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Strong Tits Alternative for Subgroups of Coxeter Groups

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Abstract. It is proved that any subgroup of a Coxeter group has a finite index subgroup which either is abelian, or has a non-abelian free quotient.

1. Introduction

A group is called *large* if a subgroup of finite index in it has a non-abelian free quotient. Largeness of some classes of groups was proved in [1], [7]. [6], [14], [4]. [9]. [8], [2], [10]. In particular, in [10] (see also [5]) it was proved that any non-affine infinite indecomposable Coxeter group of finite rank is large. We prove

Theorem 1.1. *Any subgroup Γ of a Coxeter group W of finite rank is either large or virtually abelian.*

We make use of the technique of trees of [10] and the Moussong complex of a Coxeter group [12].

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2. Piecewise Euclidean cell complexes

A *Euclidean cell* is a compact intersection of a finite number of affine half-spaces in $\mathbb{R}^n$. A *piecewise Euclidean cell complex* X is a connected locally finite cell complex made up by gluing together Euclidean cells via isometries of their faces. The canonical metric in each cell allows to measure the lengths of finite polygonal paths in X. The *path metric* d on X is defined by setting the distance between $x, y \in X$ to be the infimum of the lengths of polygonal paths joining x to y. Under assumption that there are only finitely many isometry types of cells in X, the path metric is proper, that is any closed bounded subset in X is compact and, hence, there is a length minimizing path between any two points in X ([3], Chapter I.7).

Let Δ be a geodesic triangle in X. The *comparison triangle* for Δ is the Euclidean triangle Δ' with the same side lengths as Δ. We say that X is

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nonpositively curved, if for any geodesic triangle Δ in X and two points x, y on Δ, the distance between x and y in X is less than or equal to the Euclidean distance between the corresponding points x', y' on the comparison triangle Δ'. The nonpositive curvature condition implies the uniqueness of geodesics and the geodesicity of local geodesics.

Let X be a nonpositively curved piecewise Euclidean cell complex with finitely many isometry types of cells and g be an isometry of X without fixed points. An *axis* of g is a g-invariant bi-infinite geodesic (on which g acts as a non-trivial translation). The existence of an axis can be proved as follows. Let $d_g : X \to \mathbb{R}_+$ be the *displacement function* of g defined by $d_g(x) = d(gx, x)$. Assume that it attains a (positive) minimum at some point $x \in X$. Then the union of the geodesic segments $[g^n x, g^{n+1} x]_{n \in \mathbb{Z}}$ is an axis of g [3].

It is easy to see that the translation length is one and the same for all axes of g [3]. It follows that, if the function d_g attains a minimum, any axis is obtained in the way described above. The existence of a minimum for the displacement function can be proved by means of the following

Lemma 2.1. ([3], Proposition 6.10(2)) *Let Γ be a cocompact discrete group of isometries of a proper metric space X. Then, for any $g \in \Gamma$, the displacement function d_g attains a minimum.*

Proof. Let $|g| = \inf\{d_g(x) : x \in X\}$, and let D be a fundamental domain of Γ. Then there exist $x_n \in D$ and $g_n \in \Gamma$ $(n \in \mathbb{N})$ such that $d_g(g_n x_n) \to |g|$. Since D is compact, we may assume that x_n tends to some $x_0 \in D$. Then $d_g(g_n x_0) \to |g|$. We have $d_g(g_n x_0) = d(g_n^{-1} g g_n x_0, x_0)$. Since the points $g_n^{-1} g g_n x_0$ lie in a bounded subset, there are only finitely many of them. Passing to a subsequence, we may assume that $g_n^{-1} g g_n x_0 = g_0^{-1} g g_0 x_0$ for some $g_0 \in \Gamma$ and all $n \in \mathbb{N}$. Then $d_g(g_0 x_0) = d(g_0^{-1} g g_0 x_0, x_0) = |g|$, so the function d_g attains a minimum at $g_0 x_0$. ∎

3. Moussong complex of a Coxeter group

In [12] G. Moussong constructed for any Coxeter group W of finite rank a contractible piecewise Euclidean complex $\mathcal{M}_W$ of nonpositive curvature on which W acts discretely and cocompactly by isometries. Moreover, there are only finitely many isometry types of cells in $\mathcal{M}_W$. In particular, $\mathcal{M}_W$ is proper relative to the path metric. All the cells with their Euclidean metric are isometrically embedded into $\mathcal{M}_W$.

If W is finite, then $\mathcal{M}_W$ is just one cell, which is obtained as the convex hull P of the W-orbit of a suitable point p in the standard linear representation of W as a group generated by reflections. The point p is chosen in such a way that its stabilizer in W be trivial and all the edges of P be of length 1. The faces of P are naturally identified with the Moussong complexes of the subgroups of W conjugate to standard subgroups.

In the general case, the complex $\mathcal{M}_W$ is built up of the Moussong complexes of maximal finite subgroups of W gluing together along their faces corresponding to common finite subgroups. The 1-skeleton of $\mathcal{M}_W$, considered as a combinatorial graph is isomorphic to the Cayley graph $\mathcal{C}_W$ of W with respect to the standard generating set.

The *walls* in $\mathcal{M}_W$ are the fix point sets of reflections in W. The intersections of a wall Hm with cells of $\mathcal{M}_W$ supply H with a structure of a piecewise Euclidean cell complex with finitely many isometry types of cells. Any geodesic joining two points of H entirely lies in H. It follows that the induced metric of H coincides with its path metric. Hence, H is nonpositively curved. The same is true for any non-empty intersection of walls.

Lemma 3.1. *Let $H_1, ..., H_k$ be walls such that $H_1 \cap ... \cap H_k \neq \emptyset$ and, for any i, H_i does not contain $H_1 \cap ... \cap H_{i-1}$. Then the codimension of $H_1 \cap ... \cap H_k$ in $\mathcal{M}_W$ is at least k.*

Proof. Let r_i be the reflection in H_i. It follows from the condition that $r_1, ..., r_k$ generate a finite subgroup and, for any i, r_i is not contained in the subgroup generated by $r_1, ..., r_{i-1}$. Let C be the cell defined by any maximal finite subgroup of W containing $r_1, ..., r_k$. Then, for any i, $H_i \cap C$ does not contain $H_1 \cap ... \cap H_{i-1} \cap C$. Hence, the codimension of $H_1 \cap ... \cap H_k \cap C$ in C is k. Since the dimension of any such C does not exceed the dimension of $\mathcal{M}_W$, the assertion of the lemma follows. ∎

Lemma 3.2. *Let an element $g \in W$ of infinite order leave invariant a non-empty intersection of walls $H_1, ..., H_k$. Then g has an axis lying in $H_1 \cap ... \cap H_k$.*

Proof. Consider an axis of the restriction of g to $H_1 \cap ... \cap H_k$. It is a geodesic in $\mathcal{M}_W$ and hence an axis of g in $\mathcal{M}_W$. ∎

The following lemma is well-known: see, e.g., [11], Lemma 2.1.

Lemma 3.3. *Let W_0 be a normal torsionfree subgroup in W. Then for any $g \in W_0$ and any wall H either $gH = H$ or $gH \cap H = \emptyset$.*

Proof. Let $r \in W$ be the reflection in H. Then grg^{-1} is the reflection in gH. If $gH \cap H \neq \emptyset$, the element $grg^{-1}r$ fixes $gH \cap H$ pointwise. By the choice of W_0 it is trivial and thus g commutes with r, so $grg^{-1} = r$ and $gH = H$. ∎

Any wall in the Moussong complex is "totally geodesic" in the following sense [13].

Lemma 3.4. *Any geodesic containing at least two points of a wall H entirely lies in H.*

Proof. Let two consecutive segments σ_1, σ_2 of a geodesic lie in some cells C_1, C_2 respectively, and their common point p lie in the intersection C_0 of C_1 and C_2. Let us prove that the orthogonal projections σ_1', σ_2' of σ_1, σ_2 (or, maybe, some smaller segments) to C_0 continue one another. If it is not so, then moving p along the bisector of the angle formed by σ_1' and σ_2', we obtain a shorter path which contradicts the geodesicity. Now, if σ_1 lies on a wall H, then (since H is perpendicular to C_0) σ_1' and, hence, σ_2' lie on H as well. But then σ_2 also lies on H. This proves the lemma. ∎

Any wall divides $\mathcal{M}_W$ into two connected components [13]. All the walls yield a decomposition of $\mathcal{M}_W$ into (closed) convex sets called *chambers*. The set of all chambers with an appropriate adjacency relation is isomorphic to the Cayley graph $\mathcal{C}_W$. The Cayley graph distance between two chambers equals the number of walls separating these chambers.

We say that a wall H and a bi-infinite geodesic A are *transversal* if $H \cap A$ is a one point set.

Lemma 3.5. *For any bi-infinite geodesic A there is a wall, transversal to A.*

Proof. Since the geodesic A cannot lie in one chamber, there are two points of it that do not belong to one chamber. These points are separated by a wall, which is thereby transversal to A by Lemma 3.4. ■

4. Trees

Let Γ be a subgroup in a Coxeter group W and let H be a wall in $\mathcal{M}_W$ such that for any $g \in \Gamma$ either $gH = H$ or $gH \cap H = \emptyset$. The walls $gH, g \in \Gamma$, yield a partition of $\mathcal{M}_W$ into (closed) convex sets, which we shall call Γ-*chambers*. Consider the graph $\mathcal{T} = \mathcal{T}(\Gamma, H)$, whose vertices (resp. edges) are the Γ-chambers (resp. the walls $gH, g \in \Gamma$) and the incidence is defined by inclusion. Clearly, $\mathcal{T}$ is a tree and Γ acts naturally on $\mathcal{T}$. By construction this action is transitive on the set of edges. The following assertion may be applied to prove that Γ is large.

Proposition 4.1. ([10], Proposition 2) *Let a group Γ act on a tree $\mathcal{T}$ which is not a star or a line. Suppose that Γ does not reverse edges and its action on the set of edges is transitive and residually finite. Then Γ is large.*

Recall that an action of a group Γ on the set X is called *residually finite*, if for any different $x, x' \in X$ there exist an action of Γ on a finite set F and a Γ-equivariant map $f : X \to F$ such that $f(x) \neq f(x')$. One sees easily that if an action $\Gamma : X$ is residually finite then the induced actions $\Delta : X$, where Δ is a subgroup of Γ, and $\Gamma : Y$, where Y is an invariant subset of X, are also residually finite. These properties and the fact that the action of a Coxeter group of finite rank on the set of reflections (= the action on the set of walls) is residually finite [10] imply the following

Proposition 4.2. *Let Γ be a subgroup in a Coxeter group W of finite rank and let H be a wall in $\mathcal{M}_W$ such that for any $g \in \Gamma$ either $gH = H$ or $gH \cap H = \emptyset$. Then the action of Γ on the set of edges of the tree $\mathcal{T}(\Gamma, H)$ is residually finite.*

5. Proof of Theorem

Since the group W is linear, it contains a torsionfree normal subgroup W_0 of finite index. Replacing Γ with $\Gamma \cap W_0$, we may assume that $\Gamma \subset W_0$. Then by Lemma 3.3 for any wall H and any $g \in \Gamma$ either $gH = H$ or $gH \cap H = \emptyset$.

Take any $g \in \Gamma, g \neq e$. Let A be an axis of g. By Lemma 3.5 there is a wall H transversal to A. If H meets A at a point p, then the points $g^{-1}p, gp$

and, hence, the walls $g^{-1}H, gH$ are on different sides of H. It follows that the corresponding edges of the tree $\mathcal{T} = \mathcal{T}(\Gamma, H)$ cannot have a common vertex. This shows that $\mathcal{T}$ is not a star. Possibly passing to a subgroup of index 2, we may assume that Γ does not reverse edges of $\mathcal{T}$ (see Lemma 3 in [10]). If $\mathcal{T}$ is not a line, then Γ is large by Proposition 4.1.

Let us assume now that the tree $\mathcal{T} = \mathcal{T}(\Gamma, H)$ *is a line for any wall* H *transversal to an axis of some element of* Γ.

Take any nontrivial element $g_1 \in \Gamma$. Let A_1 be an axis of it, and H_1 a wall transversal to A_1. If the stabilizer Γ_1 of H_1 in Γ is not trivial, take any nontrivial element $g_2 \in \Gamma_1$. By Lemma 3.2 it has an axis A_2 contained in H_1. Let H_2 be a wall transversal to A_2. Obviously, H_2 does not contain H_1. Let Γ_2 be the intersection of the stabilizers of H_1 and H_2 in Γ. If Γ_2 is not trivial, take any nontrivial element $g_3 \in \Gamma_2$. Let $A_3 \subset H_1 \cap H_2$ be an axis of it, and H_3 a wall transversal to A_3. Obviously, H_3 does not contain $H_1 \cap H_2$. Let Γ_3 be the intersection of the stabilizers of H_1, H_2, and H_3 in Γ. If Γ_3 is not trivial, take any nontrivial element $g_4 \in \Gamma_3$, and so on. The procedure must terminate due to Lemma 3.1.

Thus, we obtain walls $H_1, H_2, ..., H_k$ transversal to axes of some elements of Γ such that the intersection of their stabilizers in Γ is trivial. According to our assumption, the tree $\mathcal{T}(\Gamma, H_i)$ is a line for any i. The natural action of Γ on it defines a homomorphism ϕ_i of Γ to the infinite dihedral group. The intersection of the kernels of these homomorphisms lies in the intersection of the stabilizers of the walls $H_1, H_2, ..., H_k$ and hence is trivial. Possibly passing to a subgroup of finite index, we may assume that the group $\phi_i(\Gamma)$ is (infinite) cyclic for any i. Then Γ is abelian.

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