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Remark on the Complexified Iwasawa Decomposition

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Abstract. Let $G_\mathbb{R}$ be a real form of a complex semisimple Lie group $G_\mathbb{C}$. We identify the complexification $K_\mathbb{C}A_\mathbb{C}N_\mathbb{C} \subset G_\mathbb{C}$ of an Iwasawa decomposition $G_\mathbb{R} = K_\mathbb{R}A_\mathbb{R}N_\mathbb{R}$ as $\{g \in G_\mathbb{C} \mid gB \in \Omega\}$ where $B \subset G_\mathbb{C}$ is a Borel subgroup of $G_\mathbb{C}$ that contains $A_\mathbb{R}N_\mathbb{R}$ and Ω is the open $K_\mathbb{C}$–orbit on $G_\mathbb{C}/B$. This is done in the context of subsets $K_\mathbb{C}R_\mathbb{C} \subset G_\mathbb{C}$, where $R_\mathbb{C}$ is a parabolic subgroup of $G_\mathbb{C}$ defined over $\mathbb{R}$, and the open $K_\mathbb{C}$–orbits on complex flag manifolds $G_\mathbb{C}/Q$.

1. Details

Let $G_\mathbb{R}$ be a real form of a complex semisimple Lie group $G_\mathbb{C}$. For simplicity we assume that $G_\mathbb{C}$ is connected and that the adjoint representation of $G_\mathbb{R}$ maps $G_\mathbb{R}$ into the group of inner automorphisms of the Lie algebra of $G_\mathbb{C}$. Choose an Iwasawa decomposition $G_\mathbb{R} = K_\mathbb{R}A_\mathbb{R}N_\mathbb{R}$ and a corresponding minimal parabolic subgroup $P_\mathbb{R} = M_\mathbb{R}A_\mathbb{R}N_\mathbb{R}$ of $G_\mathbb{R}$. Every parabolic subgroup of $G_\mathbb{R}$ is conjugate to one that contains $P_\mathbb{R}$. Let $R_\mathbb{C} \subset G_\mathbb{C}$ be the complexification of a (real) parabolic subgroup $R_\mathbb{R} \subset G_\mathbb{R}$. Passing to a $G_\mathbb{R}$–conjugate we may, and do, assume $P_\mathbb{R} \subset R_\mathbb{R}$. Let $K_\mathbb{C} \subset G_\mathbb{C}$ be the complexification of the maximal compactly embedded subgroup $K_\mathbb{R} \subset G_\mathbb{R}$. Then $G_\mathbb{R} = K_\mathbb{R}R_\mathbb{R}$. In particular the Lie algebras satisfy $\mathfrak{g}_\mathbb{R} = \mathfrak{k}_\mathbb{R} + \mathfrak{r}_\mathbb{R}$, so their complexifications satisfy $\mathfrak{g}_\mathbb{C} = \mathfrak{k}_\mathbb{C} + \mathfrak{r}_\mathbb{C}$, and thus $K_\mathbb{C}R_\mathbb{C}$ is open in $G_\mathbb{C}$. Since the projection $G_\mathbb{C} \to G_\mathbb{C}/R_\mathbb{C}$ is an open map, $\Omega_X := K_\mathbb{C}(1R_\mathbb{C})$ is an open $K_\mathbb{C}$–orbit in $X = G_\mathbb{C}/R_\mathbb{C}$. If $g \in G_\mathbb{C}$ then $g \in K_\mathbb{C}R_\mathbb{C}$ if and only if $gR_\mathbb{C} \in K_\mathbb{C}(1R_\mathbb{C}) \subset G_\mathbb{C}/R_\mathbb{C}$. Now

Lemma 1.1. *An element $g \in G_\mathbb{C}$ is contained in $K_\mathbb{C}R_\mathbb{C}$ if and only if $gR_\mathbb{C} \in \Omega_X$.*

Let Q be a parabolic subgroup of $G_\mathbb{C}$ and write Ω_Z for the open orbit $K_\mathbb{C}(1Q)$ on $Z = G_\mathbb{C}/Q$. The argument of Lemma 1.1 shows:

Lemma 1.2. *Suppose $K_\mathbb{C}Q = K_\mathbb{C}R_\mathbb{C}$. Then $g \in K_\mathbb{C}R_\mathbb{C}$ if and only if $gQ \in \Omega_Z$.*

There is a unique closed $G_\mathbb{R}$–orbit on X, and the other $G_\mathbb{R}$–orbits are finite in number and higher in dimension. See [4]. By $(G_\mathbb{R}, K_\mathbb{C})$–duality ([3]; see [2] for a geometric proof), there is a unique open $K_\mathbb{C}$–orbit on X (which thus must be Ω_X), and the other $K_\mathbb{C}$–orbits are finite in number and lower in dimension. Thus Ω_X is a dense open subset of X. Now Lemma 1.1 gives us

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Lemma 1.3. *$K_{\mathbb{C}}R_{\mathbb{C}}$ is a dense open subset of $G_{\mathbb{C}}$.*

Let $M_{\mathbb{C}}$, $A_{\mathbb{C}}$ and $N_{\mathbb{C}}$ denote the respective complexifications of $M_{\mathbb{R}}$, $A_{\mathbb{R}}$ and $N_{\mathbb{R}}$. Here $M_{\mathbb{C}}$ is the centralizer of $A_{\mathbb{C}}$ in $K_{\mathbb{C}}$. Let B denote a Borel subgroup of $G_{\mathbb{C}}$ that contains $A_{\mathbb{R}}N_{\mathbb{R}}$; in other words $B = B_M A_{\mathbb{C}} N_{\mathbb{C}}$ where B_M is a Borel subgroup of $M_{\mathbb{C}}$. The phenomenon $K_{\mathbb{C}}Q = K_{\mathbb{C}}R_{\mathbb{C}}$ of Lemma 1.2 occurs, for example, when $R_{\mathbb{R}}$ is the minimal parabolic subgroup $P_{\mathbb{R}}$ of $G_{\mathbb{R}}$ and $Q = B$. In that case $M_{\mathbb{R}} \subset K_{\mathbb{R}}$, so $B_M \subset M_{\mathbb{C}} \subset K_{\mathbb{C}}$, and thus $K_{\mathbb{C}} = K_{\mathbb{C}}B_M$. As $B = B_M A_{\mathbb{C}} N_{\mathbb{C}}$ now $K_{\mathbb{C}}A_{\mathbb{C}}N_{\mathbb{C}} = K_{\mathbb{C}}B$. We have proved

Proposition 1.4. *Let B be a Borel subgroup of $G_{\mathbb{C}}$ that contains $A_{\mathbb{R}}N_{\mathbb{R}}$. Let $g \in G_{\mathbb{C}}$. Then g is contained in $K_{\mathbb{C}}A_{\mathbb{C}}N_{\mathbb{C}}$ if and only if gB belongs to the open $K_{\mathbb{C}}$–orbit on $G_{\mathbb{C}}/B$. In particular $K_{\mathbb{C}}A_{\mathbb{C}}N_{\mathbb{C}}$ is a dense open subset of $G_{\mathbb{C}}$.*

Proposition 1.4 describes the complexified Iwasawa decomposition set $K_{\mathbb{C}}A_{\mathbb{C}}N_{\mathbb{C}}$ in terms of the open $K_{\mathbb{C}}$–orbit on the full flag manifold $G_{\mathbb{C}}/B$. Usually it is easy to decide whether a given group element belongs to $K_{\mathbb{C}}A_{\mathbb{C}}N_{\mathbb{C}}$, and that then describes the open $K_{\mathbb{C}}$–orbit on $G_{\mathbb{C}}/B$:

Example 1.5. Let $G_{\mathbb{R}} = SL_n(\mathbb{R})$ and $G_{\mathbb{C}} = SL_n(\mathbb{C})$. For $g \in G_{\mathbb{C}}$, let s_k be the $k \times k$ submatrix in the upper left corner of ${}^t gg$. Then $g \in K_{\mathbb{C}}A_{\mathbb{C}}N_{\mathbb{C}}$ if and only if $\det(s_k) \neq 0$ for $1 \leqq k \leqq n$.

Added in proof. After this note was accepted for publication, we learned that there is a nontrivial overlap with material in [1, Appendix B], specifically [1, Theorem B.1.2] and some of the immediately preceding discussion.

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