Journal of Lie Theory
Volume **12** (2002) 583–596

A Leibniz Algebra Structure on the Second Tensor Power

R. Kurdiani and T. Pirashvili

Communicated by K.-H. Neeb

Abstract. For any Lie algebra $\mathfrak{g}$, the bracket
$$[x \otimes y, a \otimes b] := [x, [a, b]] \otimes y + x \otimes [y, [a, b]]$$
defines a Leibniz algebra structure on the vector space $\mathfrak{g} \otimes \mathfrak{g}$. We let $\mathfrak{g}\underline{\otimes}\mathfrak{g}$ be the maximal Lie algebra quotient of $\mathfrak{g} \otimes \mathfrak{g}$. We prove that this particular Lie algebra is an abelian extension of the Lie algebra version of the nonabelian tensor product $\mathfrak{g} \boxtimes \mathfrak{g}$ of Brown and Loday [1] constructed by Ellis [2], [3]. We compute this abelian extension and Leibniz homology of $\mathfrak{g} \otimes \mathfrak{g}$ in the case, when $\mathfrak{g}$ is a finite dimensional semi-simple Lie algebra over a field of characteristic zero.

0. Introduction

Let $\mathfrak{g}$ be a Lie algebra. We define the following bracket

$$[x \otimes y, a \otimes b] := [x, [a, b]] \otimes y + x \otimes [y, [a, b]] \tag{1}$$

on $\mathfrak{g} \otimes \mathfrak{g}$. It turns out that $\mathfrak{g} \otimes \mathfrak{g}$ equipped with this bracket is not in general a Lie algebra but only a Leibniz algebra. Let us recall that Leibniz algebras are non-anti-commutative generalization of Lie algebras [10], [11]. More precisely a Leibniz algebra $\mathfrak{h}$ is a vector space equipped with a bracket

$$[-,-] : \mathfrak{h} \otimes \mathfrak{h} \to \mathfrak{h},$$

satisfying the *Leibniz identity*:

$$[x, [y, z]] = [[x, y], z] - [[x, z], y]. \tag{2}$$

Clearly any Lie algebra is a Leibniz algebra, and conversely any Leibniz algebra $\mathfrak{h}$ with property $[x, x] = 0$, $x \in \mathfrak{h}$ is a Lie algebra.

Any Leibniz algebra $\mathfrak{h}$ gives rise to a Lie algebra $\mathfrak{h}_{\mathsf{Lie}}$, which is obtained as the quotient of $\mathfrak{h}$ by the relation $[x, x] = 0$. For a Lie algebra $\mathfrak{g}$ we put

$$\mathfrak{g}\underline{\otimes}\mathfrak{g} := (\mathfrak{g} \otimes \mathfrak{g})_{\mathsf{Lie}}.$$

We prove that there exist a central extension of Lie algebras

$$0 \to \gamma(\mathfrak{g}) \to \mathfrak{g}\underline{\otimes}\mathfrak{g} \to \mathfrak{g} \boxtimes \mathfrak{g} \to 0,$$

ISSN 0949–5932 / $2.50

where $\mathfrak{g} \boxtimes \mathfrak{g}$ is the Lie algebra version of the nonabelian tensor product of Brown and Loday [1] constructed by Ellis [2], [3].

Among other things, we prove that the dimension of the abelian Lie algebra $\gamma(\mathfrak{g})$ is equal to $\mathsf{dim}_{\mathsf{U}\mathfrak{g}}\mathsf{Hom}(\mathfrak{g}, \mathfrak{g})$ provided $\mathfrak{g}$ is a finite dimensional semi-simple Lie algebra over a field of characteristic zero. In this case we also compute the Leibniz homology of $\mathfrak{g} \otimes \mathfrak{g}$.

The authors were partially supported by the grant TMR network K-theory and algebraic groups, ERB FMRX CT-97-0107. The second author is grateful to MPI at Bonn for hospitality. He was also partially supported by the grant INTAS-99-00817

1. Preliminaries on Leibniz algebras

All vector spaces are defined over a field K. We write $\otimes$ instead of $\otimes_K$.

It follows from the Leibniz identity (2) that in any Leibniz algebra one has

$$[x,[y,y]] = 0,\ \ [x,[y,z]] + [x,[z,y]] = 0. \tag{3}$$

Let $\mathfrak{g}$ be a Leibniz algebra. A subspace $\mathfrak{h} \subset \mathfrak{g}$ is called left (resp. right) ideal if for any $a \in \mathfrak{h}$ and $x \in \mathfrak{g}$ one has $[x, a] \in \mathfrak{h}$ (resp. $[a, x] \in \mathfrak{h}$). If $\mathfrak{h}$ is both left and right ideal, then $\mathfrak{h}$ is called two-sided ideal. For a Leibniz algebra $\mathfrak{g}$ one puts

$$\mathsf{Z}^r(\mathfrak{g}) = \{a \in \mathfrak{g} \mid [x, a] = 0,\ x \in \mathfrak{g}\}.$$

Moreover, we let $\mathfrak{g}^{\mathsf{ann}}$ be the subspace of $\mathfrak{g}$ spanned by elements of the form $[x, x]$, $x \in \mathfrak{g}$. Clearly for any $x, y \in \mathfrak{g}$ one has

$$\mathsf{ann}(x, y) := [x, y] + [y, x] \in \mathfrak{g}^{\mathsf{ann}}.$$

By (3) one has also

$$\mathfrak{g}^{\mathsf{ann}} \subset \mathsf{Z}^r(\mathfrak{g}).$$

Lemma 1.1. *$\mathsf{Z}^r(\mathfrak{g})$ and $\mathfrak{g}^{\mathsf{ann}}$ are two-sided ideals of $\mathfrak{g}$. Moreover*

$$[\mathsf{Z}^r(\mathfrak{g}), \mathfrak{g}] \subset \mathfrak{g}^{\mathsf{ann}}.$$

Proof. Since $\mathfrak{g}^{\mathsf{ann}} \subset \mathsf{Z}^r(\mathfrak{g})$ and $[\mathfrak{g}, \mathsf{Z}^r(\mathfrak{g})] = 0$ it suffices to show only the last inclusion. To prove this one observes that for any $u \in \mathsf{Z}^r(\mathfrak{g})$ and $x \in \mathfrak{g}$ by definition one has $[x, u] = 0$ and so

$$[u, x] = [u, x] + [x, u] = \mathsf{ann}(u, x) \in \mathfrak{g}^{\mathsf{ann}}. \qquad \blacksquare$$

It is clear that the quotient

$$\mathfrak{g}_{\mathsf{Lie}} := \mathfrak{g}/\mathfrak{g}^{\mathsf{ann}}$$

is a Lie algebra, which satisfies the following universal property: any Leibniz homomorphism $\mathfrak{g} \to \mathfrak{h}$ into a Lie algebra $\mathfrak{h}$ factors trough $\mathfrak{g}_{\mathsf{Lie}}$. Since $\mathfrak{g}^{\mathsf{ann}} \subset \mathsf{Z}^r(\mathfrak{g})$ we see that

$$\mathfrak{g}^{\mathsf{Lie}} := \mathfrak{g}/\mathsf{Z}^r(\mathfrak{g})$$

is also a Lie algebra. Thus by definition one has a central extension of Lie algebras

$$0 \to \mathfrak{g}^a \to \mathfrak{g}_{\mathsf{Lie}} \to \mathfrak{g}^{\mathsf{Lie}} \to 0,$$

where $\mathfrak{g}^a = \mathsf{Z}^r(\mathfrak{g})/\mathfrak{g}^{\mathsf{ann}}$.

The *lower central series* or *descending central series* of $\mathfrak{g}$ is the sequence

$$\cdots \subset \mathcal{C}^n\mathfrak{g} \subset \cdots \subset \mathcal{C}^2\mathfrak{g} \subset \mathcal{C}^1\mathfrak{g}$$

of two-sided ideals of $\mathfrak{g}$ defined inductively as follows

$$\mathcal{C}^1\mathfrak{g} = \mathfrak{g} \text{ and } \mathcal{C}^{n+1}\mathfrak{g} = [\mathcal{C}^n\mathfrak{g}\ , \mathfrak{g}], n > 0.$$

A Leibniz algebra $\mathfrak{g}$ is called *nilpotent* if $\mathcal{C}^n\mathfrak{g} = 0$ for some n. Let n be the smallest integer such that $\mathcal{C}^n\mathfrak{g} = 0$. Then $n-1$ is called *class of nilpotency* of $\mathfrak{g}$. It is clear that $\mathfrak{g}$ is nilpotent of class $n-1$ iff for any $x_1, \cdots, x_n \in \mathfrak{g}$ one has

$$[x_1, [x_2, \cdots, [x_{n-1}, x_n] \cdots] = 0.$$

In this case $[\cdots[x_1, x_2], x_3], \cdots, x_n] = 0$ as well.

The *derived series* of a Leibniz algebra $\mathfrak{g}$ is the sequence

$$\cdots \subset \mathcal{D}^n\mathfrak{g} \subset \cdots \subset \mathcal{D}^2\mathfrak{g} \subset \mathcal{D}^1\mathfrak{g}$$

of two-sided ideals of $\mathfrak{g}$ defined inductively as follows

$$\mathcal{D}^1\mathfrak{g} = \mathfrak{g} \text{ and } \mathcal{D}^{n+1}\mathfrak{g} = [\mathcal{D}^n\mathfrak{g}\ , \mathcal{D}^n\mathfrak{g}].$$

It is clear that, then $\mathcal{D}^m\mathfrak{g} \subset \mathcal{C}^m(\mathfrak{g})$ for any m. A Lie algebra $\mathfrak{g}$ is called *solvable* if $\mathcal{D}^n\mathfrak{g} = 0$ for some n.

Example 1.2. As was observed in [12] if $\mathfrak{g}$ is a Lie algebra, M is a (right) $\mathfrak{g}$-module and

$$f : M \to \mathfrak{g}$$

is a $\mathfrak{g}$-homomorphism from M to the adjoint representation of $\mathfrak{g}$, then

$$[m_1, m_2] := [m_1, f(m_2)],\ m_1, m_2 \in M \tag{4}$$

defines a Leibniz algebra structure on M. We let $\mathsf{Le}(f)$ be this particular Leibniz algebra. One observes that $\mathsf{Le}(f)$ is a Lie algebra iff

$$[x, f(y)] + [y, f(x)] = 0, \quad x, y \in M.$$

Here we assume that $\mathsf{Char}(K) \neq 2$. In this case M is called *crossed* $\mathfrak{g}$*-module.* It is easy to check that this definition is equivalent to the original one due to Kassel and Loday [7].

We recall also (see [12]) that in this way one gets any Leibniz algebra. More precisely, by (3) the bracket

$$[-,-] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g},$$

factors through the map
$$[-,-] : \mathfrak{g} \otimes \mathfrak{g}_{\mathsf{Lie}} \to \mathfrak{g},$$
and hence yields a $\mathfrak{g}_{\mathsf{Lie}}$-module structure on $\mathfrak{g}$. Thus the projection $p : \mathfrak{g} \to \mathfrak{g}_{\mathsf{Lie}}$ is a $\mathfrak{g}_{\mathsf{Lie}}$-homomorphism and it follows that one has an isomorphism of Leibniz algebras $\mathfrak{g} \cong \mathsf{Le}(p)$. We also recall that (see [11]) the class $ch(\mathfrak{g})$ of the extension
$$0 \to \mathfrak{g}^{\mathsf{ann}} \to \mathfrak{g} \to \mathfrak{g}_{\mathsf{Lie}} \to 0$$
in $\mathsf{Ext}^1_{U(\mathfrak{g}_{\mathsf{Lie}})}(\mathfrak{g}_{\mathsf{Lie}}, \mathfrak{g}^{\mathsf{ann}}) = \mathsf{H}^1(\mathfrak{g}_{\mathsf{Lie}}, \mathsf{Hom}(\mathfrak{g}_{\mathsf{Lie}}, \mathfrak{g}^{\mathsf{ann}}))$ is called *the characteristic element* of $\mathfrak{g}$. Here $U(\mathfrak{g}_{\mathsf{Lie}})$ is the classical universal enveloping algebra of the Lie algebra $\mathfrak{g}_{\mathsf{Lie}}$ and H^* denotes the Lie algebra cohomology. It follows that the triple $(\mathfrak{g}_{\mathsf{Lie}}, \mathfrak{g}^{\mathsf{ann}}, ch(\mathfrak{g}))$ characterizes $\mathfrak{g}$ completely.

Let $\mathfrak{g}$ be a Leibniz algebra. We let $\delta_n : \mathfrak{g}^{\otimes n} \to \mathfrak{g}^{\otimes n-1}$ be the linear map given by
$$\delta_n(x_1 \otimes \cdots \otimes x_n) = \sum_{i<j} (-1)^{n-j} x_1 \otimes \cdots \otimes [x_i, x_j] \otimes \cdots \otimes \hat{x}_j \otimes \cdots \otimes x_n.$$
For any $n \geq 0$ the vector space $\mathfrak{g}^{\otimes n}$ has a natural $\mathfrak{g}_{\mathsf{Lie}}$-module structure, which is defined by
$$[x_1 \otimes \cdots \otimes x_n, x] = \sum_i x_1 \otimes \cdots \otimes [x_i, x] \otimes \cdots \otimes x_n.$$
One observes that δ_n is a $\mathfrak{g}_{\mathsf{Lie}}$-homomorphism (see also [10]). Note that $\delta_3 : \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ is given by
$$\delta_3(x \otimes y \otimes z) = -[x, y] \otimes z + [x, z] \otimes y + x \otimes [y, z]$$
and $\delta_2 = [-,-] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$. By [10] $\delta_{n-1}\delta_n = 0$. So one has the complex
$$\cdots \to \mathfrak{g}^{\otimes n} \xrightarrow{\delta_n} \mathfrak{g}^{\otimes n-1} \to \cdots \to \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g} \to k.$$
Homology of this complex is called *Leibniz homology* of $\mathfrak{g}$ and it is denoted by $\mathsf{HL}_*\mathfrak{g}$. If $\mathfrak{g}$ is a Lie algebra, then clearly $\mathsf{HL}_1\mathfrak{g} = \mathsf{H}_1\mathfrak{g}$.

Let us return to the homomorphism $f : M \to \mathfrak{g}$ of $\mathfrak{g}$-modules. Then one has the inclusions:
$$\mathsf{Le}(f)^{\mathsf{ann}} \subset \mathsf{Ker}(f) \subset \mathsf{Z}^r(\mathsf{Le}(f)).$$
Thus
$$0 \to \mathsf{Ker}(f)/\mathsf{Le}(f)^{\mathsf{ann}} \to \mathsf{Le}(f)_{\mathsf{Lie}} \to \mathsf{Im}(f) \to 0$$
and
$$0 \to \mathsf{Z}^r(\mathsf{Le}(f))/\mathsf{Ker}(f) \to \mathsf{Im}(f) \to \mathsf{Le}(f)^{\mathsf{Lie}} \to 0$$
are central extensions of Lie algebras. One observes that
$$\mathsf{Z}^r(\mathsf{Le}(f)) = \{a \in M \mid [x, fa] = 0,\ x \in M\}.$$

Let us also note that
$$0 \to \mathsf{Ker}(f) \to \mathsf{Le}(f) \to \mathsf{Im}(f) \to 0$$
is an abelian extension of Leibniz algebras. Moreover $[\mathsf{Le}(f), \mathsf{Ker}(f)] = 0$, in other words $\mathsf{Ker}(f)$ is an antisymmetric representation of $\mathsf{Im}(f)$ in the sense of [11].

2. Elementary properties of $\mathfrak{g} \otimes \mathfrak{g}$

Let $\mathfrak{g}$ be a Lie algebra. As was mentioned in the previous section $\mathfrak{g} \otimes \mathfrak{g}$ is a $\mathfrak{g}$-module via

$$[a \otimes b, x] = [a, x] \otimes b + a \otimes [b, x].$$

Then one easily observes that the commutator map

$$[-,-] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$$

is $\mathfrak{g}$-linear. Thus by the previous section the tensor product $\mathfrak{g} \otimes \mathfrak{g}$ carries a Leibniz algebra structure. One observes that this structure is nothing else but the one given by (1).

Lemma 2.1. *The map given by*

$$x \otimes y \mapsto -y \otimes x$$

is an automorphism of the Leibniz algebra $\mathfrak{g} \otimes \mathfrak{g}$.
The proof is obvious.

Lemma 2.2. *Let* $\mathfrak{g}$ *be a nilpotent Lie algebra, then* $\mathfrak{g} \otimes \mathfrak{g}$ *is a nilpotent Leibniz algebra. More precisely, if the class of nilpotency of* $\mathfrak{g}$ *is* $\le 2n$, *then the class of nilpotency of* $\mathfrak{g} \otimes \mathfrak{g}$ *is* $\le 2n-1$.

Proof. Clearly

$$[\mathcal{C}^i\mathfrak{g} \otimes \mathcal{C}^j\mathfrak{g}, \mathcal{C}^k\mathfrak{g} \otimes \mathcal{C}^l\mathfrak{g}] \subset \mathcal{C}^{i+k+l}\mathfrak{g} \otimes \mathcal{C}^j\mathfrak{g} + \mathcal{C}^i\mathfrak{g} \otimes \mathcal{C}^{j+k+l}\mathfrak{g}.$$

It follows that $[\mathcal{C}^i\mathfrak{g} \otimes \mathcal{C}^j\mathfrak{g}, \mathfrak{g} \otimes \mathfrak{g}] \subset \mathcal{C}^{i+2}\mathfrak{g} \otimes \mathcal{C}^j\mathfrak{g} + \mathcal{C}^i\mathfrak{g} \otimes \mathcal{C}^{j+2}\mathfrak{g}$. By induction we get

$$\mathcal{C}^m(\mathfrak{g} \otimes \mathfrak{g}) \subset \sum_{i+j=m-1} \mathcal{C}^{2i+1}\mathfrak{g} \otimes \mathcal{C}^{2j+1}\mathfrak{g}.$$

If $\mathcal{C}^{2n+1}\mathfrak{g} = 0$ and $i + j = 2n - 1$, then $2i + 1 \ge 2n + 1$ or $2j + 1 \ge 2n + 1$ and therefore $\mathcal{C}^{2n}(\mathfrak{g} \otimes \mathfrak{g}) = 0$. ■

Lemma 2.3. *Let* $\mathfrak{g}$ *be a solvable Lie algebra of class* m, *then* $\mathfrak{g} \otimes \mathfrak{g}$ *is a solvable Leibniz algebra of class* m. *More generally, if* $f : M \to \mathfrak{g}$ *is a* $\mathfrak{g}$*-homomorphism into the adjoint representation of a Lie algebra* $\mathfrak{g}$ *with property* $f(M) \subset \mathcal{D}^k\mathfrak{g}$ *and* $\mathcal{D}^n\mathfrak{g} = 0$. *Then* $\mathcal{D}^{n+2-k}(\mathsf{Le}(f)) = 0$.

Proof. The first part indeed follows from the second one. One needs to take $M = \mathfrak{g} \otimes \mathfrak{g}$, $k = 2$ and f to be the commutator map. Since $f : \mathsf{Le}(f) \to \mathfrak{g}$ is a homomorphism of Leibniz algebras and $f(\mathsf{Le}(f)) \subset \mathcal{D}^k\mathfrak{g}$ we have $f(\mathcal{D}^m(\mathsf{Le}(f)) \subset \mathcal{D}^{k+m-1}\mathfrak{g}$. Hence

$$f(\mathcal{D}^{n-k+1}\mathsf{Le}(f)) \subset \mathcal{D}^n\mathfrak{g} = 0.$$

By definition we have

$$\begin{aligned}\mathcal{D}^{n+2-k}(\mathsf{Le}(f)) &= [\mathcal{D}^{n+1-k}(\mathsf{Le}(f)), \mathcal{D}^{n+1-k}(\mathsf{Le}(f))] = \\ &= [\mathcal{D}^{n-k+1}(\mathsf{Le}(f)), f(\mathcal{D}^{n-k+1}\mathsf{Le}(f))] = 0.\end{aligned}$$ ■

Remark 2.4. We leave as an exercise to the interested reader to show that, if $\mathfrak{g}$ is a Leibniz algebra, then the bracket (1) still makes $\mathfrak{g} \otimes \mathfrak{g}$ into Leibniz algebra, as well as the facts that Lemma 2.1, Lemma 3.1, Lemma 3.2, Corollary 3.3 and the fact that sequence (5) is exact are still true in this generality.

3. Relation with nonabelian tensor product

Lemma 3.1. *The image of the map* $\delta_3 : \mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{g}\otimes\mathfrak{g}$ *is an abelian two-sided ideal in* $\mathfrak{g}\otimes\mathfrak{g}$*. Moreover* $\mathsf{Im}(\delta_3)\subset \mathsf{Z}^r(\mathfrak{g}\otimes\mathfrak{g})$*.*

Proof. $\mathsf{Im}(\delta_3)$ is a right ideal because δ_3 is a $\mathfrak{g}$-homomorphism. The inclusion $\mathsf{Im}(\delta_3)\subset \mathsf{Z}^r(\mathfrak{g}\otimes\mathfrak{g})$ follows from the fact that $[-,-]\circ\delta_3=0$. This equality implies that $\mathsf{Im}(\delta_3)$ is also a left ideal. ∎

Thanks to 3.1 the vector space

$$\mathfrak{g}*\mathfrak{g}=\mathsf{Coker}(\delta_3:\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{g}\otimes\mathfrak{g})$$

has the Leibniz algebra structure. We let $x*y$ be the image of $x\otimes y\in\mathfrak{g}\otimes\mathfrak{g}$ into $\mathfrak{g}*\mathfrak{g}$. Since $[x,y]*z=[x,z]*y+x*[y,z]$, we see that

$$[x*y,a*b]=[x,y]*[a,b].$$

By the definition of Leibniz homology one has an exact sequence of Leibniz algebras

$$0\to\mathsf{HL}_2\mathfrak{g}\to\mathfrak{g}*\mathfrak{g}\xrightarrow{[-,-]}\mathfrak{g}\to\mathsf{HL}_1\mathfrak{g}\to 0. \tag{5}$$

Here $\mathsf{HL}_2\mathfrak{g}$ and $\mathsf{HL}_1\mathfrak{g}$ are abelian Lie algebras. Moreover one can show that $\mathsf{HL}_2\mathfrak{g}$ is a central subalgebra of $\mathfrak{g}*\mathfrak{g}$. Therefore $\mathfrak{g}*\mathfrak{g}$ is a variation of the "non-abelian Leibniz tensor product" given in [6]: the both operations gives the same result for perfect Leibniz algebras but not for abelian Lie algebras.

As usual we let $\Gamma(\mathfrak{g})$ be the subspace of $\mathfrak{g}\otimes\mathfrak{g}$ spanned by the symmetric tensors $x\otimes x$, $x\in\mathfrak{g}$. Clearly $x\otimes y+y\otimes x\in\Gamma(\mathfrak{g})$, $x,y\in\mathfrak{g}$.

Lemma 3.2. $\Gamma(\mathfrak{g})$ *is an abelian two-sided ideal of* $\mathfrak{g}\otimes\mathfrak{g}$*. Moreover* $\Gamma(\mathfrak{g})\subset \mathsf{Z}^r(\mathfrak{g}\otimes\mathfrak{g})$*.*

Proof. Take $b=a$ in (1) and use (3) to get $[x\otimes y,a\otimes a]=0$. This shows $\Gamma(\mathfrak{g})\subset\mathsf{Z}^r(\mathfrak{g}\otimes\mathfrak{g})$ and hence $\Gamma(\mathfrak{g})$ is a left ideal. If one puts $y=x$ in (1) one gets that $[x\otimes x,a\otimes b]=z\otimes x+x\otimes z\in\Gamma(\mathfrak{g})$, where $z=[x,[a,b]]$. ∎

Corollary 3.3. *One has an abelian extension of Leibniz algebras*

$$0\to\Gamma(\mathfrak{g})\to\mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{g}\wedge\mathfrak{g}\to 0,$$

where $\mathfrak{g}\wedge\mathfrak{g}$ *is a Leibniz algebra under the bracket*

$$[x\wedge y,a\wedge b]:=[x,[a,b]]\wedge y-[y,[a,b]]\wedge x.$$

Let $\widetilde{\Gamma}(\mathfrak{g})$ be the subspace of $\mathfrak{g}*\mathfrak{g}$ spanned by the images of the elements

$$x\otimes[y,z]+[y,z]\otimes x\in\Gamma(\mathfrak{g})\subset\mathfrak{g}\otimes\mathfrak{g},\ x,y,z\in\mathfrak{g}$$

and $[u,v]\otimes[u,v]$, $u,v\in\mathfrak{g}$ under the homomorphism $\mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{g}*\mathfrak{g}$.

Lemma 3.4. *One has inclusions* $(\mathfrak{g} * \mathfrak{g})^{\mathsf{ann}} \subset \widetilde{\Gamma}(\mathfrak{g}) \subset \mathsf{Z}^r(\mathfrak{g} * \mathfrak{g})$. *Hence* $\widetilde{\Gamma}(\mathfrak{g})$ *is a two-sided ideal of* $\mathfrak{g} * \mathfrak{g}$.

Proof. In $\mathfrak{g} * \mathfrak{g}$ we have $[x * y, x * y] = [x, y] * [x, y]$, which implies the first inclusion. The second one follows from the previous lemma. ∎

Corollary 3.5. *The quotient* $\mathfrak{g} \boxtimes \mathfrak{g} := (\mathfrak{g} * \mathfrak{g})/\widetilde{\Gamma}(\mathfrak{g})$ *is a Lie algebra.*

Remark 3.6. The Lie algebra $\mathfrak{g} \boxtimes \mathfrak{g}$ is nothing else but the "non-abelian tensor product of Lie algebras" as it is defined in [3], while the quotient $\mathfrak{g} * \mathfrak{g}$ by the image of $\Gamma(\mathfrak{g})$ under the projection $\mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g} * \mathfrak{g}$ is nothing else but the "second non-abelian exterior power of Lie algebras" as it is defined in [2]. We let $\mathfrak{g} \curlywedge \mathfrak{g}$ be this quotient.

Summarizing the above statements we have the following

Proposition 3.7. *The underlying vector space of the non-abelian tensor product* $\mathfrak{g} \boxtimes \mathfrak{g}$ *is isomorphic to the quotient of the tensor product* $\mathfrak{g} \otimes \mathfrak{g}$ *by the subspace generated by the elements*

$$x \otimes [y, z] + [y, z] \otimes x, \qquad [x, y] \otimes [x, y],$$

$$[x, y] \otimes z + [y, z] \otimes x + [z, x] \otimes y,$$

where $x, y, z \in \mathfrak{g}$.

Comparing the definitions one sees that there is a commutative diagram of Leibniz algebras

$$\begin{array}{ccccc} \mathfrak{g} \otimes \mathfrak{g} & \longrightarrow & \mathfrak{g} * \mathfrak{g} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ \mathfrak{g} \wedge \mathfrak{g} & \longrightarrow & \mathfrak{g} \curlywedge \mathfrak{g} & \longrightarrow & 0, \end{array}$$

which implies that the sequence

$$\mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g} \xrightarrow{\delta_3} \mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{g} \curlywedge \mathfrak{g} \to 0$$

is also exact. Thus one has also the following diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathsf{HL}_2\mathfrak{g} & \longrightarrow & \mathfrak{g} * \mathfrak{g} & \xrightarrow{[-,-]} & \mathfrak{g} & \longrightarrow & \mathsf{HL}_1\mathfrak{g} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \| & & \\ 0 & \longrightarrow & \mathsf{H}_2\mathfrak{g} & \longrightarrow & \mathfrak{g} \curlywedge \mathfrak{g} & \xrightarrow{[-,-]} & \mathfrak{g} & \longrightarrow & \mathsf{H}_1\mathfrak{g} & \longrightarrow & 0, \end{array}$$

compare also with [2] and [3].

Let us also recall that there are exact sequences

$$0 \to \Gamma(\mathsf{H}_1\mathfrak{g}) \to \mathfrak{g} \boxtimes \mathfrak{g} \to \mathfrak{g} \curlywedge \mathfrak{g} \to 0, \tag{6}$$

$$\mathsf{HL}_3\mathfrak{g} \to \mathsf{H}_3\mathfrak{g} \xrightarrow{\alpha} \mathsf{H}_0(\mathfrak{g}, \Gamma(\mathfrak{g})) \to \mathsf{HL}_2\mathfrak{g} \to \mathsf{H}_2\mathfrak{g} \to 0.$$

The first one is constructed in [3], while the second one in [13]. Let us recall that the homomorphism α is the dual of the classical homomorphism of Koszul (see [8])

$$\{\text{Invariant quadratic forms on } \mathfrak{g}\} \to \mathsf{H}_3\mathfrak{g}.$$

For the further results we refer to [5].

For a Lie algebra $\mathfrak{g}$ we let $\mathfrak{g}\underline{\otimes}\mathfrak{g}$ be the Liezation of $\mathfrak{g} \otimes \mathfrak{g}$, that is

$$\mathfrak{g}\underline{\otimes}\mathfrak{g} := (\mathfrak{g} \otimes \mathfrak{g})_{\mathsf{Lie}}.$$

Since $\mathfrak{g} \boxtimes \mathfrak{g}$ is also a Lie algebra quotient of $\mathfrak{g} \otimes \mathfrak{g}$, we have a canonical surjective homomorphism

$$\bar{p} : \mathfrak{g}\underline{\otimes}\mathfrak{g} \to \mathfrak{g} \boxtimes \mathfrak{g}.$$

We let $\gamma(\mathfrak{g})$ be the kernel of $\bar{p}$. Since $\gamma(\mathfrak{g})$ is contained in the image of

$$\mathsf{Ker}([-,-] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g})$$

under the canonical projection $\mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}\underline{\otimes}\mathfrak{g}$, we see that

$$0 \to \gamma(\mathfrak{g}) \to \mathfrak{g}\underline{\otimes}\mathfrak{g} \xrightarrow{\bar{p}} \mathfrak{g} \boxtimes \mathfrak{g} \to 0 \tag{7}$$

is a central extension of Lie algebras. Here p is the canonical projection $\mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g} \boxtimes \mathfrak{g}$.

Lemma 3.8. *If $\mathfrak{g}$ is a perfect Lie algebra, then $\bar{p}$ has a section, thus $\mathfrak{g}\underline{\otimes}\mathfrak{g}$ is a product of $\mathfrak{g} \boxtimes \mathfrak{g}$ and an abelian Lie algebra $\gamma(\mathfrak{g})$.*

Proof. If $\mathsf{H}_1\mathfrak{g} = 0$, then (6) shows that $\mathfrak{g} \boxtimes \mathfrak{g} \cong \mathfrak{g} \curlywedge \mathfrak{g}$. Thus one has an universal central extension (compare with [2] and [3])

$$0 \to \mathsf{H}_2\mathfrak{g} \to \mathfrak{g} \boxtimes \mathfrak{g} \to \mathfrak{g} \to 0$$

and therefore $\mathfrak{g}\boxtimes\mathfrak{g}$ is a super perfect Lie algebra (that is $\mathsf{H}_1(\mathfrak{g}\boxtimes\mathfrak{g}) = \mathsf{H}_2(\mathfrak{g}\boxtimes\mathfrak{g}) = 0$ see [4]) and $\bar{p}$ has a section. ∎

Let us recall that $\mathcal{D}^2\mathfrak{g}$ denotes the commutator subalgebra $[\mathfrak{g}, \mathfrak{g}]$ of $\mathfrak{g}$.

Lemma 3.9. $[\Gamma\mathfrak{g}, \mathcal{D}^2\mathfrak{g}] \subset (\mathfrak{g} \otimes \mathfrak{g})^{\mathsf{ann}}$

Proof. Take $a \in \mathfrak{g}$ and $x = [b, c] \in \mathcal{D}^2\mathfrak{g}$. Then

$$[a \otimes a, x] = [a \otimes a, b \otimes c] =$$

$$= [a \otimes a, b \otimes c] + [b \otimes c, a \otimes a] \in (\mathfrak{g} \otimes \mathfrak{g})^{\mathsf{ann}}.$$ ∎

Lemma 3.10. *There is a well-defined homomorphism*

$$\mathsf{H}_0(\mathcal{D}^2\mathfrak{g}, \Gamma\mathfrak{g}) \to \mathfrak{g}\underline{\otimes}\mathfrak{g}.$$

Moreover, the image of the composite

$$\mathsf{H}_0(\mathcal{D}^2\mathfrak{g}, \Gamma\mathcal{D}^2\mathfrak{g}) \to \mathsf{H}_0(\mathcal{D}^2\mathfrak{g}, \Gamma\mathfrak{g}) \to \mathfrak{g}\underline{\otimes}\mathfrak{g}$$

lies in $\gamma(\mathfrak{g})$.

Proof. Since $\mathsf{H}_0(\mathcal{D}^2\mathfrak{g}, \Gamma\mathfrak{g}) \cong (\Gamma\mathfrak{g})/[\Gamma\mathfrak{g}, \mathcal{D}^2\mathfrak{g}]$ Lemma 3.9 shows that the composite

$$\Gamma\mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}\underline{\otimes}\mathfrak{g}$$

factors trough $\mathsf{H}_0(\mathcal{D}^2\mathfrak{g}, \Gamma\mathfrak{g})$ and the expected homomorphism indeed exists. Thanks to Proposition 3.7 the image of $[x, y] \otimes [x, y]$ into $\mathfrak{g} \boxtimes \mathfrak{g}$ is zero and we obtain the second part of Lemma. ∎

Thus we obtain the homomorphism

$$\beta : \mathsf{H}_0(\mathcal{D}^2\mathfrak{g}, \Gamma\mathcal{D}^2\mathfrak{g}) \to \gamma(\mathfrak{g}).$$

We let

$$\bar{\alpha} : \mathsf{H}_3(\mathcal{D}^2\mathfrak{g}) \to \gamma(\mathfrak{g})$$

be the composite of $\alpha_{\mathcal{D}^2\mathfrak{g}}$ with β. We recall that

$$\alpha_{\mathfrak{g}} : \mathsf{H}_3(\mathfrak{g}) \to \mathsf{H}_0(\mathfrak{g}, \Gamma\mathfrak{g})$$

was defined in (6).

4. $\mathfrak{g} \otimes \mathfrak{g}$ for a semisimple Lie algebra $\mathfrak{g}$

Lemma 4.1. *Let $\mathfrak{g}$ be a perfect Lie algebra. Then $(\mathfrak{g} \otimes \mathfrak{g})^{ann}$ is a $\mathfrak{g}$-submodule of $\mathfrak{g} \otimes \mathfrak{g}$.*

Proof. Let us recall that for any Leibniz algebra $\mathfrak{h}$ the vector space $\mathfrak{h}^{ann}$ is a $\mathfrak{h}_{Lie}$-submodule of $\mathfrak{h}$. We take $h = \mathfrak{g} \otimes \mathfrak{g}$. Then (1) shows that the action of $\mathfrak{h}_{Lie} = \mathfrak{g}\underline{\otimes}\mathfrak{g}$ on $\mathfrak{g} \otimes \mathfrak{g}$ factors trough the commutator map $\mathfrak{g}\underline{\otimes}\mathfrak{g} \to \mathfrak{g}$. By the assertion the last map is an epimorphism and hence the result. ∎

In the rest of this section we assume that $\mathfrak{g}$ is a finite dimensional semisimple Lie algebra over a field of characteristic zero. By the classical results of Whitehead $\mathfrak{g}$ is a superperfect Lie algebra, that is $\mathsf{H}_1\mathfrak{g} = \mathsf{H}_2\mathfrak{g} = 0$. Thanks to [13] in fact $\mathsf{HL}_i\mathfrak{g} = 0$ for all $i > 0$. It is a well known fact that the category of finite dimensional $\mathfrak{g}$-modules is a semi-simple abelian category. As a consequence any such module M has a functorial decomposition $M = M^t \oplus M^{nt}$, where M^t is a $\mathfrak{g}$-module with trivial $\mathfrak{g}$-action, while M^{nt} is a $\mathfrak{g}$-submodule with $\mathsf{H}^0(\mathfrak{g}, M^{nt}) = 0$. Clearly $M^t = \mathsf{H}^0(\mathfrak{g}, M)$.

Lemma 4.2. *Let $f : M \to \mathfrak{g}$ be a $\mathfrak{g}$-homomorphism from a finite dimensional $\mathfrak{g}$-module to the adjoint representation of a finite dimensional semisimple Lie algebra $\mathfrak{g}$. Then $(\mathsf{Le}(f))^{\mathsf{ann}} \subset M^{nt}$. Moreover, if f^t is the restriction of f on M^t, then the composite*

$$M^t \subset M \to \mathsf{Le}(f)_{\mathsf{Lie}}$$

yields an isomorphism from $\mathsf{Le}(f^t)$ to a central subalgebra of $\mathsf{Le}(f)_{\mathsf{Lie}}$.

Proof. Since $\mathfrak{g}^t = 0$ and f is a $\mathfrak{g}$-homomorphism it follows that $f(M^t) = 0$. Thus $M^t \subset \mathsf{Z}^r(\mathsf{Le}(f))$. Take $a, b \in M$ and write $a = a^t + a^{nt}$, $b = b^t + b^{nt}$ according to the decomposition $M = M^t \oplus M^{nt}$. Then $f(a) = f(a^{nt})$ and $f(b) = f(b^{nt})$. Hence

$$[a,b] + [b,a] = [a, f(b)] + [b, f(a)] = [a, f(b^{nt})] + [b, f(a^{nt})] =$$
$$= [a^{nt}, f(b^{nt})] + [b^{nt}, f(a^{nt})] \in M^{nt}.$$

Since $(\mathsf{Le}(f))^{\mathsf{ann}}$ is spanned by elements of the form $[a,b] + [b,a]$ we see that $(\mathsf{Le}(f))^{\mathsf{ann}} \subset M^{nt}$. Therefore $(\mathsf{Le}(f))^{\mathsf{ann}} \cap M^t = 0$ and hence the restriction of the canonical projection $\mathsf{Le}(f) \to \mathsf{Le}(f)_{\mathsf{Lie}}$ to M^t is a monomorphism and the result follows. ■

For any finite dimensional semisimple Lie algebra $\mathfrak{g}$ we put

$$\mathsf{r}(\mathfrak{g}) := \mathsf{dimHom}_{U\mathfrak{g}}(\mathfrak{g}, \mathfrak{g}).$$

The following is well-known

Lemma 4.3. *Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra. Then* $\mathsf{dimH}^0(\mathfrak{g}, \mathfrak{g} \otimes \mathfrak{g}) = \mathsf{r}(\mathfrak{g})$*. Moreover, if $\mathfrak{g}$ is a simple Lie algebra over an algebraically closed field, then* $\mathsf{r}(\mathfrak{g}) = 1$.

Proof. The Killing form yields an isomorphism $\mathfrak{g}^* \cong \mathfrak{g}$ of $\mathsf{U}\mathfrak{g}$-modules and hence $\mathfrak{g} \otimes \mathfrak{g} \cong \mathsf{Hom}(\mathfrak{g}, \mathfrak{g})$. It follows that

$$\mathsf{H}^0(\mathfrak{g}, \mathfrak{g} \otimes \mathfrak{g}) \cong \mathsf{Hom}_{U\mathfrak{g}}(\mathfrak{g}, \mathfrak{g}).$$

The last assertion follows from the Schur lemma, because the adjoint representation of a simple Lie algebra is a simple $\mathfrak{g}$-module. ■

Lemma 4.4. *Let $f : M \to \mathfrak{g}$ be a $\mathfrak{g}$-homomorphism to the adjoint representation of a Lie algebra $\mathfrak{g}$. Assume $M \cong M_1 \oplus M_2$ as $\mathfrak{g}$-modules, $f(M_1) = 0$ and the action of $\mathfrak{g}$ on M_1 is trivial. Then $\mathsf{Le}(f)$ as a Leibniz algebra is isomorphic to the product of the abelian Leibniz algebra M_1 and $\mathsf{Le}(f_2)$, where f_2 is the restriction of f on M_2.*

The proof is obvious.

Theorem 4.5. *Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra. Then $\mathfrak{g} \otimes \mathfrak{g}$ is isomorphic as a Leibniz algebra to the direct product of an abelian Lie algebra of dimension $\mathsf{r}(\mathfrak{g})$ and a Leibniz algebra $\mathfrak{h}$. Moreover $\mathfrak{h}_{\mathsf{Lie}} \cong \mathfrak{g}$. Thus one has an isomorphism of Lie algebras*

$$\mathfrak{g} \underline{\otimes} \mathfrak{g} \cong k^{\mathsf{r}(\mathfrak{g})} \times \mathfrak{g}.$$

Proof. The first part is a consequence of Lemma 4.4. Indeed we take $f = [-,-] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ to be the commutator map and $M_1 = (\mathfrak{g} \otimes \mathfrak{g})^t$, $M_2 = (\mathfrak{g} \otimes \mathfrak{g})^{nt}$. Since $\mathfrak{g}^t = 0$, we have $f(M_1) = 0$. Therefore the conditions of Lemma 4.4 hold and hence

$$\mathfrak{g} \otimes \mathfrak{g} \cong (\mathfrak{g} \otimes \mathfrak{g})^t \times \mathfrak{h},$$

where $(\mathfrak{g} \otimes \mathfrak{g})^t$ is an abelian Lie algebra and h is a Leibniz algebra which is isomorphic to $\mathsf{Le}((\mathfrak{g} \otimes \mathfrak{g})^{nt} \to \mathfrak{g})$. By Lemma 4.3 we know that $(\mathfrak{g} \otimes \mathfrak{g})^t \cong k^{\mathsf{r}(\mathfrak{g})}$. Thus we proved the first part of the theorem. Since $\mathfrak{g}$ is superperfect, we have $\mathfrak{g} \boxtimes \mathfrak{g} \cong \mathfrak{g}$ and by Lemma 3.8 $\mathfrak{g} \underline{\otimes} \mathfrak{g}$ is a product of the Lie algebra $\mathfrak{g}$ and an abelian Lie algebra $\mathfrak{a}$. We have to prove that $\mathfrak{a} \cong k^{\mathsf{r}(\mathfrak{g})}$. By Lemma 4.1 $\mathfrak{g} \underline{\otimes} \mathfrak{g}$ is a $\mathfrak{g}$-module and the quotient map $\mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g} \underline{\otimes} \mathfrak{g}$ is a $\mathfrak{g}$-homomorphism. Hence it yields an epimorphism

$$(\mathfrak{g} \otimes \mathfrak{g})^t \to (\mathfrak{g} \underline{\otimes} \mathfrak{g})^t,$$

which is also a monomorphism thanks to Lemma 4.2. It follows from Lemma 4.3 that $(\mathfrak{g} \otimes \mathfrak{g})^t \cong k^{\mathsf{r}(\mathfrak{g})}$ and hence the result. ■

Remark. The same proof shows that if $\mathfrak{g}$ is semisimple, then

$$(\mathfrak{g} \wedge \mathfrak{g})_{\mathsf{Lie}} \cong \mathfrak{g}.$$

Lemma 4.6. *Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra. Then $ch(\mathfrak{g} \otimes \mathfrak{g}) = 0$ and the map*

$$\bar{\alpha} : \mathsf{H}_3(\mathcal{D}^2\mathfrak{g}) \to \gamma(\mathfrak{g})$$

is an isomorphism.

Proof. We know that $\mathfrak{g} \otimes \mathfrak{g} \cong k^{\mathsf{r}(\mathfrak{g})} \times \mathfrak{h}$, with $\mathfrak{h}_{\mathsf{Lie}} \cong \mathfrak{g}$. Thus $ch(\mathfrak{g} \otimes \mathfrak{g})$ is the pullback of $ch(\mathfrak{h})$ along the projection $\mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{h}$. Since $ch(\mathfrak{h}) \in \mathsf{H}^1(\mathfrak{g}, \mathfrak{h}^{ann}) = 0$ we obtain $ch(\mathfrak{g} \otimes \mathfrak{g}) = 0$ as well. To proof the second assertion, one observes that for a semisimple Lie algebra $\mathfrak{g}$, we have $\mathcal{D}^2\mathfrak{g} = \mathfrak{g}$ and the map α is an isomorphism thanks to [13], thus the result follows from 4.5. ■

Corollary 4.7. *Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra. Then*

$$\mathsf{HL}_n(\mathfrak{g} \otimes \mathfrak{g}) \cong k^{\mathsf{r}(\mathfrak{g}) \times n}, \quad n > 0.$$

Proof. We know that $\mathfrak{g} \otimes \mathfrak{g} \cong k^{\mathsf{r}(\mathfrak{g})} \times \mathfrak{h}$. By Proposition 4.3 [13] we know that $\mathsf{HL}_n\mathfrak{h} = 0$ for $n > 0$ and $\mathsf{HL}_0\mathfrak{h} \cong k$. Thus the result follows from the Loday-Künneth theorem for Leibniz homology [9]. ■

Example 4.8. Let $\mathfrak{g} = \mathfrak{sl}_n$. By Lemma 4.3 $\mathsf{r}(\mathfrak{g}) = 1$. Hence

$$\mathsf{HL}_n(\mathfrak{g} \otimes \mathfrak{g}) \cong k^n, \quad n > 0.$$

5. An example

In this section we consider the case, when $\mathfrak{g}$ is a Lie algebra of upper triangular matrices of order n, $n \geq 2$. We will assume that $\mathsf{char}(K) \neq 2$. We let E_{ij} denote the elementary matrice which is zero everywhere except at the place (i,j), where it is 1. By definition $\mathfrak{g}$ is spanned on E_{ij}, where $1 \leq i < j \leq n$. One has $[E_{ij}, E_{jk}] = E_{ik}$ and $[E_{ij}, E_{km}] = 0$ provided $i \neq m$ and $j \neq k$. For $n = 2$ and $n = 3$ the Leibniz algebra $\mathfrak{g} \otimes \mathfrak{g}$ is abelian. Therefore we will assume that $n \geq 4$.

It is well known that $\mathfrak{g}$ is a graded Lie algebra. More precisely, we let $\mathfrak{g}_s$ denote the subspace of $\mathfrak{g}$ spanned on E_{ij} with $j = i + s$. Then $\mathsf{dim}\mathfrak{g}_s = n - s$,

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{n-1}$$

and $[\mathfrak{g}_s, \mathfrak{g}_t] \subset \mathfrak{g}_{s+t}$. The goal of this section is to describe the Lie algebra $\mathfrak{g}\underline{\otimes}\mathfrak{g}$. We set $\mathfrak{g}_{ij} = \mathfrak{g}_i \otimes \mathfrak{g}_j$. Then one has

$$[\mathfrak{g}_{ij}, \mathfrak{g}_{st}] \subset \mathfrak{g}_{i+s+t,j} \oplus \mathfrak{g}_{i,j+s+t}.$$

We first consider the case $n = 4$. We claim that the Lie algebra $\mathfrak{g}\underline{\otimes}\mathfrak{g}$ is abelian of dimension 25. To this end, one observes that

$$(\mathfrak{g} \otimes \mathfrak{g})^{\mathsf{ann}} \subset [\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g} \otimes \mathfrak{g}] \subset \bigoplus_{i=1}^{3} (\mathfrak{g}_{i3} \oplus \mathfrak{g}_{3i}).$$

On the other hand $(\mathfrak{g}\otimes\mathfrak{g})^{\mathsf{ann}}$ is closed with respect of the involution $x \otimes y \mapsto y \otimes x$ thanks to 2.1. Furthermore we have

$$E_{14} \otimes E_{13} = \mathsf{ann}(E_{12} \otimes E_{13}, E_{23} \otimes E_{34}),$$

$$E_{14} \otimes E_{14} = \mathsf{ann}(E_{12} \otimes E_{14}, E_{23} \otimes E_{34}),$$

$$E_{14} \otimes E_{34} = \mathsf{ann}(E_{12} \otimes E_{34}, E_{23} \otimes E_{34}),$$

$$E_{14} \otimes E_{24} = -\mathsf{ann}(E_{12} \otimes E_{23}, E_{34} \otimes E_{24})$$

$$E_{14} \otimes E_{12} = -\mathsf{ann}(E_{34} \otimes E_{12}, E_{12} \otimes E_{23}),$$

$$E_{14} \otimes E_{23} = -\mathsf{ann}(E_{34} \otimes E_{23}, E_{12} \otimes E_{23}).$$

Since $\mathfrak{g}_3$ is spanned on E_{14} it follows that

$$(\mathfrak{g} \otimes \mathfrak{g})^{\mathsf{ann}} = \bigoplus_{i=1}^{3} (\mathfrak{g}_{i3} \oplus g_{3i})$$

and

$$[\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g} \otimes \mathfrak{g}] = (\mathfrak{g} \otimes \mathfrak{g})^{\mathsf{ann}}.$$

Therefore

$$\mathfrak{g}\underline{\otimes}\mathfrak{g} = (\mathfrak{g} \otimes \mathfrak{g})/(\mathfrak{g} \otimes \mathfrak{g})^{\mathsf{ann}} \cong (\mathfrak{g}_1 + \mathfrak{g}_2)^{\otimes 2}$$

is an abelian Lie algebra and hence the claim.

This example lieds to the problem of more close description of Lie algebras $\mathfrak{g}$ for which $\mathfrak{g}\underline{\otimes}\mathfrak{g}$ is abelian. One easily shows that any such type Lie algebra must be metaabelian that is $\mathcal{D}^3\mathfrak{g} = 0$.

Now we assume that $n \geq 5$. We will prove that as a vector space $\mathfrak{g}\underline{\otimes}\mathfrak{g}$ is isomorphic to

$$M = (\mathfrak{g}_1 \oplus \mathfrak{g}_2)^{\otimes 2} \oplus \mathfrak{g}_4 \oplus \cdots \oplus \mathfrak{g}_{n-1}.$$

Moreover, let $f : M \to \mathfrak{g}$ be the linear map, which is the inclusion on $\oplus_{s=4}^{n-1}\mathfrak{g}_s$ and is induced by the commutator map on $(\mathfrak{g}_1 \oplus \mathfrak{g}_2)^{\otimes 2}$.

Proposition 5.1. *The linear map $f : M \to \mathfrak{g}$ is a crossed module and one has an isomorphism of Lie algebras*

$$\mathfrak{g}\underline{\otimes}\mathfrak{g} \cong \mathsf{Le(f)}.$$

Proof. We set

$$\mathfrak{h} = \bigoplus_{i,j} \mathfrak{g}_{ij}$$

where the sum is taken over all (i, j) such that $i > 2$ or $j > 2$. Then

$$\mathfrak{g} \otimes \mathfrak{g} = (\mathfrak{g}_1 \oplus \mathfrak{g}_2)^{\otimes 2} \oplus \mathfrak{h}.$$

It is clear that

$$(\mathfrak{g} \otimes \mathfrak{g})^{\mathsf{ann}} \subset [\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g} \otimes \mathfrak{g}] \subset \mathfrak{h}.$$

Hence

$$\mathfrak{g}\underline{\otimes}\mathfrak{g} \cong (\mathfrak{g}_1 \oplus \mathfrak{g}_2)^{\otimes 2} \oplus \mathfrak{h}/(\mathfrak{g} \otimes \mathfrak{g})^{\mathsf{ann}}.$$

We let $h : \mathfrak{h} \to \mathfrak{g}_4 \oplus \cdots \oplus \mathfrak{g}_{n-1}$ be the restriction of the commutator map $\mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ to $\mathfrak{h}$. Since h is surjective, it suffices to show $\mathsf{Ker}(h) = (\mathfrak{g} \otimes \mathfrak{g})^{\mathsf{ann}}$. Since the inclusion $(\mathfrak{g} \otimes \mathfrak{g})^{\mathsf{ann}} \subset \mathsf{Ker}(h)$ is obvious, we have to show that $\mathsf{Ker}(h) \subset (\mathfrak{g} \otimes \mathfrak{g})^{\mathsf{ann}}$. It is clear that

$$E_{ij} \otimes E_{st} \in \mathsf{Ker}(h), \quad j \neq s,\ i \neq t,$$

$$E_{ij} \otimes E_{jk} + E_{jk} \otimes E_{ij} \in \mathsf{Ker}(h),$$

$$E_{ij} \otimes E_{jk} - E_{il} \otimes E_{lk} \in \mathsf{Ker}(h).$$

By comparing the dimensions one can show that these elements acually generate the vector space $\mathsf{Ker}(h)$. Thus we have to show that each such elements lie in $(\mathfrak{g} \otimes \mathfrak{g})^{\mathsf{ann}}$. The varification of this fact is quite similar to above formulas and therefore we omit it. ∎

References

[1] Brown, R. and J.-L. Loday, *Van Kampen theorems for diagrams of spaces. With an appendix by M. Zisman*, Topology **26** (1987), 311–335.

[2] Ellis, G. J., *Non-abelian exterior product of Lie algebras and an exact sequence in the homology of Lie algebras*, J. Pure Appl. Algebra **46** (1987), 111–115.

[3] —, *A non-abelian tensor product of Lie algebras*, Glasgow Math. J. **33** (1991), 101–120.

[4] Garland, H., *The arithmetic theory of free loop groups*, Publ. Math. IHES **52** (1980), 181–312.

[5] Gnedbaye, A. V., *Third homology groups of universal central extensions of a Lie algebra*, Afrika Mat. (3) **10** (1999), 46–63.

[6] —, *A non-abelian tensor product of Leibniz algebras*, Ann. Inst. Fourier (Grenoble) **49** (1999), 1149–1177.

[7] Kassel, C., and J.-L. Loday, *Extensions centrales d'algébres de Lie*, Ann. Inst. Fourier (Grenoble) **32** (1982), 119–142.

[8] Koszul, J.-L., *Homologie et cohomologie des algèbres de Lie*, Bull. Soc. Math. France **78** (1950), 65–127.

[9] Loday, J.-L., *Künneth-style formula for the homology of Leibniz algebras*, Math. Z. **221** (1996), 41–47.

[10] —, "Cyclic Homology," Grund. Math. Wiss. **301**, 2nd edition. Springer-Verlag Berlin etc., 1998.

[11] Loday, J.-L., and T. Pirashvili, *Universal enveloping algebras of Leibniz algebras and (co)homology*, Math. Ann. **296** (1993), 139–158.

[12] —, *The tensor category of linear maps and Leibniz algebras*, Georgian Math. J. **5** (1998), 263–276.

[13] Pirashvili T., *On Leibniz homology*, Ann. Inst. Fourier (Grenoble) **44** (1994), 401–411.

R. Kurdiani and T. Pirashvili
A. Razmadze Mathematical Institute
Aleksidze str. 1, Tbilisi, 380093,
Georgia
rezo@rmi.acnet.ge
pira@rmi.acnet.ge

Received September 7, 2001
and in final form November 8, 2001