Journal of Lie Theory
Volume **12** (2002) 461–481

Moore–Penrose Inverse, Parabolic Subgroups, and Jordan Pairs

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Communicated by E. B. Vinberg

Abstract. A Moore–Penrose inverse of an arbitrary complex matrix A is defined as a unique matrix A^+ such that $AA^+A = A$, $A^+AA^+ = A^+$, and AA^+, A^+A are hermitian matrices. We show that this definition has a natural generalization in the context of shortly graded simple Lie algebras corresponding to parabolic subgroups with *aura* (abelian unipotent radical) in simple complex Lie groups, or equivalently in the context of simple complex Jordan pairs. We give further generalizations and applications.

Introduction

The nice notion of a generalized inverse of an arbitrary matrix (possibly singular or even non-square) has been discovered independently by Moore [Mo] and Penrose [Pe]. The following definition belongs to Penrose (Moore's definition is different but equivalent):

Definition. A matrix A^+ is called *a MP-inverse* of a matrix A if

$$AA^+A = A, \quad A^+AA^+ = A^+,$$

and AA^+, A^+A are Hermite matrices.

It is quite surprising but a MP-inverse always exists and is unique. Since the definition is symmetric with respect to A and A^+ it follows that $(A^+)^+ = A$. If A is a non-singular square matrix then A^+ coincides with an ordinary inverse matrix A^{-1}. The theory of MP-inverses and their numerous modifications becomes now a separate subfield of Linear Algebra [CM] with various applications. The aim of this paper is to demonstrate that this notion quite naturally arises in the theory of shortly graded simple Lie algebras. To explain this connection let us first give another definition of a MP-inverse.

Equivalent definition of a MP-inverse. Suppose that $A \in \mathrm{Mat}_{n,m}(\mathbb{C})$. Then a matrix $A^+ \in \mathrm{Mat}_{m,n}(\mathbb{C})$ is called *a MP-inverse* of A if there exist

ISSN 0949–5932 / $2.50

Hermite matrices $B_1 \in \mathrm{Mat}_{n,n}(\mathbb{C})$ and $B_2 \in \mathrm{Mat}_{m,m}(\mathbb{C})$ such that the following matrices form an $\mathfrak{sl}_2$-triple in $\mathfrak{sl}_{n+m}(\mathbb{C})$:

$$E = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ A^+ & 0 \end{pmatrix}.$$

Remark. By an $\mathfrak{sl}_2$-triple $\langle e, h, f\rangle$ in a Lie algebra $\mathfrak{g}$ we mean a collection of (possibly zero) vectors such that

$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f.$$

In other words, an $\mathfrak{sl}_2$-triple is a homomorphic image of canonical generators of $\mathfrak{sl}_2$ with respect to some homomorphism of Lie algebras $\mathfrak{sl}_2 \to \mathfrak{g}$.

This definition admits an immediate generalization. In the sequel we shall use various facts about shortly graded simple Lie algebras without specific references to original papers, the reader may consult, for example, papers [RRS], [Pa], or [MRS] for explanations and further references. All necessary facts about complex and real Lie groups, Lie algebras, and algebraic groups can be found in [VO].

Suppose that $\mathfrak{g}$ is a simple complex Lie algebra, G is a corresponding simple simply-connected Lie group. Suppose further that P is a parabolic subgroup of G with abelian unipotent radical (with *aura*). Then $\mathfrak{g}$ admits a short grading

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

with only three nonzero parts. Here $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a Lie algebra of P and $\exp \mathfrak{g}_1$ is the abelian unipotent radical of P. Let $\mathfrak{k}_0$ be a compact real form of $\mathfrak{g}_0$.

Remark. In this paper we shall permanently consider compact real forms of reductive subalgebras of simple Lie algebras. These subalgebras will always be Lie algebras of algebraic reductive subgroups of a corresponding simple complex algebraic group. Their compact real forms will always be understood as Lie algebras of compact real forms of corresponding algebraic groups. For example, a Lie algebra of an algebraic torus has a unique compact real form.

Suppose now that $e \in \mathfrak{g}_1$. It is well-known that there exists a *homogeneous* $\mathfrak{sl}_2$-triple $\langle e, h, f\rangle$ such that $h \in \mathfrak{g}_0$ and $f \in \mathfrak{g}_{-1}$.

Definition. An element $f \in \mathfrak{g}_{-1}$ is called *a MP-inverse* of $e \in \mathfrak{g}_1$ if there exists a homogeneous $\mathfrak{sl}_2$-triple $\langle e, h, f\rangle$ with $h \in i\mathfrak{k}_0$.

MP-inverses of elements $f \in \mathfrak{g}_{-1}$ are defined in the same way. It is clear that if f is a MP-inverse of e then e is a MP-inverse of f.

Example. Suppose that $G = \mathrm{SL}_{n+m}$ and $P \subset G$ is a maximal parabolic subgroup of block triangular matrices of the form

$$\begin{pmatrix} B_1 & A \\ 0 & B_2 \end{pmatrix}, \quad \text{where} \quad B_1 \in \mathrm{Mat}_{n,n},\ A \in \mathrm{Mat}_{n,m},\ B_2 \in \mathrm{Mat}_{m,m}.$$

The graded components of the correspondent grading consist of matrices of the following form:

$$\mathfrak{g}_{-1} = \begin{pmatrix} 0 & 0 \\ A' & 0 \end{pmatrix}, \quad \mathfrak{g}_0 = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}, \quad \mathfrak{g}_1 = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix},$$

where $A' \in \mathrm{Mat}_{m,n}$, $B_1 \in \mathrm{Mat}_{n,n}$, $B_2 \in \mathrm{Mat}_{m,m}$, and $A \in \mathrm{Mat}_{n,m}$. One can take $\mathfrak{k}_0$ to be a real Lie algebra of block diagonal skew-Hermite matrices with zero trace. Then $i\mathfrak{k}_0$ is a vector space of block diagonal Hermite matrices with zero trace. Therefore in this case we return to the previous definition of a Moore–Penrose inverse.

Our first result is the following

Theorem 1. *For any $e \in \mathfrak{g}_1$ there exists a unique MP-inverse $f \in \mathfrak{g}_{-1}$.*

It obviously follows that for any non-zero $f \in \mathfrak{g}_{-1}$ there exists a unique MP-inverse $e \in \mathfrak{g}_1$. So taking a MP-inverse is a well-defined involutive operation. In general, it is not equivariant with respect to a Levi subgroup $L \subset P$ with Lie algebra $\mathfrak{g}_0$, but only with respect to its maximal compact subgroup $K_0 \subset L$.

Theorem 1 will be proved in §1 by a general argument, without using case-by-case considerations. But the classification of parabolic subgroups with aura in simple groups is, of course, well-known. We have tried to give an intrinsic description of the Moore-Penrose inverse in all arising cases. The calculation of Moore–Penrose inverses arising from short gradings of classical simple Lie algebras is quite straightforward, so we shall give here only the summary of these calculations and avoid proofs.

Linear maps. This is, of course, the classical Moore–Penrose inverse. Let us recall its intrinsic description. Suppose that $\mathbb{C}^n$ and $\mathbb{C}^m$ are vector spaces equipped with standard Hermite scalar products. For any linear map $F : \mathbb{C}^n \to \mathbb{C}^m$ its Moore-Penrose inverse is a linear map $F^+ : \mathbb{C}^m \to \mathbb{C}^n$ defined as follows. Let $\mathrm{Ker} F \subset \mathbb{C}^n$ and $\mathrm{Im} F \subset \mathbb{C}^m$ be the kernel and the image of F. Let $\mathrm{Ker}^\perp F \subset \mathbb{C}^n$ and $\mathrm{Im}^\perp F \subset \mathbb{C}^m$ be their orthogonal complements with respect to the Hermite scalar products. Then F defines via restriction a bijective linear map $\tilde{F} : \mathrm{Ker}^\perp F \to \mathrm{Im} F$. Then $F^+ : \mathbb{C}^m \to \mathbb{C}^n$ is a unique linear map such that $F^+|_{\mathrm{Im}^\perp F} = 0$ and $F^+|_{\mathrm{Im} F} = \tilde{F}^{-1}$. This MP-inverse corresponds to short gradings of $\mathfrak{sl}_{n+m}$.

Symmetric and skew-symmetric bilinear forms. Suppose that $V = \mathbb{C}^n$ is a vector space equipped with a standard Hermite scalar product. For any symmetric (resp. skew-symmetric) bilinear form ω on V its Moore-Penrose inverse is a symmetric (resp. skew-symmetric) bilinear form ω^+ on V^* defined as follows. Let $\mathrm{Ker}\,\omega \subset V$ be the kernel of ω. Then ω induces a non-degenerate bilinear form $\tilde{\omega}$ on $V/\mathrm{Ker}\,\omega$. Let $\mathrm{Ann}(\mathrm{Ker}\,\omega) \subset V^*$ be an annihilator of $\mathrm{Ker}\,\omega$. Then $\mathrm{Ann}(\mathrm{Ker}\,\omega)$ is canonically isomorphic to the dual of $V/\mathrm{Ker}\,\omega$. Therefore the form $\tilde{\omega}^{-1}$ on $\mathrm{Ann}(\mathrm{Ker}\,\omega)$ is well-defined. The form ω^+ is defined as a unique form such that its restriction on on $\mathrm{Ann}(\mathrm{Ker}\,\omega)$ coincides with $\tilde{\omega}^{-1}$ and its kernel is $\mathrm{Ann}(\mathrm{Ker}\,\omega)^\perp$, the orthogonal complement with respect to a standard Hermite scalar product on V^*. This MP-inverse corresponds to the short grading of $\mathfrak{sp}_{2n}$

(resp. $\mathfrak{so}_{2n}$) which can be described as follows. $\mathfrak{g}_0 = \mathfrak{gl}(V) = V \otimes V^*$, $\mathfrak{g}_1 = S^2 V$ (resp. $\mathfrak{g}_1 = \Lambda^2 V$), $\mathfrak{g}_1 = S^2 V^*$ (resp. $\mathfrak{g}_1 = \Lambda^2 V^*$). Commutators coincide (up to a sign) with obvious tensor contractions.

Vectors in a vector space with scalar product. Let $V = \mathbb{C}^n$ be a vector space with standard bilinear scalar product $(\cdot,\cdot)$. For any vector $v \in V$ its Moore-Penrose inverse v^+ is again a vector in V defined as follows:

$$v^+ = \begin{cases} \dfrac{2v}{(v,v)}, & \text{if } (v,v) \neq 0 \\ \dfrac{\overline{v}}{(\overline{v},v)}, & \text{if } (v,v) = 0,\ v \neq 0 \\ 0, & \text{if } v = 0. \end{cases}$$

Here $\overline{v}$ denotes the complex conjugate vector. This MP-inverse corresponds to the short grading of $\mathfrak{g} = \mathfrak{so}_{n+2}$ defined as follows: $\mathfrak{g}_{-1} = \mathfrak{g}_1 = V$, $\mathfrak{g}_0 = \mathfrak{so}(V) \oplus \mathbb{C}$. The commutators have the following form. For $\xi = (A,\lambda) \in \mathfrak{g}_0$, $\eta = v \in \mathfrak{g}_1$ we have $[\xi,\eta] = Av + \lambda v$. For $\xi = (A,\lambda) \in \mathfrak{g}_0$, $\zeta = u \in \mathfrak{g}_{-1}$ we have $[\xi,\zeta] = Au - \lambda u$. For $\eta = v \in \mathfrak{g}_1$, $\zeta = u \in \mathfrak{g}_{-1}$ we have $[\eta,\zeta] = \big((u,\cdot)v - (v,\cdot)u, (u,v)\big)$.

The short gradings of exceptional Lie algebras E_6 and E_7 deserve more detailed considerations. This is done in §3. It is well-known that the theory of shortly graded simple Lie algebras is equivalent to the theory of finite-dimensional simple Jordan pairs. It turns out that the Moore–Penrose inversion has a very simple interpretation in this alternative language. We describe this connection also in §3.

In §2 we consider shortly graded real simple Lie algebras. It turns out that the analogue of Theorem 1 is also true in this case.

It is quite natural to ask whether it is possible to extend the notion of the Moore–Penrose inverse from parabolic subgroups with aura to arbitrary parabolic subgroups. It is also interesting to consider the "non-graded" situation. Let us start with it. Suppose G is a simple connected simply-connected Lie group with Lie algebra $\mathfrak{g}$. We fix a compact real form $\mathfrak{k} \subset \mathfrak{g}$.

Definition. A nilpotent orbit $\mathcal{O} \subset \mathfrak{g}$ is called *a Moore–Penrose orbit* if for any $e \in \mathcal{O}$ there exists an $\mathfrak{sl}_2$-triple $\langle e,h,f\rangle$ such that $h \in i\mathfrak{k}$.

It turns out that it is quite easy to find all Moore-Penrose orbits. Recall that the height $\mathrm{ht}(\mathcal{O})$ of a nilpotent orbit $\mathcal{O} = \mathrm{Ad}(G)e$ is equal to the maximal integer k such that $\mathrm{ad}(e)^k \neq 0$. Clearly $\mathrm{ht}(\mathcal{O}) \geq 2$.

Theorem 2. *$\mathcal{O}$ is a Moore–Penrose orbit if and only if $ht(\mathcal{O}) = 2$. In this case for any $e \in \mathcal{O}$ there exists a unique $\mathfrak{sl}_2$-triple $\langle e,h,f\rangle$ such that $h \in i\mathfrak{k}$.*

This theorem will be proved in §1. It is worthy to mention here the following result of Panyushev [Pa1]: $\mathrm{ht}(\mathcal{O}) \leq 3$ if and only if $\mathcal{O}$ is a spherical G-variety (that is, a Borel subgroup $B \subset G$ has an open orbit in $\mathcal{O}$). Therefore, all Moore–Penrose orbits are spherical. If $G = \mathrm{SL}_n$ or $G = \mathrm{Sp}_n$ then the converse is also true.

Now let us turn to the graded situation. Suppose that $\mathfrak{g}$ is a $\mathbb{Z}$-graded simple Lie algebra, $\mathfrak{g} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_k$. Let $P \subset G$ be a parabolic subgroup with the

Lie algebra $\mathfrak{p} = \underset{k\geq 0}{\oplus}\, \mathfrak{g}_k$. Let $L \subset P$ be a Levi subgroup with Lie algebra $\mathfrak{g}_0$. We choose a compact real form $\mathfrak{k}_0$ of $\mathfrak{g}_0$. Suppose now that $e \in \mathfrak{g}_k$. It is well-known that there exists a homogeneous $\mathfrak{sl}_2$-triple $\langle e, h, f\rangle$ with $h \in \mathfrak{g}_0$ and $f \in \mathfrak{g}_{-k}$.

Definition. Take any $k > 0$ and any L-orbit $\mathcal{O} \subset \mathfrak{g}_k$. Then $\mathcal{O}$ is called *a Moore-Penrose orbit* if for any $e \in \mathcal{O}$ there exists a homogeneous $\mathfrak{sl}_2$-triple $\langle e, h, f\rangle$ such that $h \in i\mathfrak{k}_0$. In this case f is called a MP-inverse of e. A grading is called *a Moore–Penrose grading in degree* $k > 0$ if all L-orbits in $\mathfrak{g}_k$ are Moore-Penrose. A grading is called *a Moore–Penrose grading* if it is a Moore–Penrose grading in any positive degree. A parabolic subgroup $P \subset G$ is called *a Moore–Penrose parabolic subgroup* if there exists a Moore–Penrose grading $\mathfrak{g} = \underset{k\in\mathbb{Z}}{\oplus}\, \mathfrak{g}_k$ such that $\mathfrak{p} = \underset{k\geq 0}{\oplus}\, \mathfrak{g}_k$ is the Lie algebra of P.

One should be careful comparing graded and non-graded situation: if $\mathcal{O} \subset \mathfrak{g}_k$ is a Moore–Penrose L-orbit then $\mathrm{Ad}(G)\mathcal{O} \subset \mathfrak{g}$ is not necessarily a Moore–Penrose G-orbit. Let us give a criterion for an L-orbit to be Moore–Penrose. Suppose $\mathcal{O} = \mathrm{Ad}(L)e \subset \mathfrak{g}_k$. Take any homogeneous $\mathfrak{sl}_2$-triple $\langle e, h, f\rangle$. Then h defines a grading $\mathfrak{g} = \underset{n\in\mathbb{Z}}{\oplus}\, \mathfrak{g}^n$, such that $\mathrm{ad}(h)|_{\mathfrak{g}^n} = n\cdot\mathrm{Id}$. Since $h \in \mathfrak{g}_0$, in fact we get a bigrading $\mathfrak{g} = \underset{n,k\in\mathbb{Z}}{\oplus}\, \mathfrak{g}_k^n$.

Theorem 3. *$\mathcal{O}$ is a Moore–Penrose orbit if and only if $ad(e)\mathfrak{g}_0^n = 0$ for any $n > 0$. In this case for any $e' \in \mathcal{O}$ there exists a unique homogeneous $\mathfrak{sl}_2$-triple $\langle e', h', f'\rangle$ such that $h' \in i\mathfrak{k}_0$.*

This Theorem will be proved in §1. It gives a characterization of Moore–Penrose orbits independent on the choice of a compact form and also provides an algorithm for checking the Moore–Penrose property. For example, for a homogeneous $\mathfrak{sl}_2$-triple $\langle e, h, f\rangle$, $e \in \mathfrak{g}_k$, the orbit Le is Moore–Penrose iff $\mathrm{ad}(e)\mathfrak{g}_0^n = 0$ for any $n > 0$ (Theorem 3) iff $\mathfrak{g}_k^n = 0$ for any $n > 2$ ($\mathfrak{sl}_2$-theory) iff $\mathfrak{g}_{-k}^{-n} = 0$ for any $n > 2$ (non-degeneracy of the Killing form) iff $\mathrm{ad}(f)\mathfrak{g}_0^{-n} = 0$ for any $n > 0$ (again $\mathfrak{sl}_2$-theory) iff the orbit Lf is Moore–Penrose (again Theorem 3). In particular, a grading is Moore–Penrose in degree k iff it is Moore–Penrose in degree $-k$.

It is easy to see that in the graded situation a Moore–Penrose orbit is not necessarily spherical. However, some interesting orbits are both spherical and Moore-Penrose. Let us give several examples.

Example 1. If P is a parabolic subgroup with aura then all L-orbits in $\mathfrak{g}_1$ are Moore–Penrose by Theorem 1. It is well-known that all of them are also spherical. More generally, take any grading of $\mathfrak{g}$ and suppose that d is equal to the maximal k such that $\mathfrak{g}_k \neq 0$ (the *height* of grading). Then all L-orbits in $\mathfrak{g}_k$ are both spherical and Moore-Penrose for $k > d/2$. This fact easily follows from the previous remark. (Consider the short-graded Lie algebra $\mathfrak{g}_{-k} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_k$. Of course it is not necessarily simple but this is not essential.)

Example 2. Suppose that G is a simple group of type G_2. We fix a root decomposition. There are two simple roots α_1 and α_2 such that α_1 is short and α_2 is long. There are 3 proper parabolic subgroups: Borel subgroup B and two maximal parabolic subgroups P_1 and P_2 such that a root vector of α_i belongs to a Levi subgroup of P_i. Then the following is an easy application of Theorem 3. B is a Moore–Penrose parabolic subgroup (actually Borel subgroups in all simple groups are Moore–Penrose parabolic subgroups with respect to any grading). P_1 is not Moore–Penrose, but it is a Moore–Penrose parabolic subgroup in degree 2 (with respect to the natural grading of height 2). P_2 is a Moore–Penrose parabolic subgroup.

Example 3. Suppose $G = \mathrm{SL}_n$. We fix positive integers $d_1, \ldots, d_k$ such that $n = d_1 + \ldots + d_k$. Any $(n \times n)$-matrix A has the block decomposition $A = (A_{ij})_{i,j=1\ldots k}$, where A_{ij} is a $(d_i \times d_j)$-matrix. We consider the parabolic subgroup $P(d_1, \ldots, d_k) \subset \mathrm{SL}_n$ that consists of all upper-triangular block matrices. We take the standard grading of $\mathfrak{g}$ such that $A \in \mathfrak{g}_p$ iff $A_{ij} = 0$ for $j - i \neq p$. Then $\mathfrak{g}_1$ is identified with the linear space of all tuples of linear maps $\{f_1, \ldots, f_k\}$,

$$\mathbb{C}^{d_1} \xleftarrow{f_1} \mathbb{C}^{d_2} \xleftarrow{f_2} \ldots \xleftarrow{f_{k-1}} \mathbb{C}^{d_k},$$

$\mathfrak{g}_{-1}$ is identified with the linear space of all tuples of linear maps $\{g_1, \ldots, g_k\}$,

$$\mathbb{C}^{d_1} \xrightarrow{g_1} \mathbb{C}^{d_2} \xrightarrow{g_2} \ldots \xrightarrow{g_{k-1}} \mathbb{C}^{d_k},$$

and Levi subgroup $L(d_1, \ldots, d_k)$ is just a group of all k-tuples

$$(A_1, \ldots, A_k) \in \mathrm{GL}_{d_1} \times \ldots \times \mathrm{GL}_{d_k} \quad \text{such that} \quad \det(A_1) \cdot \ldots \cdot \det(A_k) = 1,$$

acting on these spaces of linear maps in an obvious way. The most important among L-orbits are varieties of complexes. To define them, let us fix in addition non-negative integers $m_1, \ldots, m_{k-1}$ such that $m_{i-1} + m_i \leq d_i$ (we set $m_0 = m_k = 0$), and consider the subvariety of all tuples $\{f_1 \ldots, f_{k-1}\}$ as above such that $\mathrm{rk} f_i = m_i$ and $f_{i-1} \circ f_i = 0$ for any i. These tuples form a single L-orbit $\mathcal{O}$ called a variety of complexes. It is well-known that $\mathcal{O}$ is spherical. For any tuple $\{f_1, \ldots, f_{k-1}\} \in \mathcal{O}$ consider the tuple $\{f_1^+, \ldots, f_{k-1}^+\} \in \mathfrak{g}_{-1}$, where f_i^+ is a classical "matrix" Moore–Penrose inverse of f_i. The reader may check that this new tuple is again a complex, moreover, this complex is a Moore–Penrose inverse (in our latest meaning of this word) of an original complex. In particular, orbits of complexes are Moore–Penrose orbits.

From the first glance only few parabolic subgroups are Moore–Penrose. But this is scarcely true. For example, we have the following Theorem:

Theorem 4. *Any parabolic subgroup in SL_n is Moore–Penrose.*

This Theorem will be proved in §4. We shall also describe an algorithm there which shows that in order to find all Moore–Penrose parabolic subgroups in some simple group G it is sufficient to determine all Moore–Penrose maximal parabolic subgroups in simple components of Levi subgroups of G. In particular in order to find all Moore–Penrose parabolic subgroups in classical simple groups it suffices to do this job only for maximal parabolic subgroups. We shall do this also in §4.

To explain our interest in Moore–Penrose parabolic subgroups let us reproduce a conjecture from [Te]. Suppose once again that G is a simple connected simply-connected Lie group, P is its parabolic subgroup, $\mathfrak{p} \subset \mathfrak{g}$ are their Lie algebras. We take any irreducible G-module V. There exists a unique maximal proper P-submodule M_V of V. We have the inclusion $i : M_V \to V$, the projection $\pi : V \to V/M_V$ and the map $R_V : \mathfrak{g} \to \mathrm{End}(V)$ defining the representation. Therefore we have a linear map $\tilde{R}_V : \mathfrak{g} \to \mathrm{Hom}(M_V, V/M_V)$, namely $\tilde{R}_V(x) = \pi \circ R_V(x) \circ i$. Clearly $\mathfrak{p} \subset \mathrm{Ker}\tilde{R}_V$. Therefore, we finally have a linear map $\Psi_V : \mathfrak{g}/\mathfrak{p} \to \mathrm{Hom}(M_V, V/M_V)$.

Conjecture. *There exists an algebraic stratification* $\mathfrak{g}/\mathfrak{p} = \bigsqcup\limits_{i=1}^{n} X_i$ *such that for any* V *the function* $\mathrm{rk}\,\Psi_V(\cdot)$ *is constant along each* X_i.

These stratifications were used in [Te] in order to solve some geometric problems similar to the classical problem of determining the maximal dimension of a projective subspace contained in a generic hypersurface of a given degree in a projective space.

It is clear that all functions $\mathrm{rk}\,\Psi_V(\cdot)$ are P-invariant. Therefore if P has finitely many orbits in $\mathfrak{g}/\mathfrak{p}$ then the conjecture is true. By Pyasetsky theorem [P] this holds if and only if P has finitely many orbits in the dual module $(\mathfrak{g}/\mathfrak{p})^*$, or, equivalently, in the unipotent radical of P. All parabolic subgroups with this property are now completely classified [HR]. There are not too many of them. It turns out that there is another case when the conjecture is true.

Theorem 5. *Suppose that a grading* $\mathfrak{g} = \bigoplus\limits_{k\in\mathbb{Z}} \mathfrak{g}_k$ *is a Moore–Penrose grading in all positive degrees except at most one. Then the Conjecture is true for the corresponding parabolic subgroup* P.

This Theorem is proved in §5. For example, combining Theorem 4, Theorem 5, and Example 2 we get the following corollary. The proof of the Conjecture for other simple groups will be given elsewhere.

Corollary. *The conjecture is true for any parabolic subgroup in* SL_n *or* G_2.

This paper was written during my stay in the Mathematical Institute in Basel. I would like to thank prof. H. Kraft for the warm hospitality. I am grateful to the referee for various corrections and improvements. The research was supported by the grant INTAS-OPEN-97-1570 of the INTAS foundation.

§1. Moore–Penrose orbits in periodically graded simple Lie algebras

In this section we shall derive Theorems 1, 2, and 3 from the more general theorem, which describes Moore–Penrose orbits in periodically graded simple Lie algebras.

Suppose that A is either the group of integers $\mathbb{Z}$ or the group $\mathbb{Z}_m$ of residues modulo m. Let $\mathfrak{g}$ be an A-graded simple Lie algebra, $\mathfrak{g} = \bigoplus\limits_{k\in A} \mathfrak{g}_k$, and let G be a corresponding simple simply-connected group. $L \subset G$ is a connected reductive subgroup with the Lie algebra $\mathfrak{g}_0$, $\mathfrak{k}_0$ is a compact real form of $\mathfrak{g}_0$.

We fix $k \in A$, let $\mathcal{O} \subset \mathfrak{g}_k$ be a nilpotent L=orbit. It is well-known (see [Vi]) that for any $e \in \mathcal{O}$ there exists a homogeneous $\mathfrak{sl}_2$-triple $\langle e, h, f\rangle$. Then $\mathcal{O}$ is called *a Moore-Penrose orbit* if for any $e \in \mathcal{O}$ there exists a homogeneous $\mathfrak{sl}_2$-triple $\langle e, h, f\rangle$ such that $h \in i\mathfrak{k}_0$.

Let us give a criterion for a nilpotent L-orbit $\mathcal{O} = \mathrm{Ad}(L)e \subset \mathfrak{g}_k$ to be Moore–Penrose. Take any homogeneous $\mathfrak{sl}_2$-triple $\langle e, h, f\rangle$. Then h defines a grading $\mathfrak{g}_0 = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_0^n$, such that $\mathrm{ad}(h)|_{\mathfrak{g}_0^n} = n \cdot \mathrm{Id}$. Denote $\bigoplus_{n>0} \mathfrak{g}_0^n$ by $\mathfrak{n}_+$ and $\bigoplus_{n<0} \mathfrak{g}_0^n$ by $\mathfrak{n}_-$.

Theorem 6. *$\mathcal{O}$ is a Moore–Penrose orbit if and only if $ad(e)\mathfrak{n}_+ = 0$. In this case for any $e' \in \mathcal{O}$ there exists a unique homogeneous $\mathfrak{sl}_2$-triple $\langle e', h', f'\rangle$ such that $h' \in i\mathfrak{k}_0$.*

We shall need a few lemmas. Let $x \to \overline{x}$ denotes the complex conjugation in $\mathfrak{g}_0$ with respect to the compact form $\mathfrak{k}_0$. Therefore $x = \overline{x}$ iff $x \in \mathfrak{k}_0$ and $x = -\overline{x}$ iff $x \in i\mathfrak{k}_0$. Let $B(x,y) = \mathrm{Tr}\,\mathrm{ad}(x)\mathrm{ad}(y)$ be the Killing form of $\mathfrak{g}$. Finally, let $H(x,y) = -B(x,\overline{y})$ be a positive-definite Hermite form on $\mathfrak{g}_0$.

Lemma 1.1. *We fix a nilpotent element $e \in \mathfrak{g}_k$. Suppose that $\langle e, h, f\rangle$ is a homogeneous $\mathfrak{sl}_2$-triple such that $h \in i\mathfrak{k}_0$. Then for any other homogeneous $\mathfrak{sl}_2$-triple $\langle e, h', f'\rangle$ we have $H(h,h) < H(h',h')$. In particular, if there exists an $\mathfrak{sl}_2$-triple $\langle e, h, f\rangle$ with $h \in i\mathfrak{k}_0$ then the $\mathfrak{sl}_2$-triple with this property is unique.*

Proof. Recall that if $\langle e, h, f\rangle$ is an $\mathfrak{sl}_2$-triple then h is called *a characteristic* of e. Consider the subset $\mathcal{H} \subset \mathfrak{g}_0$ consisting of all possible homogeneous characteristics of e. It is well-known that $\mathcal{H}$ is an affine subspace in $\mathfrak{g}_0$ such that the corresponding linear subspace is precisely the unipotent radical $\mathfrak{z}^u_{\mathfrak{g}_0}(e)$ of the centralizer $\mathfrak{z}_{\mathfrak{g}_0}(e)$ in $\mathfrak{g}_0$ of the element e. Since $H(h',h')$ is a strongly convex function on $\mathcal{H}$, there exists a unique element $h_0 \in \mathcal{H}$ such that $H(h_0,h_0) < H(h',h')$ for any $h' \in \mathcal{H}$, $h' \neq h_0$. We need to show that $h_0 = h$. It is clear that an element $h_0 \in \mathcal{H}$ minimizes $H(h',h')$ on $\mathcal{H}$ iff $H(h_0, \mathfrak{z}^u_{\mathfrak{g}_0}(e)) = 0$ iff $B(\overline{h}_0, \mathfrak{z}^u_{\mathfrak{g}_0}(e)) = 0$. If $h \in \mathcal{H} \cap i\mathfrak{k}_0$ then $\overline{h} = -h$ and we have

$$B(\overline{h}, \mathfrak{z}^u_{\mathfrak{g}_0}(e)) = -B(h, \mathfrak{z}^u_{\mathfrak{g}_0}(e)) = -B([e,f], \mathfrak{z}^u_{\mathfrak{g}_0}(e)) = B(f, [e, \mathfrak{z}^u_{\mathfrak{g}_0}(e)]) = 0.$$

Therefore $h = h_0$. ■

Lemma 1.2. *If $ad(e)\mathfrak{n}_+ = 0$ then $\mathcal{O} = Ad(L)e$ is a Moore–Penrose orbit.*

Proof. We need to prove that for any element $e' \in \mathcal{O}$ there exists a homogeneous $\mathfrak{sl}_2$-triple $\langle e', h', f'\rangle$ such that $h' \in i\mathfrak{k}_0$, where $\mathfrak{k}_0$ is a *fixed* compact real form of $\mathfrak{g}_0$. Clearly it is sufficient to prove that for an *arbitrary* compact real form $\mathfrak{k}_0$ of $\mathfrak{g}_0$ there exists a homogeneous $\mathfrak{sl}_2$-triple $\langle e, h, f\rangle$ with $h \in i\mathfrak{k}_0$. Let us start with an arbitrary homogeneous $\mathfrak{sl}_2$-triple $\langle e, h, f\rangle$. The space of all homogeneous characteristics is an affine space $h + \mathfrak{z}^u_{\mathfrak{g}_0}(e) = h + \mathfrak{n}_+$. Arguing as in the proof of Lemma 1.1, let us change a characteristic h in such a way that $B(\overline{h}, \mathfrak{n}_+) = 0$, where $x \to \overline{x}$ denotes a complex conjugation in $\mathfrak{g}_0$ with respect to the compact form $\mathfrak{k}_0$. It remains to prove that $h \in i\mathfrak{k}_0$. Since B is a non-degenerate ad-invariant scalar product on $\mathfrak{g}_0$ it follows that $\overline{h} \in \mathfrak{q}$, where $\mathfrak{q} = \mathfrak{g}_0^0 \oplus \mathfrak{n}_+$. Let $\mathfrak{l}_0$

be some "standard" compact real form of $\mathfrak{g}_0$ such that $h \in i\mathfrak{l}_0$ and $\tilde{\mathfrak{n}}_\pm = \mathfrak{n}_\mp$, where $x \to \tilde{x}$ denotes a complex conjugation in $\mathfrak{g}_0$ with respect to the compact form $\mathfrak{l}_0$. Let $Q \subset L$ be a parabolic subgroup of L with the Lie algebra $\mathfrak{q}$, let $H \subset Q$ be its Levi subgroup with the Lie algebra $\mathfrak{g}_0^0$. There exists $g \in Q$ such that $\mathrm{Ad}(g)\mathfrak{k}_0 = \mathfrak{l}_0$. (The conjugation theorem is usually stated only for semi-simple Lie algebras, while $\mathfrak{g}_0$ is only reductive. But according to our conventions (see Remark in the Introduction), the conjugation theorem holds for $\mathfrak{g}_0$ as well.) Therefore

$$\widetilde{\mathrm{Ad}(g)}h = \mathrm{Ad}(g)\overline{h} \subset \mathrm{Ad}(g)(\mathfrak{q}) = \mathfrak{q}.$$

We can express g as a product uz, where $u \in \exp(\mathfrak{n}_+)$, $\mathrm{Ad}(z)h = h$. Then $\widetilde{\mathrm{Ad}(g)}h = \widetilde{\mathrm{Ad}(u)}h$. If u is not the identity element of G then $\mathrm{Ad}(u)h = h + \xi$, where $\xi \in \mathfrak{n}_+$ and $\xi \neq 0$. Therefore $\widetilde{\mathrm{Ad}(u)}h = -h + \tilde{\xi}$. But $\tilde{\xi} \in \mathfrak{n}_-$ and hence $\widetilde{\mathrm{Ad}(u)}h \notin \mathfrak{q}$, contradiction. Therefore u is trivial and since $\mathrm{Ad}(z)h = h$ we finally get

$$\overline{h} = \tilde{h} = -h. \qquad \blacksquare$$

Lemma 1.3. *Suppose that $\mathcal{O} = Ad(L)e$ is a Moore–Penrose orbit. Then $ad(e)\mathfrak{n}_+ = 0$.*

Proof. We choose a standard compact real form $\mathfrak{l}_0$ as in the proof of the previous Lemma. Clearly, $\mathfrak{z}^u_{\mathfrak{g}_0}(e)$ is a graded subalgebra of $\mathfrak{n}_+ = \bigoplus\limits_{k>0} \mathfrak{g}_0^k$. Suppose, on the contrary, that $\mathfrak{z}^u_{\mathfrak{g}_0}(e) \neq \mathfrak{n}_+$. Let $\xi \in \mathfrak{g}_0^p$, $p > 0$, be a homogeneous element that does not belong to $\mathfrak{z}^u_{\mathfrak{g}_0}(e)$. Let $u = \exp(\xi)$. Let $e' = \mathrm{Ad}(u)e$. We claim that all characteristics of e' don't belong to $i\mathfrak{l}_0$. Indeed, all characteristics of e' have a form $\mathrm{Ad}(u)h + \mathrm{Ad}(u)x$, where $x \in \mathfrak{z}^u_{\mathfrak{g}_0}(e)$. Suppose that for some x we have $\mathrm{Ad}(u)h + \mathrm{Ad}(u)x \in i\mathfrak{l}_0$. Since $h \in i\mathfrak{l}_0$, $\tilde{\mathfrak{n}}_\pm = \mathfrak{n}_\mp$, and $\mathrm{Ad}(u)(h+x) - h \in \mathfrak{n}_+$ it follows that $\mathrm{Ad}(u)(h+x) = h$. In $\mathfrak{n}_+$ modulo $\bigoplus\limits_{k>p} \mathfrak{g}_0^k$ we obtain the equation $[\xi, h] + x = 0$, but $[h, \xi] = p\xi$ and therefore $\xi \in \mathfrak{z}^u_{\mathfrak{g}_0}(e)$. Cotradiction. $\blacksquare$

Proof of Theorem 6. Combining Lemma 1.2 and 1.3 we see that $\mathcal{O}$ is a Moore–Penrose orbit if and only if $\mathrm{ad}(e)\mathfrak{g}_0^n = 0$ for any $n > 0$. In this case for any $e' \in \mathcal{O}$ there exists a unique homogeneous $\mathfrak{sl}_2$-triple $\langle e', h', f' \rangle$ such that $h' \in i\mathfrak{k}_0$ by Lemma 1.1. $\blacksquare$

Proof of Theorem 3. This is a particular case of Theorem 6 for $\mathbb{Z}$-gradings. $\blacksquare$

Proof of Theorem 1. We should show that the condition $\mathrm{ad}(e)\mathfrak{g}_0^n = 0$ for any $n > 0$ is satisfied always if $\mathfrak{g}_k = 0$ for $|k| > 1$. Suppose that $x \in \mathfrak{g}_0^n$, $n > 0$. If $\mathrm{ad}(e)x \neq 0$ then there exists an element $y \in \mathfrak{g}_1$ such that $\mathrm{ad}(h)y = (n+2)y$. Since $\mathrm{ad}(e)\mathfrak{g}_i \subset \mathfrak{g}_{i+1}$ it follows from $\mathfrak{sl}_2$-theory that there exists a non-zero element $z \in \mathfrak{g}_{1-(n+2)}$. But $1 - (n+2) < -1$. Contradiction. $\blacksquare$

Proof of Theorem 2. We take the trivial grading $\mathfrak{g} = \mathfrak{g}_0$. By Lemmas 1.2 and 1.3 we see that a nilpotent orbit $\mathcal{O} = \mathrm{Ad}(G)e \subset \mathfrak{g}$ is a Moore–Penrose orbit if and only if $\mathfrak{z}^u_{\mathfrak{g}}(e) = \mathfrak{n}_+$. It follows from the $\mathfrak{sl}_2$-theory that $\dim \mathfrak{z}^u_{\mathfrak{g}}(e) =$

$\dim \mathfrak{g}^1 + \dim \mathfrak{g}^2$. Therefore $\mathfrak{z}^u_{\mathfrak{g}}(e) = \mathfrak{n}_+$ if and only if $\mathfrak{g}^p = 0$ for $p > 2$. Clearly, this is precisely equivalent to $\mathrm{ht}(\mathcal{O}) = 2$. In this case for any $e \in \mathcal{O}$ there exists a unique $\mathfrak{sl}_2$-triple $\langle e, h, f\rangle$ such that $h \in i\mathfrak{k}$ by Lemma 1.1. ∎

§2. Shortly graded simple real Lie algebras

Let $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a real simple shortly graded Lie algebra. We suppose that $\mathfrak{g}$ does not admit a complex structure, so its complexification $\mathfrak{g}^c$ is a simple complex Lie algebra. There exists a unique element $c \in \mathfrak{g}_0$ such that $ad(c)|_{\mathfrak{g}_k} = k \cdot \mathrm{Id}$. Then $\mathfrak{c} = \mathbb{R}c$ is a center of $\mathfrak{g}_0$ and $\mathfrak{g}_0 = \mathfrak{c} \oplus \mathfrak{g}_0'$, where $\mathfrak{g}_0' = [\mathfrak{g}_0, \mathfrak{g}_0]$. We fix a maximal compact subalgebra $\mathfrak{k}_0 \subset \mathfrak{g}_0'$ and a Cartan decomposition $\mathfrak{g}_0' = \mathfrak{k}_0 \oplus \mathfrak{p}_0'$. Let $\mathfrak{p}_0 = \mathfrak{p}_0' \oplus \mathfrak{c}$. Then we may call $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ a Cartan decomposition of $\mathfrak{g}_0$.

Definition. Let $e \in \mathfrak{g}_1$. An element $f \in \mathfrak{g}_{-1}$ is called a Moore–Penrose inverse of e if there exists an element $h \in \mathfrak{p}_0$ such that $\langle e, h, f\rangle$ is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$.

The following is an easy consequence of Theorem 1:

Theorem 7. *For any $e \in \mathfrak{g}_1$ its MP-inverse exists and is unique.*

Proof. The complexification $\mathfrak{g}^c$ of $\mathfrak{g}$ is a shortly graded complex simple Lie algebra

$$\mathfrak{g}^c = \mathfrak{g}^c_{-1} \oplus \mathfrak{g}^c_0 \oplus \mathfrak{g}^c_1, \text{ where } \mathfrak{g}^c_k = \mathfrak{g}_k \otimes \mathbb{C}.$$

Then $\hat{\mathfrak{k}}_0 = \mathfrak{k}_0 \oplus i\mathfrak{p}$ is a compact real form of $\mathfrak{g}^c_0$. By Theorem 1 there exists a unique $\mathfrak{sl}_2$-triple $\langle e, h, f\rangle$ in $\mathfrak{g}^c$ such that $h \in i\hat{\mathfrak{k}}_0 = i\mathfrak{k}_0 \oplus \mathfrak{p}$. It suffices to show that, in fact, $h \in \mathfrak{p}$. Indeed, $\langle e, \overline{h}, \overline{f}\rangle$ is an $\mathfrak{sl}_2$-triple such that $\overline{h} \in i\mathfrak{k}_0 \oplus \mathfrak{p}$, where bar denotes the complex conjugation in $\mathfrak{g}^c$ with respect to $\mathfrak{g}$. By Theorem 1 it follows that $h = \overline{h}$. Therefore, $h \in \mathfrak{p}$. ∎

The list of shortly graded simple real Lie algebras is well-known. It can be easily obtained from [D], where it is shown that there exists a bijection of $\mathbb{Z}$-graded real simple Lie algebras and weighted Satake diagrams with certain natural restrictions. In the rest part of this section we describe arising Moore–Penrose inverses. The proofs are quite straightforward, so they are omitted. We do this job only for classical real Lie algebras. For two real forms of E_6 and two real forms of E_7 that admit short gradings the answer is quite similar to one obtained in §3. We only need to change split complex Cayley numbers to either split real Cayley numbers $\mathbb{C}\mathrm{a}(\mathbb{R})$ or the division algebra of octonions $\mathbb{O}$.

Real and quaternionic linear maps. Suppose that $U = \mathbb{R}^n$ and $V = \mathbb{R}^m$ (resp. $U = \mathbb{H}^n$ and $V = \mathbb{H}^m$) are real vector spaces (resp. right quaternionic vector spaces) equipped with standard Euclidean scalar products (resp. with standard Hermite scalar products $\sum_i \overline{q_i'} q_i$, where bar denotes the standard quaternionic involution). For any linear map $F : U \to V$ its Moore-Penrose inverse is a linear map $F^+ : V \to U$ defined as follows. Let $\mathrm{Ker} F \subset U$ and $\mathrm{Im} F \subset V$ be the kernel and the image of F. Let $\mathrm{Ker}^{\perp}F \subset \mathbb{R}^n$ and $\mathrm{Im}^{\perp}F \subset \mathbb{R}^m$ be their orthogonal complements with respect to the Euclidean scalar products (resp. the Hermite scalar products, notice that in this case we shall get right vector subspaces). Then F defines via restriction a bijective linear map $\tilde{F} : \mathrm{Ker}^{\perp}F \to \mathrm{Im}F$. Then $F^+ : V \to U$ is a unique linear map such that $F^+|_{\mathrm{Im}^{\perp}F} = 0$ and $F^+|_{\mathrm{Im}F} = \tilde{F}^{-1}$. This MP-inverse corresponds to short gradings of $\mathfrak{sl}_{n+m}(\mathbb{R})$ (resp. $\mathfrak{sl}_{n+m}(\mathbb{H})$).

Skew-Hermite matrices (or forms). This example is an analogue of a MP-inverse of skew-symmetric forms from the Introduction. So for diversity we give a matrix description. We consider matrices over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. The Moore–Penrose inverse of a skew-Hermite matrix A (with respect to the canonical involution, so in the real case A is just skew-symmetric) is a unique skew-Hermite matrix A^+ such that

$$AA^+A = A, \quad A^+AA^+ = A^+, \quad [A, A^+] = 0. \tag{$*$}$$

This MP-inverse corresponds to the short grading of $\mathfrak{so}_{p,p}$ (real case), $\mathfrak{su}_{p,p}$ (complex case), $\mathfrak{sp}_{p,p}$ (quaternionic case).

Hermite matrices (or forms). This example is an analogue of a MP-inverse of symmetric forms from the Introduction. So again we shall give a matrix description. We consider matrices over $\mathbb{R}$ or $\mathbb{H}$. The Moore–Penrose inverse of a Hermite matrix A is a unique Hermite matrix A^+ that satisfies equations $(*)$. This MP-inverse corresponds to the short grading of $\mathfrak{sp}_{2p}(\mathbb{R})$ (real case), $\mathfrak{u}^*_{2p}(\mathbb{H})$ (quaternionic case).

Vectors in a pseudo-Euclidean space. We take a vector space $V = \mathbb{R}^{n+m}$ with a standard Euclidean scalar product $(\cdot,\cdot)$. Let $I = \begin{pmatrix} \mathrm{Id}_n & 0 \\ 0 & -\mathrm{Id}_m \end{pmatrix}$, where Id_k is an identity matrix from $\mathrm{Mat}_{k,k}$. Let $\{u, v\} = (u, Iv)$ be a pseudo-Euclidean scalar product. The Moore–Penrose inverse takes any vector $v \in V$ to a vector v^+ defined as follows:

$$v^+ = \begin{cases} \dfrac{2v}{\{v,v\}}, & \text{if } \{v,v\} \neq 0 \\ \dfrac{Iv}{(v,v)}, & \text{if } \{v,v\} = 0,\ v \neq 0 \\ 0, & \text{if } v = 0. \end{cases}$$

This MP-inverse corresponds to the short grading of $\mathfrak{so}_{n+1.m+1}$.

§3. Jordan pairs

A Jordan pair is a pair of vector spaces (V_+, V_-) with trilinear multiplications

$$V_\pm \otimes V_\mp \otimes V_\pm \to V_\pm, \quad x \otimes y \otimes z \mapsto \{xyz\},$$

which satisfy the certain set of axioms (see [Lo]). Fortunately, there is no need to write them down due to the following fundamental observation. V_+ and V_- form a Jordan pair if and only if there exists a shortly graded Lie algebra $\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1$ such that

$$\mathfrak{g}_{-1}=V_-,\quad \mathfrak{g}_1=V_+,\quad \text{and } \{x,y,z\}=\frac{1}{2}[[x,y],z].$$

In fact, this construction provides the bijection of the set of Jordan pairs up to an isomorphism and the set of short graded Lie algebras up to a certain equivalence relation. We refer the reader to [Lo] and [Ja] for more details about Jordan pairs and Jordan algebras used throughout this section.

Example 1. Take $V_+=\mathrm{Mat}_{n,m}$, $V_-=\mathrm{Mat}_{m,n}$. Then (V_+,V_-) is a Jordan pair with respect to trilinear maps $\{ABC\}=\frac{1}{2}(ABC+CBA)$ (matrix multiplication). This Jordan pair corresponds to a short grading of $\mathfrak{sl}_{n+m}$.

Example 2. Suppose that A is a Jordan algebra, that is, an algebra with a unit such that the bilinear multiplication in A satisfies two axioms

$$ab=ba \text{ (commutativity)},\quad ((aa)b)a=(aa)(ba) \text{ (Jordan axiom)}.$$

For example, we can take any associative algebra and define a new multiplication by the formula $a*b=\frac{1}{2}(ab+ba)$. This will be a (special) Jordan algebra. Any Jordan algebra A corresponds to a Jordan pair (V_+,V_-) defined as follows:

$$V_+=V_-=A,\quad \{abc\}=(ab)c+(bc)a-(ac)b\quad \text{(Jordan triple product)}.$$

Simple Jordan algebras correspond to shortly graded simple Lie algebras such that the corresponding Hermite homogeneous space G/P (recall that P is a parabolic subgroup with the Lie algebra $\mathfrak{g}_0\oplus\mathfrak{g}_1$) has a tube type, or, equivalently, if there exists an L-invariant hypersurface in $\mathfrak{g}_1$, where L is a Levi subgroup of P, or, equivalently, if the action of $[L,L]$ on $\mathfrak{g}_1$ has not an open orbit.

We shall be interested only in Jordan pairs arising from shortly graded complex simple Lie algebras. It can be shown that these Jordan pairs are precisely simple complex Jordan pairs. For simplicity we shall use the term 'Jordan pairs' only for these pairs.

In the Introduction we defined a MP-inverse for shortly graded simple Lie algebras. In the language of Jordan pairs a MP-inverse is some map $V_\pm\to V_\mp$. The first aim of this section is to define a MP-inverse entirely in terms of trilinear maps $\{\cdot,\cdot,\cdot\}$. First let us give some definitions and lemmas. For any Jordan pair (V_+,V_-) we denote the corresponding shortly graded simple Lie algebra by $\mathfrak{g}=\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1$.

Definition. *A Killing pairing* $B(\cdot,\cdot)$ of a Jordan pair is a bilinear map

$$V_\pm\otimes V_\mp\to\mathbb{C} \text{ given by } x\otimes y\mapsto \mathrm{Tr}\{x,y,\cdot\}.$$

Example. Suppose that the Jordan pair corresponds to a Jordan algebra A. Then

$$\begin{aligned}B(x,y)&=\mathrm{Tr}\{x,y,\cdot\}=\mathrm{Tr}\big\{(ab)\cdot+(b\cdot)a-(a\cdot)b\big\}\\&=\mathrm{Tr}\big\{(ab)\cdot\big\}+\mathrm{Tr}\big\{a(b\cdot)-b(a\cdot)\big\}=\mathrm{Tr}\big\{(ab)\cdot\big\}.\end{aligned}$$

This is the usual definition of a scalar product in a Jordan algebra (up to a positive multiple).

Proposition 3.1. *A Killing pairing is symmetric and non-degenerate. It coincides up to a positive multiple with a restriction of a Killing form of* $\mathfrak{g}$ *on* $\mathfrak{g}_{-1} \oplus \mathfrak{g}_1$.

Proof. Since $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ are dual $\mathfrak{g}_0$-modules it follows that for any $\xi \in \mathfrak{g}_0$ we have

$$\mathrm{Tr}[\xi, \cdot]|_{\mathfrak{g}_{-1}} = -\mathrm{Tr}[\xi, \cdot]|_{\mathfrak{g}_1}.$$

Therefore, for any $x \in V_+$, $y \in V_-$, we get

$$B(x, y) = \mathrm{Tr}\{x, y, \cdot\} = \frac{1}{2}\mathrm{Tr}[[x, y], \cdot]|_{\mathfrak{g}_1}$$

$$= -\frac{1}{2}\mathrm{Tr}[[x, y], \cdot]|_{\mathfrak{g}_{-1}} = \frac{1}{2}\mathrm{Tr}[[y, x], \cdot]|_{\mathfrak{g}_{-1}} = B(y, x).$$

This proves symmetry. To show that B is non-degenerate it is sufficient to prove that $B(\cdot, \cdot)$ coincides up to a positive multiple with a restriction of a Killing form $(\cdot, \cdot)$ of $\mathfrak{g}$ on $\mathfrak{g}_{-1} \oplus \mathfrak{g}_1$. We choose a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_0$. Let Δ be the corresponding root system. For any root $\alpha \in \Delta$ we choose a root vector $e_\alpha \in \mathfrak{g}$ in such a way that $[e_\alpha, e_{-\alpha}] = h_\alpha$, where $h_\alpha \in \mathfrak{h}$ is a coroot of α, so for any $\beta \in \Delta$ we have $\beta(h_\alpha) = \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$. We have a decomposition $\Delta = \Delta_{-1} \cup \Delta_0 \cup \Delta_1$, where $\alpha \in \Delta_k$ if and only if $e_\alpha \in \mathfrak{g}_k$. We also take a set of simple roots $\Pi_0 \subset \Delta_0$ and a root $\gamma \in \Delta_1$ such that the system $\Pi_0 \cup \{\gamma\}$ is a set of simple roots for Δ. Let Δ_0^+ be a set of positive roots of Δ_0 corresponding to Π_0. Then $\Delta^+ = \Delta_0^+ \cup \Delta_1$ is a set of positive roots for Δ. It suffices to show that for any $\alpha \in \mathfrak{g}_1$, $\beta \in \mathfrak{g}_{-1}$ we have $B(e_\alpha, e_\beta) = c(e_\alpha, e_\beta)$, where $c > 0$ does not depend on α and β. If $\alpha + \beta \neq 0$ then clearly $(e_\alpha, e_\beta) = 0$. But in this case $B(e_\alpha, e_\beta)$ is also equal to zero, because $[[e_\alpha, e_\beta], \cdot]$ is a nilpotent operator. Suppose now that $\beta = -\alpha$. Then $(e_\alpha, e_{-\alpha}) = \frac{2}{(\alpha,\alpha)}$. On the other hand,

$$B(e_\alpha, e_{-\alpha}) = \frac{1}{2}\mathrm{Tr}[h_\alpha, \cdot]|_{\mathfrak{g}_1} = \sum_{\beta \in \Delta_1} \frac{2(\beta, \alpha)}{(\alpha, \alpha)} = \frac{2(\rho_1, \alpha)}{(\alpha, \alpha)}, \text{ where } \rho_1 = \sum_{\beta \in \Delta_1} \beta.$$

So it suffices to prove that (ρ_1, α) is positive and does not depend on the choice of $\alpha \in \Delta_1$. In fact, the first claim will follow from the second, because then $(\rho_1, \alpha) = \frac{1}{\#\Delta_1}(\rho_1, \rho_1) > 0$. So let us prove the second claim. Any root $\alpha \in \Delta_1$ is a sum of γ and of a positive linear combination of some simple roots from Π_0. Therefore we need to prove that for any $\delta \in \Pi_0$ we have $(\rho_1, \delta) = 0$. But

$$\rho_1 = \rho - \rho_0, \text{ where } \rho = \sum_{\beta \in \Delta^+} \beta, \ \rho_0 = \sum_{\beta \in \Delta_0^+} \beta.$$

Therefore

$$(\rho_1, \delta) = (\rho, \delta) - (\rho_0, \delta) = (\delta, \delta) - (\delta, \delta) = 0. \qquad \blacksquare$$

Definition. A pair of antilinear maps $\omega : V_\pm \to V_\mp$ is called a Cartan involution of a Jordan pair if

$$\omega^2 = \mathrm{Id}, \quad \{\omega(x), \omega(y), \omega(z)\} = \omega\{x, y, z\},$$
$$\text{Hermite form } H(x) = B(x, \omega(x)) \text{ is positive definite on } V_\pm.$$

Suppose that $\tilde{\omega}$ is a Cartan involution of $\mathfrak{g}$ (so $\mathfrak{g}^{\tilde{\omega}}$ is a compact real form of $\mathfrak{g}$) such that $\tilde{\omega}(\mathfrak{g}_k) = \mathfrak{g}_{-k}$. Let $\sigma \in \mathrm{Aut}(\mathfrak{g})$ be defined as follows:

$$\sigma|_{\mathfrak{g}_0} = \mathrm{Id}, \quad \sigma|_{\mathfrak{g}_{-1}\oplus\mathfrak{g}_1} = -\mathrm{Id}.$$

Clearly, $\hat{\omega} = \sigma\tilde{\omega}$ is an antilinear involution of $\mathfrak{g}$. Then $\hat{\omega}$ restricted to $\mathfrak{g}_{-1}\oplus\mathfrak{g}_1$ is a Cartan involution ω of a corresponding Jordan pair. In particular, any Jordan pair has a Cartan involution.

Proposition 3.2. *The correspondence $\tilde{\omega} \to \omega$ is bijective.*

Proof. The set of commutators $[x, y]$ for $x \in \mathfrak{g}_1$, $y \in \mathfrak{g}_{-1}$ spans $\mathfrak{g}_0$. Indeed, it easily follows from the Jacoby identity that the linear subspace $\mathfrak{g}_{-1}\oplus[\mathfrak{g}_{-1}, \mathfrak{g}_1]\oplus\mathfrak{g}_1$ is an ideal in $\mathfrak{g}$ hence it coincides with $\mathfrak{g}$ since $\mathfrak{g}$ is simple. Since $\tilde{\omega}([x, y]) = [\omega(x), \omega(y)]$ it follows that this correspondence is injective. Suppose now that ω is a Cartan involution of a Jordan pair. Then we define $\tilde{\omega}$ on $\mathfrak{g}_{-1} \oplus \mathfrak{g}_1$ as $-\omega$ and we define $\tilde{\omega}$ on $\mathfrak{g}_0$ by setting $\tilde{\omega}([x, y]) = [\omega(x), \omega(y)]$. Let us show that $\tilde{\omega}$ is well-defined. Suppose that $\xi = [x_1, y_1] + \ldots + [x_k, y_k] = 0$ for $x_i \in \mathfrak{g}_1$, $y_i \in \mathfrak{g}_{-1}$. We need to show that $\xi_\omega = [\omega(x_1), \omega(y_1)] + \ldots + [\omega(x_k), \omega(y_k)] = 0$. The representation of $\mathfrak{g}_0$ on $\mathfrak{g}_1$ is faithful. Therefore it suffices to prove that for any $z \in \mathfrak{g}_{-1}$ we have $[\xi_\omega, \omega(z)] = 0$. But

$$[\xi_\omega, \omega(z)] = 2\sum_{i=1}^{k}\{\omega(x_i), \omega(y_i), \omega(z)\} = 2\sum_{i=1}^{k}\omega\{x_i, y_i, z\} = \omega[\xi, z] = 0.$$

Therefore $\tilde{\omega}$ is well-defined. Clearly $\tilde{\omega}^2 = \mathrm{Id}$ and $[\tilde{\omega}(x), \tilde{\omega}(y)] = \tilde{\omega}[x, y]$. It remains to prove that the Hermite form $\tilde{H}(x) = -(x, \tilde{\omega}(x))$ is positive definite on $\mathfrak{g}$. Since the decomposition $\mathfrak{g} = \mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1$ is orthogonal with respect to $\tilde{H}$ and the restriction of $\tilde{H}$ on $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ coincides with H up to a positive multiple, $\tilde{H}$ is positive definite on $\mathfrak{g}_{-1} \oplus \mathfrak{g}_1$. There exists a Cartan involution τ compatible with $\tilde{\omega}$ and such that $\tau(\mathfrak{g}_0) = \mathfrak{g}_0$ (see [VO]). Then, clearly, $\tau(\mathfrak{g}_{\pm1}) = \mathfrak{g}_{\mp1}$. Therefore, $\tau\tilde{\omega}$ is an involution preserving the grading. We need to show that $\tau\tilde{\omega} = \mathrm{Id}$. It is sufficient to show that $\tau\tilde{\omega}|_{\mathfrak{g}_1} = \mathrm{Id}$. Suppose, on the contrary, that there exists $x \in \mathfrak{g}_1$, $x \neq 0$, such that $\tau\tilde{\omega}(x) = -x$. Then $\tilde{\omega}(x) = -\tau(x)$ implies $0 > (x, \tilde{\omega}(x)) = -(x, \tau(x)) > 0$. Contradiction. ∎

Now we are ready to give a new definition of a Moore–Penrose inverse.

Definition. Suppose that (V_+, V_-) is a Jordan pair. Then for any $A \in V_\pm$ its MP-inverse is an element $A^+ \in V_\mp$ such that

$$\{AA^+A\} = A, \quad \{A^+AA^+\} = A^+, \tag{$*$}$$

$$\{A, A^+, \cdot\},\ \{A^+, A, \cdot\} \text{ are Hermite operators with respect to } H. \tag{$**$}$$

In fact we shall see later that if one of operators in $(**)$ is Hermite then another one is a Hermite operator automatically.

Example. Suppose that the Jordan pair corresponds to a Jordan algebra A. An element $b \in A$ is called a (usual) inverse of an element a if $ab = 1$, $(aa)b = a$. This definition does not look very symmetric, but in fact it is. So automatically $(bb)a = b$. Let us show that in this case b coincides with a MP-inverse a^+. Indeed, $\{aba\} = (ab)a + (ba)a - (aa)b = a + a - a = a$. Similarly $\{bab\} = b$. Moreover, one can show that operators $(a\cdot)$ and $(b\cdot)$ commute (see [Ja]), and, therefore, $\{ab\cdot\}$ and $\{ba\cdot\}$ are identity operators and, hence, Hermite operators with respect to H.

Proposition 3.3. *New definition of a MP-inverse coincides with given in the Introduction. In particular, MP-inverse in Jordan pairs exists and is unique.*

Proof. Indeed, conditions $(*)$ mean that $\langle A, [A, A^+], A^+\rangle$ is a homogeneous $\mathfrak{sl}_2$-triple in $\mathfrak{g}$. We choose a compact form $\mathfrak{k}_0$ in $\mathfrak{g}_0$ such that $\mathfrak{k}_0 = \mathfrak{g}_0^{\tilde{\omega}}$. Since the representations of $\mathfrak{g}_0$ in $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ are faithful, any of operators in $(**)$ is Hermite if and only if $[A, A^+] \in i\mathfrak{k}_0$. ∎

Now we can describe a MP-inverse arising from short gradings of E_6 and E_7. We start with E_7. Let $\mathbb{C}\mathrm{a}$ denote the algebra of split Cayley numbers over $\mathbb{C}$. Let $\mathbb{O} \subset \mathbb{C}\mathrm{a}$ be the Cayley division algebra of octonions. Let $\mathbb{A}$ be the Albert algebra of Hermite (3×3)-matrices over $\mathbb{C}\mathrm{a}$ (with respect to the canonical anti-involution in $\mathbb{C}\mathrm{a}$). This is a complex Jordan algebra with respect to the symmetrisation of matrix multiplication. The corresponding Jordan pair arises from the short grading of E_7. Let $\mathbb{A}(\mathbb{O}) \subset \mathbb{A}$ be the real subalgebra of matrices with entries in $\mathbb{O}$. It is well-known that the scalar product $\mathrm{Tr}((ab)\cdot)$ is positive definite on $\mathbb{A}(\mathbb{O})$. Therefore the complex conjugation ω of $\mathbb{A}$ with respect to $\mathbb{A}(\mathbb{O})$ is a Cartan involution and we have all the information necessary to write down equations $(*, **)$.

The Jordan pair corresponding to the short grading of E_6 is, in fact, 'a subpair' of a previous one. Namely, $V_+ = V_- = \mathbb{C}\mathrm{a} \oplus \mathbb{C}\mathrm{a}$. We can consider both of them as matrices from $\mathbb{A}$ of the form

$$\begin{pmatrix} 0 & c_1 & c_2 \\ \overline{c}_1 & 0 & 0 \\ \overline{c}_2 & 0 & 0 \end{pmatrix}.$$

Then the Jordan triple product in $\mathbb{A}$ defines the trilinear maps for this Jordan pair. So the Moore–Penrose inverse in this case is the restriction of a previous MP-inverse.

§4. Moore–Penrose parabolic subgroups

Though the definition of Moore–Penrose parabolic subgroups given in the Introduction does not depend on the grading, in order to apply this definition, one has to sort out all gradings compatible with P. Let us give a more transparent equivalent definition. Suppose that G is a simple simply–connected Lie group, $P \subset G$ is a parabolic subgroup, $L \subset P$ is a Levi subgroup, $Z \subset L$ is a connected component of its center. Then a Lie algebra $\mathfrak{g}$ of G admits a natural

$\mathbb{Z}^m$-grading, where $\mathbb{Z}^m$ is a character group of Z. Clearly the Lie algebra $\mathfrak{l}$ of L is just a zero component of this grading. In fact any $\mathbb{Z}$-grading of $\mathfrak{g}$ "compatible" with P can be obtained from this $\mathbb{Z}^m$-grading via some homomorphism $\mathbb{Z}^m \to \mathbb{Z}$. Moreover, for generic homomorphism non-zero homogeneous components will be the same for $\mathbb{Z}$- and $\mathbb{Z}^m$-grading. Let $\mathfrak{k}_0$ be a compact real form of $\mathfrak{l}$. It is easy to see that the definition of Moore–Penrose parabolic subgroups given in the Introduction is equivalent to the following:

Definition. P is called *a Moore–Penrose parabolic subgroup* if for any $e \in \mathfrak{g}_\alpha$, $\alpha \in \mathbb{Z}^m$, $\alpha \neq 0$, there exists a $\mathbb{Z}^m$-homogeneous $\mathfrak{sl}_2$-triple $\langle e, h, f\rangle$ such that $h \in i\mathfrak{k}_0$. In this case f is called a MP-inverse of e.

For any $\alpha \in \mathbb{Z}^m$, $\alpha \neq 0$, let us consider a reductive subalgebra $\mathfrak{g}^\alpha = \bigoplus_{\beta \in \mathbb{R}\alpha} \mathfrak{g}_\beta$. Then $\mathfrak{g}^\alpha$ is a Levi subalgebra of $\mathfrak{g}$ and, clearly, P is a Moore–Penrose parabolic subgroup if and only if the maximal parabolic subalgebra $\bigoplus_{\beta \in \mathbb{R}_{\geq 0}\alpha} \mathfrak{g}_\beta$ of $\mathfrak{g}^\alpha$ is a Moore–Penrose parabolic subalgebra for any α. In particular, in order to describe all Moore–Penrose parabolic subgroups of G it suffices to find all Moore–Penrose maximal parabolic subgroups in simple components of Levi subgroups of G.

Proof of Theorem 4. All simple components of Levi subgroups of SL_n are again simple groups of type SL_k. All maximal parabolic subgroups in SL_k have aura and therefore are Moore–Penrose parabolic subgroups by Theorem 1. ■

In the rest part of this section we shall find all Moore–Penrose maximal parabolic subgroups in remaining classical groups SO_n and Sp_n. Let G be one of this groups, P be its maximal parabolic subgroup. We denote their Lie algebras by $\mathfrak{g}$ and $\mathfrak{p}$. The corresponding grading has a form $\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_k$. Recall that L is a Levi subgroup of P such that $\mathfrak{g}_0$ is a Lie algebra of L. It easy to see that $\mathfrak{g}_k = 0$ either for $|k| > 1$ or for $|k| > 2$. In the first case P has an aura and therefore is Moore-Penrose by Theorem 1. We take a usual bijection between maximal parabolic subgroups and simple roots of the corresponding algebra. In Bourbaki-numbering of simple roots P has an aura if and only if P corresponds to one of simple roots α_1 (B_n-case); α_n (C_n-case); α_1, α_{n-1}, α_n (D_n-case). Now let us consider other possibilities. Clearly all L-orbits in $\mathfrak{g}_2$ are Moore–Penrose (see Example 1 in the Introduction). Now we shall classify all Moore–Penrose L-orbits in $\mathfrak{g}_1$. But first we shall reformulate the problem in linear–algebraic terms.

Suppose that $U = \mathbb{C}^k$, $V = \mathbb{C}^n$ are complex vector spaces with standard Hermite scalar products. In the symplectic case we assume that n is even. We choose a symmetric (resp. skew-symmetric) 2-form ω in V with matrix $I = E$ (resp. $I = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$), where E is an identity matrix. Let $G(\omega)$ be a special orthogonal (resp. symplectic) group corresponding to ω. Let $\mathfrak{g}(\omega)$ be a corresponding Lie algebra. Then L, $\mathfrak{g}_0$, $\mathfrak{g}_1$, $\mathfrak{g}_{-1}$ have the following interpretation:

$$L = \mathrm{SL}(U) \times G(\omega) \times \mathbb{C}^*, \quad \mathfrak{g}_0 = \mathfrak{sl}(U) \oplus \mathfrak{g}(\omega) \oplus \mathbb{C},$$

$$\mathfrak{g}_1 = \mathrm{Hom}(U,V), \quad \mathfrak{g}_{-1} = \mathrm{Hom}(V,U).$$

The action of $\mathrm{SL}(U)\times G(\omega)$ on $\mathrm{Hom}(U,V)$ is standard, $\mathbb{C}^*$ acts by homotheties. So the action of L on $\mathfrak{g}_1$ has finitely many orbits $\mathcal{O}(a,b)$ indexed by $a=\mathrm{rk}(F)$ and $b=\dim\mathrm{Ker}\,\omega|_{\mathrm{Im}F}$ for $F\in\mathrm{Hom}(U,V)$. For any $F\in\mathrm{Hom}(U,V)$ its Moore–Penrose inverse (if exists) is defined as a unique $G\in\mathrm{Hom}(V,U)$ such that

$$GF,\ FG-(FG)^{\#} \text{ are Hermite operators,} \tag{$*$}$$

$$2F = 2FGF-(FG)^{\#}F, \quad 2G=2GFG-G(FG)^{\#}, \tag{$**$}$$

where for any $A\in\mathrm{Hom}(V,V)$ we denote by $A^{\#}$ its adjoint operator with respect to ω. Now let us prove the following proposition

Proposition. *An L-orbit $\mathcal{O}(a,b)\subset\mathfrak{g}_1$ is Moore–Penrose if and only if either $b=0$ or $a=b$.*

Proof. Suppose first that $b=0$, $F\in\mathcal{O}(a,b)$. Then the restriction of ω on $\mathrm{Im}(F)$ is non degenerate. Let $\mathrm{Im}(F)^{\perp}$ denote its orthogonal complement with respect to ω. Let $\mathrm{Ker}(F)^{\perp}$ denote the orthogonal complement of $\mathrm{Ker}(F)$ with respect to the Hermite form on U. Let $\tilde F\in\mathrm{Hom}(\mathrm{Ker}(F)^{\perp},\mathrm{Im}(F))$ be an operator induced by F. Let $G\in\mathrm{Hom}(V,U)$ be an operator defined as follows: $G|_{\mathrm{Im}(F)^{\perp}}=0$, $G|_{\mathrm{Im}(F)}=2\tilde F^{-1}$. Then GF is a Hermite projector on $\mathrm{Ker}(F)^{\perp}$. Since FG is an orthogonal projector on $\mathrm{Im}(F)$ with respect to ω, $(FG)^{\#}=FG$, therefore $FG-(FG)^{\#}=0$ is a Hermite operator, now relations $(**)$ are obvious.

Suppose now that $b=a$, $F\in\mathcal{O}(a,b)$. Then the restriction of ω on $\mathrm{Im}(F)$ is equal to 0. Let $\mathrm{Im}(F)^0=I\overline{\mathrm{Im}(F)}$ (recall that I is a matrix of ω, bar denotes the complex conjugation). Then $\mathrm{Im}(F)\cap\mathrm{Im}(F)^0=0$, the restriction of ω on $\mathrm{Im}(F)\oplus\mathrm{Im}(F)^0$ is non–degenerate and the orthogonal complement V' of $\mathrm{Im}(F)\oplus\mathrm{Im}(F)^0$ with respect to ω coincides with the orthogonal complement of $\mathrm{Im}(F)\oplus\mathrm{Im}(F)^0$ with respect to a Hermite form. Let $\mathrm{Ker}(F)^{\perp}$ denote the orthogonal complement of $\mathrm{Ker}(F)$ with respect to the Hermite form on U. Let $\tilde F\in\mathrm{Hom}(\mathrm{Ker}(F)^{\perp},\mathrm{Im}(F))$ be an operator induced by F. Let $G\in\mathrm{Hom}(V,U)$ be an operator defined as follows: $G|_{V'}=G|_{\mathrm{Im}(F)^0}=0$, $G|_{\mathrm{Im}(F)}=\tilde F^{-1}$. Then GF is a Hermite projector on $\mathrm{Ker}(F)^{\perp}$. It is clear that FG is equal to 0 on V' and on $\mathrm{Im}(F)^{\circ}$ and is an identity operator on $\mathrm{Im}(F)$. Therefore its adjoint operator $(FG)^{\#}$ is equal to 0 on V' and on $\mathrm{Im}(F)$ and is an identity operator on $\mathrm{Im}(F)^0$. Therefore $FG-(FG)^{\#}$ is a Hermite operator since $\mathrm{Im}(F)$ is Hermite othogonal to $\mathrm{Im}(F)^0$. Now relations $(**)$ are obvious.

It remains to prove that if $0<b<a$ then $\mathcal{O}(a,b)$ is not a Moore–Penrose orbit. Choose a subspace $L\subset V$ such that $\dim L=a$, $\dim\mathrm{Ker}\,\omega|_L=b$, and let $L_0=\mathrm{Ker}\,\omega|_L$. Let $U=U_0\oplus U_1\oplus U_2$ be an orthogonal (with respect to the Hermite form) direct sum of subspaces such that $\dim U_0=b$, $\dim U_0+\dim U_1=a$. Let $F\in\mathrm{Hom}(U,V)$ be a linear operator such that $F|_{U_2}=0$, $F(U_0)=L_0$, $F(U_0\oplus U_1)=L$, $F(U_1)$ is not orthogonal to L_0 with respect to the Hermite form. We set $L_1=F(U_1)$. We claim that F does not have a Moore–Penrose inverse. Suppose, on the contrary, that G is a Moore–Penrose inverse of F. Since GF is Hermite, we see that $G(L)\subset U_0\oplus U_1$. If $v\in L_0$, $v'\in V$ then $\omega((FG)^{\#}v,v')=\omega(v,FGv')=0$, because $\omega(L_0,L)=0$.

Therefore, $(FG)^{\#}|_{L_0} = 0$. It follows that $F|_{U_0} = FGF|_{U_0}$. So, $FG|_{L_0} = E$, $GF|_{U_0} = E$. Since GF is Hermite, it should preserve U_1, therefore $G(L_1) \subset U_1$. Finally, we see that FG preserves both L_0 and L_1. To obtain a contradiction it suffices to show that FG is a Hermite operator on L (because we know that L_0 and L_1 are not orthogonal). But since $2F = 2FGF - (FG)^{\#}F$, it follows that $2FG - (FG)^{\#} = 2E$ on L. Therefore $FG = 2E - (FG - (FG)^{\#})$ is Hermite, because we know that $FG - (FG)^{\#}$ is Hermite. ∎

If ω is orthogonal then all L-orbits are Moore–Penrose if and only if either $k = 1$ or $n \leq 2$. If ω is symplectic then all L-orbits are Moore–Penrose if and only if either and either $k \leq 2$ or $n = 2$. Therefore we have the following Corollary

Corollary. *1) All Moore–Penrose maximal parabolic subgroups in $SO(n)$ have aura, except (B_l, α_l).*

2) Moore–Penrose maximal parabolic subgroups without aura in $Sp(2n)$ correspond to simple roots α_1, α_2, α_{n-1}.

§5. Moore–Penrose inversion and rank stratification

In this section we prove Theorem 5. Let us start with some lemmas.

Suppose that G is a connected reductive group with a Lie algebra $\mathfrak{g}$. For any elements $x_1, \dots, x_r \in \mathfrak{g}$ let $\langle x_1, \dots, x_r \rangle_{alg}$ denote the minimal algebraic Lie subalgebra of $\mathfrak{g}$ that contains $x_1, \dots, x_r$. By a theorem of Richardson [Ri1] $\langle x_1, \dots, x_r \rangle_{alg}$ is reductive if and only if an orbit of the r-tuple $(x_1, \dots, x_r)$ in $\mathfrak{g}^r$ is closed with respect to the diagonal action of G. Suppose now that $h_1, \dots, h_r$ are semi-simple elements of $\mathfrak{g}$. Consider the closed variety $\hat{\mathcal{O}} = \mathrm{Ad}(G)h_1 \times \dots \times \mathrm{Ad}(G)h_r \subset \mathfrak{g}^r$. For any closed G-orbit $\mathcal{O} \subset \hat{\mathcal{O}}$ let us denote by $\mathfrak{g}(\mathcal{O})$ the conjugacy class of the reductive subalgebra $\langle x_1, \dots, x_r \rangle_{alg}$ for $(x_1, \dots, x_r) \in \mathcal{O}$.

Lemma 5.1. *There are only finitely many conjugacy classes $\mathfrak{g}(\mathcal{O})$.*

Proof. We shall use induction on $\dim \mathfrak{g}$. Suppose that the claim of Lemma 5.1 is true for all reductive groups H with $\dim H < \dim G$. Let $\mathfrak{z} \subset \mathfrak{g}$ be the center of $\mathfrak{g}$, $\mathfrak{g}' \subset \mathfrak{g}$ be its derived algebra. Consider two canonical homomorphisms

$$\mathfrak{g} \xrightarrow{\pi} \mathfrak{g}' \quad \text{and} \quad \mathfrak{g} \xrightarrow{\pi'} \mathfrak{z}.$$

We take any closed G-orbit $\mathcal{O} \subset \hat{\mathcal{O}}$. Let $(x_1, \dots, x_r) \in \mathcal{O}$, $y_i = \pi(x_i)$ for $i = 1, \dots, r$. Then $\langle y_1, \dots, y_r \rangle_{alg} = \pi(\langle x_1, \dots, x_r \rangle_{alg})$ and, therefore, is reductive. Let us consider two cases.

Suppose first, that $\langle y_1, \dots, y_r \rangle_{alg} = \mathfrak{g}'$. Then $\mathfrak{g}'$ is a derived algebra of $\langle x_1, \dots, x_r \rangle_{alg}$ and, therefore, $\langle x_1, \dots, x_r \rangle_{alg} = \langle \pi'(h_1), \dots, \pi'(h_r) \rangle \oplus \mathfrak{g}'$. In this case we get one conjugacy class.

Suppose now, that $\langle y_1, \dots, y_r \rangle_{alg} \neq \mathfrak{g}'$. Then $\langle y_1, \dots, y_r \rangle_{alg}$ is contained in some maximal reductive Lie subalgebra of $\mathfrak{g}'$. It is well-known (and not difficult to prove) that in a semisimple Lie algebra there are only finitely many conjugacy classes of maximal reductive subalgebras. Let $\mathfrak{h}'$ be one of them, $\mathfrak{h} = \mathfrak{z} \oplus \mathfrak{h}' \subset \mathfrak{g}$. Let H be a corresponding reductive subgroup of G. It is sufficient to prove that for any closed G-orbit $\mathcal{O}$ of $\hat{\mathcal{O}}$ that meets $\mathfrak{h}^r$ there are only finitely many possibilities for $\mathfrak{g}(\mathcal{O})$. It easily follows from Richardson's Lemma [Ri] that for any i the intersection $\mathrm{Ad}(G)h_i \cap \mathfrak{h}$ is a union of finitely many closed H-orbits, say $\mathrm{Ad}(H)h_i^1, \ldots, \mathrm{Ad}(H)h_i^{s_i}$. It remains to prove that if for some r-tuple $(x_1, \ldots, x_r) \in \mathrm{Ad}(H)h_1^{k_1} \times \ldots \mathrm{Ad}(H)h_r^{k_r}$ the corresponding subalgebra $\langle x_1, \ldots, x_r \rangle_{alg}$ is reductive then there are only finitely many possibilities for its conjugacy class. But this is precisely the claim of Lemma for the group H, which is true by the induction hypothesis. ∎

Suppose that $\mathfrak{k}$ is a compact real form of $\mathfrak{g}$.

Lemma 5.2. *If r-tuple $(x_1, \ldots, x_r)$ belongs to $(i\mathfrak{k})^r$, then its G-orbit is closed in $\mathfrak{g}^r$.*

Proof[1]. Indeed, let B be a non-degenerate ad-invariant scalar product on $\mathfrak{g}$, which is negative-definite on $\mathfrak{k}$. Let $H(x) = -B(\overline{x}, x)$ be a positive-definite $\mathfrak{k}$-invariant Hermite quadratic form on $\mathfrak{g}$, where the complex conjugation is taken with respect to $\mathfrak{k}$. Let H^r be a corresponding Hermite quadratic form on $\mathfrak{g}^r$. More precisely, $H^r(x_1, \ldots, x_r) = H(x_1) + \ldots + H(x_r)$. By the Kempf–Ness criterion [PV] in order to prove that the G-orbit of $(x_1, \ldots, x_r)$ is closed it is sufficient to prove that the real function $H^r(\cdot)$ has a critical point on this orbit. Let us show that $(x_1, \ldots, x_r)$ is this critical point. Indeed, for any $g \in \mathfrak{g}$

$$-B(\overline{x}_1, [g, x_1]) - \ldots - B(\overline{x}_r, [g, x_r]) = B(x_1, [g, x_1]) + \ldots + B(x_r, [g, x_r]) = 0. \blacksquare$$

Now let G be a simple simply-connected Lie group, let $\mathfrak{g}$ be its Lie algebra with a $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_k$. Let r be a maximal integer such that $\mathfrak{g}_r \neq 0$. We are going to change slightly our habits and denote the non-positive part of the grading $\bigoplus_{k\leq 0} \mathfrak{g}_k$ by $\mathfrak{p}$. Let $P \subset G$ be a parabolic subgroup with the Lie algebra $\mathfrak{p}$. We shall identify $\mathfrak{g}/\mathfrak{p}$ with $\bigoplus_{k>0} \mathfrak{g}_k$. Let $L \subset G$ be a connected reductive subgroup with Lie algebra $\mathfrak{g}_0$. Let V be an irreducible G-module. It is easy to see that there exists a $\mathbb{Z}$-grading $V = \bigoplus_{k\in\mathbb{Z}} V_k$ such that $\mathfrak{g}_i V_j \subset V_{i+j}$. Let R be a maximal integer such that $V_R \neq 0$. It is easy to see that $M_V = \bigoplus_{k<R} V_k$ (notice that V_R is an irreducible L-module). Now we shall prove Theorem 5.

Proof of Theorem 5. It is sufficient to prove that there exists a finite set of points $\{x_1, \ldots, x_N\} \subset \mathfrak{g}/\mathfrak{p}$ such that for any $x \in \mathfrak{g}/\mathfrak{p}$ and for any V we have $\mathrm{rk}\,\Psi_V(x) = \mathrm{rk}\,\Psi_V(x_i)$ for some i. Recall that L has finitely many orbits on each $\mathfrak{g}_k$ [Ri,Vi]. We pick some L-orbit $\mathcal{O}_i$ in each $\mathfrak{g}_i$. Then it is sufficient to find a finite set

[1] A short alternative proof was suggested by the referee. It is easy to verify that $\langle x_1, \ldots, x_r \rangle_{alg} = \langle ix_1, \ldots, ix_r \rangle_{\mathfrak{k}} \otimes \mathbb{C}$, where $\langle ix_1, \ldots, ix_r \rangle_{\mathfrak{k}}$ is the minimal real algebraic (or Malčev closed) Lie subalgebra of $\mathfrak{k}$ containing $ix_1, \ldots, ix_r$. Therefore, $\langle x_1, \ldots, x_r \rangle_{alg}$ is reductive and Lemma follows from the Richardson's theorem [Ri1].

of points as above only for points $x \in \mathfrak{g}/\mathfrak{p}$ of a form $x = x_1 + \ldots + x_r$, where $x_i \in \mathcal{O}_i$. For any orbit $\mathcal{O}_i$ let $\mathcal{H}_i$ denote the set of all possible homogeneous characteristics of all elements from $\mathcal{O}_i$. Clearly $\mathcal{H}_i$ is a closed $\mathrm{Ad}(L)$-orbit. Let $\hat{\mathcal{O}} = \mathcal{H}_1 \times \cdots \times \mathcal{H}_r \subset \mathfrak{g}_0^r$. Then by Lemma 5.1 the set of conjugacy classes of subalgebras $\mathfrak{g}(\mathcal{O})$ for closed L-orbits $\mathcal{O}$ in $\hat{\mathcal{O}}$ is finite. Let us show that for any r-tuple $(x_1, \ldots, x_r) \in \mathcal{O}_1 \times \ldots \times \mathcal{O}_r$ there exists an r-tuple $(h_1, \ldots, h_r) \in \hat{\mathcal{O}}$ such that h_i is a homogeneous characteristic of x_i and an L-orbit $\mathrm{Ad}(L)(h_1, \ldots, h_r)$ is closed. Indeed, after simultaneous conjugation of elements x_j by some element $g \in L$ we may suppose that any x_j has a homogeneous characteristic $h_j \in i\mathfrak{k}_0$ (in all degrees except at most one no conjugation is needed because of Moore–Penrose property, for one degree this is obvious). Then by Lemma 5.2 an orbit $\mathrm{Ad}(L)(h_1, \ldots, h_r)$ is closed. Since all functions $\mathrm{rk}\,\Psi_V$ are L-invariant, we may restrict ourselves to the points $x = \sum_i x_i \in \mathfrak{g}/\mathfrak{p}$ such that $x_i \in \mathcal{O}_i$, any x_i has a homogeneous characteristic h_i, and a conjugacy class of $\langle h_1, \ldots, h_r \rangle_{alg}$ is fixed. We claim that any function $\mathrm{rk}\,\Psi_V$ is constant along the set of these points. Moreover, we shall prove that

$$\mathrm{rk}\,\Psi_V(x) = \mathrm{codim}_{V^*_{-R}} (V^*_{-R})^{\langle h_1, \ldots, h_r \rangle_{alg}}. \tag{$*$}$$

Indeed,

$$\mathrm{rk}\,\Psi_V(x) = \dim \sum_i \mathrm{Im}\left(\mathrm{ad}(x_i)|_{V_{R-i}}\right).$$

Clearly V_R is $\mathrm{ad}(h_i)$-invariant and is killed by $\mathrm{ad}(x_i)$, therefore from the $\mathfrak{sl}_2$-theory we get that $V_R = \bigoplus_{k \geq 0} V_R^k$, where $\mathrm{ad}(h_i)|_{V_R^k} = k \cdot \mathrm{Id}$. Moreover,

$$\mathrm{Im}\left(\mathrm{ad}(x_i)|_{V_{R-i}}\right) = \bigoplus_{k>0} V_R^k.$$

Since V_R and V^*_{-R} are naturally dual to each other, we get that $\bigoplus_{k>0} V_R^k = \mathrm{Ann}((V^*_{-R})^0)$. Therefore,

$$\sum_i \mathrm{Im}\left(\mathrm{ad}(x_i)|_{V_{R-i}}\right) = \mathrm{Ann}\left(\cap_i (V^*_{-R})^{h_i}\right) = \mathrm{Ann}\left(\left(V^*_{-R}\right)^{\langle h_1, \ldots, h_r \rangle_{alg}}\right).$$

The formula $(*)$ follows. ∎

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Received May 13, 2001
and in final form August 27, 2001