Journal of Lie Theory
Volume **12** (2002) 357–368

The Outer Automorphism Group of Normalizers of Maximal Tori in Connected Compact Lie Groups

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Communicated by A. Valette

Abstract. Let G be a connected compact Lie group, T a maximal torus of G, $N = N_G(T)$ its normalizer and $W = N/T$ the Weyl group of G. We show that the outer automorphism group of N canonically decomposes as a semidirect product, where the normal subgroup is given by the cohomology group $H^1(W;T)$ and the complement subgroup is the outer automorphism group of G.

1. Introduction

The origin of the present work lies in a well-known theorem of Curtis, Wiederhold and Williams [3], saying that *two connected compact Lie groups are isomorphic if and only if the normalizers of their maximal tori are isomorphic* (see the paper by Osse [10] for a proof of the general, non-semisimple, case). Here, we are interested in the following natural question, formulated by Osse and closely related to this theorem:

> *What is the relationship between the automorphisms of the normalizer of a maximal torus and those of the corresponding connected compact Lie group?*

Every automorphism of a connected compact Lie group preserves some maximal torus (see [4] or [2]), thus its normalizer, and yields, therefore, an automorphism of the normalizer by restriction. Conversely, can every automorphism of the normalizer be extended to an automorphism of the whole group? The following motivating example shows that the answer is no!

Example 1.1. Let $G = \mathrm{SU}(2m)$ with $m \geqslant 2$ and let $T \subset G$ be a maximal torus. The normalizer $N = N_G(T)$ of T fits into the extension $T \hookrightarrow N \overset{\pi}{\twoheadrightarrow} \Sigma_{2m}$, where Σ_{2m} denotes the symmetric group on $2m$ letters. Then the map

$$\psi : N \longrightarrow N\,,\ n \longmapsto (-\mathbb{1})^{\epsilon(\pi(n))}\cdot n\,,$$

* Supported by the Japanese Society for the Promotion of Science.

ISSN 0949–5932 / \$2.50

where $-\mathbb{1} = \mathrm{diag}(-1,\dots,-1) \in \mathrm{SU}(2m)$ and $\epsilon : \Sigma_{2m} \to \mathbb{Z}/2 = \{0,1\}$ denotes the signature homomorphism, defines an automorphism of the normalizer. We investigate ψ for the case of the standard maximal torus consisting of the diagonal matrices of determinant 1. By contradiction, suppose that ψ can be extended to $\mathrm{SU}(2m)$. As it is trivial on the maximal torus, this extension is a conjugation by an element $t \in T$ [2, Proposition 9, p. 30]. We now check what it implies on the maximal torus and on an element $n \in N$ that exchanges the first two elements on the diagonal, i.e.

$$n \cdot \mathrm{diag}(z_1, z_2, z_3, \dots, z_{2m}) \cdot n^{-1} = \mathrm{diag}(z_2, z_1, z_3, \dots, z_{2m}) .$$

We have $t \cdot n \cdot t^{-1} = \psi(n) = -n$, which is equivalent to $n^{-1} \cdot t \cdot n = n \cdot t \cdot n^{-1} = -t$. As $m \geqslant 2$, this is clearly in contradiction with how n is acting on T.

Our main theorem gives a complete description of the automorphisms of the normalizer that are not extendable, identifying them as a cohomology group of degree 1, and answers the above question in terms of outer automorphism groups.

Theorem 1.2. *Let G be a connected compact Lie group, T a maximal torus in G, and N the normalizer of T in G. Let $W = N/T$ be the Weyl group of G. Then the outer automorphism group of the Lie group N canonically decomposes as the semidirect product*

$$\mathrm{Out}(N) \cong H^1(W;T) \rtimes \mathrm{Out}(G) ,$$

where the W-module T is endowed with the natural action induced by the extension $T \hookrightarrow N \twoheadrightarrow W$.

In section 2 we introduce the material on root diagrams that is needed for our proof of theorem 1.2. Section 3 deals with some properties of the automorphisms of N that induce the identity on T. It then recalls how the cohomology group $H^1(W;T)$ can be identified as the subgroup of $\mathrm{Out}(N)$ consisting of elements that have a representative in $\mathrm{Aut}(N)$ inducing the identity on both T and W, as typically depicted in example 1.1. Finally section 4 is devoted to the proof of the main theorem. Once $\mathrm{Out}(G)$ has also been identified as a subgroup of $\mathrm{Out}(N)$ in proposition 4.1, it shows in particular that its action on $H^1(W;T)$, which is not a priori clear, can be interpreted as conjugation in $\mathrm{Out}(N)$.

Remark 1.3. The question addressed in the present work has a natural formulation in the context of p-compact groups; we refer to [9] for related results and to [5] for an account concerning the importance of normalizers of maximal tori in the theory of p-compact groups.

2. Root diagrams

The notion of root diagram generalizes that of root system and allows to treat the case of connected compact Lie group that are *not* semisimple. Root diagrams are introduced and studied in Bourbaki [2, §4, No. 8-9], where it is shown that the classification of connected compact Lie groups can be done in this purely algebraic setting. The idea of the slightly modified version of the definition of root diagram that we present here is due to Osse; its advantage is twofold: it is more tractable than the definition in [2], and it more clearly shows that a root diagram is a generalization of a root system. We will see that the two definitions are basically equivalent.

Definition 2.1. A *(reduced) root diagram* is a triple $D=(V,M,\Phi)$ satisfying:

1. V is a euclidean space, M is a lattice in it, and Φ is a finite subset of $M\setminus\{0\}$;
2. for all $\alpha\in\Phi$ and for all $k\in\mathbb{Z}$, $k\alpha\in\Phi$ if and only if $k=\pm1$;
3. for all $\alpha\in\Phi$, the map
$$s_\alpha : V\longrightarrow V\,,\ x\longmapsto x-\frac{2(x\,,\alpha)}{(\alpha\,,\alpha)}\,\alpha\,,$$
called *reflection of vector* α , leaves the set Φ globally invariant;
4. for all $\alpha\in\Phi$ and $x\in M$, $\dfrac{2(x\,,\alpha)}{(\alpha\,,\alpha)}\in\mathbb{Z}$.

The group of automorphisms of V generated by the reflections s_α is called the *Weyl group of* D and is denoted by $W(D)$. The inner product $(.\,,.)$ is obviously $W(D)$-invariant. Hence the set Φ is a reduced root system in the euclidean space $\mathbb{R}\Phi$, whose Weyl group is precisely $W(D)$.

The following lemma shows that the definition presented here is equivalent to Bourbaki's, once, without loss of generality, a $W(D)$-invariant inner product has been fixed in the latter one.

Lemma 2.2. *Let $D=(V,\ M;\ \Phi)$ be a root diagram. Consider the set*
$$M_o=\{x\in M : (x\,,\alpha)=0,\ \ \forall\,\alpha\in\Phi\}\,.$$
Then M_o is the subgroup $M^{W(D)}$ of $W(D)$-invariants of M . Moreover it is a direct summand in M and the set $M_o\cup\Phi$ spans the vector space V .

Proof. It is clear that $M_o=M^{W(D)}$.

Since M_o is a subgroup of the free abelian group M of rank n , it is itself a free abelian group of rank $r\leqslant n$. By the elementary divisors theorem, there exist bases $\{e_1,\ldots,e_n\}$ of M and $\{\epsilon_1,\ldots,\epsilon_r\}$ of M_o , and integers $d_1,\ldots,d_r$ such that $\epsilon_j=d_j{\cdot}e_j$, for all $j=1,\ldots,r$. Let $j\in\{1,\ \ldots\ ,\ r\}$; by definition of M_o , we have $(\epsilon_j\,,\ \alpha)=0$ for all $\alpha\in\Phi$, hence the same holds for e_j , and therefore $e_j\in M_o$ and $d_j=1$. This proves that M_o is a direct summand in M .

Let us now show that $M_o\cup\Phi$ spans V . Let $x\in M$; for all $\alpha\in\Phi$, we have $s_\alpha(x)\equiv x\pmod{\mathbb{Z}\Phi}$. As $W=W(D)$ is generated by the s_α's and $\mathbb{Z}\Phi$ is globally W-invariant, it implies $w\cdot x\equiv x\pmod{\mathbb{Z}\Phi}$ for all $w\in W$. Define $y=\sum_{w\in W}w\cdot x\in M^W$; obviously, $y\equiv|W|\cdot x\pmod{\mathbb{Z}\Phi}$, and therefore $x=(1/|W|)\cdot y+z$ with $y\in M_o$ and $z\in\mathbb{Z}\Phi$. ∎

The example that will interest us is the *(covariant) root diagram* of a connected compact Lie group G with respect to a fixed maximal torus T: it is given by $D(G) = (LT, \Gamma(T), R^\vee)$, where $\Gamma(T)$ is the integral lattice in the Lie algebra LT of the maximal torus, and $R^\vee$ is the set of coroots of G. The Weyl group $W(D(G))$ is canonically isomorphic to the Weyl group W of G. In this particular case, M_o is equal to the intersection $\Gamma(T) \cap \mathfrak{Z}(LG)$ of the integral lattice with the center of the Lie algebra LG of G, i.e. the non-semisimple part of the integral lattice.

Let us now define the notion of isomorphism of root diagrams (compare with [10, Def. 4.2]).

Definition 2.3. Let $D = (V, M, \Phi)$ and $D' = (V', M', \Phi')$ be root diagrams. An *isomorphism of root diagrams* between D and D' is an isomorphism of vector spaces $f : V \longrightarrow V'$ (which is not necessarily an isometry) mapping M bijectively onto M', Φ bijectively onto Φ', and satisfying, for all $x \in M$ and all $\alpha \in \Phi$, the compatibility condition

$$\frac{2(x\,,\alpha)}{(\alpha\,,\alpha)} = \frac{2(f(x)\,,f(\alpha))}{(f(\alpha)\,,f(\alpha))}\,.$$

As an immediate consequence of this definition, the inner product on V is more or less irrelevant. More precisely, two $W(D)$-invariant inner products on V yield two isomorphic root diagrams.

The *automorphism group* of a root diagram will be denoted by $\mathrm{Aut}(D)$. We end up this section by recalling a key result on $\mathrm{Aut}(D)$, which will be needed to prove theorem 1.2 (see [2, Prop. 18, p. 42]).

Proposition 2.4. (Bourbaki) *For any connected compact Lie group G, there is a canonical isomorphism*

$$\mathrm{Aut}(D(G))/W(D(G)) \cong \mathrm{Out}(G)\,.$$

3. Automorphisms of N inducing the identity on T

We start by fixing some notations for the last two sections. The Lie algebra of a Lie group K will be denoted by LK and the differential at the unit element of a Lie group homomorphism $\varphi : K_1 \to K_2$ by $\varphi_* : LK_1 \to LK_2$. For an element g in K, we will write c_g for conjugation by g, i.e. $c_g(x) = gxg^{-1}$, for all x in K. We also assume the notations of theorem 1.2.

As T is the component of the identity of N, an automorphism $\psi \in \mathrm{Aut}(N)$ of the normalizer induces two automorphisms $\psi_{|T}$ and $\bar\psi$ fitting in the commutative diagram

$$\begin{tikzcd} T \arrow[r, hook] \arrow[d, "\cong"', "\psi_{|T}"] & N \arrow[r, two heads, "\pi"] \arrow[d, "\cong"', "\psi"] & W \arrow[d, "\cong"', "\bar\psi"] \\ T \arrow[r, hook] & N \arrow[r, two heads, "\pi"] & W \end{tikzcd}$$

The automorphism ψ also induces a vector space automorphism $\psi_* : LT \to LT$ on the Lie algebra of the torus.

Before stating the first lemma in this section, let us recall that for a root $\alpha \in R$, its associated coroot $\alpha^\vee \in LT$ and reflection s_α, the Lie algebra of the torus decomposes as $LT = \mathbb{R}\alpha^\vee \oplus H_\alpha$, where H_α denotes the hyperplane fixed by s_α, i.e. the hyperplane orthogonal to $\alpha^\vee$.

Lemma 3.1. *The automorphism $\bar{\psi}$ permutes the reflections in W. Moreover for $s_\beta = \bar{\psi}(s_\alpha)$, one has*

(*i*) $s_\beta = \psi_* \circ s_\alpha \circ \psi_*^{-1}$;

(*ii*) $\psi_*(H_\alpha) = H_\beta$;

(*iii*) $\psi_*(\mathbb{R}\alpha^\vee) = \mathbb{R}\beta^\vee$.

Proof. Let $s_\alpha \in W$ be a reflection, and let $q_\alpha \in N$ be an element such that $\pi(q_\alpha) = s_\alpha$. Let us denote $w = \bar{\psi}(s_\alpha) = \pi(\psi(q_\alpha))$, and $H = \psi_*(H_\alpha)$. Let also $Y = \psi_*(X) \in H$, with $X \in H_\alpha$. The following calculation shows that w fixes H pointwise:

$$\begin{aligned} w\cdot Y &= (c_{\psi(q_\alpha)})_*(Y) \\ &= (\psi \circ c_{q_\alpha} \circ \psi^{-1})_*(Y) \\ &= \psi_* \circ (c_{q_\alpha})_* \circ \psi_*^{-1}(Y) \\ &= \psi_* \circ s_\alpha \circ \psi_*^{-1}(Y) \\ &= \psi_* \circ s_\alpha \circ \psi_*^{-1} \circ \psi_*(X) \\ &= \psi_*(X) \\ &= Y. \end{aligned}$$

Now the subgroup of W that fixes H pointwise has the form $\langle s_\beta \rangle = \{id, s_\beta\}$ for some reflection s_β [7, Theorem 1.12 (d), p.22, and Proposition 1.14, p.24]. Since w is an element of order 2, we can conclude that $w = s_\beta$. Moreover the previous calculation shows that (*i*) and (*ii*) are verified.

The last assertion follows from (*i*) and (*ii*): for all $X \in LT$ we have

$$\begin{aligned} && s_\beta \circ \psi_*(X) &= \psi_* \circ s_\alpha(X) \\ \iff && \psi_*(X) - 2\tfrac{(\psi_*(X)\,,\beta^\vee)}{(\beta^\vee\,,\beta^\vee)}\cdot\beta^\vee &= \psi_*\Big(X - 2\tfrac{(X\,,\alpha^\vee)}{(\alpha^\vee\,,\alpha^\vee)}\cdot\alpha^\vee\Big) \\ && &= \psi_*(X) - 2\tfrac{(X\,,\alpha^\vee)}{(\alpha^\vee\,,\alpha^\vee)}\cdot\psi_*(\alpha^\vee) \\ \iff && \tfrac{(\psi_*(X)\,,\beta^\vee)}{(\beta^\vee\,,\beta^\vee)}\cdot\beta^\vee &= \tfrac{(X\,,\alpha^\vee)}{(\alpha^\vee\,,\alpha^\vee)}\cdot\psi_*(\alpha^\vee). \end{aligned}$$

Considering an element $X \in LT\backslash H_\alpha$, we have $\psi_*(X) \in LT\backslash H_\beta$, and the last equality shows that $\psi_*(\alpha^\vee)$ is a (non-zero) multiple of $\beta^\vee$. ■

Corollary 3.2. *If $\psi_{|T}$ is the identity of T, then $\bar{\psi}$ is the identity of W.*

Proof. Let us write $s_\beta = \bar{\psi}(s_\alpha)$. As $\psi_{|T} = id_T$, we also have $\psi_* = id_{LT}$. Therefore $H_\beta = \psi_*(H_\alpha) = H_\alpha$, and thus $s_\beta = s_\alpha$. ■

We end up this section by recalling a relationship between the automorphisms of an extension inducing the identity, on both the kernel and the quotient, and a group of cohomology, which will clarify the appearance of the term $H^1(W;T)$ in theorem 1.2. This relationship was deduced in [11] by exploiting an exact sequence introduced by Wells [12]. Let $A \hookrightarrow E \twoheadrightarrow Q$ be a group extension, where A is abelian. Let $\theta : Q \to \mathrm{Aut}(A)$ denote the natural action induced by the extension. Let also $\mathrm{Aut}(E,A,Q)$ be the subgroup consisting of elements ψ in $\mathrm{Aut}(E)$ that fix A pointwise and such that the induced automorphism $\bar{\psi} : Q \to Q$ is also the identity, i.e.

$$\mathrm{Aut}(E,A,Q) = \{\psi \in \mathrm{Aut}(E) : \psi_{|A} = id_A \text{ and } \bar{\psi} = id_Q\}\,.$$

We summarize the results from [11, 12] that we need in the following theorem. Another approach can be found in [1, pp. 87-88].

Theorem 3.3. *Let $A \hookrightarrow E \twoheadrightarrow Q$ a group extension with A abelian.*

(1) There is an identification

$$H^1(Q;A) = \mathrm{Aut}(E,A,Q)/\,\mathrm{Inn}_E(A)\,,$$

where $\mathrm{Inn}_E(A) = \{\psi \in \mathrm{Aut}(E,A,Q) : \psi = c_a\,,\ \textit{with } a \in A\}$.

(2) Moreover if A is a maximal abelian normal subgroup of E (this is the case if and only if the restriction of θ to the center of Q is an injection), then the induced application

$$H^1(Q;A) \longrightarrow \mathrm{Out}(E)$$

is injective.

Since T is self-centralizing in G, it is a maximal abelian normal subgroup of N . Thus this theorem says that the cohomology group $H^1(W;T)$ can be seen as the subgroup of $\mathrm{Out}(N)$ given by

$$H^1(W;T) = \mathrm{Aut}(N,T,W)/\,\mathrm{Inn}_N(T) \cong \mathrm{Aut}(N,T,W)/\big(T/Z(N)\big)\,,$$

where $Z(N)$ denotes the center of N .

Remark 3.4. Let $s : W \to N$ be a set-theoretic splitting of the canonical projection $\pi : N \twoheadrightarrow W$. Then the bijection between the set of normalized cocycles $Z^1(W;T) = \{f : W \to T \mid x{\cdot}f(y)f(xy)^{-1}f(x) = e\}$ and $\mathrm{Aut}(N,T,W)$ is explicitly given by

$$f \longmapsto \Big(\psi : s(x)t \longmapsto f(x)s(x)t\Big)\,.$$

As W is finite, this ensures that the elements of $\mathrm{Aut}(N,T,W)$ really are *Lie group* automorphisms of N .

4. Proof of the main theorem

We will prove theorem 1.2 by showing that there exists a split short exact sequence

$$H^1(W;T) \hookrightarrow \mathrm{Out}(N) \twoheadrightarrow \mathrm{Out}(G)\,.$$

The first step will consist in showing that there is a canonical injection of the outer automorphism group $\mathrm{Out}(G)$ into $\mathrm{Out}(N)$. Then, in the most important step, we will construct a surjective homomorphism $\mathrm{Out}(N) \twoheadrightarrow \mathrm{Out}(G)$, for which the injection in the first step is a splitting. Finally, we will identify the kernel of this surjection with the cohomology group $H^1(W;T)$. We subdivide these three steps in as many propositions.

Proposition 4.1. *There is a canonical inclusion*

$$i : \mathrm{Out}(G) \hookrightarrow \mathrm{Out}(N)\,.$$

of the outer automorphism group of G into that of N.

Proof. Let us consider $\mathrm{Aut}(G,T) = \{\varphi \in \mathrm{Aut}(G) : \varphi(T) = T\}$, the subgroup of automorphisms of G preserving T. As $\varphi(N) = N_G(\varphi(T)) = N$ for all $\varphi \in \mathrm{Aut}(G,T)$, we get a canonical homomorphism $\psi : \mathrm{Aut}(G,T) \to \mathrm{Aut}(N) \twoheadrightarrow \mathrm{Out}(N)$. On the other hand it is known [2, Proposition 18, p. 42] that

$$\mathrm{Aut}(G,T)/\Big(\mathrm{Aut}(G,T) \cap \mathrm{Inn}(G)\Big) \cong \mathrm{Out}(G)\,,$$

where $\mathrm{Aut}(G,T) \cap \mathrm{Inn}(G) = \{\varphi \in \mathrm{Aut}(G) : \exists n \in N \text{ with } \varphi = c_n\}$. Thus we get an induced map

$$\begin{tikzcd} \mathrm{Aut}(G,T) \arrow[r, "\psi"] \arrow[d] & \mathrm{Out}(N) \\ \mathrm{Out}(G) \arrow[ur, "\bar\psi = i"'] & \end{tikzcd}$$

We conclude by showing that i is injective. Let $\varphi \in \mathrm{Aut}(G,T)$; we have $[\varphi] \in \ker i$ if and only if $\varphi|_N \in \mathrm{Inn}(N)$. Therefore we find $n \in N$ such that $\varphi|_N = c_n$. So $c_n^{-1} \circ \varphi|_T = \mathrm{id}_T$ and thus the automorphism $c_n^{-1} \circ \varphi$ is a conjugation by an element $t \in T$ [2, Proposition 9, p. 30]. Hence $[\varphi] = [c_{nt}]$ is trivial in $\mathrm{Aut}(G,T)/(\mathrm{Aut}(G,T) \cap \mathrm{Inn}(G)) \cong \mathrm{Out}(G)$ and i is injective. ■

Proposition 4.2. *There exists a surjective homomorphism*

$$p : \mathrm{Out}(N) \twoheadrightarrow \mathrm{Out}(G)$$

for which the injection i is a splitting, i.e. such that $p \circ i$ is the identity of $\mathrm{Out}(G)$.

Before proving this proposition, we first introduce two lemmas.

Lemma 4.3. *Let W be a finite real reflection group and let $W = W_1 \times W_2 \times \ldots \times W_n$ be its decomposition into irreducible components. Let φ be an automorphism of W that preserves the set of reflections in W (i.e. for any reflection s_α, the element $\varphi(s_\alpha)$ is a reflection). Then φ preserves the decomposition of W into irreducible components, i.e. for all i we have $\varphi(W_i) = W_j$, for some j.*

Proof. By definition of an irreducible component, and by the fact that any irreducible component is generated by the reflections it contains, two reflections s_α and $s_{\alpha'}$ are in the same component if and only if there exists m reflections $s_{\alpha_1} = s_\alpha, \dots, s_{\alpha_m}$ such that s_{α_i} and $s_{\alpha_{i+1}}$ do not commute, for all $1 \leqslant i \leqslant m-1$, and such that $s_{\alpha'} = s_{\alpha_m} \dots s_{\alpha_2} s_{\alpha_1}$. As φ preserves reflections, the previous equivalence implies that $\varphi(s_\alpha)$ and $\varphi(s_{\alpha'})$ are in the same component if and only if s_α and $s_{\alpha'}$ are in the same component. ∎

Remark 4.4. One cannot drop the hypothesis on φ in this lemma: the Weyl group of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ has two irreducible components; it is isomorphic to the Klein group whose automorphisms permute the three non-trivial elements, but it contains only two reflections.

Direct factors isomorphic to $\mathrm{SO}(2\ell+1)$ in G will play a special role in the proof of proposition 4.2 (see [10, 8] for related special appearances of this direct factors). The following lemma asserts that the corresponding direct factors in N are well-behaved with respect to automorphisms.

Lemma 4.5. *Let $\psi \in \mathrm{Aut}(N)$. Then ψ permutes the maximal tori of the direct factors in N that correspond to direct factors isomorphic to $\mathrm{SO}(2\ell+1)$ in G.*

Proof. We assume that G has at least one direct factor isomorphic to $\mathrm{SO}(2\ell+1)$. By the decomposition theorem of connected compact Lie groups, we can write $G = H \times G_s$, where $G_s = \mathrm{SO}(2\ell+1)$, and $H = (S \times G_1 \times \dots \times G_{s-1})/K$, with S a torus, $G_1, \dots, G_{s-1}$ simple 1-connected compact Lie groups and K a finite central subgroup of $S \times G_1 \times \dots \times G_{s-1}$. This also implies the following decompositions: $T = (S \times T_1 \times \dots \times T_{s-1})/K \times T_s$, $LT = LS \oplus LT_1 \oplus \dots \oplus LT_{s-1} \oplus LT_s$ for the Lie algebra of T, $N = (S \times N_1 \times \dots \times N_{s-1})/K \times N_s$, and $W = W_1 \times \dots \times W_{s-1} \times W_s$. By lemma 3.1, the automorphism $\bar\psi : W \to W$ permutes the reflections, and thus, by lemma 4.3, it permutes the irreducible components of W. Suppose, without loss of generality, that $\bar\psi(W_1) = W_s$. By lemma 3.1, we therefore also have $\psi_*(LT_1) = LT_s$. Let us consider the composite homomorphism

$$\kappa : G_1 \hookrightarrow S \times G_1 \times \dots \times G_{s-1} \twoheadrightarrow (S \times G_1 \times \dots \times G_{s-1})/K \hookrightarrow G,$$

which factorizes through the injection $\alpha : \bar G_1 = G_1/\ker\kappa \hookrightarrow G$, with $\ker\kappa$ central. By restriction, we also have $\alpha_{|\bar N_1} : \bar N_1 = N_1/\ker\kappa \hookrightarrow N$, and $\alpha_{|\bar T_1} : \bar T_1 = T_1/\ker\kappa \hookrightarrow T$. Let us then consider the composition

$$\psi_1 : \bar N_1 \overset{\alpha}{\hookrightarrow} N \overset{\psi}{\to} N \overset{p_s}{\twoheadrightarrow} N_s,$$

where p_s denotes the canonical projection.

Claim. ψ_1 is an isomorphism. First we show that ψ_1 is surjective. ¿From the commutative diagram

$$\begin{tikzcd} LT_1 \arrow[r, "\alpha_*", "\cong"'] \arrow[d] & LT_1 \arrow[r, "\psi_{*|LT_1}", "\cong"'] & LT_s \arrow[r, "(p_s)_{*|LT_s}", "\cong"'] & LT_s \arrow[d] \\ \bar T_1 \arrow[rrr, "\psi_{1|\bar T_1}"] & & & T_s \end{tikzcd}$$

we get that $\psi_{1|\bar{T}_1} : \bar{T}_1 \to T_s$ is surjective. On the other hand, we have

$$\bar{\psi} \circ \bar{\alpha} : \bar{W}_1 = \bar{N}_1/\bar{T}_1 \xrightarrow{\cong} W_1 \xrightarrow{\cong} W_s \,,$$

and we deduce that im ψ_1 meets every connected component of N_s . So im ψ_1 is a subgroup of N_s that contains T_s and that meets all components, therefore it is equal to N_s and we get the surjectivity of ψ_1 . For the injectivity, we first observe that the situation we have just described at the level of Weyl groups implies that $\ker \psi_1 \subset \bar{T}_1$. Now $\psi \circ \alpha$ is clearly injective, and maps $\bar{T}_1$ bijectively onto T_s . As p_s maps T_s bijectively onto itself, we get that ψ_1 is injective. This proves our claim.

Then, by the theorem of Curtis-Wiederhold-Williams [3, 10], recalled in the introduction, $\bar{G}_1$ must be isomorphic to $\mathrm{SO}(2\ell+1)$. It follows that the group $\bar{G}_1$, which is of adjoint type, is a direct factor in G ; its maximal torus $\bar{T}_1$ is mapped by ψ isomorphically onto T_s . ∎

Proof of proposition 4.2. Let $\psi \in \mathrm{Aut}(N)$; the main part of the proof consists in showing that the vector space automorphism $\psi_* : LT \to LT$ is an automorphism of the root diagram $D(G)$. As $(\psi_1 \circ \psi_2)_* = (\psi_1)_* \circ (\psi_2)_*$, it will yield, by theorem 2.4 , a homomorphism $\bar{p} : \mathrm{Aut}(N) \to \mathrm{Out}(G)$. We finally check that $\bar{p}$ factorizes through a homomorphism $p : \mathrm{Out}(N) \to \mathrm{Out}(G)$ with the required properties.

Let us show that $\psi_* \in \mathrm{Aut}(D(G))$ in several steps.

Step 1: The naturality of the exponential implies that ψ_* maps $\Gamma(T)$ bijectively onto itself.

Step 2: We derive some formulae that will be useful in the next step. In particular, we get that the the compatibility condition in definition 2.3 is satisfied. Let $\alpha^\vee \in R^\vee$; fix an element $X \in LT$ and decompose it as $X = Y + Z$, with $Y \in \mathbb{R}\alpha^\vee$ and $Z \in H_\alpha$. Explicitly, we have $Y = \frac{(X\,,\alpha^\vee)}{(\alpha^\vee\,,\alpha^\vee)} \cdot \alpha^\vee$, and thus

$$\psi_*(Y) = \frac{(X\,,\alpha^\vee)}{(\alpha^\vee\,,\alpha^\vee)} \cdot \psi_*(\alpha^\vee)\,.$$

On the other hand, we have $\psi_*(X) = \psi_*(Y) + \psi_*(Z)$, with, again by lemma 3.1 , $\psi_*(Y) \in \mathbb{R}\beta^\vee$ and $\psi_*(Z) \in H_\beta$. Therefore we have

$$\psi_*(Y) = \frac{(\psi_*(X)\,,\psi_*(\alpha^\vee))}{(\psi_*(\alpha^\vee)\,,\psi_*(\alpha^\vee))} \cdot \psi_*(\alpha^\vee)\,.$$

Comparing these two expressions for $\psi_*(Y)$, we get that

$$\frac{(X\,,\alpha^\vee)}{(\alpha^\vee\,,\alpha^\vee)} = \frac{(\psi_*(X)\,,\psi_*(\alpha^\vee))}{(\psi_*(\alpha^\vee)\,,\psi_*(\alpha^\vee))}\,, \forall\, X \in LT\,, \forall\, \alpha^\vee \in R^\vee\,.$$

In particular, this equality shows that the compatibility condition in definition 2.3 is satisfied. Now setting $X = \beta^\vee \in R^\vee$, we get

$$\frac{\|\beta^\vee\|}{\|\alpha^\vee\|} \cdot \cos\theta = \frac{\|\psi_*(\beta^\vee)\|}{\|\psi_*(\alpha^\vee)\|} \cdot \cos\phi\,,$$

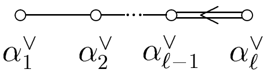
where $\theta = \measuredangle(\alpha^\vee, \beta^\vee)$ et $\phi = \measuredangle(\psi_*(\alpha^\vee), \psi_*(\beta^\vee))$. Symmetrically we also get

$$\frac{\|\alpha^\vee\|}{\|\beta^\vee\|}\cdot\cos\theta = \frac{\|\psi_*(\alpha^\vee)\|}{\|\psi_*(\beta^\vee)\|}\cdot\cos\phi\,.$$

For two non-orthogonal coroots $\alpha^\vee$ and $\beta^\vee$, these last two expressions finally yield

$$\frac{\|\alpha^\vee\|}{\|\beta^\vee\|} = \frac{\|\psi_*(\alpha^\vee)\|}{\|\psi_*(\beta^\vee)\|}\,. \tag{$*$}$$

Step 3: In this step we show that ψ_* maps the set of coroots $R^\vee$ bijectively onto itself. We first observe that the troublesome cases are those where G has direct factors of type $\mathrm{SO}(2\ell+1)$. By lemma 3.1, ψ_* permutes the directions generated by the coroots. We know from step 1 that ψ_* preserves $\Gamma(T)$. Moreover the set $R^\vee$ is contained in the integral lattice $\Gamma(T)$, and it is almost always characterized by the fact that $\alpha^\vee$ is minimal, i.e. $\mathbb{R}\alpha^\vee \cap \Gamma(T) = \mathbb{Z}\alpha^\vee$ for all $\alpha^\vee \in R^\vee$. The only exception is the following case (see [8, Proposition 3.2.] or [10, pp. 86-87]): R has an irreducible component of type B_ℓ ($\ell \geqslant 1$, $B_1 = A_1$) that corresponds in G , to a direct factor $\mathrm{SO}(2\ell+1)$, and α is a short root. For $R^\vee$, this translates into a dual irreducible component of type C_ℓ , with a long coroot $\alpha^\vee$; in this case $1/2\cdot\alpha^\vee$ is the minimal element. By minimality of coroots in all the other cases, it only remains to treat this exceptional case. Lemma 4.5 further reduces the situation to the case where N is the normalizer of a maximal torus in a single copy of $\mathrm{SO}(2\ell+1)$. The case $\ell = 1$ is straightforward: the integral lattice is isomorphic to $\mathbb{Z}$ and is preserved by ψ_*, thus the generator $1/2\cdot\alpha^\vee$ is mapped to $\pm 1/2\cdot\alpha^\vee$, and therefore ψ_* preserves $R^\vee = \{\pm\alpha^\vee\}$. If $\ell > 1$, there are two possible lengths for the coroots, and the factor between the norms of a long and a short coroot is $\sqrt{2}$. The only pathological case that could occur is the following: a long coroot $\alpha^\vee$ (i.e. a non-minimal one) maps to twice a short coroot. A short coroot could then map to either a short coroot or half a long coroot (because $\psi_*(\Gamma(T)) = \Gamma(T)$) . Now $\alpha^\vee = \alpha_\ell^\vee$ is an element of some basis $\{\alpha_1^\vee, \dots, \alpha_{\ell-1}^\vee, \alpha_\ell^\vee\}$ of $R^\vee$, with $\alpha_\ell^\vee$ and $\alpha_{\ell-1}^\vee$ non-orthogonal and $\alpha_{\ell-1}^\vee$ necessarily short, as depicted in the following Dynkin diagram of type C_ℓ:



Let us write $2\cdot\beta_\ell^\vee = \psi_*(\alpha_\ell^\vee)$, for some short coroot $\beta_\ell^\vee$. We apply $(*)$ with $\alpha^\vee = \alpha_\ell^\vee$ and $\beta^\vee = \alpha_{\ell-1}^\vee$. Distinguishing the two possible cases for the length of the image of the short coroot $\alpha_{\ell-1}^\vee$, we get

$$\sqrt{2} = \frac{\|\alpha_\ell^\vee\|}{\|\alpha_{\ell-1}^\vee\|} = \frac{\|2\cdot\beta_\ell^\vee\|}{\|\psi_*(\alpha_{\ell-1}^\vee)\|} = \frac{2\|\alpha_{\ell-1}^\vee\|}{\|\psi_*(\alpha_{\ell-1}^\vee)\|} = \begin{cases} 2\frac{\|\alpha_{\ell-1}^\vee\|}{\|\alpha_{\ell-1}^\vee\|} & = 2 \\ 2\frac{\|\alpha_{\ell-1}^\vee\|}{\frac{1}{2}\|\alpha_\ell^\vee\|} & = 2\sqrt{2}\,, \end{cases}$$

which is manifestly impossible and ends up the proof of step 3.

Steps 1,2 and 3 together show that ψ_* is an automorphism of $D(G)$. Finally, it is clear that inner automorphisms of N act as elements of the Weyl group of $D(G)$. By theorem 2.4, this implies that the homomorphism $\bar{p}$ factorizes through $p : \mathrm{Out}(N) \to \mathrm{Out}(G)$. By construction, it is also clear that the composition $p \circ i : \mathrm{Out}(G) \hookrightarrow \mathrm{Out}(N) \to \mathrm{Out}(G)$ is the identity. This shows that p is split-surjective, with i as a splitting, and completes the proof. ■

Proposition 4.6. *The kernel of p is*

$$\ker p = H^1(W;T)\,.$$

Proof. By definitions:

$$\begin{aligned}[][\psi] \in \ker p &\iff p[\psi] \in W(D) \\ &\iff \Big(\psi_* : LT \to LT\Big) \in W\,.\end{aligned}$$

So we find $n \in N$ such that $\psi_* = (c_n)_*$. Clearly $[\psi] = [c_n^{-1} \circ \psi] \in \mathrm{Out}(N)$. Now $(c_n^{-1} \circ \psi)_* = id_{LT}$ and thus $c_n^{-1} \circ \psi_{|T} = id_T$. Therefore, by corollary 3.2, the induced automorphism on W is also the identity, and we can conclude by invoking theorem 3.3. ∎

Proof of theorem 1.2. Propositions 4.1, 4.2 and 4.6 clearly constitute a proof. ∎

Remark 4.7. We mention here a few properties and results related to the group $H^1(W;T)$. These are taken from the author's PhD thesis [6] and will be the object of a later publication. Using a presentation of the normalizer due to Tits, it is shown that $H^1(W;T)$ is a (finite) elementary abelian 2-group. Explicit calculations of this group are then carried out: for example if $G = \mathrm{SU}(n)$ then $H^1(W;T) \cong \mathbb{Z}/2$ for $n = 2m \geqslant 4$, generated by the class of the automorphism of example 1.1, and trivial in the remaining cases. Together with some recent work of Suter and Matthey, these explicit calculations cover *all* simple connected compact Lie groups.

Remark 4.8. The explicit description of $\mathrm{Out}(N)$ in theorem 1.2 turns out to be a key ingredient in [6] for proving a generalization of the theorem of Curtis, Wiederhold and Williams, to *nonconnected* compact Lie groups. This generalization then allows us to get a new proof, in the *nonconnected* case, of the fact that the classifying space BG characterizes the compact Lie group G.

Acknowledgments. The material in this paper is taken from my PhD thesis at the University of Neuchâtel, Switzerland, and it is a pleasure to warmly thank my advisor Professor U. Suter for his guidance and constant encouragements. I am also greatly indebted to M. Matthey and A. Osse for many fruitful discussions and suggestions. In particular, the contents of section 2 were discussed at a seminar in which Suter, Matthey, Osse and myself participated. I also gratefully acknowledge the financial support of the Swiss National Science Foundation, under grant no. 2000-056816.99, and the current financial support of the Japan Society for the Promotion of Science.

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Received February 16, 2001
and in final form October 30, 2001