Journal of Lie Theory
Volume **12** (2002) 495–502

On Observable Subgroups of Complex Analytic Groups and Algebraic Structures on Analytic Homogenous Spaces

Nazih Nahlus

Communicated by K. H. Hofmann

Abstract. Let L be a closed analytic subgroup of a faithfully representable complex analytic group G, let $R(G)$ be the algebra of complex analytic representative functions on G, and let G_0 be the universal algebraic subgroup (or algebraic kernel) of G.

In this paper, we show many characterizations of the property that the homogenous space G/L is (representationally) *separable*, i.e, $R(G)^L$ separates the points of G/L. For example, G/L is separable if and only if $G_0 \cap L$ is an algebraic subgroup of G_0 which is (rationally) observable in G_0. These characterizations yield new characterizations for the analytic observability of L in G and new characterizations for the existence of a quasi-affine structure on G/L. For example, L is (analytically) observable in G if and only if G/L is separable and $L_0 = G_0 \cap L$.

Similarly, we discuss a weaker separability of G/L and the existence of a representative algebraic structure on G/L.

2000 Mathematics Subject Classification: 22E10, 22E45, 22F30, 20G20, 14L15.

Let L be a closed analytic subgroup of a faithfully representable complex analytic group G. Then L is called (analytically) *observable* in G if every finite-dimensional complex analytic representation of L is extendable to a finite-dimensional analytic representation of G; or more precisely, if every finite-dimensional analytic L-module is a sub L-module of a finite-dimensional analytic G-module [7, p. 166]. Similarly, we have the notion of (rational) observability for algebraic subgroups of linear algebraic groups [1]. *If there is no ambiguity, we shall simply use the term "observable".*

The homogenous space G/L will be called (representationally) *separable* if $R(G)^L$ separates the points of G/L. The homogenous space G/L is said to have a *quasi-affine structure* if G/L has the structure of a quasi-affine algebraic variety which is compatible with G and $R(G)$ [2, p. 813] (see Definition 1 below). Moreover, G/L is said to have a (representative) *algebraic structure* if G/L has

ISSN 0949–5932 / \$2.50

the structure of an algebraic variety which is compatible with G and $[R(G)]$ [3, p. 852] (see Definition 2 below).

The question of observability of L in G has been studied by Lee and Wu in [7]. But there was no explicit mention of the separability of G/L (see Theorem A below). So our Theorem 1 and Corollary 2 completely clarify the role of separability in this respect. The question of having a quasi-affine algebraic structure on G/L has been studied by Hochschild and Mostow [2] where the separability of G/L plays an essential role (see Theorem C below). Moreover, the question of having a (representative) algebraic structure on G/L has been also studied by Hochschild and Mostow [3] where the weaker separability by $[R(G)]^L$ plays an essential role (see Theorem D below). So Theorems 1 and 4 and their corollaries clarify the role of separability and weaker separability of G/L which are essential for the existence of such structures on G/L.
In addition, Remark 1 clarifies the definition of a quasi-affine structure on G/L. If G is an algebraic group and L is an algebraic subgroup of G, Remark 2 shows that if L is analytically observable in G, then L is rationally observable in G. Finally we use the proof of Theorem 1 to give a new proof, independently of [7], of the "if part" or the extension part in Thm. 4.5 of [7] stated in Theorem A below.

The proofs in this paper rely heavily on [2], [3], and [7] as well as the recent work by Magid and the author in [8] concerning observable subgroups of pro-affine algebraic groups. We shall assume that the reader is familiar with the basic theory of pro-affine algebraic groups found in [4, Section 2] as well as the basic facts about observable subgroups of linear algebraic groups found in [1] or [6, Thm. 2.1].

Notation and Conventions. Let C be the field of complex numbers, let $R(G)$ be the Hopf algebra of complex analytic representative functions on G, and let G^* be the pro-affine algebraic group associated with $R(G)$. Since G is assumed to be faithfully representable, we shall identify G with its canonical image in G^*. So $R(G) = C[G^*]$ where $C[G^*]$ is the Hopf algebra of polynomial functions on G^*. If A is a subgroup of G^*, let $R(G)^A$ be the A-fixed part of $R(G)$ under the translation action $a.f(x) = f(xa)$, and let ${}^AR(G)$ be the A-fixed part of $R(G)$ under the translation action $f.a(x) = f(ax)$. Similarly, we define $[R(G)]^A$ and ${}^A[R(G)]$, where $[R(G)]$ is the field of fractions of the integral domain $R(G)$. Note that Hochschild and Mostow in [2] and [3], worked with the right cosets $L \setminus G$ rather than the left cosets G/L and worked with $(R(G)^L)' = {}^LR(G)$ rather than $R(G)^L$ where $f'(x) = f(x^{-1})$.

We recall that the *universal algebraic subgroup* G_0 of G may be defined as the subgroup generated by $[G, G]$ and all reductive analytic subgroup of G [9, p. 623]. In fact, G_0 is the unique maximal normal subgroup of G that is algebraic under all finite-dimensional analytic representations of G. Moreover, G_0 has a unique irreducible algebraic group structure which is compatible with its analytic group structure [9]. In [7], G_0 is referred to as the *algebraic kernel* of G.

For the convenience of the reader, we recall the following (slightly reworded) definitions and results.

Definition 1. [2, top p. 813] A quasi-affine structure for $L\backslash G$ is the structure of a quasi-affine algebraic variety on $L\backslash G$ satisfying the following two conditions:

(1) The variety $L \setminus G$ is G-*homogenous* [2, top. 809] in the sense that it satisfies the following conditions.
(a) For each element x of G, the translation action of x on $L \setminus G$ is an automorphism of the algebraic variety $L \setminus G$. (Moreover, G acts transitively on $L \setminus G$.)
(b) If $P(L \setminus G)$ is the algebra of polynomial functions on $L \setminus G$, then $P(L\backslash G) \subset {}^L R(G)$ (where the elements of ${}^L R(G)$ are viewed as functions on $L \setminus G$).

(2) The variety $L \setminus G$ is a G-*variety* [2, p. 810] in the sense that the translation action of G on $L \setminus G$ staisfies (a) above, and
(c) for every polynomial f on $L \setminus G$ and every point v of $L \setminus G$, the map $f_v : G \to C$ defined by $f_v(x) = f(v.x)$ is a holomorphic function on G.

Remark 1. *The above two conditions (1) and (2) on $L \setminus G$ are equivalent.*

Proof. In fact, if $L \setminus G$ is a G-variety, then $L \setminus G$ is G-homogenous by [2, Thm. 1.3]. Converseley, suppose that $L \setminus G$ is G-homogenous. To prove condition (c) above, Let f be a polynomial function on $L \setminus G$, let $v = Lg$ be an element of $L\backslash G$, let g^* be the translation action of g on $L\backslash G$, let $\pi : G \to L\backslash G$ be the canonical projection, and let $\pi^t : P(L \setminus G) \to {}^L R(G)$ be its transpose in view of (b) above. Then $f_v(x) = f(v.x) = f(Lg.x) = f(g^*(Lgxg^{-1})) = f(g^*(\pi(gxg^{-1}))) = (\pi^t(f \circ g^*))(gxg^{-1})$. Hence f_v is a holomorphic function on G because, g^* is rational by (a), and $\pi^t(P(L \setminus G)) \subset {}^L R(G)$ by (b), so we have condition (c). Hence (1) and (2) are equivalent. ∎

Definition 2. [3, p. 852] A representative algebraic structure for $L \setminus G$ is the structure of an irreducible algebraic variety on $L \setminus G$ satisfying the following two conditions:

(a) For each element x of G, the translation action of x on $L \setminus G$ is an automorphism of the algebraic variety $L \setminus G$.
(b) If $F(L \setminus G)$ is the algebra of rational functions on $L \setminus G$, then $F(L \setminus G) \subset {}^L[R(G)]$ (where the elements of ${}^L[R(G)]$ are viewed as functions on $L \setminus G$).

Theorem A. [7, Thm. 4.5] *L is observable in G if and only if L satisfies the following conditions:*

(1) $L_0 = L \cap G_0$*;*
(2) L_0 *is observable in* G_0 *(in the categroy of algebraic groups).*

Theorem B. [7, Lemmas 4.1, 4.2] *If $L \cap G_0$ is an algebraic subgroup of G_0, then $L^* \cap G = L$ (where L^* is defined as in Theorem 1 below). In particular, $L^* \cap G_0 = L \cap G_0$.*

Theorem C. [2, Thm. 7.1] *G/L has a quasi-affine structure if and only if L satisfies the following conditions:*

(1) $R(G)^L$ *separates the points of* G/L*;*
(2) $S(L)\alpha(G) = \alpha(G)^*$ *(See [2] for terminology);*
(3) $N(L)/L$ *has only a finite number of connected components where* $N(L)$ *is the normalizer of* L *in* G*.*

Theorem D. [3, Thm. 3.3] G/L *has a representative algebraic structure if and only if* L *satisfies the following:*
(1) $[R(G)]^L$ *separates the points of* G/L*;*
(2) $S(L)\alpha(G) = \alpha(G)^*$*;*
(3) $N(L)/L$ *has only a finite number of connected components where* $N(L)$ *is the normalizer of* L *in* G*.*

Theorem 1. *Let* L *be a closed analytic subgroup of a faithfully representable complex analytic group* G*, let* $R(G)$ *be the algebra of complex analytic representative functions on* G*, let* G^* *be the pro-affine algebraic group associated with* $R(G)$*, and let* L^* *be the algebraic closure of* L *in* G^**. Let* G_0 *and* L_0 *be the universal algebraic subgroups of* G *and* L *respectively. Then the following are equivalent.*
(1) G/L *is (representationally) separabe, i.e,* $R(G)^L$ *separates the points of* G/L*.*
(2) L^* *is observable in* G^* *(in the category of pro-affine algebraic groups) and* $L \cap G_0$ *is an algebraic subgroup of* G_0*.*
(3) $L \cap G_0$ *(and hence* L_0*) is an observable algebraic subgroup of* G_0 *(in the category of algebraic groups).*
(4) $[R(G)^L] = [R(G)]^L$ *and* $L \cap G_0$ *is an algebraic subgroup of* G_0*.*

Proof. We shall need the fact that $[G,G]$ and G_0 are normal in G^* since $[G,G] = [G^*,G^*]$ [5, p. 1149 (last paragraph)]. Suppose $R(G)^L$ separates the points of G/L. Let X be the subgroup consisting of all elements of G^* that leave the elements of $R(G)^L$ fixed (under the right translation action of G^* on $R(G) = C[G^*]$) Then X is an algebraic sugroup of G^* and $X \cap G = L$ since $R(G)^L$ separates the points of G/L. Hence $L \cap G_0 = X \cap G \cap G_0 = X \cap G_0$, so $L \cap G_0$ is an algebraic subgroup of G_0. Since $R(G)^L = R(G)^{L^*}$, $R(G)^{L^*}$ separates the points of G/L, so in particular, $R(G)^{L^*}$ separates the points of $G/L^* \cap G \cong G.L^*/L^*$. Hence $R(G)^{L^*}$ separates the points of $[G,G].L^*/L^*$. So if $Y = [G,G].L^*$, then $C[Y]^{L^*}$ separates the points of Y/L^*. Hence L^* is observable in Y [8, Thm. 3]. Moreover, $Y = [G,G].L^*$ is observable in G^* for being a normal algebraic subgroup of G^* [8, Thm. 1]. Hence L^* is observable in G^*, so (1) implies (2).
Suppose (2) holds. Then $L^* \cap G_0$ is observable in L^* for being a normal algebraic subgroup [8, Thm. 1]. But $L^* \cap G_0 = L \cap G_0$ by Theorem B. Hence $L \cap G_0$ is observable in L^* . Consequently, by transitivity, $L \cap G_0$ is also observable in G^* since L^* is given to be observable in G^*, so (2) implies (3).

Now we show (3) implies (2). Since $L^* \cap G_0 = L \cap G_0$ by Theorem B, $L^* \cap G_0$ is observable in G_0. Although it can be justified using [8, section 2] that that the pro-variety $G_0.L^*/L^*$ is isomorphic to $G_0/G_0 \cap L^*$ which is quasi-affine, to deduce that L^* is observable in $G_0.L^*$, it is simpler to work with the characterization concerning separation of points. Put $Z = L^* \cap G_0$, so Z is observable in G_0. Then $C[G_0]^Z$ separates the points of G_0/Z. If $f \in C[G_0]^Z$, define $f^+ \in C[G_0.L^*]^{L^*}$ by $f^+(ab) = f(a)$ if $a \in G_0$ and $b \in L^*$. In fact, f^+ is well-defined since f is invariant under $Z = L^* \cap G_0$. Consequently, $C[G_0.L^*]^{L^*}$ separates the points of $G_0.L^*/L^*$. Hence L^* is observable in $G_0.L^*$ [8, Thm. 3]. But this last is observable in G^* for being normal in G^* [8, Thm. 1]. Hence L^* is observable in G^* by transitivity, so (3) implies (2).

Now we show that (2) implies (1). Since L^* is observable in G^*, and $C[G^*] = R(G)$, $R(G)^{L^*}$ separates the points of G^*/L^* [8, Thm. 1]. Since $R(G)^{L^*} = R(G)^L$, it follows that $R(G)^L$ separates the points of $G.L^*/L^* \cong G/L^* \cap G$. But $L^* \cap G = L$ by Theorem B. Hence $R(G)^L$ separates the points of G/L, so (2) implies (1). Thus (1)-(2)-(3) are equivalent.

Finally, suppose (2) holds. Since L^* is observable in G^* and $C[G^*] = R(G)$, $[R(G)^{L^*}] = [R(G)]^{L^*}$ ([8, Thm. 1] or [7, Thm. 2.2]) But $[R(G)^{L^*}] = [R(G)^L]$ and $[R(G)]^{L^*} = [R(G)]^L$ since L^* is the algebraic closure of L in G^*. Hence $[R(G)^L] = [R(G)]^L$, so (2) implies (4). Converseley, suppose (4) holds. Then $[R(G)^L] = [C[G^*]]^{L^*}$. But this last separates the points of G^*/L^* [8, Prop. 1]. Hence $R(G)^L$ separates the points of G^*/L^*. Consequently, $R(G)^L$ separates the points of $G.L^*/L^* \cong G/G \cap L^* = G/L$ by Theorem B above, so (4) implies (1). This proves Theorem 1. ∎

In view of Theorems A and C, we have the following corollaries.

Corollary 2. *The following are equivalent.*

(1) *L is observable in G.*
(2) *G/L is separable and $L \cap G_0 = L_0$.*
(3) *L_0 is observable in G_0 (in the category of algebraic groups) and $L \cap G_0 = L_0$.*
(4) *$[R(G)^L] = [R(G)]^L$ and $L \cap G_0 = L_0$.*

Corollary 3. *G/L has a quasi-affine structure if and only if L satisfies the following conditions:*

(1) *$R(G)^L$ separates the points of G/L, or equivalently, $L \cap G_0$ is an observable algebraic subgroup of G_0 (in the category of algebraic groups);*
(2) *$S(L)\alpha(G) = \alpha(G)^*$;*
(3) *$N(L)/L$ has only a finite number of connected components (where $N(L)$ is the normalizer of L in G).*

Theorem 4. *The following are equivalent.*

(1) *$[R(G)]^L$ separates the points of G/L.*
(2) *$L \cap G_0$ is an algebraic subgroup of G_0.*

Proof. Let X be the fixer of $[R(G)]^L$ in G^*. Then X is an algebraic subgroup of G^*. If $[R(G)]^L$ separates the points of G/L, then $X \cap G = L$. Consequently, $L \cap G_0 = X \cap G_0$. Hence $L \cap G_0$ is an algebraic subgroup of G_0, so (1) implies (2). Converseley, suppose $L \cap G_0$ is an algebraic subgroup of G_0. Then $L^* \cap G = L$ by Theorem B above. Since $[R(G)]^L = [R(G)]^{L^*}$ and this last separates the points of G^*/L^* [7, Prop. 1], it follows that $[R(G)]^L$ separates the points of $G.L^*/L^* \cong G/L^* \cap G = G/L$. This proves Theorem 4. ∎

In view of Theorem D above, we have the following.

Corollary 5. *G/L has a representative algebraic structure if and only if L satisfies the following:*

(1) *$[R(G)]^L$ separates the points of G/L, or equivalently, $L \cap G_0$ is an algebraic subgroup of G_0.*
(2) *$S(L)\alpha(G) = \alpha(G)^*$;*
(3) *$N(L)/L$ has only a finite number of connected components (where $N(L)$ is the normalizer of L in G).*

Example 1. Let $V = C \times C$ where C is the additive group of complex numbers, let $G = V.T$ be the semi-direct product group where $T = C$ acts on V by $t(x, y) = (e^t x, y)$ for every $t \in T$, and let L be the diagonal subgroup in V. Then $L_0 = (0) = L \cap G_0$. Hence L is observable in G by Theorem A. However, $N(L) = V.T^*$ where T^* consists of the elements $t \in T$ such that $e^t = 1$, i.e, of the integral multiples of $2\pi i$, so $N(L)/L$ is an infinite group. Hence, by Theorem B and Theorem C, *G/L does not have any quasi-affine structures or even representative algebraic structures although L is observable in G.*

Example 2. Let $G = SL(2, C)$ and let L be the unipotent subgroup of elements of the form $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ where $a \in C$. Then $L_0 = (1)$ while $L \cap G_0 = L$, so L is not observable in G (in the category of analytic groups) by Theorem A. However, L is a unipotent algebraic subgroup of G, so L is observable in G (in the category of algebraic groups). Hence G/L is a quasi-affine algebraic variety. *Thus G/L has a quasi-affine structure although L is not observable in G* (in the category of analytic groups).

Example 3. Let $G = C^* \times SL(2, C)$ and let L be the subgroup of elements of the form $(e^a, \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix})$ where $a \in C$. Then $G_0 = G$, so $L \cap G_0$ is L which is not an algebraic subgroup of G_0. Hence G/L *is neither separable* by Theorem 1, nor weakly separable by Theorem 4. Moreover, L is not observable in G by Corollary 2.

Remark 2. *Supoose that G is an algebraic group and L is an algebraic subgroup of G. If L is analytically observable in G, then L is rationally observable in G. However, the converse is false in general.*

Proof. The converse is false in general by Example 2. So suppose that L is analytically observable in G. Then L_0 is rationally observable in G_0 by Theorem A. Moreover, G_0 is rationally observable in G for being normal in G. Hence L_0 is rationally observable in G. Now L/L_0 is a unipotent algebraic group since $L = L_u \, . \, P$ and $L_0 = N \, . \, P$ for every maximal reductive algebraic subgroup P of L where L_u is the unipotent radical of L and N is the radical of $[L, L]$. Hence every multiplicative rational character on L/L_0 is trivial. Since L_0 is rationally observable in G, it follows that L is rationally observable in G [10, Cor. 2 (2)]. ∎

Finally, we use the proof of Theorem 1 to give a new proof independently of [7] of the "if part" or the extension part in Thm. 4.5 of [7].

Theorem A (if part). *L is observable in G if L satisfies the following conditions:*

(1) $L_0 = L \cap G_0$*;*

(2) L_0 *is observable in* G_0 *(in the categroy of algebraic groups).*

Proof. Let L^+ be the pro-affine algebraic group associated with $R(L)$, so $R(L) = C[L^+]$. Then the restriction map $R(G) \to R(L)$ yields a canonical map

$$f : L^+ \to G^*$$

and note that $f(L^+) = L^*$. Now Let V be a finite-dimensional analytic representation of L. Then V is rational representation of L^+ . To obtain the extension to G, first we show that it suffices to have that f is injective. If this is the case, then V becomes a rational representation of L^*. Our two given assumptions imply that $L \cap G_0$ is an observable algebraic subgroup of G_0 (in the category of algebraic groups). Hence L^* is observable in G^* by the implication (3) $\Rightarrow$ (2) in Theorem 1. (Note that the proof of this implication does not rely on Theorem B, so it is independent of [7]). Hence V can be extended to a finite-dimensional rational representation of G^* whose restriction to G yields the required extension.

Now we show that f is indeed injective by showing that the induced map $f^+ : L^+/L \cap G_0 \to G/G_0$ is injective. The inclusion map of L into G yields an injection of abelian analytic groups $i : L/L \cap G_0 \to G/G_0$. So every representative function on $L/L \cap G_0$ can be extended to a representative function on G/G_0. Hence the canonical map $R(G/G_0) \to R(L/L \cap G_0)$ is surjective. But $R(G/G_0) = R(G)^{G_0} = C[G^*]^{G_0} = C[G^*/G_0]$. Similarly $R(L/L \cap G_0) = C[L^+/L \cap G_0]$. Hence the canonical map (induced by f) $C[G^*/G_0] \to C[L^+/L \cap G_0]$ is surjective. So the canonical map $f^+ : L^+/L \cap G_0 \to G/G_0$ is injective. Hence $\mathrm{Ker}(f) \subset L \cap G_0$. But $L \cap G_0 = L_0$ and f is injective on L. Hence $\mathrm{Ker}(f)$ is trivial, and our proof is complete. ∎

References

[1] Bialynicki-Birula, G. Hochschild, and G. D. Mostow, *Extensions of representations of algebraic groups*, Amer. J. Math. **85** (1963), 131–144.

[2] Hochschild, G., and G. D. Mostow, *Affine embeddings of complex analytic homogenous spaces*, Amer. J. Math. **87** (1965), 807–839.

[3] —, *Representative algebraic structures on complex analytic homogenous spaces*, Amer. J. Math. **88** (1966), 847–866.

[4] —, *Pro-affine algebraic groups*, Amer. J. Math. **91** (1969), 1127–1140.

[5] —, *Complex analytic groups and Hopf algebras*, Amer. J. Math. **91** (1969), 1141–1151.

[6] Grosshans, F. D., "Algebraic homogenous spaces and invariant theory," Lecture Notes in Mathematics **1673**, Springer-Verlag, Berlin, 1997, vi+148 pp.

[7] Lee, Dong Hoon, and Wu, Ta-Sun, *On observable subgroups of complex analytic groups*, J. Algebra **173** (1995), 166–179.

[8] Magid, A. R., and Nahlus, N., *Provarieties and observable subgroups of pro-affine algebraic groups*, Journal of Group Theory, (to appear).

[9] Nahlus, N., *Representative functions on complex analytic groups*, Amer. J. Math. **116** (1994), 621–636.

[10] —, *Note on observable subgroups of linear algebraic groups and a theorem of Chevalley*, Journal of Lie Theory **12** (2002), 301–304.

Nazih Nahlus
Department of Mathematics
American University of Beirut
Beirut, Lebanon
nahlus@aub.edu.lb

Received May 29 , 2001
and in final form July 9, 2001

Editorial Remark. Through an editorial oversight, the Communicating Editor of the author's paper [10] was stated incorrectly. That article was communicated by Martin Moskowitz.