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# Resolutions of Singularities of Varieties of Lie Algebras of Dimensions 3 and 4

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**Abstract.** We will determine the singular points and a resolution of singularities of each irreducible component of the varieties of the Lie algebras of dimension 3 and 4 over $\mathbb{C}$.

## 1. Introduction

Let $\mathcal{L}_n$ be the projective variety of the Lie algebras of dimension $n$ over $\mathbb{C}$. In some recent papers many results on the irreducible components of $\mathcal{L}_n$ were found for small values of $n$. In [2] Carles and Diakité determined the open orbits and described the irreducible components of $\mathcal{L}_n$ as orbit closures for $n \leq 7$. In [6] Kirillov and Neretin determined the number of irreducible components of $\mathcal{L}_n$ and their dimension for $n \leq 6$; they also determined representatives of the generic orbits of any component of $\mathcal{L}_4$. In [1] Burde and Steinhoff gave a classification of any orbit closure of $\mathcal{L}_4$. The variety $\mathcal{L}_3$ has two irreducible components and one of them is a linear variety; the variety $\mathcal{L}_4$ has four irreducible components.
In this paper we will determine the singular points and find a resolution of singularities of each irreducible component of $\mathcal{L}_3$ and $\mathcal{L}_4$. By using the classification of the Lie algebras of dimension 3 and 4 over $\mathbb{C}$, we will describe each irreducible component by giving algebraic equations of it. The first classification is well known (see [3]); the second one may be deduced from [8] and from [9] (see [1]); nevertheless we will give a short proof of it. Each resolution of singularities is a subvariety of the product of the irreducible component with a suitable grassmannian or is a resolution of singularities of a variety of this type. We observe that the results of this paper are also true over any algebraically closed field $K$ such that char $K \neq 2$.

## 2. Preliminaries

For any $n \in \mathbb{N}$ let $\mathcal{L}_n$ be the subvariety of the projective space

$$\mathbb{P}(\mathrm{Hom}(\mathbb{C}^n \wedge \mathbb{C}^n, \mathbb{C}^n))$$

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of all $[\alpha]$ such that $\alpha(x \wedge \alpha(y \wedge z)) + \alpha(y \wedge \alpha(z \wedge x)) + \alpha(z \wedge \alpha(x \wedge y)) = 0$ for any $x, y, z \in \mathbb{C}^n$, which we regard as the variety of all the Lie algebras over $\mathbb{C}$ of dimension $n$. For any $[\alpha] \in \mathcal{L}_n$ let $L_\alpha$ be the Lie algebra defined by $\alpha$. The group $\mathrm{GL}(n, \mathbb{C})$ acts on $\mathrm{Hom}(\mathbb{C}^n \wedge \mathbb{C}^n, \mathbb{C}^n)$ by the relation $\alpha \cdot G(Gx \wedge Gy) = G(\alpha(x \wedge y))$, for any $G \in \mathrm{GL}(n, \mathbb{C})$, $\alpha \in \mathrm{Hom}(\mathbb{C}^n \wedge \mathbb{C}^n, \mathbb{C}^n)$, $x, y \in \mathbb{C}^n$ and this induces an action of $\mathrm{GL}(n, \mathbb{C})$ on $\mathcal{L}_n$; the orbits of this action are the classes of isomorphic Lie algebras. For any $n, n' \in \mathbb{N}$ let $M_{n \times n'}$, $M_n$ and $S_n$ be the vector spaces of all $n \times n'$ matrices, of all $n \times n$ matrices and of all $n \times n$ symmetric matrices respectively over $\mathbb{C}$. Let $\{e_1, \ldots, e_n\}$ be the canonical basis of $\mathbb{C}^n$ and let us order the set $\{e_i \wedge e_j \ : \ i, j = 1, \ldots, n, \ i < j\}$, writing it as $\{E_1, \ldots, E_m\}$. For any $\alpha \in \mathrm{Hom}(\mathbb{C}^n \wedge \mathbb{C}^n, \mathbb{C}^n)$ let $A_\alpha \in M_{n \times m}$ be the matrix of $\alpha$ with respect to the previous bases; then $A_{\alpha \cdot G} = G A_\alpha \widehat{G}$ where $\widehat{G} \in \mathrm{GL}(m, \mathbb{C})$ is the matrix whose $(h, k)$ entry is the determinant of the $2 \times 2$ submatrix of $G^{-1}$ obtained by choosing the rows $i, j$ with $E_h = e_i \wedge e_j$ and the columns $i', j'$ with $E_k = e_{i'} \wedge e_{j'}$. If $n = 3$ we set $E_1 = e_2 \wedge e_3$, $E_2 = e_3 \wedge e_1$, $E_3 = e_1 \wedge e_2$ and we get $A_{\alpha \cdot G} = (\det G)^{-1} G A_\alpha G^t$. Then we have

$$\mathcal{L}_3 = \{[\alpha] \in \mathbb{P}(\mathrm{Hom}(\mathbb{C}^3 \wedge \mathbb{C}^3, \mathbb{C}^3)) \ : \ \operatorname{cof} A_\alpha \in S_3\}$$

where for any $A = (a_{ij}) \in M_n$ $\operatorname{cof} A$ is the matrix whose $(i, j)$ entry is the algebraic complement of $a_{ji}$.

We recall that, up to isomorphisms, we have the following non-abelian Lie algebras of dimension 3 over $\mathbb{C}$ ([3]), which may also be obtained as in the proof of theorem 4.1:

$$\begin{aligned}
\mathbf{l}_a \quad &: \quad [e_1, e_2] = e_2, \ [e_1, e_3] = a e_3, \ [e_2, e_3] = 0, \ a \in \mathbb{C}, \\
\mathbf{n}_3 \quad &: \quad [e_1, e_2] = [e_1, e_3] = 0, \ [e_2, e_3] = e_1, \\
\mathbf{r}_3 \quad &: \quad [e_1, e_2] = e_2, \ [e_1, e_3] = e_2 + e_3, \ [e_2, e_3] = 0, \\
\mathbf{sl}(2, \mathbb{C}) \quad &: \quad [e_1, e_2] = e_3, \ [e_1, e_3] = -2e_1, \ [e_2, e_3] = 2e_2,
\end{aligned}$$

where the only pairs of isomorphic Lie algebras are $\{\mathbf{l}_a, \mathbf{l}_{a^{-1}}\}$, $a \neq 0, a^{-1}$, and $\mathbf{n}_3$, the Heisenberg Lie algebra, is the only nilpotent one. Hence the following subvarieties:

$$\begin{aligned}
\mathcal{W}_1 &= \{[\alpha] \in \mathcal{L}_3 \ : \ A_\alpha \in S_3\} \\
&= \{[\alpha] \in \mathcal{L}_3 \ : \ \text{for any } v \in L_\alpha \ \operatorname{tr} \operatorname{ad} v = 0\},
\end{aligned}$$

which is isomorphic to $\mathbb{P}(S_3)$, and

$$\begin{aligned}
\mathcal{W}_2 &= \{[\alpha] \in \mathcal{L}_3 \ : \ \operatorname{rank} A_\alpha \leq 2\} \\
&= \{[\alpha] \in \mathcal{L}_3 \ : \ L_\alpha \text{ has an abelian ideal of dimension } 2\},
\end{aligned}$$

that is the subvariety of the solvable Lie algebras, are the irreducible components of $\mathcal{L}_3$.

For any $n, n' \in \mathbb{N}$ let $G_{n', n}$ be the grassmannian of all the subspaces of $\mathbb{C}^n$ of dimension $n'$.

## 3. The variety of the Lie algebras of dimension 3

We identify $\alpha$ with $A_\alpha$ and we set $A = (a_{ij})$ for any $A \in M_3$.

**Lemma 3.1.** *We have* $\mathcal{W}_2 = \{[A] \in \mathbb{P}(M_3) : \dim(\ker A \cap \ker A^t) \geq 1\}$.

**Proof.** Since both subsets are stable with respect to the action of $\mathrm{GL}(3, \mathbb{C})$ it is sufficient to show that if $A$ is such that $a_{j1} = 0,\ j = 1, 2, 3$, the condition $\mathrm{cof}\, A \in S_3$ is equivalent to the condition $\dim(\ker A \cap \ker A^t) \geq 1$. But in this case both these conditions are equivalent to the following one: $\mathrm{rank}\, A \leq 1$ or $a_{1j} = 0,\ j = 2, 3$; hence we get the claim. The result also follows from the classification of the Lie algebras of dimension 3 over $\mathbb{C}$. ∎

Let

$$\mathcal{W}_2' = \{(H, [A]) \in \mathbb{P}^2(\mathbb{C}) \times \mathbb{P}(M_3) : H \subseteq \ker A \cap \ker A^t\}$$

and let $\pi$, $\pi'$ be the canonical projections of $\mathcal{W}_2'$ onto $\mathbb{P}^2(\mathbb{C})$ and $\mathcal{W}_2$ respectively.

**Proposition 3.2.** $\mathcal{W}_2$ *is irreducible,* $\dim \mathcal{W}_2 = 5$ *and* $\pi'$ *is a resolution of singularities of* $\mathcal{W}_2$. *The set of the singular points of* $\mathcal{W}_2$ *is* $\mathcal{Z} = \{[A] \in \mathbb{P}(M_3) : \dim(\ker A \cap \ker A^t) = 2\}$, *that is the orbit of* $\mathbf{n}_3$, *and* $\dim \mathcal{Z} = 2$.

**Proof.** For $i = 1, 2, 3$ let $\mathcal{U}_i$ be the open subset of $\mathbb{P}^2(\mathbb{C})$ given by the condition that the $i$-th coordinate doesn't vanish and let $\mathcal{F}_i$ be the subset of $\mathbb{P}(M_3)$ of all $[A]$ such that the $i$-th row and column of $A$ vanish. Let $G_i \in \mathrm{GL}(3, \mathbb{C})$ be such that $G_i(e_i) \in \langle e_i \rangle$ and let $G_i^1, G_i^2, G_i^3$ be the columns of $G_i$. Let $\phi_i : \mathcal{U}_i \to \mathrm{GL}(3, \mathbb{C})$ be such that for any $H = \langle (x_1, x_2, x_3) \rangle \in \mathcal{U}_i$ the $i$-th column of $\phi_i(H)$ is $G_i^i - \sum_{j \neq i} x_j (x_i)^{-1} G_i^j$, the others are equal to those of $G_i$; then $\phi_i(H)(H) = \langle e_i \rangle$. If $\mathcal{A}_i = \pi^{-1}(\mathcal{U}_i)$ the map $(H, [A]) \mapsto (H, [(\phi_i(H)^{-1})^t A \phi_i(H)^{-1}])$ from $\mathcal{A}_i$ to $\mathcal{U}_i \times \mathcal{F}_i$ is an isomorphism. Hence $\mathcal{W}_2'$, with the map $\pi$, is a vector bundle on $\mathbb{P}^2(\mathbb{C})$ with fibers isomorphic to $\mathbb{P}(M_2)$.
The map $([A]) \mapsto (\ker A \cap \ker A^t, [A])$ from $\mathcal{W}_2$ to $\mathcal{W}_2'$ is regular except in the points of $\mathcal{Z}$, where the fibers of $\pi'$ have dimension 1, and is a birational inverse of $\pi'$. Let $\mathcal{Z}' = \{(H, [A]) \in G_{2,3} \times \mathbb{P}(M_3) : H \subseteq \ker A \cap \ker A^t\}$. If $\pi_1$ and $\pi_2$ are the canonical projections of $\mathcal{Z}'$ on $G_{2,3}$ and $\mathcal{Z}$ respectively, $\pi_2$ is a birational morphism and the fibers of $\pi_1$ have only one point. Hence $\mathcal{Z}'$ and $\mathcal{Z}$ are irreducible of dimension 2 and $(\pi')^{-1}(\mathcal{Z})$ is irreducible of dimension 3. Then by Theorem 2 of chap. II, §4 of [10] we get the claim. ∎

**Corollary 3.3.** *The set of the singular points of* $\mathcal{L}_3$ *is* $\mathcal{W}_1 \cap \mathcal{W}_2$, *that is the union of the orbits of* $\mathbf{n}_3$ *and* $\mathbf{l}_{-1}$.

For any $[\alpha] \in \mathcal{L}_n$ the tangent space in $[\alpha]$ to $\mathcal{L}_n$ is $\mathbb{P}(V_\alpha)$, where $V_\alpha$ is the vector space of 2-cocycles in the cohomology of $L_\alpha$ as $L_\alpha$-module ([5]). By the equations of the space of 2-cocycles of a Lie algebra we have found that the dimensions of the tangent spaces to $\mathcal{L}_3$ in $\mathbf{n}_3$ and $\mathbf{l}_{-1}$ are 7 and 6 respectively.

## 4. Classification of the Lie algebras of dimension 4 over $\mathbb{C}$

For any $(\beta, \gamma) \in \mathbb{C}^2$ let $[[\beta, \gamma]]$ and $[[\beta]]$ be the orbit in $\mathbb{P}^2(\mathbb{C})$ of $[1, \beta, \gamma]$ and $[1, \beta, 1-\beta]$ respectively with respect to the action of the group of the permutations of the coordinates of $\mathbb{P}^2(\mathbb{C})$.

**Theorem 4.1.** *We have $[\alpha] \in \mathcal{L}_4$ if and only if $L_\alpha$ is isomorphic to one and only one of the following Lie algebras (where we omit $[e_i, e_j]$, $i, j \in \{1, \dots, 4\}$, if it is 0):*

$$\begin{array}{rcl}
\mathbf{g}_{[[\beta,\gamma]]} & : & [e_4, e_1] = e_1, \quad [e_4, e_2] = \beta e_2, \quad [e_4, e_3] = \gamma e_3, \ \beta, \gamma \in \mathbb{C}; \\
\mathbf{g}_{[[\beta]]} & : & [e_2, e_3] = e_1, \quad [e_4, e_1] = e_1, \quad [e_4, e_2] = \beta e_2, \\
 & & [e_4, e_3] = (1-\beta) e_3, \ \beta \in \mathbb{C}; \\
\mathbf{g}_c & : & [e_4, e_1] = c e_1, \quad [e_4, e_2] = e_2, \quad [e_4, e_3] = e_2 + e_3, \quad c \in \mathbb{C}; \\
\mathbf{a}_1 & : & [e_2, e_3] = e_1, \quad [e_4, e_1] = 2e_1, \quad [e_4, e_2] = e_2, \quad [e_4, e_3] = e_2 + e_3; \\
\mathbf{a}_2 & : & [e_4, e_1] = e_1, \quad [e_4, e_2] = e_1 + e_2, \quad [e_4, e_3] = e_2 + e_3; \\
\mathbf{a}_3 & : & [e_3, e_2] = e_2, \quad [e_4, e_1] = e_1; \\
\mathbf{a}_4 & : & [e_4, e_1] = e_1, \quad [e_4, e_2] = -e_2, \quad [e_1, e_2] = e_4; \\
\mathbf{a}_5 & : & [e_1, e_2] = e_3, \quad [e_4, e_1] = e_1, \quad [e_4, e_2] = -e_2; \\
\mathbf{a}_6 & : & [e_4, e_1] = e_1, \quad [e_4, e_2] = e_3; \\
\mathbf{a}_7 & : & [e_2, e_3] = e_1; \\
\mathbf{a}_8 & : & [e_2, e_3] = e_1, \quad [e_4, e_3] = e_2.
\end{array}$$

**Proof.** Let $L$ be a Lie algebra over $\mathbb{C}$ of dimension 4. Let $H$ be a Cartan subalgebra of $L$, $h \in H$ be such that $H = L_0(\operatorname{ad} h) = \{v \in L : \exists\, n \in \mathbb{N} : (\operatorname{ad} h)^n v = 0\}$ and $\operatorname{ad} h$, if not nilpotent, has the eigenvalue 1, $H'$ be a subspace of $L$ such that $H \oplus H' = L$, $[h, H'] = H'$.
Let $\dim H = 1$. Then $H' = [L, L]$. Let $\{x, y, z\}$ be a basis of $H'$ such that the matrix of $\operatorname{ad}_{H'} h$ with respect to it is in Jordan canonical form. From the Jacobi's relations between $h$ and the pairs of elements of $\{x, y, z\}$, when $\operatorname{ad}_{H'} h$ is represented by a diagonal matrix with diagonal entries $1, \beta, \gamma$ respectively, $\beta, \gamma \neq 0$, we get

$$(\beta+1)[x, y] = [h, [x, y]], \quad (\gamma+1)[x, z] = [h, [x, z]], \quad (\beta+\gamma)[y, z] = [h, [y, z]],$$

hence either $H'$ is abelian or, permuting $x, y, z$ and multiplying them by a scalar if necessary, $\beta + \gamma = 1$ and $H'$ is a Heisenberg Lie algebra with $x = [y, z]$. We get the Lie algebras $\mathbf{g}_{[[\beta,\gamma]]}$, $\beta, \gamma \neq 0$, and $\mathbf{g}_{[[\beta]]}$, $\beta \neq 0, 1$, respectively. If $\operatorname{ad}_{H'} h$ is represented by two Jordan blocks, the first one of order 2 and eigenvalue 1, the second one of eigenvalue $c \neq 0$, we get

$$(c+1)[z, x] = [h, [z, x]], \quad [z, x] + (c+1)[z, y] = [h, [z, y]], \quad 2[x, y] = [h, [x, y]],$$

hence $[z, x] = [z, y] = 0$ and either $H'$ is abelian or $c = 2$ and $H'$ is a Heisenberg Lie algebra, with (multiplying $x$ and $y$ by a scalar) $[x, y] = z$. We get the Lie algebras $\mathbf{g}_c$, $c \neq 0$, and $\mathbf{a}_1$ respectively. If $\operatorname{ad}_{H'} h$ is represented by only one Jordan block we get

$$2[x, y] = [h, [x, y]], \quad 2[x, z] = [h, [x, z]] - [x, y], \quad 2[y, z] = [h, [y, z]] - [x, z],$$

hence $H'$ is abelian and we get the Lie algebra $\mathbf{a}_2$.
Let $\dim H = 2$. Then, since $H$ is abelian, $\operatorname{ad}_L H$ is abelian and $H' = [H, H'] = [H, L]$. Let $\{x, y\}$ be a basis of $H'$ such that the matrix of $\operatorname{ad}_{H'} h$ with respect to $\{x, y\}$ is in Jordan canonical form. We have to require

$$[h, [x, y]] = [x, [h, y]] + [y, [x, h]] = (\operatorname{tr} \operatorname{ad}_{H'} h)[x, y],$$

hence either $[x,y]=0$ or for any $v \in H$ $\mathrm{ad}_{H'} v$ has the eigenvalues $1,-1$ and $\dim \mathrm{ad}\, H \le 1$. If $\dim \mathrm{ad}\, H = 2$ and there exist in $H$ elements $v$ such that $\mathrm{ad}_{H'} v$ has two different eigenvalues, we may choose $w, z \in H$ such that with respect to the basis $\{x,y\}$ $\mathrm{ad}_{H'} w$ and $\mathrm{ad}_{H'} z$ are represented by two diagonal matrices with diagonal entries $1,0$ and $0,1$ respectively, hence we get the Lie algebra $\mathbf{a}_3$. If $\dim \mathrm{ad}\, H = 2$ but for any $v \in H$ $\mathrm{ad}_{H'} v$ has only one eigenvalue we may choose $h, z \in H$ such that with respect to the basis $\{x,y\}$ $\mathrm{ad}_{H'} h$ and $\mathrm{ad}_{H'} z$ are represented respectively by the identity matrix and by the nilpotent Jordan block of order 2, hence we get the Lie algebra $\mathbf{g}_{[[0]]}$. If $\dim \mathrm{ad}\, H = 1$ let $z \in H \setminus \{0\}$ be such that $\mathrm{ad}\, z = 0$. If the Jordan form of $\mathrm{ad}_{H'} h$ is diagonal and $[x,y]=0$ we get the Lie algebras $\mathbf{g}_{[[0,\gamma]]}$, $\gamma \in \mathbb{C} \setminus \{0\}$. If the Jordan form of $\mathrm{ad}_{H'} h$ is diagonal and $[x,y] \notin \langle z \rangle$ we may assume $h = [x,y]$ and we get the Lie algebra $\mathbf{a}_4$. If the Jordan form of $\mathrm{ad}_{H'} h$ is diagonal and $[x,y] \in \langle z \rangle \setminus \{0\}$ we may assume $[x,y]=z$ getting the Lie algebra $\mathbf{a}_5$. If the Jordan form of $\mathrm{ad}_{H'} h$ has only one Jordan block we get the Lie algebra $\mathbf{g}_0$.

Let $\dim H = 3$. If $H$ is abelian, since $\dim \mathrm{ad}\, H = 1$ there exist $y, z \in H$ linearly independent such that $\mathrm{ad}\, y = \mathrm{ad}\, z = 0$, hence we get the Lie algebra $\mathbf{g}_{[[0,0]]}$. If $H$ is a Heisenberg Lie algebra, since the subset of all $v \in H$ such that $H = L_0(\mathrm{ad}\, v)$ is open in $H$, we may assume $H = \langle h, y, z \rangle$ with $[h,y]=z$, $[h,x]=x$, $x \notin H$. Since $\mathrm{ad}\, h$ and $\mathrm{ad}\, z$ commute, $[z,x] \in \langle x \rangle$. Since $\mathrm{ad}\, y$ and $\mathrm{ad}\, z$ commute, if we had $[z,x] \neq 0$ we would have $[y,x] \in \langle x \rangle$ and then, since $\mathrm{ad}_H y$ and $\mathrm{ad}_H h$ commute, $\mathrm{ad}\, y$ and $\mathrm{ad}\, h$ would commute; but this holds if and only if $\mathrm{ad}\, z = 0$. Hence $[z,x]=0$ and $[y,x] \in \langle x \rangle$. Since $\dim[H,x]=1$ we may choose $y$ such that $[y,x]=0$; we get the Lie algebra $\mathbf{a}_6$.

Let $\dim H = 4$, that is $L$ nilpotent. If $L$ isn't abelian there exists $x \neq 0$ such that $x \in Z(L) \cap [L,L]$. If $x = [y,z]$, since $H'' = \langle x, y, z \rangle$ is a nilpotent subalgebra, $\dim H'' = 3$ and $H''$ is a Heisenberg Lie algebra. Since $L$ is nilpotent $[h, H''] \subseteq H''$ for any $h \in L$. Since $[h,x]=0$ it is possible to choose $h, x, y, z$ such that $h \notin H''$, the matrix of $\mathrm{ad}_{H''} h$ with respect to the basis $\{x,y,z\}$ is in Jordan canonical form and $[h,y]=0$ (in fact if $[h,y]=x$ then $[h+z,y]=0$). We get the Lie algebras $\mathbf{a}_7$ and $\mathbf{a}_8$. ∎

## 5. The variety of the Lie algebras of dimension 4

For any Lie algebra $L$ let $Z(L)$ be the center of $L$.

**Proposition 5.1.** *$\mathcal{L}_4$ is the union of the following closed subsets:*

$$\begin{aligned}
\mathcal{C}_1 &= \{[\alpha] \in \mathcal{L}_4 : L_\alpha \textit{ has an abelian ideal of dimension 3}\},\\
\mathcal{C}_2 &= \{[\alpha] \in \mathcal{L}_4 : L_\alpha \textit{ has a nilpotent ideal } J_\alpha \textit{ of dimension 3 such}\\
&\qquad \textit{that } \tfrac{1}{2}\,\mathrm{tr}\,\mathrm{ad}\, v \textit{ is eigenvalue of } \mathrm{ad}_{J_\alpha} v \textit{ for any } v \in L_\alpha\},\\
\mathcal{C}_3 &= \{[\alpha] \in \mathcal{L}_4 : \dim[L_\alpha, L_\alpha] \le 2,\ \mathrm{ad}_{[L_\alpha,L_\alpha]} L_\alpha \textit{ is abelian}\},\\
\mathcal{C}_4 &= \{[\alpha] \in \mathcal{L}_4 : Z(L_\alpha) \neq \{0\},\ \mathrm{tr}\,\mathrm{ad}\, v = 0 \textit{ for any } v \in L_\alpha\}
\end{aligned}$$

*and* $\mathcal{C}_i \not\subseteq \bigcup_{j \neq i} \mathcal{C}_j$ *for* $i,j = 1, \ldots, 4$.

**Proof.** Since by Theorem 4.1 each one of these subsets is the union of the orbits of the following Lie algebras:

$$\begin{array}{ll} \mathcal{C}_1 : & \mathbf{g}_{[[\beta,\gamma]]},\ \mathbf{g}_c,\ \mathbf{a}_2,\ \mathbf{a}_6,\ \mathbf{a}_7,\ \mathbf{a}_8 \\ \mathcal{C}_2 : & \mathbf{g}_{[[\gamma+1,\gamma]]},\ \mathbf{g}_{[[\beta]]},\ \mathbf{g}_0, \mathbf{g}_2,\ \mathbf{a}_1,\ \mathbf{a}_5,\ \mathbf{a}_7,\ \mathbf{a}_8 \\ \mathcal{C}_3 : & \mathbf{g}_{[[0,\gamma]]},\ \mathbf{g}_{[[0]]},\ \mathbf{g}_0,\ \mathbf{a}_3,\ \mathbf{a}_6,\ \mathbf{a}_7,\ \mathbf{a}_8 \\ \mathcal{C}_4 : & \mathbf{g}_{[[0,-1]]},\ \mathbf{a}_4,\ \mathbf{a}_5,\ \mathbf{a}_7,\ \mathbf{a}_8 \end{array}$$

where $\beta, \gamma, c \in \mathbb{C}$, we get the claim. ■

For any $i = 1, \dots, 4$ let $\mathcal{A}_i = \{J \in G_{3,4} \ : \ e_i \notin J\}$ and let $\{i_1, i_2, i_3\} = \{1, \dots, 4\} \setminus \{i\}$, $i_1 < i_2 < i_3$. If $J \in \mathcal{A}_i$ let $J = \langle e^J_{i_1}, e^J_{i_2}, e^J_{i_3}\rangle$, where, with respect to the basis $\{e_{i_1}, e_{i_2}, e_{i_3}, e_i\}$, for $j = 1,2,3$ the $j$-th coordinate of $e^J_{i_j}$ is 1 and for $k \in \{1,2,3\}$, $k \neq j$ the $k$-th coordinate of $e^J_{i_j}$ is 0. Let

$$\mathcal{C}'_1 = \{(J, [\alpha]) \in G_{3,4} \times \mathcal{C}_1 \ : \ J \text{ is an abelian ideal of } L_\alpha\}$$

and let $p_1$, $p'_1$ be the canonical projections of $\mathcal{C}'_1$ onto $G_{3,4}$ and $\mathcal{C}_1$ respectively.

**Proposition 5.2.** *$\mathcal{C}_1$ is irreducible, $\dim \mathcal{C}_1 = 11$ and $p'_1$ is a resolution of singularities of $\mathcal{C}_1$. The set of the singular points of $\mathcal{C}_1$ is $\mathcal{Z}_1 = \{[\alpha] \in \mathcal{C}_1 \ : \ L_\alpha$ is nilpotent and $\dim[L_\alpha, L_\alpha] \leq 1\}$, that is the orbit of $\mathbf{a}_7$, and $\dim \mathcal{Z}_1 = 5$.*

**Proof.** Let $\mathcal{A}'_i := (p_1)^{-1}(\mathcal{A}_i)$, $i = 1, \dots, 4$. The map $\xi_i \ : \ \mathcal{A}_i \times \mathbb{P}(M_3) \to \mathcal{A}'_i$ defined by $\xi_i(J, [A]) = (J, [\alpha])$ where $[\alpha]$ is such that in $L_\alpha$ $\mathrm{ad}_J\, e_i$ is represented by $A$ with respect to the basis $\{e^J_{i_1}, e^J_{i_2}, e^J_{i_3}\}$ is an isomorphism, hence $\mathcal{C}'_1$, with the map $p_1$, is a vector bundle and $\dim \mathcal{C}'_1 = 11$.
The map $p'_1$ is birational and $(p'_1)^{-1}$ is regular in the open subset of $\mathcal{C}_1$ of all $[\alpha]$ such that $L_\alpha$ is not nilpotent or there exists $x \in L_\alpha$ such that $\dim[x, L_\alpha] \geq 2$ (we set $(p'_1)^{-1}([\alpha]) = (J, [\alpha])$ where $J$ is the subspace of all the nilpotent elements $x$ of $L_\alpha$ such that $\dim[x, L_\alpha] \leq 1$). It isn't regular in the points of $\mathcal{Z}_1 = \{[\alpha] \in \mathcal{C}_1 \ : \ L_\alpha$ is nilpotent and $\dim[L_\alpha, L_\alpha] \leq 1\}$, that is the orbit of $\mathbf{a}_7$, since the fibers of $p'_1$ on the elements of $\mathcal{Z}_1$ have dimension 1. The variety $\mathcal{Z}'_1 := (p'_1)^{-1}(\mathcal{Z}_1)$, with the map $p_1|_{\mathcal{Z}'_1}$, is a bundle on $G_{3,4}$ whose fibers are isomorphic to $\mathbb{P}(N'_3)$, where $N'_3$ is the variety of all the nilpotent $3 \times 3$ matrices over $\mathbb{C}$ of rank less or equal 1; hence it is irreducible of dimension 6, which shows the claim. ■

Let

$$\begin{aligned} \mathcal{C}'_2 \ = \ & \{(J, [\alpha]) \in G_{3,4} \times \mathcal{C}_2 \ : \ J \text{ is a nilpotent ideal of } L_\alpha \text{ and} \\ & \text{for any } v \in L_\alpha \ \tfrac{1}{2}\operatorname{tr} \operatorname{ad} v \text{ is eigenvalue of } \operatorname{ad}_J v\}. \end{aligned}$$

**Lemma 5.3.** *If $(J, [\alpha]) \in \mathcal{C}'_2$ and $v \in L_\alpha$ then $[J, J]$ is contained in the eigenspace of $\mathrm{ad}_J\, v$ corresponding to $\frac{1}{2}\operatorname{tr}\operatorname{ad} v$.*

**Proof.** Let $y \neq 0$ belong to the previous eigenspace but $[J, J] \not\subseteq \langle y \rangle$. Then we may choose a basis $\{y, x, z\}$ of $J$ such that $[J, J] \subseteq \langle x \rangle$. Since $[x, v] \in \langle x \rangle$ (in fact $0 = [x, [y, v]] = [y, [x, v]]$, hence $[x, v] \in \langle x, y \rangle$, in the same way $[x, v] \in \langle x, z \rangle$), there exist $a, b, c, d \in \mathbb{C}$ such that $[v, y] = ay$, $[v, x] = bx$, $[v, z] = (a-b)z + cx + dy$, hence by the condition $[y, [z, v]] = [z, [y, v]] + [v, [z, y]]$ we get $a = b$. ■

Let

$$\begin{aligned}\mathcal{S}' &= \{(H,[(A,B)]) \in \mathbb{P}^2(\mathbb{C}) \times \mathbb{P}(S_3 \times M_3) \,:\, \operatorname{Im} A \subseteq H,\\ &\qquad H \subseteq \ker\big(B - \big(\tfrac{1}{2}\operatorname{tr} B\big) I_3\big)\};\end{aligned}$$

let $\mathcal{S}$ be the image of the canonical projection of $\mathcal{S}'$ on $\mathbb{P}(S_3 \times M_3)$ and let $s, s'$ be the canonical projections of $\mathcal{S}'$ on $\mathbb{P}^2(\mathbb{C})$ and $\mathcal{S}$ respectively.

**Lemma 5.4.** *$\mathcal{S}$ is irreducible, $\dim \mathcal{S} = 8$ and $s'$ is a resolution of singularities of $\mathcal{S}$. The set of the singular points of $\mathcal{S}$ is*

$$\widehat{\mathcal{S}} = \{[(A,B)] \in \mathcal{S} \,:\, A = 0,\ \dim\ker\big(B - \big(\tfrac{1}{2}\operatorname{tr} B\big) I_3\big) \geq 2\},$$

*which is irreducible of dimension 4.*

**Proof.** The variety $\mathcal{S}'$ with the map $s$ is a vector bundle on $\mathbb{P}^2(\mathbb{C})$ with fibers of dimension 6. The map $s'$ is birational and $(s')^{-1}$ is regular in the open subset of all $[(A,B)]$ such that $A \neq 0$ or $\dim\ker\big(B - \big(\frac{1}{2}\operatorname{tr} B\big) I_3\big) = 1$. It isn't regular in the points of $\widehat{\mathcal{S}}$, where the generic fiber of $s'$ has dimension 1, and $\widehat{\mathcal{S}}' := (s')^{-1}(\widehat{\mathcal{S}})$ is irreducible of dimension 5 (the fiber of $s|_{\widehat{\mathcal{S}}'}$ in $H$ is birational to $\{(V,[B]) \in G_{2,3} \times \mathbb{P}(M_3) \,:\, H \subset V \subseteq \ker\big(B - \big(\frac{1}{2}\operatorname{tr} B\big) I_3\big)\}$, hence has dimension 3), which shows the claim. ■

Let $p_2$ and $p_2'$ be the canonical projections of $\mathcal{C}_2'$ on $G_{3,4}$ and $\mathcal{C}_2$ respectively.

**Lemma 5.5.** *$\mathcal{C}_2'$, with the map $p_2$, is a bundle on $G_{3,4}$ with fibers isomorphic to $\mathcal{S}$.*

**Proof.** Let $\mathcal{U}_i = (p_2)^{-1}(\mathcal{A}_i)$, $i = 1, \ldots, 4$. For any $(J,[\alpha]) \in \mathcal{C}_2'$ let $\alpha_J \in \operatorname{Hom}(J \wedge J, J)$ be defined by $\alpha_J(v \wedge v') = \alpha|_{J \wedge J}(v \wedge v')$ for any $v, v' \in J$. The map $\nu_i \,:\, \mathcal{A}_i \times \mathcal{S} \to \mathcal{U}_i$ such that $\nu_i(J,[(A,B)]) = (J,[\alpha])$ where $\alpha$ is such that the matrix of $\alpha_J$ with respect to the bases $\{e^J_{i_2} \wedge e^J_{i_3}, e^J_{i_3} \wedge e^J_{i_1}, e^J_{i_1} \wedge e^J_{i_2}\}$ and $\{e^J_{i_1}, e^J_{i_2}, e^J_{i_3}\}$ is $A$ and in $L_\alpha$ the matrix of $\operatorname{ad}_J e_i$ with respect to the basis $\{e^J_{i_1}, e^J_{i_2}, e^J_{i_3}\}$ is $B$ is an isomorphism, which shows the claim. ■

For any $i = 1, \ldots, 4$ and $J \in \mathcal{A}_i$ let $B^J_i = \{e^J_{i_1}, e^J_{i_2}, e^J_{i_3}, e_i\}$. Let $J \in \mathcal{A}_i \cap \mathcal{A}_{i'}$ and let $G_J$ be the matrix whose columns are the coordinates of the elements of $B^J_i$ with respect to $B^J_{i'}$. Let $\delta \,:\, S_3 \times M_3 \to M_{4\times 6}$ be the isomorphism such that, by regarding $\delta((A,B))$ as a block matrix, we have

$$\delta((A,B)) = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}.$$

Then, by using the notations of the proof of Lemma 5.5, we have that the automorphism $(\nu_{i'})^{-1} \circ \nu_i$ of $(\mathcal{A}_i \cap \mathcal{A}_{i'}) \times \mathcal{S}$ is given by

$$(\nu_{i'})^{-1} \circ \nu_i(J,[(A,B)]) = (J,[\delta^{-1}(G_J \delta((A,B))\widehat{G_J})]).$$

Let $\mathcal{C}_2''$ be the vector bundle on $G_{3,4}$ which is the union of open subsets $\mathcal{U}_i'$, $i = 1, \ldots, 4$, with isomorphisms $\nu_i' \,:\, \mathcal{A}_i \times \mathcal{S}' \to \mathcal{U}_i'$ such that

$$(\nu_{i'}')^{-1} \circ \nu_i'(J,(H,[(A,B)])) = (J,(H_J,[\delta^{-1}(G_J \delta((A,B))\widehat{G_J})]))$$

where if $H = [h_1, h_2, h_3]$ then $H_J = [h^J_1, h^J_2, h^J_3]$ is such that $(h^J_1, h^J_2, h^J_3, 0) = G_J^{-1}(h_1, h_2, h_3, 0)$. Let $p'' \,:\, \mathcal{C}_2'' \to \mathcal{C}_2'$ be the morphism such that $p''(\mathcal{U}_i') = \mathcal{U}_i$ and, if $p_i''$ is $p''|_{\mathcal{U}_i'}$ as map onto $\mathcal{U}_i$, we have $\nu_i \circ (\operatorname{id}_{\mathcal{A}_i} \times s') = p_i'' \circ \nu_i'$ for any $i = 1, \ldots, 4$. Then $p''$ is a resolution of singularities of $\mathcal{C}_2'$.

**Proposition 5.6.** *$\mathcal{C}_2$ is irreducible,* $\dim \mathcal{C}_2 = 11$ *and* $p_2' \circ p''$ *is a resolution of singularities of* $\mathcal{C}_2$. *The set of the singular points of* $\mathcal{C}_2$ *is* $\mathcal{Z}_2 = \widehat{\mathcal{Z}}_2 \cup \widetilde{\mathcal{Z}}_2$ *where* $\widehat{\mathcal{Z}}_2 = \{[\alpha] \in \mathcal{C}_2 \,:\, L_\alpha$ *is nilpotent*$\}$ *and* $\widetilde{\mathcal{Z}}_2 = \{[\alpha] \in \mathcal{C}_2 \,:\, L_\alpha$ *has an abelian ideal of dimension 3 and for any* $v \in L_\alpha$ $\dim \mathrm{Im}(\mathrm{ad}\, v - (\frac{1}{2}\, \mathrm{tr}\, \mathrm{ad}\, v)\, \mathrm{id}) \leq 1\}$. *We have that* $\widehat{\mathcal{Z}}_2$ *is irreducible of dimension 8 and is the union of the orbits of* $\mathbf{a}_7$ *and* $\mathbf{a}_8$; $\widetilde{\mathcal{Z}}_2$ *is irreducible of dimension 7 and is the union of the orbits of* $\mathbf{g}_{[[0,1]]}$, $\mathbf{g}_0$ *and* $\mathbf{a}_7$.

**Proof.** The map $p_2'$ is birational and the subset of $\mathcal{C}_2$ in which $(p_2')^{-1}$ isn't regular is $\widehat{\mathcal{Z}}_2$, since $(p_2')^{-1}([\alpha]) = (J, [\alpha])$ where $J$ is the subspace of $L_\alpha$ of all the nilpotent elements and the generic fiber of $p_2'$ on $\widehat{\mathcal{Z}}_2$ has dimension 1. Let $\widehat{\mathcal{Z}}_2' := (p_2')^{-1}(\widehat{\mathcal{Z}}_2)$. If we set $\overline{\mathcal{S}} = \{[(A,B)] \in \mathcal{S} \,:\, B$ is nilpotent$\}$ we have that $\overline{\mathcal{S}}$ is irreducible and $\dim \overline{\mathcal{S}} = 6$ (in fact, by Lemma 5.4, $(s')^{-1}(\overline{\mathcal{S}})$ has these properties). Since the fibers of $p_2|_{\widehat{\mathcal{Z}}_2'}$ are isomorphic to $\overline{\mathcal{S}}$ we get that $\widehat{\mathcal{Z}}_2'$ is irreducible of dimension 9 and $\widehat{\mathcal{Z}}_2$ is irreducible of dimension 8, hence by Theorem 2 of chap. II, §4 of [10] the points of $\widehat{\mathcal{Z}}_2$ are singular for $\mathcal{C}_2$. By Lemma 5.4 and Lemma 5.5 $\widetilde{\mathcal{Z}}_2$ is irreducible of dimension 7 and the points of $\widetilde{\mathcal{Z}}_2 \setminus \widehat{\mathcal{Z}}_2$ are singular for $\mathcal{C}_2$, hence we get the claim. ∎

For any $n \in \mathbb{N}$ let $C_n = \{(A,B) \in M_n \times M_n \,:\, [A,B] = 0\}$. If $(x_0, \ldots, x_7)$ are coordinates of $\mathbb{C}^8$, we set

$$A = \begin{pmatrix} x_0 & x_2 \\ x_4 & x_0 + x_6 \end{pmatrix}, \quad B = \begin{pmatrix} x_1 & x_3 \\ x_5 & x_1 + x_7 \end{pmatrix}$$

and we regard $C_2$ as a subvariety of $\mathbb{C}^8$. Let $\mathcal{V}' = \{(x_0, \ldots, x_7) \in C_2 \,:\, (x_2, \ldots, x_7) \neq (0, \ldots, 0)\}$; then the map $u \,:\, \mathcal{V}' \to \mathbb{P}^5(\mathbb{C})$ such that $u((x_0, \ldots, x_7)) = [x_2, \ldots, x_7]$ is a morphism. Let $\mathcal{V} = u(\mathcal{V}')$, let:

$$\mathcal{W} = \{((x_0, \ldots, x_7), [z_2, \ldots, z_7]) \in C_2 \times \mathcal{V} \,:\, x_i z_j = z_i x_j,\ i,j = 2, \ldots, 7\}$$

and let $r$ be the canonical projection of $\mathcal{W}$ on $C_2$.

**Lemma 5.7.** *$C_2$ is irreducible,* $\dim C_2 = 6$ *and*

$$\mathcal{Y} = \{(A,B) \in C_2 \,:\, A, B \in \langle I_2 \rangle\}$$

*is the set of the singular points of* $C_2$. *The variety* $\mathcal{W}$ *is irreducible and* $r$ *is a resolution of singularities of* $C_2$.

**Proof.** For any $n \in \mathbb{N}$ $C_n$ is irreducible of dimension $n^2 + n$ ([7], [4]). If $X = (x_{ij})$, $Y = (y_{ij})$ are the coordinates of $M_n \times M_n$ and $(A,B) \in C_n$ then $[A,X] + [B,Y] = 0$ are equations of the tangent space to $C_n$ in $(A,B)$. Hence the points $(A,B)$ such that $A$ or $B$ is regular, that is has centralizer of minimum dimension $n$, are non singular for $C_n$, which shows the first claim. Since $\mathcal{V}$ and $C_2$ have the same equations, $\mathcal{V}$ is an irreducible nonsingular variety of dimension 3. The map $r$ is birational, since for any $(x_0, \ldots, x_7) \in C_2$ such that $(x_2, \ldots, x_7) \neq (0, \ldots, 0)$ we may set $r^{-1}((x_0, \ldots, x_7)) = ((x_0, \ldots, x_7), [x_2, \ldots, x_7])$; if $(x_2, \ldots, x_7) = (0, \ldots, 0)$ we have $r^{-1}(\{(x_0, \ldots, x_7)\}) = \{(x_0, \ldots, x_7)\} \times \mathcal{V}$. Since for any $x_0, x_1, t \in \mathbb{C}$ and $[z_2, \ldots, z_7] \in \mathcal{V}$ we have that $((x_0, x_1, tz_2, \ldots, tz_7), [z_2, \ldots, z_7]) \in \mathcal{W}$, $\mathcal{W}$ is irreducible. Since $\mathcal{V}$ has the same equations as $C_2$ the tangent space to $\mathcal{W}$ in a point such that $(x_2, \ldots, x_7) = (0, \ldots, 0)$ has the same dimension as in a point of $\mathcal{W} \setminus r^{-1}(\mathcal{Y})$, hence we get the claim. ∎

Let

$$\begin{aligned}\mathcal{G}' &= \{([y_1, y_2, x_0, \dots, x_7], [z_2, \dots, z_7]) \in \mathbb{P}^9(\mathbb{C}) \times \mathbb{P}^5(\mathbb{C}) : \\ &: ((x_0, \dots, x_7), [z_2, \dots, z_7]) \in \mathcal{W}\};\end{aligned}$$

let $\mathcal{G}$ be the image of the canonical projection of $\mathcal{G}'$ onto $\mathbb{P}^9(\mathbb{C})$ and let $r'$ be the canonical projection of $\mathcal{G}'$ on $\mathcal{G}$.

**Corollary 5.8.** *The map $r'$ is a resolution of singularities of $\mathcal{G}$.*

Let

$$\mathcal{C}_3' = \{(W, [\alpha]) \in G_{2,4} \times \mathcal{C}_3 \,:\, [L_\alpha, L_\alpha] \subseteq W,\ \mathrm{ad}_W L_\alpha \text{ is abelian}\},$$

and let $p_3$, $p_3'$ be the canonical projections of $\mathcal{C}_3'$ on $G_{2,4}$ and $\mathcal{C}_3$ respectively. For any $i, j \in \{1, \dots, 4\}$, $i < j$ let $\mathcal{A}_{ij} = \{W \in G_{2,4} \,:\, W \cap \langle e_i, e_j \rangle = \{0\}\}$. Let $\{i_0, j_0\} = \{1, \dots, 4\} \setminus \{i, j\}$, $i_0 < j_0$; if $W \in \mathcal{A}_{ij}$ let $W = \langle e_{i_0}^W, e_{j_0}^W \rangle$ where the first two coordinates of $e_{i_0}^W$ and $e_{j_0}^W$ with respect to the basis $\{e_{i_0}, e_{j_0}, e_i, e_j\}$ are $1, 0$ and $0, 1$ respectively.

**Lemma 5.9.** *$\mathcal{C}_3'$ with the map $p_3$ is a bundle on $G_{2,4}$ with fibers isomorphic to $\mathcal{G}$.*

**Proof.** Let $\mathcal{U}_{ij} = (p_3)^{-1}(\mathcal{A}_{ij})$, $i, j \in \{1, \dots, 4\}$, $i < j$. The map $\eta_{ij} : \mathcal{A}_{ij} \times \mathcal{G} \to \mathcal{U}_{ij}$ defined by $\eta_{ij}(W, [(y_1, y_2, A, B)]) = (W, [\alpha])$ where $\alpha$ is such that in $L_\alpha$ $[e_i e_j] = y_1 e_{i_0}^W + y_2 e_{j_0}^W$ and $\mathrm{ad}_W e_i$, $\mathrm{ad}_W e_j$ are represented, with respect to the basis $\{e_{i_0}^W, e_{j_0}^W\}$, respectively by $A$ and $B$ is an isomorphism, which shows the claim. ■

For any $i, j \in \{1, \dots, 4\}$, $i < j$, and $W \in \mathcal{A}_{ij}$ let $B_{ij}^W = \{e_{i_0}^W, e_{j_0}^W, e_i, e_j\}$. Let $W \in \mathcal{A}_{ij} \cap \mathcal{A}_{i'j'}$ and let $G_W$ be the matrix whose columns are the coordinates of the elements of $B_{ij}^W$ with respect to $B_{i'j'}^W$. Let $\zeta : \mathbb{C}^2 \times M_2 \times M_2 \to M_{4\times 6}$ be the isomorphism such that, by regarding $\zeta((y_1, y_2, A, B))$ as a block matrix, we have:

$$\zeta((y_1, y_2, A, B)) = \begin{pmatrix} 0 & Y & A & B \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Then, by using the notations of the proof of Lemma 5.9, we have that the automorphism $(\eta_{i'j'})^{-1} \circ \eta_{ij}$ of $(\mathcal{A}_{ij} \cap \mathcal{A}_{i'j'}) \times \mathcal{G}$ is given by

$$\begin{aligned}(\eta_{i'j'})^{-1} \circ \eta_{ij} \quad & (W, [y_1, y_2, x_0, \dots, x_7]) = \\ = \quad & (W, [\zeta^{-1}(G_W \zeta((y_1, y_2, x_0, \dots, x_7))\widehat{G_W})]).\end{aligned}$$

Let $\overline{u} : \mathbb{C}^2 \times \mathcal{V}' \to \mathcal{V}$ be defined by $\overline{u}((y_1, y_2, x_0, \dots, x_7)) = [x_2, \dots, x_7]$. Let $\mathcal{C}_3''$ be the vector bundle on $G_{2,4}$ which is the union of open subsets $\mathcal{U}_{ij}'$, $i, j \in \{1, \dots, 4\}$, $i < j$, with isomorphisms $\eta_{ij}' : \mathcal{A}_{ij} \times \mathcal{G}' \to \mathcal{U}_{ij}'$, such that

$$\begin{aligned}(\eta_{i'j'}')^{-1} \circ \eta_{ij}' \quad & (W, ([y_1, y_2, x_0, \dots, x_7], [z_2, \dots, z_7])) = \\ = \quad & (W, ([\zeta^{-1}(G_W \zeta((y_1, y_2, x_0, \dots, x_7))\widehat{G_W})], \\ & \overline{u} \circ \zeta^{-1}(G_W \zeta((0, \dots, 0, z_2, \dots, z_7))\widehat{G_W}))\end{aligned}$$

and let $q'' : \mathcal{C}_3'' \to \mathcal{C}_3'$ be the morphism such that $q''(\mathcal{U}_{ij}') = \mathcal{U}_{ij}$ and, if $q_{ij}''$ is $q''|_{\mathcal{U}_{ij}'}$ as map onto $\mathcal{U}_{ij}$, we have $\eta_{ij} \circ (\mathrm{id}_{\mathcal{A}_{ij}} \times r') = q_{ij}'' \circ \eta_{ij}'$ for any $i, j \in \{1, \dots, 4\}$, $i < j$. Then $q''$ is a resolution of singularities of $\mathcal{C}_3'$.

**Proposition 5.10.** *$\mathcal{C}_3$ is irreducible,* $\dim \mathcal{C}_3 = 11$ *and $p_3' \circ q''$ is a resolution of singularities of $\mathcal{C}_3$. The set of the singular points of $\mathcal{C}_3$ is $\mathcal{Z}_3 = \{[\alpha] \in \mathcal{C}_3 : \mathrm{ad}_{[L_\alpha, L_\alpha]} L_\alpha \subseteq \langle \mathrm{id} \rangle\}$, that is the union of the orbits of $\mathbf{g}_{[[0,0]]}$, $\mathbf{g}_{[[0,1]]}$ and $\mathbf{a}_7$, which is irreducible of dimension 7.*

**Proof.** The map $p_3'$ is birational and the subset of $\mathcal{C}_3$ in which $(p_3')^{-1}$ isn't regular is $\widehat{\mathcal{Z}}_3 := \{[\alpha] \in \mathcal{C}_3 : \dim[L_\alpha, L_\alpha] < 2\}$ (since $(p_3')^{-1}([\alpha]) = ([L_\alpha, L_\alpha], [\alpha])$ and the generic fiber of $p_3'$ on $\widehat{\mathcal{Z}}_3$ has dimension 2). By Theorem 4.1 we have $\widehat{\mathcal{Z}}_3 \subset \mathcal{Z}_3$ and by Lemma 5.7 and Lemma 5.9 the points of $\mathcal{Z}_3 \setminus \widehat{\mathcal{Z}}_3$ are singular for $\mathcal{C}_3$. If $\mathcal{Z}_3' := (p_3')^{-1}(\mathcal{Z}_3)$, the fibers of $p_3|_{\mathcal{Z}_3'}$ are isomorphic to $\mathbb{P}^3(\mathbb{C})$, hence $\mathcal{Z}_3$ is irreducible of dimension 7. Since the subset of the singular points is closed this shows the claim. ∎

Let

$$\mathcal{C}_4' = \{(T, [\alpha]) \in \mathbb{P}^3(\mathbb{C}) \times \mathcal{C}_4 : T \subseteq Z(L_\alpha)\}$$

and let $p_4'$ be the canonical projections of $\mathcal{C}_4'$ on $\mathcal{C}_4$.

**Proposition 5.11.** *$\mathcal{C}_4$ is irreducible,* $\dim \mathcal{C}_4 = 11$ *and $p_4'$ is a resolution of singularities of $\mathcal{C}_4$. The set of the singular points of $\mathcal{C}_4$ is $\mathcal{Z}_4 = \{[\alpha] \in \mathcal{C}_4 : \dim Z(L_\alpha) \geq 2\}$, that is the orbit of $\mathbf{a}_7$.*

**Proof.** Let $\mathcal{C}_4'' = \{(J, T, [\alpha]) \in G_{3,4} \times \mathcal{C}_4' : J$ is an ideal of $L_\alpha\}$ and let $q_1$, $q_2$ be the canonical projections of $\mathcal{C}_4''$ on $G_{3,4} \times \mathbb{P}^3(\mathbb{C})$ and on $\mathcal{C}_4'$ respectively. If $(J, T) \in G_{3,4} \times \mathbb{P}^3(\mathbb{C})$ is such that $T \not\subseteq J$ the fiber of $q_1$ in $(J, T)$ is isomorphic to $\mathbb{P}(S_3)$. If $(J, T) \in G_{3,4} \times \mathbb{P}^3(\mathbb{C})$ is such that $T \subset J$ then $J$ is a nilpotent ideal such that $[J, J] \subset T$ for any $L_\alpha$ such that $(J, T, [\alpha]) \in \mathcal{C}_4''$, hence the fiber of $q_1$ in $(J, T)$ is also a projective subspace of dimension 5. This proves that $\mathcal{C}_4'$ is irreducible and $\dim \mathcal{C}_4' = 11$, since $q_2$ is birational ($(q_2)^{-1}$ is regular in the open subset of all the elements $(T, [\alpha])$ such that $\dim[L_\alpha, L_\alpha] = 3$).
For any $i \in \{1, \ldots, 4\}$ let $\mathcal{A}^i = \{T \in \mathbb{P}^3(\mathbb{C}) : T \cap \langle e_{i_1}, e_{i_2}, e_{i_3} \rangle = \{0\}\}$; for any $T \in \mathcal{A}^i$ we set $T = \langle e^T \rangle$ where the first coordinate of $e^T$ with respect to the basis $\{e_i, e_{i_1}, e_{i_2}, e_{i_3}\}$ is 1. Let

$$\mathcal{U}^i = \{(T, [\alpha]) \in \mathcal{C}_4' : T \in \mathcal{A}^i,\ \alpha(e_{i_1} \wedge e_{i_3}) \neq 0\},$$
$$\mathcal{U}'' = \{[x_1, \ldots, x_8] \in \mathbb{P}^7(\mathbb{C}) : (x_1, \ldots, x_4) \neq (0, \ldots, 0)\}.$$

The map $\psi : \mathcal{A}^i \times \mathcal{U}'' \to \mathcal{U}^i$ defined by $\psi(T, [x_1, \ldots, x_8]) = (T, [\alpha])$ where $\alpha$ is such that $\alpha(e_{i_1} \wedge e_{i_2}) = x_5 e^T + x_1 e_{i_2} + x_6 e_{i_3}$, $\alpha(e_{i_1} \wedge e_{i_3}) = x_2 e^T + x_3 e_{i_1} + x_4 e_{i_2} - x_1 e_{i_3}$, $\alpha(e_{i_2} \wedge e_{i_3}) = x_7 e^T + x_8 e_{i_1} - x_3 e_{i_2}$ is an isomorphism, hence $\mathcal{C}_4'$ is nonsingular. The map $p_4'$ is birational and the subset of $\mathcal{C}_4$ in which $(p_4')^{-1}$ isn't regular is $\mathcal{Z}_4 = \{[\alpha] \in \mathcal{C}_4 : \dim Z(L_\alpha) \geq 2\}$, that is the orbit of $\mathbf{a}_7$, where the fibers of $p_4'$ have dimension 1. By Proposition 5.2 $(p_4')^{-1}(\mathcal{Z}_4)$ is irreducible of dimension 6, which shows the claim. ∎

**Corollary 5.12.** *The varieties $\mathcal{C}_i$, $i = 1, \ldots, 4$, are the irreducible components of $\mathcal{L}_4$ and the set of the singular points of $\mathcal{L}_4$ is $\bigcup_{i \neq j} \mathcal{C}_i \cap \mathcal{C}_j$, $i, j = 1, \ldots, 4$, that is the union of the orbits of the following Lie algebras: $\mathbf{g}_{[[0,\gamma]]}$, $\gamma \in \mathbb{C}$, $\mathbf{g}_{[[\gamma+1,\gamma]]}$, $\gamma \in \mathbb{C}$, $\mathbf{g}_{[[0]]}$, $\mathbf{g}_c$, $c = 0, 2$, $\mathbf{a}_5$, $\mathbf{a}_6$, $\mathbf{a}_7$, $\mathbf{a}_8$.*

By the equations of the space of 2-cocycles of a Lie algebra we have found that the dimension of the tangent space to $\mathcal{L}_4$ in $\mathbf{g}_{[[\beta,\gamma]]}$, $\beta = 0$ or $\beta = \gamma + 1$, $[[\beta, \gamma]] \neq [[0, 1]]$, $[[0, -1]]$, $[[0, 0]]$, $\mathbf{g}_{[[0]]}$, $\mathbf{g}_c$, $c = 0, 2$, $\mathbf{a}_5$, $\mathbf{a}_6$ is 12. It is 13 in $\mathbf{g}_{[[0,1]]}$, $\mathbf{g}_{[[0,-1]]}$, $\mathbf{g}_{[[0,0]]}$. In $\mathbf{a}_7$ and $\mathbf{a}_8$ it is 18 and 14 respectively.

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