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Two Observations on Irreducible Representations of Groups

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Abstract. For an irreducible representation of a connected affine algebraic group G in a vector space V of dimension at least 2, it is shown that the intersection of any orbit $\pi(G)x$ (with $x \in V$) and any hyperplane of V is non-empty. The question is raised to decide whether an analogous fact holds for irreducible continuous representations of connected compact groups, for example of SU(2).
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By definition, a linear representation $\pi : G \longrightarrow GL(V)$ of a group G in a vector space V is *irreducible* if, for any vector $x \neq 0$ in V and for any hyperplane H of V, the orbit $\pi(G)x$ does *not* lie inside H. The purpose of this note is to record how irreducibility may imply other geometrical properties of the orbits, either in general as in the most elementary Proposition 1 below about "affine irreducibility", or for representations of algebraic groups as in Proposition 2. We provide also examples which show that Proposition 2 has no analogue for noncompact semisimple *real* Lie groups, but we leave open the question to decide if it has for *compact* semisimple Lie groups.

Proposition 1. *Let G be a group, V a vector space over some field, and $\pi : G \longrightarrow GL(V)$ an irreducible linear representation distinct from the unit representation. If A is an* affine *subspace of V which is invariant by G, then $A = 0$ or $A = V$.*

Proof. If an affine subspace A is $\pi(G)$-invariant, so is the linear space H of differences of vectors in A, so that H is one of 0 or V, and the same holds for A. ∎

Proposition 2. *Let G be a connected algebraic group over some algebraically closed field $\mathbb{K}$, let V a finite dimensional vector space of dimension at least*

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two, and let $\pi : G \longrightarrow GL(V)$ be a rational irreducible representation. For any linear hyperplane H of V and any $x \in V$, the intersection of H with the orbit $X = \pi(G)x$ is non empty.

Proof. Consider first the case of a group G which is semisimple. Choose a linear form $f \neq 0$ on V such that $H = \ker(f)$. Define a regular function $\phi : G \longrightarrow \mathbb{K}$ by $\phi(g) = f(\pi(g)x)$.

Assume (ab absurdo) that the intersection of H and X is empty. Then ϕ does not have any zero on G. A theorem of Rosenlicht (see below) implies that there exists a constant $c \neq 0$ such that $c\phi$ is a group homomorphism $G \longrightarrow \mathbb{K}^*$; this implies that ϕ is constant since G is perfect. Thus X is contained in an affine hyperplane of V. The affine hull of X is non-trivial and invariant by G; this is absurd by Proposition 1, so that the proposition is proved in the semisimple case.

Consider now the general case. Let R_u denote the unipotent radical of G. By a theorem of Kolchin (see e.g. 4.8 in [1]), the subspace $V^u = \{v \in V \mid \pi(r)v = v \text{ for all } r \in R_u\}$ is not reduced to zero. This space being $\pi(G)$-invariant, because R_u is normal in G, and π being irreducible, we have $V^u = V$. Consequently, we may replace G by G/R_u, namely we may assume that G is reductive.

Let T denote the solvable radical of G, which is a torus (11.21 in [1]). Let $V = \oplus V^\chi$ denote the weight space decomposition of the restriction $\pi|T$, where $V^\chi = \{v \in V \mid \pi(t)v = \chi(t)v \text{ for all } t \in T\}$ for $\chi \in \mathrm{Hom}(T, \mathbb{K}^*)$. We may choose $\psi \in \mathrm{Hom}(T, \mathbb{K}^*)$ such that $V^\psi \neq \{0\}$. Since T is normal in G and since the abelian group $\mathrm{Hom}(T, \mathbb{K}^*)$ is finitely generated (8.2 in [1]), there is a natural action of the connected group G on $\mathrm{Hom}(T, \mathbb{K}^*)$ and this action is trivial. Hence V^ψ is $\pi(G)$-invariant, and indeed is equal to V by irreducibility of π. Thus π coincides on T with some $\psi \in \mathrm{Hom}(T, \mathbb{K}^*)$.

Now G is a product of its derived group DG and of T, and DG is semisimple (14.2 in [1]). Thus the equality $\pi|T = \psi \otimes id_V$ and the irreducibility of π imply that the restriction of π to the semisimple group DG is irreducible. This ends the proof of the reduction of the general case to the semisimple case.■

Reminder of Rosenlicht's result [6]. If Y, Z are two irreducible affine algebraic varieties, any scalar-valued function on the product $Y \times Z$ which is regular and without zero is a product of a regular function on Y by a regular function on Z. Thus, if ϕ is a regular function without zero on a linear algebraic group G, there exist regular functions ψ, χ such that $\phi(gh) = \psi(g)\chi(h)$ for all $g, h \in G$. Set $c = \phi(1)^{-1}$ and let φ denote the function $c\phi$; the previous relation implies that $\varphi = \psi(1)^{-1}\psi = \chi(1)^{-1}\chi$ and that $\varphi(gh) = \varphi(g)\varphi(h)$ for all $g, h \in G$, namely that φ is a character on G, by which we mean here a homomorphism of groups $G \longrightarrow \mathbb{K}^*$. For an exposition of Rosenlicht's result, see [4]; see also [2].

Corollary. *Let G be a reductive connected complex Lie group, let V a finite dimensional complex vector space of dimension at least two, and let $\pi : G \longrightarrow GL(V)$ be an irreducible holomorphic representation. For any linear hyperplane H of V and any $x \in V$, the intersection of H with the orbit $X = \pi(G)x$ is non empty.*

Proof. This is a straightforward consequence of Proposition 2, since a connected reductive complex Lie group G has a unique algebraic structure, and a holomorphic representation of such a group is necessarily algebraic. See e.g. Theorem 6.4 of Chapter 1 and Theorem 2.8 of Chapter 4 in [5]. ∎

Observations. There are no analogues of Proposition 2 for finite groups and for simple connected real Lie groups, as the following examples show.

(i) If G is a finite group, G-orbits in $V \setminus \{0\}$ and hyperplanes are generically disjoint.

(ii) Let π be the 2-dimensional irreducible representation of the group $SL_2(\mathbb{R})$ in the space $\mathbb{C}^2$. For a vector $x \in \mathbb{R}^2$, $x \neq 0$, and the linear span H of the vector $(1, i) \in \mathbb{C}^2$, the $SL_2(\mathbb{R})$-orbit of x and the hyperplane H are disjoint.

For another example, consider the 3-dimensional irreducible representation of $SL_2(\mathbb{R})$ in the space V of homogeneous polynomials of degree 2 with complex coefficients in 2 variables ξ, η. If $x \in V$ is the polynomial $\xi\eta$, its $SL_2(\mathbb{R})$-orbit X is a surface of equation $\rho^2 - 4\sigma\tau = 1$ (with respect to appropriate coordinates (ρ, σ, τ) on V), and its intersection with the complex hyperplane of equation $\sigma = \sqrt{i}\,\tau$ is empty.

(iii) Consider more generally an integer $n \geq 2$, the connected component G of the group $SO(n, 1)$, and the natural irreducible representation π of G in $\mathbb{C}^{n+1}$. For a non-zero vector $x \in \mathbb{R}^{n+1}$ inside and a real hyperplane $H_0 \subset \mathbb{R}^{n+1}$ outside the light cone, it is clear that the orbit $\pi(G)x$ is disjoint from H_0; it follows that $\pi(G)x$ is also disjoint from the complexified hyperplane $H_0 \otimes_{\mathbb{R}} \mathbb{C}$ in $\mathbb{C}^{n+1}$.

Question. In which situations does some Proposition 2 hold? what about a connected compact group and an irreducible continuous representation? what about the irreducible representation of $SU(n)$? of $SU(2)$? We spell out explicitely the last particular case of the question:

Let π_k be the natural representation of $SU(2)$ in the space $\mathcal{P}_k$ of complex polynomials in two variables which are homogeneous of degree k, for some $k \geq 1$, let H be a complex hyperplane in $\mathcal{P}_k$ and let $P \in \mathcal{P}_k$; is it always true that $\pi_k(SU(2))P \cap H \neq \emptyset$?

Remarks. (i) Let G be a compact topological group, V an Hermitian space, and $\pi : G \longrightarrow U(V)$ an irreducible continuous unitary representation distinct from the unit representation. It is known [3] that, for any vector $x \in V$ of norm 1, the diameter $\max_{g \in G} \|\pi(g)x - x\|$ of the orbit $\pi(G)x$ is *strictly* larger than $\sqrt{2}$.

(ii) Let G be a compact connected topological group, V a finite dimensional *real* vector space, $\pi : G \longrightarrow GL(V)$ an irreducible continuous representation distinct from the unit representation, $X = \pi(G)x$ the G-orbit of a vector $x \neq 0$ in V, and H an hyperplane of V, say $H = \ker(f)$ for some linear form $f \neq 0$ on V. *Then the intersection of H and X is non empty.*

Indeed, define as above $\phi : G \longrightarrow \mathbb{R}$ by $\phi(g) = f(\pi(g)x)$. If $X \cap H = \emptyset$, then ϕ is either strictly positive or strictly negative on G, so that $\int_G \phi(g)dg = f(y) \neq 0$ for $y = \int_G \pi(g)x\,dg$, and in particular $y \neq 0$; but this is impossible because y is $\pi(G)$-invariant by invariance of the Haar measure dg on G.

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