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On a Family of Operators and their Lie Algebras

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Abstract. An infinite family of differential operators is constructed. Each of these operators defines a Lie bracket and the operator is a homomorphism from the new Lie algebra to the standard Lie algebra. An interesting feature of these operators is that they factorize into first order operators with integer coefficients. This generalizes recent results of Zhiber and Sokolov

1. Introduction

In a recent paper Zhiber and Sokolov [6] study integrable hyperbolic equations of Liouville type. They found a family of special operators with the property that they define a new Lie bracket and are homomorphisms from the Lie algebra with the newly induced bracket to the original Lie algebra. These operators have the form

$$\begin{aligned}
\pounds^1_{..} &= D \\
\pounds^2_{..} &= D(D+\mathbf{u}) \\
\pounds^3_{..} &= D(D+\mathbf{u})(D+\mathbf{u}) \\
\pounds^4_{0.} &= D(D+\mathbf{u})(D+\mathbf{u})(D+\mathbf{u}) \\
\pounds^4_{1.} &= D(D+\mathbf{u})(D+\mathbf{u})(D+2\mathbf{u}) \\
\pounds^5_{00} &= D(D+\mathbf{u})(D+\mathbf{u})(D+\mathbf{u})(D+\mathbf{u}) \\
\pounds^5_{01} &= D(D+\mathbf{u})(D+\mathbf{u})(D+2\mathbf{u})(D+3\mathbf{u}) \\
\pounds^6_{00} &= D(D+\mathbf{u})(D+\mathbf{u})(D+\mathbf{u})(D+\mathbf{u})(D+\mathbf{u}) \\
\pounds^6_{01} &= D(D+\mathbf{u})(D+\mathbf{u})(D+\mathbf{u})(D+\mathbf{u})(D+2\mathbf{u}) \\
\pounds^6_{11} &= D(D+\mathbf{u})(D+\mathbf{u})(D+2\mathbf{u})(D+3\mathbf{u})(D+3\mathbf{u}) \\
\pounds^6_{10} &= D(D+\mathbf{u})(D+\mathbf{u})(D+2\mathbf{u})(D+3\mathbf{u})(D+4\mathbf{u}),
\end{aligned}$$

where D is the total derivative operator with respect to the independent variable x and $\mathbf{u}$ is the dependent variable. They are the polynomial homogeneous operators

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with D and $\mathbf{u}$ of equal weight of order < 7 with the property that for all P and Q, one has

$$[\pounds^n P, \pounds^n Q] \in im\ \pounds^n, \tag{1}$$

where P, Q are functions of x and the derivatives of $\mathbf{u}$. Here the bracket is given by

$$[P, Q] = Q'P - P'Q,$$

where P' is the Fréchet derivative of P. According to [6] these operators form a complete list all homogeneous operators of the weigth $(1,1)$ and of order < 7 satisfying the homomorphism condition. In particular, contrary to our somewhat restricted approach, no a priori assumption is made on the factorization of these operators.

The Lie algebra of such P and Q with this bracket is denoted by $\mathfrak{g}$. These operators are claimed to lead to a not necessarily anti-symmetric generalization of Hamiltonian operators. Of course, the occurrence of all these regular sequences of integers in a problem that starts from a rather innocent looking condition (1) on the operator is rather startling and the authors of [6] suspect that there is some deep mathematics behind this.

In this paper we show how this sequence continues by explicitly producing the family of operators and the corresponding Lie brackets. We show that for $n > 4$ there exist operators $\pounds^n_{\alpha\beta}$ where $\alpha = 0, 1$ and $\beta = 0$ for n odd, and $\beta = 0, 1$ for n even, see Figure 1. We do this by studying multiplicative deformations, that is, when we have an operator that works, we multiply it from the right with a first order operator and derive the properties of this first order operator assuming the product is in the family. Thus our study does not answer the question of Zhiber and Sokolov whether these operators are unique. But we do prove that they are unique under multiplicative deformations. No deep mathematics is needed to get to these results, which of course does not imply that there is no deep mathematics behind it. Although the proofs in this paper are fairly straightforward, the reader should appreciate the amount of work that went into guessing the right form of the operators and Lie brackets. This was done using Maple V [1] and Form 3.0 [5] programs.

2. Main results

We consider the Lie algebra of evolutionary vectorfields based on a the independent variable x and the dependent variable $\mathbf{u}$. Let D be the total derivative operator with respect to x. We write $\mathbf{u}_i = D^i\mathbf{u}$. Given any two expressions P and Q depending on x and a finite number of the $\mathbf{u}_i, i = 0, \cdots$, their bracket is defined to be $[P, Q] = Q'P - P'Q$, where P' is the Fréchet derivative of P, cf. [3] for the theoretical foundations.

We study a kind of special local differential operators $\Re$, which produce a Lie-subalgebra, that is, for any P and Q, there is a $\mathbb{B}_{\Re}(P, Q)$ such that

$$[\Re P, \Re Q] = \Re \mathbb{B}_{\Re}(P, Q) \tag{2}$$

and so the domain of $\Re$ is a subalgebra. If we view $\mathbb{B}_{\Re}$ as a 2-form on the domain of $\Re$, then this form is antisymmetric, and obeys the Jacobi identity modulo the

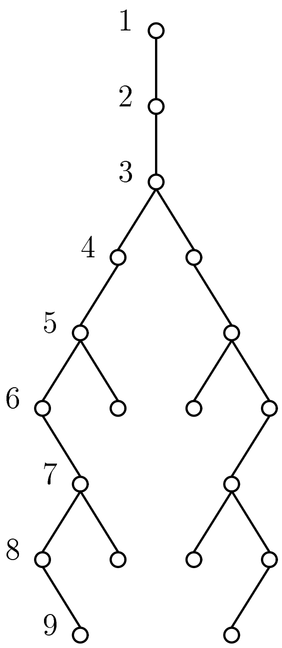

Figure 1: The operators to order 9

kernel of $\Re$, so it defines a (possibly new) Lie bracket on a suitable domain. Before we give the results, we define the following notation, which will use throughout the paper.

Definition 2.1. Let $\Gamma_i = D + \alpha_i \mathbf{u}$, with $\alpha_i \in \mathbb{C}$. Define $\mathbb{L}_k^l = \prod_{i=k}^l \Gamma_i$, $\Delta_n^0 = \alpha_n$, $\Delta_n^1 = \Delta_n^0 - \Delta_{n-1}^0$ and $\Delta_n^2 = \Delta_n^1 - \Delta_{n-1}^1$.

Theorem 2.2. *For every even n there exist 4 and for every odd n there exist 2 operators of the type $\Re = \mathbb{L}_1^n$, as long as the order n is larger than 4, such that for every P, Q there is a $\mathbb{B}_n(P,Q)$ satisfying $[\mathbb{L}_1^n P, \mathbb{L}_1^n Q] = \mathbb{L}_1^n \mathbb{B}_n(P,Q)$. The sequences of $\alpha_i, i = 1, \cdots, n$ are*

$$
\begin{array}{ll}
\pounds_{00}^{n+2} : 0,1,1,1,1,\cdots,1,1, & n \in \mathbb{Z}, \\
\pounds_{01}^{n+2} : 0,1,1,1,1,\cdots,1,2, & n \in 2\mathbb{Z}, \\
\pounds_{11}^{n+2} : 0,1,1,2,3,\cdots,n-1,n-1, & n \in 2\mathbb{Z}, \\
\pounds_{10}^{n+2} : 0,1,1,2,3,\cdots,n-1,n, & n \in \mathbb{Z}.
\end{array}
$$

Here the first subindex is Δ_4^1, the second one is $|\Delta_{n+2}^2|$. Moreover, the bracket $\mathbb{B}_n(P,Q) = \mathbb{Z}_n(P,Q) + D_Q[\mathbb{L}_1^n P] - D_P[\mathbb{L}_1^n Q]$, where, for any odd n,

$$
\mathbb{Z}_n(P,Q) \;=\; \Delta_n^0(\mathbb{L}_2^n P \cdot Q - \mathbb{L}_2^n Q \cdot P) - \Delta_n^1(\mathbb{L}_3^n P \cdot \mathbb{L}_n^n Q - \mathbb{L}_3^n Q \cdot \mathbb{L}_n^n P), \quad (3)
$$

and for any even $n = 2m$,

$$
\begin{aligned}
\mathbb{Z}_n(P,Q) = {} & \Delta_n^0(\mathbb{L}_2^n P \cdot Q - \mathbb{L}_2^n Q \cdot P) - \Delta_n^1(\mathbb{L}_3^n P \cdot \mathbb{L}_n^n Q - \mathbb{L}_3^n Q \cdot \mathbb{L}_n^n P) \\
& + \Delta_n^2 \sum_{i=2}^{m-1} (-1)^i (\mathbb{L}_{i+2}^n P \cdot \mathbb{L}_{n-i+1}^n Q - \mathbb{L}_{i+2}^n Q \cdot \mathbb{L}_{n-i+1}^n P). \quad (4)
\end{aligned}
$$

Proof. The existence of the operators and Lie brackets is proved in section 3, combining the results in the Lemmas. The uniqueness under multiplication, that is to say, proving that given $\Re$ there exist one (if the order of $\Re$ is even) or two (if the order of $\Re$ is odd) uniquely determined operators Γ such that $\Re\Gamma$ again satisfies the conditions, is the subject of section 4, as well as proving the nonexistence of any such Γ in case the sequence of $\Re$ ends with $1,1,2$ or $n-4, n-3, n-3$. ■

Corollary 2.3. *The generating function for the number of operators of given order is*

$$\frac{1+t^4}{1-t}+\frac{2t^6}{1-t^2}.$$

3. Existence of the operators

Lemma 3.1. *$\Re$ is an operator satisfying (2) if and only if for any $P,Q\in\mathfrak{g}$, there is a $\mathbb{Z}_\Re\in C^2(\mathfrak{g},\mathfrak{g})$ such that*

$$\Re'[\Re P]Q-\Re'[\Re Q]P=\Re\mathbb{Z}_\Re(P,Q).$$

Proof Let us compute the following bracket

$$\begin{aligned}[][\Re P,\Re Q] &= D_{\Re Q}[\Re P]-D_{\Re P}[\Re Q]\\ &= \Re(D_Q[\Re P]-D_P[\Re Q])+\Re'[\Re P]Q-\Re'[\Re Q]P.\end{aligned}$$

So, if and only if the lemma is valid, there exists $\mathbb{B}_\Re\in C^2(\mathfrak{g},\mathfrak{g})$ satisfying (2) with $\mathbb{B}_\Re(P,Q)=\mathbb{Z}_\Re(P,Q)+D_Q[\Re P]-D_P[\Re Q]$.

Lemma 3.2. *Assume $\Re$ is an operator satisfying (2). For any operator Γ, $\Re\Gamma$ is also such an operator if and only if*

$$\mathcal{O}(P,Q)=\mathbb{Z}_\Re(\Gamma P,\Gamma Q)+\Gamma'[\Re\Gamma P]Q-\Gamma'[\Re\Gamma Q]P \tag{5}$$

is in the image of the operator Γ for all $P,Q\in\mathfrak{g}$.

Proof. According to Lemma 3.1, we need to compute

$$\begin{aligned}&(\Re\Gamma)'[\Re\Gamma P]Q-(\Re\Gamma)'[\Re\Gamma Q]P\\ =\;&\Re'[\Re\Gamma P]\Gamma Q-\Re'[\Re\Gamma Q]\Gamma P+\Re(\Gamma'[\Re\Gamma P]Q-\Gamma'[\Re\Gamma Q]P)\\ =\;&\Re\left(\mathbb{Z}_\Re(\Gamma P,\Gamma Q)+\Gamma'[\Re\Gamma P]Q-\Gamma'[\Re\Gamma Q]P\right)\end{aligned}$$

Therefore, we only need to check whether the expression $\mathcal{O}(P,Q)$ is in the image of the operator Γ. ■

Lemma 3.3. *Let $\alpha_1=0$ and $\alpha_2=\alpha_3=1$. Suppose $n\geq 3$. Define $\Re=\mathbb{L}_1^n$ and*

$$\mathbb{Z}_n(P,Q) = \Delta_n^0(\mathbb{L}_2^nP\cdot Q-\mathbb{L}_2^nQ\cdot P)-\Delta_n^1(\mathbb{L}_3^nP\cdot\Gamma_nQ-\mathbb{L}_3^nQ\cdot\Gamma_nP),$$

Then, if $\Delta_{n+1}^1=\Delta_n^1=\alpha$, with $\alpha=0$ or 1 (and therefore $\Delta_{n+1}^2=0$),

$$\mathbb{Z}_n(\Gamma_{n+1}P,\Gamma_{n+1}Q)+\Gamma_{n+1}'[\mathbb{L}_1^n\Gamma_{n+1}P]Q-\Gamma_{n+1}'[\mathbb{L}_1^n\Gamma_{n+1}Q]P=\Gamma_{n+1}\mathbb{Z}_{n+1}(P,Q).$$

Proof. We directly compute

$$
\begin{aligned}
&\Gamma_{n+1}\mathbb{Z}_{n+1}(P,Q)-\Gamma'_{n+1}[\mathbb{L}_1^{n+1}P]Q+\Gamma'_{n+1}[\mathbb{L}_1^{n+1}Q]P-\mathbb{Z}_n(\Gamma_{n+1}P,\Gamma_{n+1}Q)\\
&= \Gamma_{n+1}(\Delta^0_{n+1}(\mathbb{L}_2^{n+1}P\cdot Q-\mathbb{L}_2^{n+1}Q\cdot P))\\
&\quad -\Gamma_{n+1}(\Delta^1_{n+1}(\mathbb{L}_3^{n+1}P\cdot\Gamma_{n+1}Q-\mathbb{L}_3^{n+1}Q\cdot\Gamma_{n+1}P))\\
&\quad -\Gamma'_{n+1}[\mathbb{L}_1^{n+1}P]Q+\Gamma'_{n+1}[\mathbb{L}_1^{n+1}Q]P\\
&\quad -\Delta^0_n(\mathbb{L}_2^{n+1}P\cdot\Gamma_{n+1}Q-\mathbb{L}_2^{n+1}Q\cdot\Gamma_{n+1}P)\\
&\quad +\Delta^1_n(\mathbb{L}_3^{n+1}P\cdot\Gamma_n\Gamma_{n+1}Q-\mathbb{L}_3^{n+1}Q\cdot\Gamma_n\Gamma_{n+1}P)\\
&= \Delta^1_{n+1}(\mathbb{L}_2^{n+1}P\cdot\Gamma_{n+1}Q-\mathbb{L}_2^{n+1}Q\cdot\Gamma_{n+1}P)\\
&\quad -\Delta^1_{n+1}(\mathbb{L}_2^{n+1}P\cdot\Gamma_{n+1}Q-\mathbb{L}_2^{n+1}Q\cdot\Gamma_{n+1}P)\\
&\quad -\Delta^1_{n+1}(\mathbb{L}_3^{n+1}P\cdot\Gamma_n\Gamma_{n+1}Q-\mathbb{L}_3^{n+1}Q\cdot\Gamma_n\Gamma_{n+1}P)\\
&\quad +\Delta^1_n(\mathbb{L}_3^{n+1}P\cdot\Gamma_n\Gamma_{n+1}Q-\mathbb{L}_3^{n+1}Q\cdot\Gamma_n\Gamma_{n+1}P)\\
&= -\Delta^2_{n+1}(\mathbb{L}_3^{n+1}P\cdot\Gamma_n\Gamma_{n+1}Q-\mathbb{L}_3^{n+1}Q\cdot\Gamma_n\Gamma_{n+1}P)\\
&= 0,
\end{aligned}
$$

where we used the fact that either $\Delta^0_{n+1}=\Delta^0_n+1$ or $\Delta^0_{n+1}=1=\alpha_2$. This shows that we have an inductive relation for such a family of $\mathbb{L}_1^n$ and $\mathbb{Z}_n$ for $\alpha=0,1$. We have not yet shown how the induction starts, but for this we can refer to [6], as listed in section 1. ■

Suppose now all the Γ_i satisfy the relations given in Lemma 3.3 up till Γ_n, but Γ_{n+1} does not, i.e., $\Delta^2_{n+1}\neq 0$.
We say that Γ_{n+1} *splits over* $\mathbb{L}_1^n$ if

$$\Gamma'_{n+1}=\Gamma'_{i+1}+\Gamma'_{n-i+1},\ \forall i=2,\cdots,n-2.$$

Lemma 3.4. *If Γ_{n+1} splits over $\mathbb{L}_1^n$ then*

$$\Gamma_{n+1}(P\cdot Q)=\Gamma_{i+1}P\cdot Q+P\cdot\Gamma_{n-i+1}Q.$$

Proof. This is obvious, since $\alpha_{n+1}\mathbf{u}=\alpha_{i+1}\mathbf{u}+\alpha_{n-i+1}\mathbf{u}$. ■

Lemma 3.5. *Suppose $n\geq 5$ and Γ_{n+1} splits over $\mathbb{L}_1^n$. If $\mathbb{L}_1^n=\pounds_{00}^n$ this implies $\Gamma'_{n+1}=\Gamma'_n+1=2$ and if $\mathbb{L}_1^n=\pounds_{10}^n$ then $\Gamma'_{n+1}=\Gamma'_n=n-2$. Let, for $n=2m+1$ odd and $m\geq 2$,*

$$
\begin{aligned}
\mathbb{Z}_{n+1}(P,Q) &= \Delta^0_{n+1}(\mathbb{L}_2^{n+1}P\cdot Q-\mathbb{L}_2^{n+1}Q\cdot P)\\
&\quad -\Delta^1_{n+1}(\mathbb{L}_3^{n+1}P\cdot\Gamma_{n+1}Q-\mathbb{L}_3^{n+1}Q\cdot\Gamma_{n+1}P)\\
&\quad +\Delta^2_{n+1}\sum_{i=2}^{m}(-1)^i(\mathbb{L}_{i+2}^{n+1}P\cdot\mathbb{L}_{n-i+2}^{n+1}Q-\mathbb{L}_{i+2}^{n+1}Q\cdot\mathbb{L}_{n-i+2}^{n+1}P)
\end{aligned}
$$

Then

$$\mathbb{Z}_n(\Gamma_{n+1}P,\Gamma_{n+1}Q)+\Gamma'_{n+1}[\mathbb{L}_1^{n+1}P]Q-\Gamma'_{n+1}[\mathbb{L}_1^{n+1}Q]P=\Gamma_{n+1}\mathbb{Z}_{n+1}(P,Q).$$

Proof. We compute

$$
\begin{aligned}
&\Gamma_{n+1}\mathbb{Z}_{n+1}(P,Q)-\Gamma'_{n+1}[\mathbb{L}_1^{n+1}P]Q+\Gamma'_{n+1}[\mathbb{L}_1^{n+1}Q]P-\mathbb{Z}_n(\Gamma_{n+1}P,\Gamma_{n+1}Q)=\\
&=\ \Gamma_{n+1}(\Delta^0_{n+1}(\mathbb{L}_2^{n+1}P\cdot Q-\mathbb{L}_2^{n+1}Q\cdot P))\\
&\quad -\Gamma_{n+1}(\Delta^1_{n+1}(\mathbb{L}_3^{n+1}P\cdot\Gamma_{n+1}Q-\mathbb{L}_3^{n+1}Q\cdot\Gamma_{n+1}P))\\
&\quad +\Gamma_{n+1}(\Delta^2_{n+1}\sum_{i=2}^{m}(-1)^i(\mathbb{L}_{i+2}^{n+1}P\cdot\mathbb{L}_{n-i+2}^{n+1}Q-\mathbb{L}_{i+2}^{n+1}Q\cdot\mathbb{L}_{n-i+2}^{n+1}P))\\
&\quad -\Gamma'_{n+1}[\mathbb{L}_1^{n+1}P]Q+\Gamma'_{n+1}[\mathbb{L}_1^{n+1}Q]P\\
&\quad -\Delta^0_n(\mathbb{L}_2^n\Gamma_{n+1}P\cdot\Gamma_{n+1}Q-\mathbb{L}_2^n\Gamma_{n+1}Q\cdot\Gamma_{n+1}P)\\
&\quad +\Delta^1_n(\mathbb{L}_3^n\Gamma_{n+1}P\cdot\Gamma_n\Gamma_{n+1}Q-\mathbb{L}_3^n\Gamma_{n+1}Q\cdot\Gamma_n\Gamma_{n+1}P)\\
&=\ +\Delta^1_n(\mathbb{L}_3^{n+1}P\cdot\Gamma_n\Gamma_{n+1}Q-\mathbb{L}_3^{n+1}Q\cdot\Gamma_n\Gamma_{n+1}P)\\
&\quad -\Delta^1_{n+1}(\mathbb{L}_3^{n+1}P\cdot\Gamma_n\Gamma_{n+1}Q-\mathbb{L}_3^{n+1}Q\cdot\Gamma_n\Gamma_{n+1}P)\\
&\quad +\Delta^2_{n+1}\sum_{i=2}^{m}(-1)^i(\mathbb{L}_{i+1}^{n+1}P\cdot\mathbb{L}_{n-i+2}^{n+1}Q-\mathbb{L}_{i+1}^{n+1}Q\cdot\mathbb{L}_{n-i+2}^{n+1}P)\\
&\quad +\Delta^2_{n+1}\sum_{i=2}^{m}(-1)^i(\mathbb{L}_{i+2}^{n+1}P\cdot\mathbb{L}_{n-i+1}^{n+1}Q-\mathbb{L}_{i+2}^{n+1}Q\cdot\mathbb{L}_{n-i+1}^{n+1}P)\\
&=\ -\Delta^2_{n+1}(\mathbb{L}_3^{n+1}P\cdot\Gamma_n\Gamma_{n+1}Q-\mathbb{L}_3^{n+1}Q\cdot\Gamma_n\Gamma_{n+1}P)\\
&\quad +\Delta^2_{n+1}\sum_{i=2}^{m}(-1)^i(\mathbb{L}_{i+1}^{n+1}P\cdot\mathbb{L}_{n-i+2}^{n+1}Q-\mathbb{L}_{i+1}^{n+1}Q\cdot\mathbb{L}_{n-i+2}^{n+1}P)\\
&\quad -\Delta^2_{n+1}\sum_{i=3}^{m+1}(-1)^i(\mathbb{L}_{i+1}^{n+1}P\cdot\mathbb{L}_{n-i+2}^{n+1}Q-\mathbb{L}_{i+1}^{n+1}Q\cdot\mathbb{L}_{n-i+2}^{n+1}P)\\
&=\ -\Delta^2_{n+1}(\mathbb{L}_3^{n+1}P\cdot\Gamma_n\Gamma_{n+1}Q-\mathbb{L}_3^{n+1}Q\cdot\Gamma_n\Gamma_{n+1}P)\\
&\quad +\Delta^2_{n+1}(\mathbb{L}_3^{n+1}P\cdot\mathbb{L}_n^{n+1}Q-\mathbb{L}_3^{n+1}Q\cdot\mathbb{L}_n^{n+1}P)\\
&=\ 0,
\end{aligned}
$$

where we used the fact that either $\Gamma'_{n+1}=1+\Gamma'_n$ or $\Delta^1_{n+1}=0$. ■

4. Uniqueness of the operators under multiplicative deformation

In this section we show, using the symbolic method [2, 4], which choice we have if we want to go from $\Re=\mathbb{L}_1^n$, with $n\geq 4$, to $\Re\Gamma$, with $\Gamma=\Gamma_{n+1}=D+\alpha_{n+1}\mathbf{u}$. According to Lemma 3.2 we need to check whether the expression (5) is in the image of the operator Γ.
We do this term by term. First we pick out the terms in $\mathcal{O}(P,Q)$ without $\mathbf{u}$ and its derivatives, denoted by H_0^{n+1} and integrate it. The image part is denoted by Z_0 and the rest is put as zero to obtain the condition on α_{n+1}. On the next step we pick out the term in $\mathcal{O}(P,Q)-\Gamma Z_0$ linear in u or its derivatives, denoted by H_1^{n+1} and treat it the same way as before. We continue this procedure until we either get the obstruction, meaning no such operator exists or the operator $\Re\Gamma$.
To prove the uniqueness of the operators, we start to show there is at most one solution for each case from even order to odd order (cf. Lemma 4.1) and two solutions for each case from odd order to even order (cf. Lemma 4.2). Then we

prove that there are no such operators starting from $\pounds^n_{\alpha 1}$, where $\alpha = 0, 1$ and $n \in 2\mathbb{Z}$ by a long computation (cf. after Lemma 4.3), see Figure 1.
We carry out the above procedure by the symbolic method by mapping $\mathbf{u}_i \mapsto z^i$, $P_i \mapsto x^i$ and $Q_i \mapsto y^i$.
Notice that $\mathbb{L}^n_k = D^{l-k+1} + \sum_{j=k}^n \alpha_j D^{j-k}\mathbf{u}D^{n-j} + O(\mathbf{u}^2)$. The symbolic expression of $\mathbf{u}$-linear terms in $\mathbb{L}^n_k P$ is

$$\psi^n_k(x,z) = \sum_{j=k}^{n} \alpha_j (x+z)^{j-k} x^{n-j}. \tag{6}$$

From formula (3) and (4), we obtain, for any n,

$$H^{n+1}_0(x,y) = \alpha_{n+1}(x^{n+1} - y^{n+1}) + \alpha_n(x^n y - x y^n) - \Delta^1_n(x^{n-1}y^2 - x^2 y^{n-1})$$

Lemma 4.1. *The function $H^{2m+1}_0(x,y)$ has a factor $x+y$ if and only if*

$$\Delta^2_{2m+1} = 0.$$

Proof. It is easy to see that

$$H^{2m+1}_0(x,-x) = 2(\alpha_{2m+1} - \alpha_{2m} - \Delta^1_{2m})x^{2m+1} = 2\Delta^2_{2m+1}x^{2m+1}.$$

This has to be zero. ∎

Notice that the function $H^{2m}_0(x,y)$ has a factor $x+y$. We go to the next step. Let us write out the symbolic expression of the linear term in $\mathbf{u}$.

$$H^{2m}_1(x,y,z) = \wp_{2m}(x,y,z) - \wp_{2m}(y,x,z) - \alpha_{2m}\frac{H^{2m}_0(x,y)}{x+y},$$

where (cf. formula (6))

$$\wp_{2m}(x,y,z) = \alpha_{2m}\psi^{2m}_1(x,z) + \alpha_{2m}\alpha_{2m-1}x^{2m-1} + \alpha_{2m-1}\psi^{2m}_2(x,z)y \\ -\Delta^1_{2m-1}\left(x^{2m-2}(\alpha_{2m}(y+z) + \alpha_{2m-1}y) + \psi^{2m}_3(x,z)y^2\right).$$

Lemma 4.2. *The function $H^{2m}_1(x,y,z)$ has a factor $x+y+z$ if and only if*

1. *$\alpha_{2m} = 1$ or 2 when $\alpha_1 = 0$ and $\alpha_j = 1$ for $j = 2, \cdots, 2m-1$.*

2. *$\alpha_{2m} = 2m-2$ or $2m-3$ when $\alpha_1 = 0$, $\alpha_2 = 1$ and $\alpha_j = j-2$ for $j = 3, \cdots, 2m-1$.*

Proof. Let us compute $H^{2m}_1(x,y,-x-y)$ in both cases. For the first case

$$\begin{aligned}\psi^{2m}_k(x,-x-y) &= \sum_{j=k}^{2m} \alpha_j(-y)^{j-k}x^{2m-j} \\ &= \sum_{j=k}^{2m-1}(-y)^{j-k}x^{2m-j} + \alpha_{2m}(-y)^{2m-k} - \delta^1_k x^{2m-1} \\ &= \frac{x^{2m+1-k} - x(-y)^{2m-k}}{x+y} + \alpha_{2m}(-y)^{2m-k} - \delta^1_k x^{2m-1}.\end{aligned}$$

Therefore

$$
\begin{aligned}
&H_1^{2m}(x,y,-x-y) = \\
&= (\alpha_{2m}-1)\frac{xy^{2m-1}-yx^{2m-1}}{x+y} - (\alpha_{2m}^2-\alpha_{2m})y^{2m-1} + \alpha_{2m}x^{2m-1} \\
&\quad -(\alpha_{2m}-1)\frac{yx^{2m-1}-xy^{2m-1}}{x+y} + (\alpha_{2m}^2-\alpha_{2m})x^{2m-1} - \alpha_{2m}y^{2m-1} \\
&\quad -\alpha_{2m}\frac{\alpha_{2m}(x^{2m}-y^{2m}) + (x^{2m-1}y - xy^{2m-1})}{x+y} \\
&= (\alpha_{2m}-1)(\alpha_{2m}-2)\frac{xy(x^{2m-2}-y^{2m-2})}{x+y}.
\end{aligned}
$$

This proves the first case of the lemma. For the second case

$$
\begin{aligned}
&\psi_k^{2m}(x,-x-y) = \sum_{j=k}^{2m} \alpha_j (-y)^{j-k} x^{2m-j} \\
&= \sum_{j=k}^{2m-1} (j-2)(-y)^{j-k}x^{2m-j} + \alpha_{2m}(-y)^{2m-k} \\
&\quad +\delta_k^1(x^{2m-1} - yx^{2m-2}) + \delta_k^2 x^{2m-2} \\
&= \frac{x^{2m+1-k}(kx-2x+ky-3y) - x(-y)^{2m-k}(2mx-2x+2my-3y)}{(x+y)^2} \\
&\quad +\alpha_{2m}(-y)^{2m-k} + \delta_k^1(x^{2m-1}-yx^{2m-2}) + \delta_k^2 x^{2m-2}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&H_1^{2m}(x,y,-x-y) = \\
&= (\alpha_{2m}-2m+2)\frac{y^2x^{2m-1} + xy^{2m-1}(2mx-2x+2my-3y)}{(x+y)^2} \\
&\quad -(\alpha_{2m}-2m+2)\frac{x^2y^{2m-1} + yx^{2m-1}(2my-2y+2mx-3x)}{(x+y)^2} \\
&\quad +\alpha_{2m}^2(x^{2m-1}-y^{2m-1}) + A_{2m}(xy^{2m-2}-yx^{2m-2}) \\
&\quad -\alpha_{2m}\frac{\alpha_{2m}(x^{2m}-y^{2m}) + (2m-3)(x^{2m-1}y - xy^{2m-1}) - (x^{2m-2}y^2 - x^2y^{2m-2})}{x+y} \\
&= (\alpha_{2m}-2m+2)(2m-3)\frac{xy^{2m-1}-yx^{2m-1}}{x+y} \\
&\quad +\alpha_{2m}\frac{(\alpha_{2m}-2m+2)(x^{2m-1}y - xy^{2m-1})}{x+y} \\
&= ((\alpha_{2m}-2m+2)(\alpha_{2m}-2m+3)\frac{x^{2m-1}y - xy^{2m-1}}{x+y}.
\end{aligned}
$$

This completes the proof of the Lemma. ∎

Now we prove that for the first case, starting from $\mathbb{L}_1^{2m}$ with $\alpha_{2m} = 2$ and for the second case, with $\alpha_{2m} = 2m-3$, we can not find such $\mathbb{L}_1^{2m+1}$. To do so, we need to write out $H_0^{2m-1}(x,y)$ using formula (4).

$$\begin{aligned} &H_0^{2m+1}(x,y) = \\ &= \alpha_{2m+1}(x^{2m+1}-y^{2m+1}) + \alpha_{2m}(x^{2m}y - xy^{2m}) \\ &\quad -\Delta^1_{2m}(x^{2m-1}y^2 - x^2y^{2m-1}) + \Delta^2_{2m}\sum_{i=2}^{m-1}(-1)^i(x^{2m-i}y^{i+1} - x^{i+1}y^{2m-i}). \end{aligned}$$

Lemma 4.3. *The function $H_0^{2m+1}(x,y)$ has a factor $x+y$ if and only if*

$$\Delta^2_{2m+1} = (m-2)\Delta^2_{2m}.$$

Proof. It is easy to see that

$$H_0^{2m+1}(x,-x) = 2(\Delta^2_{2m+1} - (m-2)\Delta^2_{2m})x^{2m+1}.$$

This has to be zero. The proof of the lemma follows. ∎

Now let write out the the possible sequences of α_j $(m>1)$:

1. $\alpha_1 = 0$, $\alpha_j = 1$ for $j = 2, \cdots, 2m-1$, $\alpha_{2m} = 2$ and $\alpha_{2m+1} = m+1$.
2. $\alpha_1 = 0$, $\alpha_2 = 1$ and $\alpha_j = j-2$ for $j = 3, \cdots, 2m-1$ $\alpha_{2m} = 2m-3$ and $\alpha_{2m+1} = m-1$.

We show that for such α_j's one has $H_1^{2m+1}(x,y,-x-y) \neq 0$, that is, such operators do not exist.

$$\begin{aligned} &H_1^{2m+1}(x,y,z) = \\ &= \wp_{2m+1}(x,y,z) - \wp_{2m+1}(y,x,z) - \alpha_{2m+1}\frac{H_0^{2m+1}(x,y)}{x+y}, \end{aligned}$$

where (cf. formula (4))

$$\begin{aligned} &\wp_{2m+1}(x,y,z) = \\ &= \alpha_{2m+1}\psi_1^{2m+1}(x,z) + \alpha_{2m}\alpha_{2m+1}x^{2m} + \alpha_{2m}\psi_2^{2m+1}(x,z)y \\ &\quad -\Delta^1_{2m}\left(x^{2m-1}\psi_{2m}^{2m+1}(y,z) + \psi_3^{2m+1}(x,z)y^2\right) \\ &\quad +\Delta^2_{2m}\sum_{i=2}^{m-1}(-1)^i(x^{2m-i}\psi_{2m-i+1}^{2m+1}(y,z) + \psi_{i+2}^{2m+1}(x,z)y^{i+1}). \end{aligned}$$

For the first case,

$$\begin{aligned} \psi_k^{2m+1}(x,-x-y) &= \sum_{j=k}^{2m+1}\alpha_j(-y)^{j-k}x^{2m+1-j} \\ &= \sum_{j=k}^{2m}(-y)^{j-k}x^{2m+1-j} + (m+1)(-y)^{2m+1-k} - \delta_k^1x^{2m} + (-y)^{2m-k}x \\ &= \frac{x^{2m+2-k} - x(-y)^{2m+1-k}}{x+y} + (m+1)(-y)^{2m+1-k} - \delta_k^1x^{2m} + (-y)^{2m-k}x. \end{aligned}$$

So
$$
\begin{aligned}
&\wp_{2m+1}(x,y,-x-y) = \\
= \ & (m+1)\frac{x^{2m+1}-xy^{2m}}{x+y} + (m+1)^2y^{2m} - (m-2)y^{2m-1}x \\
& +2(m+1)x^{2m} + 2\frac{x^{2m}y+xy^{2m}}{x+y} - 3(m+1)y^{2m} \\
& -2x^{2m-1}y - \frac{x^{2m-1}y^2-xy^{2m}}{x+y} \\
& +\sum_{i=2}^{m-1}\Big(\frac{y^{i+1}(-x)^{2m-i}-yx^{2m}}{x+y} + (m+1)x^{2m} - x^{2m-1}y\Big) \\
& -\sum_{i=2}^{m-1}\Big(\frac{x^{2m-i}(-y)^{i+1}-xy^{2m}}{x+y} + (m+1)y^{2m} - y^{2m-1}x\Big) \\
= \ & \frac{(m+1)^2x^{2m+1} + (m^2-m+4)x^{2m}y - (m+1)x^{2m-1}y^2}{x+y} \\
& +2\frac{y^3x^{2m-1}+y^{m+1}(-x)^{m+1}}{(x+y)^2}.
\end{aligned}
$$

Finally,
$$
\begin{aligned}
&H_1^{2m+1}(x,y,-x-y) = \\
= \ & \frac{(m+1)^2x^{2m+1} + (m^2-m+4)x^{2m}y - (m+1)x^{2m-1}y^2}{x+y} \\
& +2\frac{y^3x^{2m-1}-x^3y^{2m-1}}{(x+y)^2} \\
& -\frac{(m+1)^2y^{2m+1} + (m^2-m+4)y^{2m}x - (m+1)y^{2m-1}x^2}{x+y} \\
& -(m+1)\frac{(m+1)(x^{2m+1}-y^{2m+1}) + 2(x^{2m}y-xy^{2m}) - (x^{2m-1}y^2-x^2y^{2m-1})}{x+y} \\
& -(m+1)\frac{x^3y^3(x^{2m-4}-y^{2m-4})}{(x+y)^2} \\
= \ & (m-1)\left((m-2)\frac{(x^{2m}y-xy^{2m})}{x+y} - \frac{x^3y^3(x^{2m-4}-y^{2m-4})}{(x+y)^2}\right). \qquad (7)
\end{aligned}
$$

For the second case,
$$
\begin{aligned}
\psi_k^{2m+1}(x,-x-y) &= \sum_{j=k}^{2m+1}\alpha_j(-y)^{j-k}x^{2m+1-j} \\
&= \sum_{j=k}^{2m}(j-2)(-y)^{j-k}x^{2m+1-j} + \alpha_{2m+1}(-y)^{2m+1-k} \\
&\quad -(-y)^{2m-k}x + \delta_k^1(x^{2m}-yx^{2m-1}) + \delta_k^2x^{2m-1} \\
&= \frac{x^{2m+2-k}\,(kx-2x+ky-3y) - x(-y)^{2m+1-k}\,(2mx-x+2my-2y)}{(x+y)^2} \\
&\quad +(m-1)(-y)^{2m+1-k} - (-y)^{2m-k}x + \delta_k^1(x^{2m}-yx^{2m-1}) + \delta_k^2x^{2m-1}.
\end{aligned}
$$

So

$$
\begin{aligned}
&\wp_{2m+1}(x,y,-x-y) = \\
= \;& (m-1)\frac{x^{2m+1}(-x-2y) - xy^{2m}(2mx-x+2my-2y)}{(x+y)^2} \\
&+(m-1)\left((m-1)y^{2m} + y^{2m-1}x + x^{2m} - yx^{2m-1}\right) \\
&+(2m-3)y\frac{-x^{2m}y + xy^{2m-1}(2mx-x+2my-2y)}{(x+y)^2} \\
&+(2m-3)\left(yx^{2m-1} - (m-1)y^{2m} - y^{2m-1}x + (m-1)x^{2m}\right) \\
&-\sum_{i=2}^{m-1}\frac{(-x)^{2m-i}y^{i+1}((2m-i-1)y+(2m-i-2)x)}{(x+y)^2} \\
&+\sum_{i=2}^{m-1}(\frac{yx^{2m}(2my-y+2mx-2x)}{(x+y)^2} - (m-1)x^{2m} - x^{2m-1}y) \\
&+\sum_{i=2}^{m-1}\frac{x^{2m-i}(-y)^{i+1}(ix+iy-y)}{(x+y)^2} \\
&-\sum_{i=2}^{m-1}(\frac{xy^{2m}(2mx-x+2my-2y)}{(x+y)^2} - (m-1)y^{2m} - y^{2m-1}x) \\
= \;& \frac{(m-1)^2x^{2m+2} + (m-1)(2m-3)x^{2m+1}y + (m-1)(3m-5)x^{2m}y^2}{(x+y)^2} \\
&+(2-2m)\frac{y^3x^{2m-1} + (-x)^{m+1}y^{m+1}}{(x+y)^2}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&H_1^{2m+1}(x,y,-x-y) = \\
= \;& \frac{(m-1)^2x^{2m+2} + (m-1)(2m-3)x^{2m+1}y + (m-1)(3m-5)x^{2m}y^2}{(x+y)^2} \\
&+(2-2m)\frac{y^3x^{2m-1} - x^3y^{2m-1}}{(x+y)^2} \\
&-\frac{(m-1)^2y^{2m+2} + (m-1)(2m-3)y^{2m+1}x + (m-1)(3m-5)y^{2m}x^2}{(x+y)^2} \\
&-(m-1)\frac{(m-1)(x^{2m+1}-y^{2m+1}) + (2m-3)(x^{2m}y - xy^{2m})}{x+y} \\
&+(m-1)\frac{x^3y^3(x^{2m-4}-y^{2m-4})}{(x+y)^2} \\
= \;& (m-1)\frac{(m-2)(x^{2m+1}y - y^{2m+1}x) + (m-2)(x^{2m}y^2 - y^{2m}x^2)}{(x+y)^2} \\
&-(m-1)\frac{x^3y^3(x^{2m-4}-y^{2m-4})}{(x+y)^2} \\
= \;& (m-1)\left((m-2)\frac{(x^{2m}y - xy^{2m})}{x+y} - \frac{x^3y^3(x^{2m-4}-y^{2m-4})}{(x+y)^2}\right). \qquad (8)
\end{aligned}
$$

5. Open problems

We have shown in this paper the existence of a certain infinite family of operators. Notice that the $\pounds^n_{0\alpha}$–family, $\alpha = 0, 1$, behaves exactly as the $\pounds^n_{1\alpha}$–family, and the obstruction expressions (7) and (8) are in both cases equal. However, we have no explanation for this symmetry.
Another question is whether the Lie brackets that are defined are mutually compatible.
Finally it remains to be shown that these operators, characterized by the property (1) are truly unique, not only up to multiplicative deformations. This would imply that they always factorize into first order operators, as is the case for order < 7, c.f. [6]

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References

[1] Char, B. W., K. O. Geddes, G. H. Gonnet, B. L. Leong, M. B. Monagan, and S. M. Watt, "Maple V Language Reference Manual," Springer–Verlag, Berlin, 1991.

[2] Gel'fand, I. M., and L. A. Dikiĭ, *Asymptotic properties of the resolvent of Sturm-Liouville equations, and the algebra of Korteweg-de Vries equations*, Uspehi Mat. Nauk, **30**(5(185)) (1975), 67–100. English translation: Russian Math. Surveys, **30** no. 5 (1975), 77–113.

[3] Olver, P. J., "Applications of Lie Groups to Differential Equations," Graduate Texts in Mathematics **107** Springer-Verlag, New York, second edition, 1993.

[4] Sanders, J. A., and J. P. Wang, *On the integrability of homogeneous scalar evolution equations*, J. Differential Equations **147**(2) (1998), 410–434.

[5] Vermaseren, J. A. M., "New Features of FORM," Technical report, Nikhef, Amsterdam, 2000; math-ph/0010025.

[6] Zhiber, A. V., and V. V. Sokolov, *Exactly integrable hyperbolic equations of Liouville type*, Uspihi mat. Nauk **56**(1) (2001), 63–106. English translation: Russian Math. Surveys **56** (2001).

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