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Vanishing of the First Cohomologies for Lattices in Lie Groups

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Abstract. We prove the following "maximal" theorem on vanishing of the first cohomologies. Let G be a connected semisimple Lie group with a lattice Γ. Assume that there is no epimorphism $\phi{:}G\to H$ onto a Lie group H locally isomorphic to $\mathrm{SO}(1,n)$ or $\mathrm{SU}(1,n)$ such that $\phi(\Gamma)$ is a lattice in H. Then $H^1(\Gamma,\rho)=0$ for any finite-dimensional representation ρ of Γ over $\mathbb{R}$. This generalizes Margulis' Theorem on vanishing of the first cohomologies for lattices in higher rank semisimple Lie groups. Some applications for proving general results on the structure of lattices in arbitrary Lie groups, are given.

Introduction

In his in-depth study of lattices in semisimple Lie groups, Margulis proved the following theorem on vanishing of the first cohomologies:

Theorem 1. (See [4].) *Let Γ be a lattice in a connected semisimple Lie group G. Assume that*

a) G has no compact factors,

b) G has finite center,

c) Γ is irreducible in G,

d) $\mathrm{rk}_{\mathbb{R}}\, G \geq 2$.

Then $H^1(\Gamma,\rho)=0$ for any finite-dimensional representation ρ of Γ over local field of characteristic 0.

We will avoid the restrictions a)–c) and relax d) thus proving a general result for representations over $\mathbb{R}$ (and hence over $\mathbb{C}$). To formulate it, we give a definition. Let G be a connected Lie group with a lattice $\Gamma \subset G$. Let us say that the pair (G,Γ) *satisfies condition* $(*)$ if there is no epimorphism $\phi\colon G\to H$ onto a Lie group H locally isomorphic to $\mathrm{SO}(1,n)$ or $\mathrm{SU}(1,n)$ such that $\phi(\Gamma)$ is a lattice in H.

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Theorem 2. *Let G be a connected semisimple Lie group with a lattice Γ. Assume that the pair (G,Γ) satisfies condition $(*)$. Then $H^1(\Gamma,\rho)=0$ for any finite-dimensional representation ρ of Γ over $\mathbb{R}$.*

In a sense, this is the "maximal" theorem on vanishing of the first cohomologies for lattices (see §4 for the discussion).

Theorem 2, in particular, claims vanishing of the first cohomologies for lattices in rank one Lie groups $\mathrm{Sp}(1,n)$, $n \geq 2$, and F_4^{-20}. This is a new result which we derive from old papers of Raghunathan [6,7] in combination with more recent Superrigidity and Arithmeticity Theorems obtained by Corlette [1] and Gromov–Schoen [3].

The paper consists of four sections. In §1 we give definitions and some immediate observations. Theorem 2 is reduced to the rank one case in §2. In §3 we study the rank one situation. In §4 we give some applications of the main result. These include: a general form of Superrigidity Theorem, a general form of the Levi–Mostow splitting of lattices in arbitrary Lie groups, and a criterion for vanishing of the first cohomologies (modulo a still open Millson problem).

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§1. Definitions and easy observations

In what follows, we work with the real field $\mathbb{R}$. All linear representations considered in this paper are assumed to be *continuous* and *finite-dimensional.* Several standard results are included for the sake of completeness.

Let F be a topological group and $\rho\colon F \to \mathrm{GL}(V)$ be a representation on a (real) vector space V. A continuous map $\alpha\colon F \to V$ is said to be a *cocycle* over ρ if

$$\alpha(gg') = \alpha(g) + \rho(g)(\alpha(g')),\ \ g, g' \in F.$$

A cocycle α is said to be a *coboundary* (or $\alpha \sim 0$) if there exists $x \in V$ such that $\alpha(g) = \rho(g)(x) - x$. The additive group $H^1(F,\rho)$ of all cocycles modulo coboundaries is called the *first cohomology group of F over ρ.*

Let τ_1 stand for the trivial representation of F on $\mathbb{R}$. Then $H^1(F,\tau_1)$ is the additive group $\mathrm{Hom}(F,\mathbb{R})$ of all homomorphisms $F \to \mathbb{R}$.

Lemma 1.1. 1) *Let α be a cocycle over ρ and let $\rho_\alpha(g) = \alpha(g)\rho(g)$. Then ρ_α is a homomorphism of F to the affine group $\mathrm{Aff}(V) = \mathrm{GL}(V){\cdot}V$, where the multiplication in $\mathrm{Aff}(V)$ is given by*

$$gx = gxg^{-1}g = g(x)g,\ \ g \in \mathrm{GL}(V), x \in V.$$

2) *Conversely, let* $\pi: \mathrm{Aff}(V) \to \mathrm{GL}(V)$ *be the natural epimorphism and let* $\rho': F \to \mathrm{Aff}(V)$ *be a homomorphism covering* ρ *(i.e.,* $\rho = \pi \circ \rho'$*). Then the formula* $\rho'(g) = \alpha(g)\rho(g)$ *defines a cocycle over* ρ.

3) *Cocycle* α *is a coboundary iff* $\rho(F)$ *and* $\rho_\alpha(F)$ *are conjugate by an element of* V.

Proof. 1) $\rho_a(gg') = \alpha(gg')\rho(gg') = (\alpha(g) + \rho(g)(\alpha(g')))\rho(gg') = \alpha(g)\rho(g)\alpha(g')\rho(g') = \rho_a(g)\rho_\alpha(g')$.

2) $\alpha(gg')\rho(gg') = \rho'(gg') = \rho'(g)\rho'(g') = \alpha(g)\rho(g)\alpha(g')\rho(g') = (\alpha(g) + \rho(g)(\alpha(g')))\rho(g)\rho(g')$.

3) $\alpha \sim 0 \iff$ there exists $x \in V$ such that $\alpha(g) = \rho(g)(x) - x = x^{-1}\rho(g)x\rho(g)^{-1} \iff \rho_\alpha(g) = \alpha(g)\rho(g) = x^{-1}\rho(g)x$. ∎

Let $G \subset \mathrm{GL}(n,\mathbb{R})$ be a linear Lie group. We say that G is $\mathbb{R}$-*algebraic group* if there exists an algebraic group $\mathbb{G} \subset \mathrm{GL}(n,\mathbb{C})$ defined over $\mathbb{R}$ with $G = \mathbb{G}_\mathbb{R}$. Clearly, given an n-dimensional vector space V, the group $\mathrm{GL}(V)$ can be identified with $\mathrm{GL}(n,\mathbb{R})$ (and hence becomes $\mathbb{R}$-algebraic) after choosing a basis in V. Also, $\mathrm{Aff}(V)$ viewed as a subgroup of $\mathrm{GL}(n+1,\mathbb{R})$ is $\mathbb{R}$-algebraic.

Given a subgroup $G \subset \mathrm{GL}(n,\mathbb{R})$ we will let $\mathcal{A}(G) \subset \mathrm{GL}(n,\mathbb{R})$ denote *the algebraic hull* of G, i.e. the smallest $\mathbb{R}$-algebraic subgroup of $\mathrm{GL}(n,\mathbb{R})$ that contains G.

Note that any $\mathbb{R}$-algebraic group has finite number of connected components. Intersection of two $\mathbb{R}$-algebraic subgroups is $\mathbb{R}$-algebraic. Factor-group $\mathbb{G}_\mathbb{R}/\mathbb{H}_\mathbb{R}$ of an $\mathbb{R}$-algebraic group $\mathbb{G}_\mathbb{R}$ over its normal $\mathbb{R}$-algebraic subgroup $\mathbb{H}_\mathbb{R}$ is a finite index subgroup of the $\mathbb{R}$-algebraic group $(\mathbb{G}/\mathbb{H})_\mathbb{R}$ (Example: $\mathrm{SL}(2,\mathbb{R})/\mathbb{Z}_2 \simeq \mathrm{SO}(1,2)_0 \subset \mathrm{SL}(3,\mathbb{R})$).

Lemma 1.2. *Let* F *be a topological group. Then all the first cohomology groups of* F *are trivial iff given any homomorphism* ϕ *from* F *to an* $\mathbb{R}$*-algebraic group* H*, the group* $\mathcal{A}(\phi(F))$ *is reductive.*

Proof. $\Rightarrow$. Let $\mathcal{A}(\phi(F)) = M{\cdot}U \subset H$ be a Levi decomposition, where M is reductive and U is the unipotent radical. Let $V = U/[U,U]$, and let $q: M \to \mathrm{GL}(V)$ be the representation given by the action of M on $U/[U,U]$ by conjugations. Take the composite homomorphism $\rho: F \to M{\cdot}U \to q(M){\cdot}V$. Let $\pi: q(M){\cdot}V \to q(M)$ be the projection. Since $H^1(F, \pi\circ\rho) = 0$, it follows that $\rho(F)$ is conjugate to $\pi(\rho(F)) \subset q(M)$. But $\rho(F)$ is Zariski dense in $q(M){\cdot}V$. We derive that V is trivial and hence so is U.

$\Leftarrow$. Take $\rho: F \to \mathrm{GL}(V)$ and let $\rho': F \to \mathrm{Aff}(V)$ be a covering homomorphism, i.e., $\rho = \pi \circ \rho'$. The groups $\mathcal{A}(\rho'(F))$ and $\mathcal{A}(\rho(F))$ are reductive and V is the unipotent radical of $\mathcal{A}(\rho'(F)){\cdot}V = \mathcal{A}(\rho(F)){\cdot}V$. Hence $\mathcal{A}(\rho'(F))$ and $\mathcal{A}(\rho(F))$ are conjugate by an element of V. ∎

Remark. In the statement of the Lemma one can say more about $M = \mathcal{A}(\phi(F))$. It is either a finite group or a semisimple group (with finite number of connected components). In fact, M is an *almost direct product* of semisimple group L and abelian group A, i.e. L and A commute, $M = LA$, and the intersection $L \cap A$ is discrete (hence finite). Note that the image of $\phi(\Gamma)$ in $M/L = A/A{\cap}L$ is Zariski dense. On the other hand, it should be finite (otherwise $H^1(\Gamma, \tau_1) \neq 0$). Hence A is finite.

Lemma 1.3. *Let F be a topological group, $F' \subset F$ be a subgroup of finite index in F, and $\rho: F \to \mathrm{GL}(V)$ be a representation. Assume that $H^1(F', \rho) = 0$. Then $H^1(F, \rho) = 0$.*

Proof. Let $\phi: F \to H$ be a homomorphism to an $\mathbb{R}$-algebraic group. Note that $\mathcal{A}(\phi(F'))$ is of finite index in $\mathcal{A}(\phi(F))$. Now the claim follows from Lemma 1.2. ∎

Remark . The converse statement is wrong. Take, for instance $F = \mathrm{PSL}(2, \mathbb{Z})$. It is well known that F is the free product of finite groups $\mathbb{Z}_2$ and $\mathbb{Z}_3$. Hence $H^1(F, \tau_1) = 0$. On the other hand, F contains a subgroup F' of index 6 which is a free group with two generators. Hence $H^1(F', \tau_1) \simeq \mathbb{R}^2$.

Lemma 1.4. *Let C be a finite normal subgroup in a topological group F, and let $F' = F/C$. Then all the first cohomology groups of F' are trivial iff all the first cohomology groups of F are trivial.*

Proof. $\Rightarrow$. Let $\phi: F \to H$ be a homomorphism to an $\mathbb{R}$-algebraic group. The group $\mathcal{A}(\phi(F))/\phi(C)$ is a finite index subgroup of some $\mathbb{R}$-algebraic group H'. Since the image of F' in H' is Zariski dense, it follows that H' is reductive. Since $\phi(C)$ is finite, it follows that $\mathcal{A}(\phi(F))$ is also reductive.

$\Leftarrow$. Obvious. ∎

Lemma 1.5. *Let F be a product of two commuting subgroups F_1 and F_2. Assume that all the first cohomology groups of F_1 and F_2 are trivial. Then all the first cohomology groups of F are trivial.*

Proof. Let $\phi: F \to H$ be a homomorphism to an $\mathbb{R}$-algebraic group. Then $\mathcal{A}(\phi(F) = \mathcal{A}(\phi(F_1))\mathcal{A}(\phi(F_2))$. Since the groups $A(\phi(F_1))$ and $\mathcal{A}(\phi(F_2))$ are reductive and commute, it follows that $\mathcal{A}(\phi(F))$ is also reductive. ∎

Lemma 1.6. *Let Z be a central subgroup of F and let $F' = F/Z$. Assume that all the first cohomology groups of F' are trivial and that $H^1(F, \tau_1) = 0$. Then all the first cohomology groups of F are trivial.*

Proof. Let $\phi: F \to H$ be a homomorphism to an $\mathbb{R}$-algebraic group. We know that the group $\mathcal{A}(\phi(F))/\mathcal{A}(\phi(Z))$ is a finite index subgroup of a reductive $\mathbb{R}$-algebraic group and hence is reductive itself. It follows that the unipotent radical U of $\mathcal{A}(\phi(F))$ lies in $\mathcal{A}(\phi(Z))$. Hence $\mathcal{A}(\phi(F))$ is the direct product of U with a reductive subgroup. Assume that U is nontrivial. The projection of $\phi(F)$ into U is Zariski dense therein, and hence is an infinite finitely generated nilpotent group. This contradicts the equality $H^1(F, \tau_1) = 0$. ∎

§2. Reduction to the rank one case

Now we are ready to apply our observations to the study of the first cohomology group for lattices in semisimple Lie groups. From now on, L is a connected semisimple Lie group and $L = SK$ is the almost direct *decomposition into noncompact and compact parts* (i.e., K is compact and all simple components of S are noncompact).

Proposition 2.1. *Let Γ be a lattice in a connected semisimple Lie group $L = SK$, $p: L \to S/S \cap K$ be the projection, and $S/S \cap K = \prod_1^n S_i$ be the (almost direct) decomposition into irreducible components with respect to $p(\Gamma)$. Assume that all the first cohomology groups of $p(\Gamma) \cap S_i$ are trivial for all i. Then all the first cohomology groups of Γ are trivial.*

Proof. By Lemma 1.5, all the first cohomology groups of $\prod_1^n (p(\Gamma) \cap S_i)$ are trivial. But $\prod_1^n (p(\Gamma) \cap S_i)$ is a normal subgroup of finite index in $p(\Gamma)$. Hence by Lemma 1.3, all the first cohomology groups of $p(\Gamma)$ are trivial. On the other hand, $p(\Gamma) = \Gamma/\Gamma \cap K$, where $\Gamma \cap K$ is a finite normal subgroup of Γ. Now by Lemma 1.4, all the first cohomology groups of Γ are trivial. ∎

Now we need the following nontrivial result which can be derived from papers of Deligne [2] and Raghunathan [8] and Margulis Arithmeticity Theorem.

Theorem 2.2. (See [4, Assertion IX.6.18 (B)]). *Let S be a connected semisimple Lie group without compact factors, and let Γ be an irreducible lattice in S. Assume that $\mathrm{rk}_{\mathbb{R}} S \geq 2$. Then the group $\Gamma/[\Gamma, \Gamma]$ is finite and hence $H^1(\Gamma, \tau_1) = 0$.*

Corollary 2.3. *Let S be a connected semisimple Lie group without compact factors and let Γ be an irreducible lattice in S. Assume that $\mathrm{rk}_{\mathbb{R}} S \geq 2$. Then all the first cohomology groups of Γ are trivial.*

Proof. Let $Z(S)$ be the center of S. Then $Z = \Gamma \cap Z(S)$ is of finite index in $Z(S)$. Hence $\Gamma' = \Gamma/Z$ is an irreducible lattice in the group $S' = S/Z$ with finite center. Now the claim follows from Theorems 1 and 2.2 and Lemma 1.6. ∎

Corollary 2.4. *Let L be a connected semisimple Lie group with a lattice Γ. Assume that there is no epimorphism $\phi: L \to H$ onto a semisimple Lie group H such that $\mathrm{rk}_{\mathbb{R}} H = 1$ and $\phi(\Gamma)$ is a lattice in H. Then all the first cohomology groups of Γ are trivial.*

Proof. Let $p: L = SK \to S/S \cap K$ be the projection along the compact part, and let $S/S \cap K = \prod_1^n S_i$ be the (almost direct) decomposition into irreducible components with respect to $p(\Gamma)$. By assumption, $\mathrm{rk}_{\mathbb{R}} S_i \geq 2$ for all i. Now the claim follows from Proposition 2.1 and Corollary 2.3. ∎

§3. Rank one case

Here we prove the following result.

Theorem 3.1. *Let G be either* $\mathrm{Sp}(1, n)$, $n \geq 2$, *or* F_4^{-20}, *and let Γ be a lattice in G. Then $H^1(\Gamma, \rho) = 0$ for any representation ρ of Γ.*

It is well known that G as above is simply connected, $\mathbb{R}$-algebraic (in the natural linear presentation) and of $\mathbb{R}$-rank one. The center of $\mathrm{Sp}(1, n)$ is isomorphic to $\mathbb{Z}_2$, and that of F_4^{-20} is trivial. Besides, both G and Γ are Kazhdan groups. In particular, $H^1(\Gamma, \tau_1) = 0$ (see [10] or [4]).

To prove Theorem 3.1, we use Superrigidity and Arithmeticity Theorems of Corlette [1] and Gromov and Schoen [3] and combine them with results of Raghunathan [6, 7].

Theorem 3.2. (Superrigidity), [1] *Let G and Γ be as in* Theorem 3.1. *Assume that*

1) H is a connected semisimple Lie group,

2) H is center-free,

3) H has no connected compact normal subgroups,

and let $\rho\colon\Gamma\to H$ be a homomorphism such that $\rho(\Gamma)$ is Zariski dense in H. Then there exists an epimorphism $R\colon G\to H$ that extends ρ.

Theorem 3.3. (Arithmeticity), [3] *Let G and Γ be as in* Theorem 3.1. *Then there exists a $\mathbb{Q}$-defined semisimple subgroup $\mathbb{H}\subset \mathrm{SL}(k,\mathbb{C})$ and an epimorphism $\phi\colon\mathbb{H}_{\mathbb{R}}\to G$ with compact kernel such that $\phi(\mathbb{H}_{\mathbb{Z}})$ and Γ are commensurable.*

Now we formulate two results of Raghunathan which (modulo the Arithmeticity Theorem) essentially say that $H^1(\Gamma,\rho)=0$ whenever ρ can be extended to a representation of G.

Theorem 3.4. [6] *Let G be a connected Lie group which has no components locally isomorphic to $\mathrm{SO}(1,n)$ and $\mathrm{SU}(1,n)$, and let $R\colon G\to \mathrm{GL}(V)$ be a nontrivial irreducible representation. Let Γ be a uniform lattice in G. Then $H^1(\Gamma,R)=0$.*

Corollary 3.5. *Let $G=S\times K$, where S is either $\mathrm{Sp}(1,n)$, $n\geq 2$, or F_4^{-20}, and K is a compact semisimple Lie group. Then given a uniform lattice $\Gamma\subset G$ and a representation $R\colon G\to\mathrm{GL}(V)$ we have $H^1(\Gamma,R)=0$.*

Proof. Since Γ is a Kazhdan group, it follows that $H^1(\Gamma,\tau_1)=0$. Now the claim follows by decomposition of R into irreducible representations. ∎

Theorem 3.6. [7] *Let $\mathbb{G}\subset\mathrm{GL}(n,\mathbb{C})$ be a $\mathbb{Q}$-defined $\mathbb{Q}$-simple algebraic Lie group with $\mathrm{rk}_{\mathbb{Q}}\,\mathbb{G}=1$. Assume that $\mathbb{G}$ has no components locally isomorphic to $\mathrm{SL}(n,\mathbb{C})$ and $\mathbb{G}_{\mathbb{R}}$ has no components locally isomorphic to $\mathrm{SO}(1,n)$. Assume that $\Gamma\subset\mathbb{G}_{\mathbb{R}}$ is an arithmetic group (i.e., Γ is commensurable with $\mathbb{G}_{\mathbb{Z}}$). Let $\mathcal{R}\colon\mathbb{G}\to\mathrm{GL}(N,\mathbb{C})$ be a nontrivial irreducible rational representation. Then $H^1(\Gamma,\mathcal{R})=0$.*

Corollary 3.7. *Let G be either $\mathrm{Sp}(1,n)$, $n\geq 2$, or F_4^{-20}, let $\Gamma\subset G$ be a non-uniform lattice, and let $R\colon G\to\mathrm{GL}(V)$ be a representation. Then $H^1(\Gamma,R)=0$.*

Proof. Since Γ is non-uniform and G is simply connected, by Arithmeticity Theorem 3.3 there exists a $\mathbb{Q}$-defined group $\mathbb{H}\subset\mathrm{GL}(k,\mathbb{C})$ such that $\mathbb{H}_{\mathbb{R}}\simeq G$ and Γ is commensurable with $\mathbb{H}_{\mathbb{Z}}$ (compact factors of $\mathbb{H}_{\mathbb{R}}$ can be avoided as in the higher rank case, cf. [12, Corollary 6.1.10]). Clearly, R can be extended from $\mathbb{H}_{\mathbb{R}}$ to a rational representation $\mathcal{R}\colon\mathbb{H}\to\mathrm{GL}(V_{\mathbb{C}})$. Since Γ is a Kazhdan group, it follows that $H^1(\Gamma,\tau_1)=0$. Now decomposing $\mathcal{R}$ into irreducible representations we derive from Theorem 3.6 that $H^1(\Gamma,\mathcal{R})=0$. Note that $\mathcal{R}|_\Gamma$ is isomporphic to the direct sum of two copies of $R|_\Gamma$. Hence $H^1(\Gamma,R)=0$. ∎

Now we proceed to the proof of Theorem 3.1. It suffices to prove that the Zariski closure $F = \mathcal{A}(\phi(\Gamma)) \subset H$ is reductive for any homomorphism $\phi\colon \Gamma \to H$ to an $\mathbb{R}$-algebraic group. Let $F = (LA){\cdot}U$ be a Levi decomposition, where U is the unipotent radical, L is a semisimple subgroup (with finite number of connected components), and A is a finite abelian subgroup (see Remark after Lemma 1.2). Let L_0 stand for the identity component of L. Replacing Γ with its finite index subgroup (see Lemma 1.3), we can assume that A is trivial and $\phi(\Gamma) \subset L_0$.

If L_0 is trivial then $\phi(\Gamma) \subset U$. But Γ is a Kazhdan group and hence $\phi(\Gamma)$ is trivial and there is nothing to prove. So we may assume that L_0 is nontrivial. We have to prove that U is trivial.

Let $V = U/[U,U]$, and let $q\colon L \to \mathrm{GL}(V)$ be the natural representation. By $\alpha\colon F = L{\cdot}U \to q(L){\cdot}V$ we denote the composite epimorphism and by $\pi\colon q(L){\cdot}V \to q(L)$ the projection. Let $\rho' = \alpha \circ \phi$ and $\rho = \pi \circ \rho'$. Then ρ' is a covering homomorpshism for the representation ρ of Γ on the vector space V.

We know that $\rho'(\Gamma)$ is Zariski dense in $q(L){\cdot}V$, and $\rho(\Gamma)$ is Zariski dense in $q(L)$. Assume that $H^1(\Gamma,\rho) = 0$. Then $\rho(\Gamma)$ is conjugate to $\rho'(\Gamma)$ and hence V is trivial, so U is also trivial. Now it suffices to prove that $H^1(\Gamma,\rho) = 0$.

Let $q(L) = SK$ be the almost direct decomposition of $q(L)$ into noncompact and compact parts.

Assume first that S is trivial and hence $q(L) = K$. The group $\widehat{\Gamma} = \{(\gamma,\rho(\gamma)),\ \gamma \in \Gamma\}$ is a lattice in $G \times K$ that projects densely into K. Hence $\widehat{\Gamma}$ is a uniform lattice in $G \times K$. Let $\widehat{\rho}\colon \widehat{\Gamma} \to \mathrm{GL}(V)$ be given by $\widehat{\rho}(\gamma,\rho(\gamma)) = \rho(\gamma)$, $\gamma \in \Gamma$. Note that $\widehat{\rho}$ extends to a representation $G \times K \to \mathrm{GL}(V)$ given by the projection onto the second factor. By Corollary 3.5 it follows that $H^1(\Gamma,\rho) = H^1(\widehat{\Gamma},\widehat{\rho}) = 0$, and we are done.

Now we assume that S is nontrivial. Let $Z(S)$ be the center of S, $\sigma\colon q(L) = SK \to \mathrm{Ad}\,S \times K$ be the natural epimorphism, and $p_S\colon \mathrm{Ad}\,S \times K \to \mathrm{Ad}\,S$ be the projection. Then $p_S \circ \sigma \circ \rho(\Gamma)$ is Zariski dense in $\mathrm{Ad}\,S$ and by Superrigidity Theorem 3.2, it follows that $p_S \circ \sigma \circ \rho$ extends to an epimorphism $R\colon G \to \mathrm{Ad}\,S$. It follows that $R(\Gamma) = p_S \circ \sigma \circ \rho(\Gamma)$ is a lattice in $\mathrm{Ad}\,S$. Since G has finite center, the kernel of R is finite and hence the intersection $K \cap \sigma \circ \rho(\Gamma)$ is finite. It follows that $\sigma \circ \rho(\Gamma)$ is a lattice in $\mathrm{Ad}\,S \times K$ and hence $\rho(\Gamma)$ is a Zariski dense lattice in $q(L)$.

Let $i\colon q(L) \to \mathrm{GL}(V)$ be the inclusion. We know that ρ has finite kernel. Now it suffices to prove that $H^1(\rho(\Gamma), i) = 0$.

Assume that $\rho(\Gamma)$ is a uniform lattice in $q(L)$. Since the natural representation of $\rho(\Gamma)$ on V extends to the natural representation of $q(L)$ on V, we can apply Corollary 3.5 and conclude that $H^1(\rho(\Gamma), i) = 0$ (to apply the Corollary directly it suffices to lift the representation of $q(L) = SK$ to that of $G \times K$).

Assume now that $\rho(\Gamma)$ is a nonuniform lattice in $q(L)$. Then $q(L) = S$ and by Corollary 3.7, $H^1(\rho(\Gamma), i) = 0$ as well. ∎

Proof of Theorem 2. Any rank one simple Lie group is locally isomorphic to one of the following: $\mathrm{Sp}(1,n)$, F_4^{-20}, $\mathrm{SO}(1,n)$, $\mathrm{SU}(1,n)$. Now Theorem 2 can be derived from Theorems 1 and 3.1 in the same manner as Corollary 2.4. ∎

§4. Applications of the main result

Superrigidity. As before, given a semisimple Lie group L, we denote by $L = SK$ an almost direct decomposition of L into noncompact and compact parts. Clearly, a lattice $\Gamma \subset G$ is Zariski dense in G (in the Ad-representation) iff $G = \overline{S\Gamma}$ (here and in what follows the horizontal bar stands for the closure of a subset in a topological space).

We recall Margulis Superrigidity Theorem for irreducible lattices in higher rank semisimple Lie groups.

Theorem 4.1. [4] *Let L be a connected semisimple Lie group with $\mathrm{rk}_{\mathbb{R}} L \geq 2$. Let $\Gamma \subset L$ be an irreducible Zariski dense lattice, and let $\phi: \Gamma \to H$ be a homomorphism such that*

1) H is a connected semisimple Lie group,

2) H is center-free,

3) H has no connected compact normal subgroups.

Then ϕ uniquely extends to a continuous homomorphism $\widetilde{\phi}: L \to H$ whenever $\phi(\Gamma)$ is Zariski dense in H.

Remark . Assume that H is an $\mathbb{R}$-algebraic group and $\phi(\Gamma)$ is Zariski dense in H. Then H is either finite or semisimple because all the first cohomologies of Γ vanish. So condition 1) can be replaced by saying that H is an $\mathbb{R}$-algebraic group and $\phi(\Gamma) \subset H_0$. Conditions 2) and 3) cannot be omitted.

To generalize this Theorem, we use a terminology from [11]. We will say that a connected Lie group G is *algebraically simply connected* if G admits a faithful finite-dimensional representation, and no proper covering group of G admits such a representation. Note that $\mathrm{SL}(2,\mathbb{R})$ is algebraically simply connected but not (topologically) simply connected. Clearly, a semisimple Lie group L is algebraically simply connected iff it decomposes into direct product of simple algebraically simply connected normal subgroups.

We say that a homomorphism $\phi: \Gamma \to H$ *virtually extends* to a continuous homomorphism $\widetilde{\phi}: G \to H$ if there exists a finite index subgroup $\Gamma' \subset \Gamma$ such that $\widetilde{\phi}|_{\Gamma'} \equiv \phi|_{\Gamma'}$.

Finally, we say that a lattice Γ in a connected Lie group G is *superrigid* if every homomorphism $\phi: \Gamma \to H$, where H is an $\mathbb{R}$-algebraic group *containing no connected simple compact normal subgroups*, virtually extends to a continuous homomorphism $\widetilde{\phi}: G \to H$ whenever $\phi(\Gamma)$ is Zariski dense in H.

In these terms we can formulate the following Superrigidity Theorem.

Theorem 4.2. *Let L be an algebraically simply connected semisimple Lie group, and Γ be a lattice in L such that (L, Γ) satisfies $(*)$. Then Γ is superrigid in L.*

Remark. If Γ is a Zariski dense irreducible lattice in L and $\mathrm{rk}_{\mathbb{R}} L \geq 2$, the result is contained in [4, Theorem IX.5.12].

Proof. Since L is algebraically simply connected, it follows that $L = S \times K$, where S is the noncompact part, and K is the compact part. Let $p_S: L \to S$ be the projection along K. Then $p_S(\Gamma)$ is a lattice in S, and $\Gamma \cap K$ is finite normal subgroup in Γ.

Let $\phi: \Gamma \to H$ be a homomorphism to an $\mathbb{R}$-algebraic group as in the definition of superrigidity. Then it follows from Theorem 2 that H_0 is either trivial or semisimple group with finite center $Z(H)$. Replacing Γ with its finite index subgroup (if necessary), we can assume that $\phi(\Gamma) \subset H_0$. Then $\phi(\Gamma \cap K) \subset Z(H)$. Let $\mathrm{Ad}_H: H \to \mathrm{Ad}\, H = H/Z(H)$ be the adjoint representation and define $\phi' = \mathrm{Ad}_H \circ \phi$. Since $p_S(\Gamma) \simeq \Gamma/\Gamma \cap K$ and $\Gamma \cap K \subset \mathrm{Ker}(\phi')$, there exists a homomorphism $\phi'': p_S(\Gamma) \to \mathrm{Ad}\, H$ such that $\phi' = \phi'' \circ p_S$.

Let $S = \prod_i S_i$ be the decomposition of S into irreducible components with respect to $p_S(\Gamma)$, and $\Gamma_i = p_S(\Gamma) \cap S_i$. Again, replacing Γ by its finite index subgroup, we can assume that $\prod_i \Gamma_i = p_S(\Gamma)$. Notice that $\mathrm{Ad}\, H$ is a center-free semisimple Lie group with no connected compact normal subgroups. Applying Theorem 4.1 to the homomorphism $\phi'': \Gamma_i \to \mathrm{Ad}\, H$ if $\mathrm{rk}_{\mathbb{R}} S_i \geq 2$ or Theorem 3.2 if $\mathrm{rk}_{\mathbb{R}} S_i = 1$, we obtain a continuous homomorphism $\widetilde{\phi}'': S \to \mathrm{Ad}\, H$ that extends ϕ''.

Since S is algebraically simply connected, it follows that there exists a continuous homomorphism $\widetilde{\phi}: S \to H$ such that $\mathrm{Ad}_H \circ \widetilde{\phi} = \widetilde{\phi}''$. Now we extend $\widetilde{\phi}$ onto $L = S \times K$ defining it trivially on K. Then $\mathrm{Ad}_H \circ \widetilde{\phi}|_\Gamma \equiv \mathrm{Ad}_H \circ \phi$. Define a map $\sigma: \Gamma \to Z(H)$ by $\sigma(g) = \widetilde{\phi}^{-1}(\gamma)\phi(\gamma)$, $\gamma \in \Gamma$. Then σ is a homomorphism and hence $\widetilde{\phi}$ agrees with ϕ on finite index subgroup $\mathrm{Ker}\, \sigma \subset \Gamma$. ∎

Now we proceed to general Lie groups. Given a connected Lie group G, we denote by R its radical and by L its Levi subgroup. Then $G = LR$ is a Levi decomposition of G. In general, L and R may intersect in a discrete central subgroup of G.

Let K be the compact part of Levi subgroup $L \subset G$. Then for any lattice $\Gamma \subset G$ the product $KR\Gamma$ is closed (see, e.g., [10]) and hence $\Gamma \cap KR$ is a uniform lattice in KR. Moreover, the product $R\Gamma$ is closed whenever $G = \overline{SR\Gamma}$. The last condition is obviously verified whenever $\mathrm{Ad}(\Gamma)$ is Zariski dense in $\mathrm{Ad}\, G$.

Remark. Let $\alpha: G \to G/KR$ be the factor map. Then $\alpha(\Gamma)$ is a lattice in semisimple Lie group G/KR. Let $\phi: G \to H$ be an epimorphism, where H is locally isomorphic to $\mathrm{SO}(1,n)$ or $\mathrm{SU}(1,n)$, such that $\phi(\Gamma)$ is a lattice. Factorizing H over its center if necessary, we can assume that H is center-free. Clearly, $\phi(\Gamma \cap KR)$ is a normal amenable subgroup of the lattice $\phi(\Gamma)$. Since H is center-free, it follows that $\phi(\Gamma \cap KR)$ is trivial. We derive that (G, Γ) satisfies $(*)$ iff $(G/KR, \alpha(\Gamma))$ does so.

One knows that G is algebraically simply connected iff the radical R of G is (topologically) simply connected and $G = L{\cdot}R$, where L is an algebraically simply connected Levi subgroup. Recall that Witte [11] proved superrigidity of Zariski dense lattices in solvable Lie groups. Now we can derive the following general result.

Theorem 4.3. *Let Γ be a lattice in an algebraically simply connected Lie group G. Assume that*

1) $\operatorname{Ad}(\Gamma)$ *is Zariski dense in* $\operatorname{Ad} G$,

2) (G,Γ) *satisfies* $(*)$.

Then Γ *is superrigid in* G.

Proof. From 1) it follows that $\Gamma \cap R$ is a lattice in R. ¿From 2) and Theorem 4.2 it follows that the image of Γ in G/R is superrigid. This together with condition 1) enables one to conclude that Γ is superrigid in G (see [11, Corollary 9.10]). ∎

Levi–Mostow Splitting. Now we apply Theorem 2 to obtain a general result on Levi–Mostow splitting of lattices. Recall a result of Mostow [5] presented as in [11].

Theorem 4.4. Cf [11]. *Let* G *be a connected Lie group with a lattice* Γ. *Assume that*

1) G *is algebraically simply connected Lie group,*
2) G *has no connected compact semisimple normal subgroups,*
3) $\operatorname{Ad}(\Gamma)$ *is Zariski dense in* $\operatorname{Ad} G$,
4) *the image of* Γ *in* G/R *is a superrigid lattice.*

Then there exists a Levi subgroup $L \subset G$ *such that* $(\Gamma \cap L)(\Gamma \cap R)$ *has finite index in* Γ.

We will prove more general result.

Theorem 4.5. *Let* G *be a connected Lie group with a lattice* Γ. *Assume that* (G,Γ) *satisfies* $(*)$. *Then there exists a Levi subgroup* $L = SK \subset G$ *such that* $(\Gamma\cap L)(\Gamma\cap KR)$ *has finite index in* Γ. *Moreover, if* $G = \overline{SR\Gamma}$ *then* $(\Gamma\cap L)(\Gamma\cap R)$ *has finite index in* Γ.

Proof. It suffices to find a Levi subgroup L that intersects Γ in a lattice. Define $G' = (\overline{SR\Gamma})_0$. Then G' is a connected Lie group such that $G' = L'R'$, where $L' = SK'$ is a Levi subgroup, $R' = TR$ is the radical, $K' \subset K$ is a compact semisimple subgroup, and $T \subset K$ is a torus. Clearly, $\Gamma' = \Gamma \cap G'$ is a lattice in G' such that (G',Γ') satisfies $(*)$ and $G' = \overline{SR'\Gamma'}$.

Therefore, we can assume that $G = \overline{SR\Gamma}$. Moreover, passing to the universal cover of G if necessary, we can assume that G is simply connected. Then $G = L{\cdot}R$ and $L = S \times K$. Let $C \subset K$ be the maximal connected compact semisimple normal subgroup of G. Then $G = G' \times C$, where $G' = L'{\cdot}R$ and $L' \subset L$. Let $p\colon G \to G'$ be the projection along C, and $\Gamma' = p(\Gamma)$. Then (G',Γ') satisfies $(*)$ and hence with no loss of generality we can assume that G contains no connected compact semisimple normal subgroups.

From the condition $G = \overline{SR\Gamma}$ it follows that $\Gamma \cap R$ is a lattice in R. It is fairly well known (see, e.g., [11, Proposition 8.2]) that one can "straighten" the radical R to make $\operatorname{Ad}(\Gamma \cap R)$ Zariski dense in $\operatorname{Ad}(R)$. More precisely, there exists a finite index subgroup $\Gamma' \subset \Gamma$ and a simply connected Lie group $G' = L{\cdot}R'$ such that Γ' is a lattice in G' and $\operatorname{Ad}(\Gamma' \cap R')$ is Zariski dense in $\operatorname{Ad}(R')$ (moreover, the nilradical of R lies in the nilradical of R').

Hence with no loss of generality we can assume that $\operatorname{Ad}(\Gamma\cap R)$ is Zariski dense in $\operatorname{Ad}(R)$. Since $G = \overline{SR\Gamma}$, it follows that $\operatorname{Ad}(\Gamma)$ is Zariski dense in $\operatorname{Ad} G$.

Hence the product of Γ with the center $Z(G)$ is closed. Let L' be the linearizer of L, i.e. the maximal algebraically simply connected factor of L. Then $L' = L/Z$, where $Z \subset L \cap Z(G)$.

In general, the product $Z\Gamma$ need not be closed. Clearly, $M = (\overline{Z\Gamma})_0$ is a connected central subgroup of G and hence $M \subset Z(G) \cap R$. Let $G' = G/ZM$, and $\alpha: G \to G' = L' \cdot R'$ be the factor map, where $R' = R/M$. Then $\Gamma' = \alpha(\Gamma)$ is a Zariski dense lattice in G' such that (G', Γ') satisfies $(*)$. Moreover, G' is algebraically simply connected, and it follows from Theorem 4.2 that G' and Γ' satisfy all the conditions 1)–4) of Theorem 4.4. Hence there exists a choice of L' such that $(\Gamma' \cap L')(\Gamma' \cap R')$ has finite index in Γ'.

It follows that $\Gamma \cap LM$ is a lattice in LM. Let $p: L \times M \to L$ be the projection. Then $p(\Gamma \cap LM)$ is a superrigid lattice in L. Hence $[\Gamma \cap LM, \Gamma \cap LM] = [p(\Gamma \cap LM), p(\Gamma \cap LM)]$ has finite index in $p(\Gamma \cap LM)$. It follows that $\Gamma \cap L$ is a lattice in L and we are done. ∎

Remark. One can bypass few steps in the proof (in particular, "straightening" of R) using a result saying that for any lattice Γ in a connected Lie group G, the intersection $\Gamma \cap LN$ is a lattice in LN, $N \subset R$ being the nilradical (cf. [9]). So one can concentrate on the Lie group LN.

Necessity of condition $(*)$ for vanishing of the first cohomologies. It is well known that for any $n \geq 2$ there exists a lattice Γ in $\mathrm{SO}(1,n)$ with its factor-group $\Gamma/[\Gamma,\Gamma]$ infinite. Hence the group $H^1(\Gamma, \tau_1) = \mathrm{Hom}(\Gamma, \mathbb{R})$ is nontrivial. The same is valid for $\mathrm{SU}(1,n)$.

Moreover, there is a conjecture (called the *Millson problem*) saying that given *any* lattice Γ in $\mathrm{SO}(1,n)$ or $\mathrm{SU}(1,n)$, there exists a finite index normal subgroup $\Gamma' \subset \Gamma$ such that the group $\Gamma'/[\Gamma',\Gamma']$ is infinite (see [10] for more details). It implies that the factor group $F = \Gamma/[\Gamma',\Gamma']$ is an infinite (finitely generated) virtually abelian group, i.e. F contains a normal subgroup $A \simeq \mathbb{Z}^k$ for some $k \geq 1$, of finite index. It is easy to construct an embedding to an $\mathbb{R}$-algebraic group $\phi: F \to H = C \cdot U$, where $U \simeq \mathbb{R}^k$ is the unipotent radical and $C \simeq F/A$, such that $\phi(F)$ is Zariski dense in H. According to Lemma 1.2, this means that one can construct a representation ρ of Γ with nontrivial $H^1(\Gamma, \rho)$.

We conclude the following.

Proposition 4.6. *Let Γ be a lattice in a connected semisimple Lie group L. If the Millson problem has positive solution, then the following conditions are equivalent:*

1) *$H^1(\Gamma, \rho) = 0$ for any representation ρ of Γ,*
2) *Γ admits no epimorphism onto an infinite virtually abelian group,*
3) *(L, Γ) satisfies $(*)$.*

Using Theorem 4.5 one can easily derive the following.

Proposition 4.7. *Let Γ be a lattice in a connected Lie group G. If the Millson problem has positive solution, then the following conditions are equivalent:*

1) *$H^1(\Gamma, \rho) = 0$ for any representation ρ of Γ,*
2) *Γ admits no epimorphism onto an infinite virtually abelian group,*

3) (G,Γ) *satisfies* $(*)$, *Levi subgroup* $L \subset G$ *is normal, and the factor-group* G/L *is compact.*

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