $\mathcal{X}$ is a bounded linear operator, we have

$$t \circ c_k = t c_k t^{\star}.$$

Thus

$$s \circ (t c_k t^{\star}) = (st) c_k (st)^{\star}.$$

Now, by linearity and continuity, the action of the group S_0 on $\mathcal{X}$ restricted to the set J can be written in the following form

$$s \circ x = (st) \left(\sum_{i=1}^{r} \eta_i c_i \right) (t^{\star} s^{\star})$$

for $s \in S_0$.

For a sequence $\eta \in \{-1, 1\}^r$, we denote by $\mathcal{O}_\eta$ the orbit of S_0 in V passing through $\sum_{i=1}^{r} \eta_i c_i$. The following lemma is an immediate consequence of Proposition 2.7.

Lemma 2.8. *The group S_0 acts on $\mathcal{O}_\eta$ simply transitively. Moreover,*

i. *$\mathcal{O}_\eta$ is open in V;*

ii. *For $\eta \neq \eta'$, $\mathcal{O}_\eta \cap \mathcal{O}_{\eta'} = \emptyset$;*

iii. $J = \bigcup_{\eta \in \{-1,1\}^r} \mathcal{O}_\eta$.

Theorem 2.9. *The mapping $x \mapsto t(x)$ restricted to $\mathcal{O}_\eta$ is a diffeomorphism.*

Proof. Clearly, the action of S_0 on $\mathcal{X}$ is C^∞. Let $x \in \mathcal{O}_\eta$. By (10)–(13), the mapping $x \mapsto t(x)$ restricted to $\mathcal{O}_\eta$ is C^∞. Moreover,

$$x \mapsto t(x) \mapsto t(x) \circ \left(\sum_{i=1}^r \eta_i c_i \right) = x,$$

which finishes the proof. ∎

A homogeneous cone Ω in a vector space V is said to be irreducible if there are no non-trivial subspaces V', V'' and homogeneous cones $\Omega' \subset V'$, $\Omega'' \subset V''$ such that V is the direct sum of V' and V'', and $\Omega = \Omega' + \Omega''$. We state an equivalent condition for irreducibility.

Proposition 2.10. *Let Ω be a homogeneous cone of rank $r > 1$ in $\mathcal{L}$ with a normal decomposition $\{\mathcal{L}_{ij}\}_{1 \le i \le j \le r}$. Then Ω is irreducible if and only if for each non-constant sequence $\eta \in \{-1,1\}^r$ there are $1 \le p < q \le r$ such that* $\dim \mathcal{L}_{pq} > 0$ *and $\eta_p \eta_q = -1$.*

Proof. Let Ω be irreducible. Let $\eta \in \{-1,1\}$ be a non-constant sequence such that for all p and q, $1 \le p < q \le r$, $\eta_p \eta_q = -1$ implies $\dim \mathcal{L}_{pq} = 0$. We introduce a partition of $\{1, \ldots, r\}$:

$$P = \{i \mid \eta_i = 1\}, \qquad Q = \{j \mid \eta_j = -1\},$$

and define

$$\mathcal{L}' = \bigoplus_{\substack{i,j \in P \\ i \le j}} \mathcal{L}_{ij}, \quad \mathcal{L}'' = \bigoplus_{\substack{i,j \in Q \\ i \le j}} \mathcal{L}_{ij}.$$

Then $\mathcal{L} = \mathcal{L}' \oplus \mathcal{L}''$. By properties of η, we get

$$\mathcal{L}' \triangle \mathcal{L}'' = 0.$$

Moreover, $\mathcal{L}'$ and $\mathcal{L}''$ are subalgebras of $\mathcal{L}$ with unit elements $e' = \sum_{i \in P} c_i$ and $e'' = \sum_{i \in Q} c_i$. Let Ω', Ω'' be homogeneous cones in $\mathcal{L}'$, $\mathcal{L}''$, respectively. Then the Lie group S_0 has a decomposition

$$S_0 = S_0' S_0'',$$

which implies $\Omega = \Omega' \oplus \Omega''$.

Assume that for all η there exist p and q such that $\eta_p\eta_q = -1$ and $\dim \mathcal{L}_{pq} > 0$. Suppose Ω is not irreducible. Then there are non-empty subspaces $\mathcal{L}'$ and $\mathcal{L}''$ and homogeneous cones $\Omega' \subset \mathcal{L}'$, $\Omega'' \subset \mathcal{L}''$ such that

$$\mathcal{L} = \mathcal{L}' \oplus \mathcal{L}'', \quad \Omega = \Omega' \oplus \Omega''.$$

Thus the Lie group S_0 has a decomposition $S_0 = S_0' S_0''$. Hence, $\mathcal{L}'$ and $\mathcal{L}''$ are subalgebras with unit elements e' and e'', respectively. We define a partition of $\{1, \dots, r\}$

$$P = \{i|\ c_i \triangle e' = c_i\}, \quad Q = \{j|\ c_j \triangle e'' = c_j\}.$$

Then $\mathcal{L}_{pq} = \{0\}$ for $p \in P$ and $q \in Q$. Taking

$$\eta_i = \begin{cases} -1 & \text{for} \quad i \in P, \\ 1 & \text{for} \quad i \in Q, \end{cases}$$

for $1 \leq i \leq r$, we get a contradiction. ■

Let Ω be an irreducible, open homogeneous cone of rank $r > 1$ in a matrix T–algebra $\mathcal{U}$. For $y \in N_0$, $\xi \in V$ and $k = 1, \dots, r$, we define

$$W_k(\xi, y) = 2\pi(\xi | y \circ c_k).$$

Let us denote by π_k, π^k the projections

$$\pi_k : \mathcal{U} \ \mapsto \bigoplus_{1 \leq i \leq j \leq k} \mathcal{U}_{ij}, \quad \pi^k : \mathcal{U} \mapsto \bigoplus_{i=1}^{k-1} \mathcal{U}_{ik}.$$

We may write

$$W_k(\xi, y) = W_k(\pi_k(\xi), e + \pi^k(y)), \tag{14}$$

since $\pi_k(\mathcal{U})$ is a subalgebra of $\mathcal{U}$ and

$$y \circ c_k = (e + \pi^k(y)) \circ c_k, \quad (\xi | y \circ c_k) = (\pi_k(\xi) | (e + \pi^k(y)) \circ c_k),$$

for $y \in N_0$, $\xi \in \mathcal{X}$.

Theorem 2.11. *Let Ω be an irreducible, open homogeneous cone of rank $r > 1$ in a matrix T–algebra $\mathcal{U}$. Then for every $\xi \in J^\star$ and $\xi \notin \overline{\Omega^\star} \cup -\overline{\Omega^\star}$ there exist $i \in \{1, \dots, r\}$ and $y_1, y_2 \in N_0$ such that*

$$W_i(\xi, y_1) > 0, \quad W_i(\xi, y_2) < 0.$$

Proof. The proof is inductive over the rank of the cone. Direct calculation shows that for cones of rank $r = 2$ the theorem is true. Now assume that the theorem holds for cones of rank $\leq r$. Let Ω be a homogeneous cone of rank $r+1$. $\pi_r(\Omega)$ is a homogeneous cone in $\pi_r(\mathcal{U})$, not necessarily irreducible. Therefore, there are non-zero subalgebras $\{\mathcal{U}^i\}_{i=1}^k$ of $\pi_r(\mathcal{U})$ and homogeneous irreducible cones $\Omega_1, \ldots, \Omega_k$ such that

$$\Omega_i \subset \mathcal{X}^i, \quad \pi_r(\Omega) = \bigoplus_{i=1}^k \Omega_i, \quad \pi_r(\mathcal{U}) = \bigoplus_{i=1}^k \mathcal{U}^i.$$

For $i \in \{1, \ldots, k\}$, we put $I_i = \{j |\ c_j \in \mathcal{U}_i\}$. Let P_i, Q_i denote the projections

$$P_i : \mathcal{U} \ \mapsto \ \mathcal{U}^i, \quad Q_i : \bigoplus_{j=1}^r \mathcal{U}_{j,r+1} \mapsto \bigoplus_{j \in I_i} \mathcal{U}_{j,r+1}.$$

Since Ω is irreducible, $\dim Q_i(\pi^{r+1}(\mathcal{U})) > 0$ for every $i = 1, \ldots, k$.

We fix $\xi \in J^\star$ and $\xi \notin \overline{\Omega^\star} \cup -\overline{\Omega^\star}$. Assume that there is i such that

$$P_i(\xi) \notin \overline{\Omega_i^\star} \cup -\overline{\Omega_i^\star}.$$

By the induction hypothesis, there exist $j \in I_i$ and $t_1, t_2 \in P_i(N_0)$ such that

$$(P_i(\xi)|t_1 \circ c_j) > 0, \quad (P_i(\xi)|t_2 \circ c_j) < 0.$$

Taking $y_l = e + \pi^j(t_l) \in N_0$ for $l = 1, 2$, we have $t_l \circ c_j = y_l \circ c_j \in \mathcal{X}_i$, and so

$$(\xi|y_l \circ c_j) = (P_i(\xi)|t_l \circ c_j).$$

Hence, the conclusion follows.

Assume now that for all $i \in \{1, \ldots, k\}$, $P_i(\xi) \in \overline{\Omega_i^\star} \cup -\overline{\Omega_i^\star}$. Since $\xi \in J^\star$,

$$P_i(\xi) \in \Omega_i^\star \cup -\Omega_i^\star.$$

Let $(\xi|c_{r+1}) > 0$. The case $(\xi|c_{r+1}) < 0$ is similar. For $y \in N_0$, we can write

$$\begin{aligned}\frac{1}{2\pi} W_{r+1}(\xi, y) &= (\xi|(e + \pi^{r+1}(y)) \circ c_{r+1}) = (\xi|(e + \sum_{i=1}^k Q_i(y)) \circ c_{r+1}) \\ &= \sum_{i=1}^k (P_i(\xi)|Q_i(y)Q_i(y)^\star) + \sum_{i=1}^k (\xi|Q_i(y) + Q_i(y)^\star) + (\xi|c_{r+1}),\end{aligned}$$

since $Q_i(y)Q_i(y)^\star \in P_i(\mathcal{U})$. Let us consider

$$F(\lambda, y) = \sum_{i=1}^k \lambda_i^2 (P_i(\xi)|Q_i(y)Q_i(y)^\star) + \sum_{i=1}^k \lambda_i (\xi|Q_i(y) + Q_i(y)^\star) + (\xi|c_{r+1})$$

for $y \in N_0$ and $\lambda \in \mathbb{R}^k$. Suppose that F does not change the sign i.e.,

$$F(\lambda, y) \geq 0. \tag{15}$$

Then, for all $i = 1, \ldots, k$, $P_i(\xi) \in \Omega_i^\star$, since $P_i(\xi) \in -\Omega_i^\star$ implies $(P_i(\xi)|Q_i(y)Q_i(y)^\star) < 0$. Thus for every $y \in N_0$ and all $i = 1, \ldots, r$

$$W_i(\xi, y) \geq 0. \tag{16}$$

Since $\xi \notin \overline{\Omega^\star} \cup -\overline{\Omega^\star}$, there exists $x \in \Omega$ such that $(\xi|x) < 0$. Writing

$$x = t \circ \left(\sum_{i=1}^{r+1} a_i c_i \right)$$

for $t \in N_0$ and $a_i > 0$, $i = 1, \ldots, r+1$, we get

$$2\pi(\xi|x) = \sum_{i=1}^{r} a_i W_i(\xi, t) + 2\pi a_{r+1}(\xi|t \circ c_{r+1}) < 0$$

i.e., by (16),

$$\frac{1}{2\pi} W_{r+1}(\xi, t) = (\xi|t \circ c_{r+1}) < 0.$$

But

$$F((1, \ldots, 1), t) = \frac{1}{2\pi} W_{r+1}(\xi, t) < 0,$$

which contradicts (15). ■

3. The Siegel domains of type II

Identification with a solvable Lie group. Let Ω be an open homogeneous irreducible cone in a real vector space V. We may assume that V is a clan with a unit element. Let $\mathcal{U}$ be the matrix T–algebra such that the homogeneous cone $\Omega(\mathcal{X})$ is isomorphic to Ω (see [11] Theorem 4) i.e., there is an isomorphism of clans with a unit element $\sigma : V \mapsto \mathcal{X}$ such that $\sigma(\Omega) = \Omega(\mathcal{X})$. We identify V with $\mathcal{X}$.

Let $V^{\mathbb{C}} = V + iV$ be the complexification of V. We extend the action of S_0 to $V^{\mathbb{C}}$.

In addition to $V^{\mathbb{C}}$, suppose that we are given a complex vector space $\mathcal{Z}$. Let $\Phi : \mathcal{Z} \times \mathcal{Z} \mapsto V^{\mathbb{C}}$ be a Hermitian symmetric sesquilinear mapping. We assume that Φ is Ω–positive i.e., $\Phi(\zeta, \zeta) \in \overline{\Omega}$ for all $\zeta \in \mathcal{Z}$ and $\Phi(\zeta, \zeta) = 0$ only if $\zeta = 0$.

The Siegel domain of type II associated with the cone Ω is defined as

$$\mathcal{D} = \{(\zeta, z) \in \mathcal{Z} \times V^{\mathbb{C}} \mid \Im z - \Phi(\zeta, \zeta) \in \Omega\}.$$

There is a representation $\sigma : S_0 \mapsto \mathrm{GL}(\mathcal{Z})$ such that

$$g\Phi(\zeta, \omega) = \Phi(\sigma(g)\zeta, \sigma(g)\omega).$$

Therefore, the transformation $(\zeta, z) \mapsto (\sigma(g)\zeta, \sigma(g)\omega)$ is a biholomorphic automorphism of $\mathcal{D}$. The elements $\zeta \in \mathcal{Z}$, $x \in V$ and $g \in S_0$ acts on $\mathcal{D}$ in the following way

$$\begin{aligned} \zeta \cdot (\omega, z) &= (\zeta + \omega, z + 2i\Phi(\omega, \zeta) + i\Phi(\zeta, \zeta)), \\ x \cdot (\omega, z) &= (\omega, z + x), \\ g \cdot (\omega, z) &= (\sigma(g)\omega, g \circ z). \end{aligned}$$

The first two actions generate a two-step nilpotent group $N(\Phi)$ (or Abelian if $\mathcal{Z} = 0$) of biholomorphic automorphisms of $\mathcal{D}$. The multiplication in $N(\Phi)$ is given by

$$(\zeta, z)(\zeta', z') = (\zeta + \zeta', z + z' + 2\Im\Phi(\zeta, \zeta')).$$

All three actions generate a solvable Lie group $S = N(\Phi)S_0$, $N(\Phi)$ being a normal subgroup of S. For $s \in S$, we use the notation $s = (\zeta, x)ya$ with $\zeta \in \mathcal{Z}$, $x \in V$, $y \in N_0$ and $a \in A$.

The action of $\sigma(\mathcal{A})$ is diagonalizable i.e.,

$$\mathcal{Z} = \bigoplus_{j=1}^{r} \mathcal{Z}_j \tag{17}$$

with $\sigma(H)\zeta = \frac{\lambda_i(H)}{2}\zeta$ for $\zeta \in \mathcal{Z}_j$ where $\lambda_1, \ldots, \lambda_r$ is dual basis to $c_1, \ldots, c_r$ (see e.g. [3]).

Given $\lambda \in V^\star$ let

$$H_\lambda(\zeta, \omega) = 4(\lambda|\Phi(\zeta, \omega)).$$

For $\lambda \in \Omega^\star$, the Hermitian form H_λ is not degenerate. If $\lambda = \sum_{j=1}^r a_j c_j$, $a_j \in \mathbb{R}$, the form H_λ decomposes nicely as

$$H_\lambda(\zeta, \omega) = \sum_{j=1}^{r} a_j H_{c_j}(\zeta_j, \omega_j)$$

where $\zeta = \sum_{j=1}^r \zeta_j$, $\omega = \sum_{j=1}^r \omega_j$, $\zeta_j, \omega_j \in \mathcal{Z}_j$. For $j \neq k$, we have

$$\begin{aligned} H_{c_j}(\zeta, \omega_k) &= 4(c_j|t_{jj} \circ \Phi(\zeta, \omega_k)) = 4(c_j|\Phi(\sigma(t_{jj})\zeta, \omega_k)) + 4(c_j|\Phi(\zeta, \sigma(t_{jj})\omega_k)) \\ &= 4\sum_{l=1}^{r}(c_j|\Phi(\frac{\lambda_l(t_{jj})}{2}\zeta_l, \omega_k)) + 4(c_j|\Phi(\zeta, \frac{\lambda_k(t_{jj})}{2}\omega_k)) = \frac{1}{2}H_{c_j}(\zeta_j, \omega_k), \end{aligned}$$

and so $H_{c_j}(\zeta, \omega_k) = 0$.

For $i = 1, \ldots, r$, we define

$$d_i = \dim \mathcal{Z}_i \geq 0, \quad m_i = n_i + \frac{d_i}{2}.$$

The Lie algebra $\mathcal{S}$ of S has the following decomposition

$$S = \bigoplus_{j=1}^{r} \mathcal{Z}_j \oplus \bigoplus_{1 \leq i \leq j \leq r} V_{ij} \oplus \bigoplus_{1 \leq i < j \leq r} \mathcal{N}_{ij} \oplus \bigoplus_{i=1}^{r} \mathcal{A}_i$$

where $\bigoplus_{1 \leq i \leq j \leq r} V_{ij}$ is the normal decomposition of the clan V. For $1 \leq i < j \leq r$, we choose an orthogonal basis $\{e_{ij}^\alpha\}$ of the subspace V_{ij} such that

$$e_{ij}^\alpha \triangle e_{ij}^\beta = \delta_{\alpha\beta} c_i.$$

Then

$$(e_{ij}^\alpha | e_{ij}^\beta) = \mathrm{tr}(e_{ij}^\alpha \triangle e_{ij}^\beta) = \delta_{\alpha\beta} n_i.$$

We identify $e_{ii}^\alpha = c_i$. Let $\{t_{ij}^\alpha\}$ be the corresponding basis for $\mathcal{U}_{ij}$ i.e.,

$$t_{ij}^\alpha = \overline{e_{ij}^\alpha}, \quad t_{ii} = e_{ii}^\alpha.$$

For $1 \le k \le r$, $1 \le i < j \le r$, $1 \le \alpha \le \dim V_{ij} = \dim \mathcal{N}_{ij}$, we define the left-invariant vector fields on S: $X_k \in V_{kk}$, $H_k \in \mathcal{A}_k$, $X_{ij}^\alpha \in V_{ij}$, $Y_{ij}^\alpha \in \mathcal{N}_{ij}$ by identifying at the identity element with $\frac{c_k}{\sqrt{m_k}}$, $\frac{t_{kk}}{\sqrt{m_k}}$, $\frac{e_{ij}^\alpha}{\sqrt{m_i}}$, $\frac{t_{ij}^\alpha}{\sqrt{m_i}}$, respectively.

In $\mathcal{Z}$ we choose coordinates compatible with the decomposition (17). Let $\{e_{j\alpha}\}$ be a basis of $\mathcal{Z}_j$ such that $H_{c_j}(e_{j\alpha}, e_{j\beta}) = \delta_{\alpha\beta}$. Then

$$H_{c_j}(\zeta, \omega) = \sum_{\alpha}^{d_j} \zeta_{j\alpha} \overline{\omega}_{j\alpha}$$

where $\zeta = \sum_{\alpha=1}^{d_j} \zeta_{j\alpha} e_{j\alpha}$ and $\omega = \sum_{\alpha=1}^{d_j} \omega_{j\alpha} e_{j\alpha}$.

Let $\zeta_{j\alpha} = x_{j\alpha} + i y_{j\alpha}$ and let $\mathcal{X}_{j\alpha}$, $\mathcal{Y}_{j\alpha}$ be the left-invariant vector fields on S corresponding to $\frac{\partial_{x_{j\alpha}}}{\sqrt{m_j}}$ and $\frac{\partial_{y_{j\alpha}}}{\sqrt{m_j}}$, respectively.

Then

$$X_k, X_{ij}^\alpha, H_k, Y_{ij}^\alpha, \mathcal{X}_{j\alpha}, \mathcal{Y}_{j\alpha} \tag{18}$$

form a basis for $\mathcal{S}$.

Admissible operators. Let T be the tangent bundle for the complex domain $\mathcal{D}$ and let $T^{\mathbb{C}}$ be the complexified tangent bundle. We extend the complex structure $\mathcal{J}$ and Bergman metric g from T to $T^{\mathbb{C}}$ by complex linearity. The space of smooth sections of T, $T^{\mathbb{C}}$ will be denoted by $\Gamma(T)$, $\Gamma(T^{\mathbb{C}})$, respectively. We extend the corresponding Riemannian connection ∇ from $\Gamma(T)$ to $\Gamma(T^{\mathbb{C}})$ by complex linearity.

For $Z, W \in \Gamma(T^{\mathbb{C}})$, we define

$$\triangle(Z, W) = Z\overline{W} - \nabla_Z \overline{W}.$$

In $T^{\mathbb{C}}$ we introduce a Hermitian scalar product

$$(Z, W) = \frac{1}{2} g(Z, \overline{W}).$$

Assume that we are given a system $z_1, \ldots, z_m$ of coordinates in $\mathcal{D}$ such that $g(\partial_{z_i}, \partial_{z_j}) = \delta_{ij}$ at the point $(0, ie)$. Let $E_1, \ldots, E_m$ be the unique S–invariant orthonormal frame such that $E_j(0, ie) = \partial_{z_j}$. Since for every $j, k \in \{1, \ldots, m\}$

$$\nabla_{\partial_{z_j}} \partial_{\overline{z}_k} = \nabla_{\partial_{\overline{z}_k}} \partial_{z_k} = 0,$$

a simple calculation proves that

$$\triangle(E_j, E_k) = \triangle_{j,k} = \sum_{p,q} b_{pq}^{jk}(z) \partial_{z_p} \partial_{\overline{z}_q}$$

and

$$\partial_{z_j} \partial_{\overline{z}_k} = \sum_{p,q} c_{pq}^{jk}(z) \triangle_{p,q}$$

for some smooth functions $b_{pq}^{jk}(z)$, $c_{pq}^{jk}(z)$, and

$$\triangle_{j,k}(0, ie) = \partial_{z_j} \partial_{\overline{z}_k}. \tag{19}$$

This shows that a second order real operator annihilating holomorphic functions can be written as $L = \sum_{j,k} a_{j,k}(z)\partial_{z_j}\partial_{\overline{z}_k}$, or $L = \sum_{j,k} b_{j,k}(z)\triangle_{j,k}$ with $a_{j,k}(z) = \overline{a_{k,j}(z)}$, $b_{j,k}(z) = \overline{b_{k,j}(z)}$, respectively. Finally, $\triangle_{j,k}$ are unique S–invariant operators with the property (19). This implies that if on top of the above assumptions we add S–invariance then $L = \sum_{j,k} b_{j,k}\triangle_{j,k}$ for $b_{j,k} \in \mathbf{C}$ with the property $b_{j,k} = \overline{b_{k,j}}$. Such operators will be called admissible (see [4], [3]).

For our purpose it will be much more convenient to consider admissible operators as operators on the group S. To do that we identify $\mathcal{D}$ with S by

$$\theta : S \mapsto \theta(s) = s \circ ie \in \mathcal{D},$$

and we transport both the Bergman metric g and the complex structure $\mathcal{J}$ from $\mathcal{D}$ to S. Although, we follows closely the calculations of [3], we keep most of them, but not all because of normalizations specific to the non-symmetric situation.

In coordinates

$$(\zeta, z) = \Big(\sum_{j,\alpha}(x_{j\alpha} + iy_{j\alpha})e_{j\alpha}, \sum_{\substack{i\le j\\ \alpha}}(x_{ij}^{\alpha} + iy_{ij}^{\alpha})e_{ij}^{\alpha}\Big)$$

the differential $d\theta$ of θ becomes

$$\begin{aligned} d\theta(X_j) &= \tfrac{1}{\sqrt{m_j}}\partial_{x_j}, & d\theta(X_{ij}^{\alpha}) &= \tfrac{1}{\sqrt{m_i}}\partial_{x_{ij}^{\alpha}},\\ d\theta(H_j) &= \tfrac{1}{\sqrt{m_j}}\partial_{y_{jj}}, & d\theta(Y_{ij}^{\alpha}) &= \tfrac{1}{\sqrt{m_i}}\partial_{y_{ij}^{\alpha}},\\ d\theta(\mathcal{X}_{j\alpha}) &= \tfrac{1}{\sqrt{m_j}}\partial_{x_{j\alpha}}, & d\theta(\mathcal{Y}_{j\alpha}) &= \tfrac{1}{\sqrt{m_j}}\partial_{y_{j\alpha}}. \end{aligned}$$

This implies the following identities

$$\begin{aligned} \mathcal{J}(X_j) &= H_j, & \mathcal{J}(X_{ij}) &= Y_{ij},\\ \mathcal{J}(H_j) &= -X_j, & \mathcal{J}(Y_{ij}) &= -X_{ij},\\ \mathcal{J}(\mathcal{X}_{j\alpha}) &= \mathcal{Y}_{j\alpha}, & \mathcal{J}(\mathcal{Y}_{j\alpha}) &= -\mathcal{X}_{j\alpha}. \end{aligned}$$

We need some commutation relations. First, we notice that the adjoint action of $\mathcal{A}$ preserves all the subspaces V_{ij}, $\mathcal{N}_{ij}$. More precisely, if $H \in \mathcal{A}$, then

$$[H, X] = \frac{\lambda_i(H) + \lambda_j(H)}{2}X \quad \text{for} \quad X \in V_{ij}, \tag{20}$$

$$[H, Y] = \frac{\lambda_i(H) - \lambda_j(H)}{2}Y \quad \text{for} \quad Y \in \mathcal{N}_{ij}, \tag{21}$$

where $\lambda_1, \dots, \lambda_r$ denote the dual basis to $c_1, \dots, c_r$. Next, for $i < j$, we have

$$[Y_{ij}^{\alpha}, X_{ij}^{\alpha}] = \frac{1}{\sqrt{m_i}}X_i, \tag{22}$$

$$[Y_{ij}^{\alpha}, X_j] = \frac{1}{\sqrt{m_j}}X_{ij}^{\alpha}, \tag{23}$$

since $[Y_{ij}^{\alpha}, X_{ij}^{\alpha}]$ is identified in e with

$$\frac{1}{m_i}\left(t_{ij}^{\alpha}e_{ij}^{\alpha} + e_{ij}^{\alpha}t_{ij}^{\alpha\star}\right) = \frac{1}{m_i}e_{ij}^{\alpha}\triangle e_{ij}^{\alpha} = \frac{1}{m_i}c_i,$$

and $[Y_{ij}^\alpha, X_j]$ with

$$\frac{1}{\sqrt{m_j m_i}}\left(t_{ij}^\alpha c_j + c_j t_{ij}^{\alpha\star}\right) = \frac{1}{\sqrt{m_j m_i}} e_{ij}^\alpha.$$

For every $j = 1, \dots, r$ with $d_j > 0$, the subgroup $\mathcal{Z}_j \oplus V_{jj}$ is a Heisenberg group in which multiplication is

$$(\zeta, x)(\omega, y) = \Big(\zeta + \omega, x + y + \frac{1}{2}\sum_{\alpha=1}^{d_j} \zeta_{j\alpha}\overline{\omega}_{j\alpha}\Big).$$

Therefore, $[\mathcal{Y}_{j\alpha}, \mathcal{X}_{j\alpha}] = \frac{1}{\sqrt{m_j}} X_j$ and for $\alpha \neq \beta$

$$[\mathcal{Y}_{j\alpha}, \mathcal{X}_{j\beta}] = [\mathcal{X}_{j\alpha}, \mathcal{X}_{j\beta}] = [\mathcal{Y}_{j\alpha}, \mathcal{Y}_{j\beta}] = 0.$$

Then

Lemma 3.1. *The basis X_j, H_j, X_{ij}^α, Y_{ij}^α, $\mathcal{X}_{j\alpha}$, $\mathcal{Y}_{j\alpha}$ is orthonormal with respect to the Riemannian structure g on $\mathcal{S}$.*

Proof. The Riemannian structure on S derived from the Bergman metric on $\mathcal{D}$ is given by the formula (see [6])

$$g(X, Y) = \frac{1}{2}\beta([\mathcal{J}X, Y])$$

where for $X \in \mathcal{S}$

$$\beta(X) = \operatorname{Tr}(\operatorname{ad}_X - \mathcal{J}\operatorname{ad}_X).$$

Using (20), (22) and (2), for every $j \in \{1, \dots, r\}$, we get

$$\beta(X_j) = 2\sqrt{m_j}$$

and $\beta \equiv 0$ on $\bigoplus_{j=1}^r \mathcal{Z}_j \bigoplus_{i<j} V_{ij} \oplus \bigoplus_{ij} \mathcal{N}_{ij}$. Thus, the lengths of the vectors are

$$g(X_j, X_j) = g(H_j, H_j) = \frac{1}{2}\beta([H_j, X_j]) = \frac{1}{2\sqrt{m_j}}\beta(X_j) = 1,$$
$$g(X_{ij}^\alpha, X_{ij}^\alpha) = g(Y_{ij}^\alpha, Y_{ij}^\alpha) = \frac{1}{2}\beta([Y_{ij}^\alpha, X_{ij}^\alpha]) = \frac{1}{2\sqrt{m_i}}\beta(X_i) = 1,$$
$$g(\mathcal{X}_{j\alpha}, \mathcal{X}_{j\alpha}) = g(\mathcal{Y}_{j\alpha}, \mathcal{Y}_{j\alpha}) = \frac{1}{2}\beta([\mathcal{Y}_{j\alpha}, \mathcal{X}_{j\alpha}]) = \frac{1}{2\sqrt{m_j}}\beta(X_j) = 1.$$

Orthogonality of the basis follows from the fact that if $\mathcal{S}_\lambda$, $\mathcal{S}_\eta$ are root spaces corresponding to roots λ and η, respectively, then $[\mathcal{S}_\lambda, \mathcal{S}_\eta] \subset \mathcal{S}_{\lambda+\eta}$. ■

The Riemannian form g and the bracket in $\mathcal{S}$ determine the invariant Riemannian connection ∇ in S.

Lemma 3.2. *The Riemannian connection is given by the formulas*

$$\nabla_{X_j} X_j = \frac{1}{\sqrt{m_j}} H_j, \quad \nabla_{H_j} H_j = 0,$$
$$\nabla_{X_{ij}^\alpha} X_{ij}^\alpha = \frac{1}{2}\left(\frac{1}{\sqrt{m_i}} H_i + \frac{1}{\sqrt{m_j}} H_j\right), \quad \nabla_{Y_{ij}^\alpha} Y_{ij}^\alpha = \frac{1}{2}\left(\frac{1}{\sqrt{m_i}} H_i - \frac{1}{\sqrt{m_j}} H_j\right),$$
$$\nabla_{\mathcal{X}_{j\alpha}} \mathcal{X}_{j\alpha} = \frac{1}{2\sqrt{m_j}} H_j, \quad \nabla_{\mathcal{Y}_{j\alpha}} \mathcal{Y}_{j\alpha} = \frac{1}{2\sqrt{m_j}} H_j.$$

Proof. Let $X \in \mathcal{S}$. Using the usual formulas for the Riemannian connection, we obtain

$$g(\nabla_X X, W) = g([W, X], X)$$

for $W \in \mathcal{S}$. The proof follows directly by (20)–(22). ∎

By Lemma 3.2, we get

Theorem 3.3. *Let*

$$\triangle_j = \triangle(X_j + iH_j, X_j + iH_j), \quad \triangle_{ij}^\alpha = \triangle(X_{ij}^\alpha + iY_{ij}^\alpha, X_{ij}^\alpha + iY_{ij}^\alpha),$$
$$\mathcal{L}_j^\alpha = \triangle(\mathcal{X}_{j\alpha} + i\mathcal{Y}_{j\alpha}, \mathcal{X}_{j\alpha} + i\mathcal{Y}_{j\alpha}).$$

Then

$$\triangle_j = X_j^2 + H_j^2 - \frac{1}{\sqrt{m_j}} H_j, \quad \triangle_{ij}^\alpha = \left(X_{ij}^\alpha\right)^2 + \left(Y_{ij}^\alpha\right)^2 - \frac{1}{\sqrt{m_i}} H_i,$$
$$\mathcal{L}_j^\alpha = (\mathcal{X}_{j\alpha})^2 + (\mathcal{Y}_{j\alpha})^2 - \frac{1}{\sqrt{m_j}} H_j.$$

The partial Fourier transform. We present some basic facts about Fourier analysis on $N(\Phi)$. All what we need has been elaborated in [9].

Let $(\cdot,\cdot)$ be the Hermitian scalar product in which the basis $\{e_{j\alpha}\}$ is orthonormal. For $\lambda \in V$, we define a Hermitian transformation $M_\lambda : \mathcal{Z} \mapsto \mathcal{Z}$ by

$$(M_\lambda \zeta, \omega) = H_\lambda(\zeta, \omega) \tag{24}$$

where $\zeta, \omega \in \mathcal{Z}$, and consider the set

$$\Lambda = \{\lambda \in V^\star |\ \operatorname{Det} M_\lambda \neq 0\} = \{\lambda \in V^\star |\ H_\lambda \text{ is not degenerate}\}.$$

The set Λ^c is closed set of measure 0, since we have $H_\lambda(\zeta,\zeta) > 0$ for $\lambda \in \Omega^\star$, and $\operatorname{Det} M_\lambda$ is a non-zero polynomial of λ. The set Λ carries the Plancherel measure (see [9]) $\rho(\lambda)d\lambda = |\operatorname{Det} M_\lambda| d\lambda$. For every $\lambda \in \Lambda$, we define a complex structure $\mathcal{J}_\lambda$ which corresponds to λ and determines the representation space $\mathcal{H}_\lambda$. Let $|M_\lambda|$ be the positive Hermitian transformation such that $|M_\lambda|^2 = M_\lambda^2$. Then

$$\mathcal{J}_\lambda = i|M_\lambda|^{-1} M_\lambda.$$

Let $B_\lambda = \Im H_\lambda$. We define a realization of the unitary irreducible representation U^λ (the Fock representation) associated with $\lambda \in \Lambda$. Let $\mathcal{H}_\lambda$ be the set of all

$C^\infty(\mathcal{Z})$ functions F which are holomorphic with respect to the complex structure $\mathcal{J}_\lambda$ and such that

$$\int_{\mathcal{Z}} |F(\zeta)|^2 \rho(\lambda) e^{-\pi B_\lambda(\mathcal{J}_\lambda \zeta, \zeta)} d\zeta < \infty.$$

The appropriate scalar product in $\mathcal{H}_\lambda$ and the representation U^λ are defined by

$$(F_1, F_2)_\lambda = \int_{\mathcal{Z}} F_1(\zeta)\overline{F_2(\zeta)} e^{-\pi B_\lambda(\mathcal{J}_\lambda \zeta, \zeta)} \rho(\lambda) d\zeta$$

and

$$U^\lambda_{(\zeta,x)} F(\omega) = e^{-2\pi(\lambda|x) - \frac{\pi}{2}|\zeta|^2 + \pi\omega\overline{\zeta}} F(\omega - \zeta)$$

with $\omega\overline{\zeta} = B_\lambda(\mathcal{J}_\lambda \omega, \zeta) + iB_\lambda(\omega, \zeta)$, $|\zeta|^2 = \zeta\overline{\zeta}$.

We will define an orthonormal basis of $\mathcal{H}_\lambda$ for $\lambda \in \Omega^\star \cup -\Omega^\star$. First, we notice that for $\lambda \in \Omega^\star \cup -\Omega^\star$, the complex structure $\mathcal{J}_\lambda$ has the form

$$\mathcal{J}_\lambda \zeta = \begin{cases} \mathcal{J}\zeta & \text{for} \quad \lambda \in \Omega^\star \\ -\mathcal{J}\zeta & \text{for} \quad \lambda \in -\Omega^\star. \end{cases}$$

Hence, inside $\Omega^\star \cup -\Omega^\star$, the action of $s^\star \in S_0^\star$ does not change the complex structure. By (24) we have $M_{s^\star \circ \lambda} = \sigma(s)^\star M_\lambda \sigma(s)$, and so

$$\rho(s^\star \circ \lambda) = |\operatorname{Det} M_{s^\star \circ \lambda}| = \rho(\lambda) \operatorname{Det} \sigma(s).$$

For $s \in S_0$ and $\xi \in \mathcal{H}_\lambda$, we put

$$s \cdot \xi(\zeta) = \xi(s \cdot \zeta). \tag{25}$$

Therefore,

$$\begin{aligned}(s \cdot \xi, s \cdot \eta)_{s^\star \circ \lambda} &= \int_{\mathcal{Z}} \xi(s \cdot \zeta)\overline{\eta(s \cdot \zeta)} e^{-\pi B_{s^\star \lambda}(\mathcal{J}_{s^\star \circ \lambda} \zeta, \zeta)} \rho(s^\star \circ \lambda) d\zeta \\ &= \int_{\mathcal{Z}} \xi(\zeta)\overline{\eta(\zeta)} e^{-\pi B_\lambda(\mathcal{J}_\lambda \zeta, \zeta)} \operatorname{Det} \sigma(s^{-1}) \rho(s^\star \lambda) d\zeta = (\xi, \eta)_\lambda \end{aligned} \tag{26}$$

i.e., the action (25) is an isometry. Moreover,

$$U^{s^\star \circ \lambda}_{(\zeta,x)} s \cdot \xi(\omega) = U^\lambda_{(\sigma(s)\zeta, s\circ x)} \xi(s \cdot \omega).$$

Hence,

$$(U^{s^\star \circ \lambda}_{(\zeta,x)} s \cdot \xi, s \cdot \eta)_{s^\star \circ \lambda} = (U^\lambda_{(\sigma(s)\zeta, s\circ x)} \xi, \eta)_\lambda. \tag{27}$$

For a multi-index $\gamma_j = (\gamma_{j1}, \dots, \gamma_{jd_j})$, we define

$$\xi_{\gamma_j}(\zeta) = \frac{\pi^{\frac{|\gamma_j|}{2}}}{\sqrt{\gamma_j!}} \prod_\alpha \zeta_{j\alpha}^{\gamma_{j\alpha}}$$

where $|\gamma_j| = \gamma_{j1} + \cdots + \gamma_{jd_j}$, $\gamma_j! = \gamma_{j1}! \cdots \gamma_{jd_j}!$. Then the polynomials

$$\xi_\gamma(\zeta) = \prod_{j=1}^r \xi_{\gamma_j}(\zeta)$$

form an orthonormal basis of $\mathcal{H}_e$. Let $\lambda \in \Omega^\star \cup -\Omega^\star$. By Proposition 2.7, there is $s(\lambda) \in N_0$ such that $\lambda = s^\star(\lambda) \circ \eta$ with $\eta = -e$ or $\eta = e$. Then putting $\xi^\lambda_\gamma(\zeta) = s(\lambda) \cdot \xi(\zeta)$, we get an orthonormal basis of $\mathcal{H}_\lambda$.

4. Poisson kernels and regularity of their Fourier transform

Poisson kernels. Let L be a second order real elliptic S–invariant differential operator which annihilates holomorphic functions. We write

$$L = \sum_{p,q} c_{pq} \triangle(Z_p, Z_q)$$

where $\{c_{pq}\}$ is a Hermitian positive-definite matrix, and

$$Z_p \in \{X_{ij}^\alpha + iY_{ij}^\alpha, X_i + iH_i, \mathcal{X}_{j\alpha} + i\mathcal{Y}_{j\alpha}\}.$$

Let π_A denote the canonical homomorphism $\pi_A : S \mapsto S/N(\Phi)N_0$. Let Y be the first order part of $\pi_A(L)$. Then

$$Y = \sum_{i=1}^{r} b_j H_j$$

with $b_j < 0$ (see [4]).

We define two subalgebras of $\mathcal{N}(\Phi) \oplus \mathcal{N}_0$

$$\mathcal{N}_1(L) = \mathcal{N}(\Phi) \oplus \bigoplus_{\substack{(\lambda_i - \lambda_j)(Y) < 0 \\ i<j}} \mathcal{N}_{ij}, \quad \mathcal{N}_0(L) = \bigoplus_{\substack{(\lambda_i - \lambda_j)(Y) \geq 0 \\ i<j}} \mathcal{N}_{ij},$$

and two subgroups $N_1(L) = \exp \mathcal{N}_1(L), N_0(L) = \exp \mathcal{N}_0(L)$. Then $N(\Phi)N_0 = N_1(L)N_0(L)$ in the sense that

$$N_1(L) \times N_0(L) \ni (x, y) \mapsto xy \in N(\Phi)N_0$$

is a diffeomorphism (see e.g. [2]). Let $\pi_1 : S \mapsto N_1(L)$ be given by $\pi_1(xya) = x$ for $x \in N_1(L)$, $y \in N_0(L)$, and $a \in A$. The space $\mathcal{H}_L$ of bounded L–harmonic functions on S is characterized in the following way.

Theorem 4.1. (see e.g. [2]) *There is a unique positive bounded smooth function ν on $N_1(L)$ with $\int_{N_1(L)} \nu(x) dx = 1$ such that the bounded L–harmonic functions F on S are in one-to-one correspondence with functions f in $L^\infty(N_1(L))$ via the Poisson integral*

$$F(s) = \int_{N_1(L)} f(\pi_1(sx)) v(x) dx.$$

$N_1(L)$ is the maximal boundary for L.

As a straightforward generalization of Lemma 2.1 from [3], we get the following

Lemma 4.2. ([3]) *There exist positive numbers $\gamma_1, \dots, \gamma_{r-1}$ such that, if Y is the $\mathcal{A}$ component of the first order part of*

$$\mathbf{L} = L + \sum_{i=1}^{r} \gamma_i \triangle_i,$$

then $(\lambda_i - \lambda_j)(Y) \geq 0$ for all $i < j$.

We fix such an operator $\mathbf{L}$. Let F be a function on $\mathcal{D}$ annihilated by $\mathbf{L}$ which satisfies

$$\sup_{s\in S_0}\int_{N(\Phi)}|F((\zeta,x)s)|^2 d\zeta dx<\infty. \qquad (\mathcal{H}^2)$$

By Lemma 4.2, $N_1(\mathbf{L})=N(\Phi)$, and so there exists $f\in L^2(N(\Phi))$ such that

$$F(s)=\int_{N(\Phi)}f(\pi_1(su))\nu(u)du=\operatorname{Det}s^{-1}\int_{N(\Phi)}f(u)\nu(s^{-1}\cdot u)du$$

where $s\in S$ and $\operatorname{Det}s$ is the determinant of the adjoint action $(\zeta,x)\mapsto s(\zeta,x)s^{-1}$ on $N(\Phi)$. Let

$$P_{ya}(\zeta,x)=P((\zeta,x)ya)=\operatorname{Det}(ya)^{-1}\check{\nu}((ya)^{-1}\cdot(\zeta,x)) \qquad (28)$$

where $\check{\nu}(\zeta,x)=\nu((\zeta,x)^{-1})$. Then

$$\mathbf{L}P((\zeta,x)ya)=0. \qquad (29)$$

For a function F on S and a fixed $s\in S$, we define the function F_s on $N(\Phi)$ by putting $F_s(\zeta,x)=F((\zeta,x)s)$. Then $F_s(\zeta,x)=f*P_s(\zeta,x)$. For F satisfying $(\mathcal{H}^2)$, the operator $U^\lambda_{F_s}$ given by

$$(U^\lambda_{F_s}\xi,\eta)_\lambda=\int_{N(\Phi)}F_s(\zeta,x)(U^\lambda_{(\zeta,x)}\xi,\eta)_\lambda d\zeta dx$$

is defined for almost every λ, and it is a Hilbert-Schmidt operator on $\mathcal{H}_\lambda$.

Let D be a left-invariant differential operator on S. By Harnack's inequality,

$$|DP((\zeta,x)ya)|\le cP((\zeta,x)ya) \qquad (30)$$

with a constant $c=c(D)$ independent of $(\zeta,x)ya\in S$. Hence,

$$DF((\zeta,x)ya)=f*DP((\zeta,x)ya), \qquad (31)$$

and so DF satisfies $(\mathcal{H}^2)$. Thus

$$U^\lambda_{(DF)_s}=U^\lambda_f U^\lambda_{(DP)_s}. \qquad (32)$$

Moreover, applying the differential operator D to the variable s, we get

$$D(U^\lambda_{F_s}\xi,\eta)_\lambda=(U^\lambda_{(DF)_s}\xi,\eta)_\lambda. \qquad (33)$$

In particular, by (29), $\mathbf{L}(U^\lambda_{P_s}\xi,\eta)_\lambda=0$. Hence, by (29), $s\mapsto(U^\lambda_{P_s}\xi,\eta)$ is real analytic for almost every $\lambda\in\Lambda$ and all $\xi,\eta\in\mathcal{H}_\lambda$.

Admissible operators on the Fourier transform side. Let $\widetilde{\mathcal{X}}_{j\alpha}$, $\widetilde{\mathcal{Y}}_{j\alpha}$, $\widetilde{X}^\alpha_{ij}$, $\widetilde{X}_j$ be a basis of the Lie algebra of $N(\Phi)$ corresponding to the vectors $\mathcal{X}_{j\alpha}$, $\mathcal{Y}_{j\alpha}$, X^α_{ij}, X_j in $\mathcal{S}$ parallel to $N(\Phi)$. For a function F on S and any $X\in N(\Phi)$, we have

$$(XF)((\zeta,x)ya)=\operatorname{Ad}_{ya}(\widetilde{X})F_{ya}(\zeta,x) \qquad (34)$$

with $(\zeta,x)\in N(\Phi)$ and $ya\in S_0$. Then

Proposition 4.3. *The Fourier transform $U^\lambda_{P_s}$ satisfies*

$$(U^\lambda_{(\triangle_j P)_{ya}}\xi,\eta)_\lambda = \left(-4\pi^2(\lambda|\operatorname{Ad}_{ya}\widetilde{X}_j)^2 + H_i^2 - \frac{1}{\sqrt{m_j}}H_j\right)(U^\lambda_{P_{ya}}\xi,\eta)_\lambda, \tag{35}$$

$$(U^\lambda_{(\triangle^\alpha_{ij} P)_{ya}}\xi,\eta)_\lambda = \left(-4\pi^2(\lambda|\operatorname{Ad}_{ya}\widetilde{X}^\alpha_{ij})^2 + \left(Y^\alpha_{ij}\right)^2 - \frac{1}{\sqrt{m_i}}H_i\right)(U^\lambda_{P_{ya}}\xi,\eta)_\lambda, \tag{36}$$

for all $\xi,\eta\in\mathcal{H}^\lambda$, $ya\in S_0$, and almost every $\lambda\in\Lambda$.

Proof. Since

$$U^\lambda_{\exp(-t\operatorname{Ad}_{ya}\widetilde{X}_j)} = e^{2\pi i t(\lambda|\operatorname{Ad}_{ya}(\widetilde{X}_j))}I,$$

so, by (30) and (34), we have

$$\begin{aligned}U^\lambda_{(X_jP)_{ya}} &= \frac{d}{dt}\int_{N(\Phi)} P_{ya}((\zeta,x)\exp(t\operatorname{Ad}_{ya}(\widetilde{X}_j)))U^\lambda_{(\zeta,x)}d\zeta dx\bigg|_{t=0}\\ &= 2\pi i(\lambda|\operatorname{Ad}_{ya}(\widetilde{X}_j))U^\lambda_{P_{ya}}.\end{aligned}$$

Using again (30) and (33) for $H_j^2 - \frac{1}{\sqrt{m_j}}H_j$, we get (35). The proof of (36) is similar. ∎

For $\lambda\in\Omega^\star\cup-\Omega^\star$, we define

$$\Phi^\lambda_{\alpha,\beta}(\zeta,x) = (U^\lambda_{(\zeta,x)}\xi^\lambda_\alpha,\xi^\lambda_\beta)_\lambda.$$

Let $j\in\{1,\dots,r\}$ be such that $d_j>0$. We consider the left invariant operator $\mathcal{L}_j$ on S given by

$$\mathcal{L}_j = \sum_{\alpha=1}^{d_j}(\mathcal{X}_{j\alpha})^2 + (\mathcal{Y}_{j\alpha})^2,$$

and the corresponding operator on $N(\Phi)$

$$L_j = \sum_{\alpha=1}^{d_j}\left(\widetilde{\mathcal{X}}_{j\alpha}\right)^2 + \left(\widetilde{\mathcal{Y}}_{j\alpha}\right)^2.$$

We state the following

Lemma 4.4. *Let $\lambda\in\Omega^\star\cup-\Omega^\star$ and $y(\lambda)\in N_0$ be such that $\lambda = y^\star(\lambda)\circ\lambda^0$ with $\lambda^0=\sum_{j=1}^r\lambda^0_jc_j$, $\lambda^0_1\dots,\lambda^0_r\in\mathbb{R}$. Then*

$$(U^\lambda_{(\mathcal{L}_jP)_{y(\lambda)^{-1}a}}\xi^\lambda_\alpha,\xi^\lambda_\beta)_\lambda = -2\pi a_j|\lambda^0_j|\frac{2|\alpha_j|+d_j}{m_j}(U^\lambda_{P_{y(\lambda)^{-1}a}}\xi^\lambda_\alpha,\xi^\lambda_\beta)_\lambda, \tag{37}$$

$$(U^\lambda_{(X_j^2P)_{y(\lambda)^{-1}a}}\xi^\lambda_\alpha,\xi^\lambda_\beta)_\lambda = -4\pi^2(\lambda^0_j)^2a_j^2(U^\lambda_{P_{y(\lambda)^{-1}a}}\xi^\lambda_\alpha,\xi^\lambda_\beta)_\lambda. \tag{38}$$

Proof. By (27) and (34), we have

$$(U^\lambda_{(\mathcal{L}_j P)_{y(\lambda)^{-1}a}}\xi^\lambda_\alpha,\xi^\lambda_\beta)_\lambda = a_j\int_{N(\Phi)}\widetilde{\mathcal{L}}_j P_{y(\lambda)^{-1}a}(\zeta,x)\Phi^{\lambda^0}_{\alpha,\beta}(y\cdot\zeta,y\circ x)d\zeta dx$$
$$= a_j(\widetilde{\mathcal{L}}_j P_{y(\lambda)^{-1}a}, y\cdot\Phi^{\lambda^0}_{\alpha,\beta})$$

where

$$\widetilde{\mathcal{L}}_j = \sum_\alpha(\mathrm{Ad}_{y(\lambda)^{-1}}\widetilde{\mathcal{X}}_{j\alpha})^2+(\mathrm{Ad}_{y(\lambda)^{-1}}\widetilde{\mathcal{Y}}_{j\alpha})^2.$$

Given a function g on $N(\Phi)$ let

$$y(\lambda)\circ g(\zeta,x) = g(\sigma(y(\lambda))\cdot\zeta, y(\lambda)\circ x).$$

For any left-invariant vector field X on S, we have

$$\mathrm{Ad}_{y(\lambda)^{-1}a}\widetilde{X}\left(y(\lambda)\cdot\Phi^{\lambda^0}_{\alpha,\beta}\right) = y(\lambda)\cdot\widetilde{X}\Phi^{\lambda^0}_{\alpha,\beta}.$$

Hence,

$$\widetilde{\mathcal{L}}_j\left(y(\lambda)\cdot\Phi^{\lambda^0}_{\alpha,\beta}\right) = y(\lambda)\cdot L_j\Phi^{\lambda^0}_{\alpha,\beta} = -2\pi a_j|\lambda^0_j|\frac{2|\alpha_j|+d_j}{m_j}y(\lambda)\cdot\Phi^{\lambda^0}_{\alpha,\beta},$$

since, by [10] Section 2.1,

$$L_j\Phi^{\lambda^0}_{\alpha,\beta} = -2\pi|\lambda^0_j|\frac{2|\alpha_j|+d_j}{m_j}\Phi^{\lambda^0}_{\alpha,\beta}$$

where the factor $\frac{1}{m_j}$ follows from our normalization (18). Moreover, $\Phi^{\lambda^0}_{\alpha,\beta}$ and all its left-invariant derivatives are bounded functions (see [3]). Thus we have

$$(\widetilde{\mathcal{L}}_j P_{y(\lambda)^{-1}a}, y(\lambda)\cdot\Phi^{\lambda^0}_{\alpha,\beta}) = (P_{y(\lambda)^{-1}a},\widetilde{\mathcal{L}}_j(y(\lambda)\cdot\Phi^{\lambda^0}_{\alpha,\beta}))$$
$$= -2\pi a_j|\lambda^0_j|\frac{2|\alpha_j|+d_j}{m_j}(P_{y(\lambda)^{-1}a}, y(\lambda)\cdot\Phi^{\lambda^0}_{\alpha,\beta})$$

which proves (37). The proof of (38) is similar. ∎

Fourier transform of the Poisson kernel.

Theorem 4.5. *For a fixed $s\in S_0$ the partial Fourier transform*

$$J^\star\ni\lambda\mapsto P_s(\hat\lambda)$$

is smooth and bounded on $J^\star$.

Proof. By (35) and (36),

$$0 = \widehat{\mathbf{L}P}(\lambda,s) = \mathcal{L}P_s(\hat\lambda)$$

where $\mathcal{L}$ is an elliptic operator with analytic coefficients applied to the variable s. Hence, for a fixed $\lambda\in V$ the function

$$s\mapsto P_s(\hat\lambda) \tag{39}$$

is real analytic.

Let $\lambda\in J^\star$. There exist $t^\star(\lambda)\in S_0^\star$ and $\epsilon\in\{-1,1\}^r$ such that $\lambda = t^\star(\lambda)\circ\eta$ with $\eta=\sum_{i=1}^r\epsilon_i c_i$. Since

$$P_s(t^{-1}(\lambda)\circ\eta) = \mathrm{Det}\, t(\lambda)P_{t(\lambda)s}(\eta),$$

we have $P_s(\hat\lambda) = P_{t(\lambda)s}(\hat\eta)$. The conclusion follows now by Theorem 2.9. ∎

5. Pluriharmonic $\mathcal{H}^2$ functions

Now we are ready to prove our main theorem.

Theorem 5.1. *Let $\mathcal{D}$ be a homogeneous Siegel domain, and let F be a real function on $\mathcal{D}$ such that*

$$\sup_{s\in S_0}\int_{N(\Phi)}|F((\zeta,x)s)|^2 d\zeta dx<\infty.$$

Assume that F is annihilated by an elliptic admissible operator L. Let

$$\mathbf{H}=\sum_{i=1}^{r}\gamma_i\triangle_i,\quad \gamma_i>0, \tag{40}$$

$$\mathcal{L}=\sum_{\substack{1\le i\le r\\ \alpha}}\gamma_i\mathcal{L}_i^{\alpha}, \tag{41}$$

be such that the maximal boundary for $\mathbf{L}=L+\mathbf{H}$ is the Shilov boundary. If $\mathbf{H}F=0$ and $\mathcal{L}F=0$, then F is the real part of a holomorphic $\mathcal{H}^2$ function.

For a tube domain, the condition $\mathcal{L}F=0$ is void.

Proof. First, we show that $U^{\lambda}_{F_s}=0$ for every $\lambda\in\overline{\Omega^\star}\cup-\overline{\Omega^\star}$ and every $s\in S_0$.

Let P be the Poisson kernel for $\mathbf{L}$ defined by (28). There exists $f\in L^2(N(\Phi))$ such that

$$F_s(\zeta,x)=f*P_s(\zeta,x).$$

Thus for almost all $\lambda\in\Lambda$ $U^{\lambda}_{F_s}=U^{\lambda}_{f}U^{\lambda}_{P_s}$, and, by (39), the mapping

$$s\mapsto(U^{\lambda}_{F_s}\xi,\eta)_\lambda \tag{42}$$

is real analytic. By Proposition 2.10 and the formula (31),

$$(U^{\lambda}_{(\mathbf{H}F)_{ya}}\xi,\eta)_\lambda=\sum_{i=1}^{r}\gamma_i\left(-4\pi^2(\lambda|\operatorname{Ad}_{ya}\widetilde{X}_i)^2+H_i^2-\frac{1}{\sqrt{m_i}}H_i\right)(U^{\lambda}_{F_{ya}}\xi,\eta)_\lambda.$$

Thus

$$\sum_{i=1}^{r}\gamma_i\left(-4\pi^2(\lambda|\operatorname{Ad}_{ya}\widetilde{X}_i)^2+H_i^2-\frac{1}{\sqrt{m_i}}H_i\right)(U^{\lambda}_{F_{ya}}\xi,\eta)_\lambda=0.$$

Writing H_i in coordinates, we get

$$H_i((\zeta,x)ya)=\frac{a_i}{\sqrt{m_i}}\partial_{a_i}. \tag{43}$$

Hence $(U^{\lambda}_{F_{ya}}\xi,\eta)_\lambda$ satisfies the following differential equation

$$\sum_{i=1}^{r}\frac{\gamma_i}{m_i}a_i^2(-W_i(\lambda,y)^2+\partial^2_{a_i})(U^{\lambda}_{F_{ya}}\xi,\eta)_\lambda=0.$$

Therefore (see e.g. [3]),

$$(U^{\lambda}_{F_{ya}}\xi,\eta)_\lambda=c(\lambda,y)e^{-\sum_{i=1}^{r}a_i|W_i(\lambda,y)|}$$

with $c(\lambda, y) = \lim_{a_j \to 0}(U^{\lambda}_{F_{ya}}\xi, \eta)_{\lambda}$ for $j = 1, \dots, r$. Since P_{ya} is an approximate identity for $a \to 0$, we get $c(\lambda, y) = (U^{\lambda}_{f}\xi, \eta)_{\lambda}$, and so

$$(U^{\lambda}_{F_{ya}}\xi, \eta) = (U^{\lambda}_{f}\xi, \eta)_{\lambda} e^{-\sum_{i=1}^{r} a_i |W_i(\lambda, y)|}. \tag{44}$$

Let $\lambda \in J^{\star}$, $\lambda \notin \overline{\Omega^{\star}} \cup -\overline{\Omega^{\star}}$ be such that $(U^{\lambda}_{f}\xi, \eta)_{\lambda} \neq 0$. Then, by Theorem 2.11, there is $1 \leq i \leq r$ such that $W_i(\lambda, y)$ changes sign. Therefore, $(U^{\lambda}_{F_{ya}}\xi, \eta)_{\lambda}$ cannot be smooth as a function of ya at the points y for which $W_i(\lambda, y) = 0$. This contradicts (42).

Thus for $\lambda \notin \overline{\Omega^{\star}} \cup -\overline{\Omega^{\star}}$ $(U^{\lambda}_{f}\xi, \eta)_{\lambda} = 0$.

We fix $\lambda \in \Omega^{\star} \cup -\Omega^{\star}$. By (40) and (41),

$$\sum_{\substack{1 \leq i \leq r \\ \alpha}} \gamma_i (U^{\lambda}_{(\mathcal{L}^{\alpha}_i F)_{ya}}\xi, \eta)_{\lambda} = 0, \quad \sum_{i=1}^{r} \gamma_i (U^{\lambda}_{(\triangle_i F)_{ya}}\xi, \eta)_{\lambda} = 0,$$

for every $\xi, \eta \in \mathcal{H}_{\lambda}$. Let $y(\lambda) \in N_0$ and $\lambda^0_1 \dots, \lambda^0_r \in \mathbb{R}$ be such that

$$\lambda = y^{\star}(\lambda) \circ \lambda^0$$

with $\lambda^0 = \sum_{j=1}^{r} \lambda^0_j c_j$. Then, by Lemma 4.4,

$$\sum_{i=1}^{r} \gamma_i \left(-2\pi a_i |\lambda^0_i| \frac{2|\alpha_i| + d_i}{m_i} - \frac{d_i}{\sqrt{m_i}} H_i \right) (U^{\lambda}_{F_{y(\lambda)^{-1}a}} \xi^{\lambda}_{\alpha}, \xi^{\lambda}_{\beta})_{\lambda} = 0,$$
$$\sum_{i=1}^{r} \gamma_i \left(-4\pi^2 (\lambda^0_i)^2 a_i^2 + H_i^2 - \frac{1}{\sqrt{m_i}} H_i \right) (U^{\lambda}_{F_{y(\lambda)^{-1}a}} \xi^{\lambda}_{\alpha}, \xi^{\lambda}_{\beta})_{\lambda} = 0.$$

Writing H_i in coordinates (43), we get

$$\sum_{i=1}^{r} \gamma_i \left(-2\pi a_i |\lambda^0_i| \frac{2|\alpha_i| + d_i}{m_i} - d_i a_i \partial_{a_i} \right) (U^{\lambda}_{F_{y(\lambda)^{-1}a}} \xi^{\lambda}_{\alpha}, \xi^{\lambda}_{\beta})_{\lambda} = 0, \tag{45}$$
$$\sum_{i=1}^{r} \gamma_i \left(-4\pi^2 |\lambda^0_i|^2 a_i^2 + \partial^2_{a_i} \right) (U^{\lambda}_{F_{y(\lambda)^{-1}a}} \xi^{\lambda}_{\alpha}, \xi^{\lambda}_{\beta})_{\lambda} = 0. \tag{46}$$

Solving (46) (see e.g. [3]), we get

$$(U^{\lambda}_{F_{y(\lambda)^{-1}a}} \xi^{\lambda}_{\alpha}, \xi^{\lambda}_{\beta})_{\lambda} = c(\lambda, y) e^{-2\pi \sum_{i=1}^{r} a_i |\lambda^0_i|}.$$

Then plugging it into the first equation, we obtain $(U^{\lambda}_{F_{y(\lambda)^{-1}a}} \xi^{\lambda}_{\alpha}, \xi^{\lambda}_{\beta})_{\lambda} = 0$ for all β and $\alpha \neq 0$. Again, since P_{ya} is an approximate identity for $a \to 0$, we get $(U^{\lambda}_{f} \xi^{\lambda}_{\alpha}, \xi^{\lambda}_{\beta})_{\lambda} = 0$. To finish the proof of Theorem 5.1, we use the following

Lemma 5.2. (see e.g. [3])

$$U^{\lambda}_{\check{P}_{ya}} \xi_0 = \left\{ \begin{array}{ll} e^{-2\pi(\lambda | ya\circ e)} \xi_0 & if \lambda \in \Omega^{\star} \quad e^{2\pi(\lambda | ya\circ e)} \xi_0 & if \lambda \in -\Omega^{\star}. \end{array} \right. \tag{47}$$

By (47) and the Fourier inversion formula (see [9]),

$$F((\zeta,x)ya) = \int_{\Omega^\star\cup-\Omega^\star} \mathrm{Tr}(U^\lambda_{(-\zeta,-x)}U^\lambda_{f*P_{ya}})\rho(\lambda)d\lambda$$
$$= \int_{\Omega^\star\cup-\Omega^\star} (U^\lambda_{f*P_{ya}}\xi_0, U^\lambda_{(\zeta,x)}\xi_0)_\lambda\rho(\lambda)d\lambda.$$

We define a function G on S by the formula

$$G((\zeta,x)ya) = \int_{\Omega^\star} e^{-2\pi(\lambda|ya\circ e)}(U^\lambda_f\xi_0, U^\lambda_{(\zeta,x)}\xi_0)_\lambda\rho(\lambda)d\lambda.$$

Then G is a holomorphic $\mathcal{H}^2$ function (see e.g. [3]). Moreover,

$$\overline{G}((\zeta,x)ya) = \int_{-\Omega^\star} e^{2\pi(\lambda|ya\circ e)}(U^\lambda_f\xi_0, U^\lambda_{(\zeta,x)}\xi_0)_\lambda\rho(\lambda)d\lambda.$$

Hence, by Lemma 5.2,

$$F((\zeta,x)ya) = G((\zeta,x)ya) + \overline{G}((\zeta,x)ya),$$

and the conclusion follows. ■

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