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On the Homology of Free Nilpotent Lie Algebras

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Abstract. Explicit computations of the homology of some complex free nilpotent Lie algebras of small rank r, as modules over the general linear group $GL(r,\mathbb{C})$, are presented. A $GL(r,\mathbb{C})$-Poincaré duality theorem and a stabilization theorem for $r \to \infty$ are also proved.

The complex free N-step nilpotent Lie algebra of rank r, $\mathcal{L}(N,r)$, has a natural polynomial structure as a $GL(r,\mathbb{C})$-module; this structure is induced to its homology groups. The description of these representations is an interesting problem.

The problem for the case of free 2-step nilpotent Lie algebras has been solved in [8], even though the same description can be deduced from the classical paper of Kostant [5] as shown in [4]. The $GL(r,\mathbb{C})$-structure of the second homology group of any free nilpotent Lie algebra is also known, since it is isomorphic to $H(N+1)$, the subspace of $(N+1)$-brackets of the free Lie algebra and the $GL(r,\mathbb{C})$-structure of the free Lie algebra was determined in [9].

We present in this paper some basic results and a list of explicit computations as a contribution to get a better perspective of the general problem and hoping they could inspire others to work on this problem. The computations have been done using Maple V.

In §2 we first prove a $GL(r,\mathbb{C})$-version of the Poincaré duality for $\mathcal{L}(N,r)$, for all N. Recall that Poincaré duality holds for any finite dimensional nilpotent Lie algebra.

Next we investigate the relation between the homologies of two free N-step nilpotent Lie algebras of different rank. If $F_i(r)$ is the family of Young diagrams describing the i-th homology group of $\mathcal{L}(N,r)$ as a $GL(r,\mathbb{C})$-representation, then there are inclusions $F_i(r) \hookrightarrow F_i(r+1) \hookrightarrow F_i(r+2) \hookrightarrow \dots$ and moreover there is an r_0 such that $F_i(r) \hookrightarrow F_i(r+1)$ is a bijection for all $r \geq r_0$.

Further restrictions on the Young diagrams that can occur in $F_i(r)$ are deduced from a Gruenberg formula for Lie algebra homology.

In §3 we display all the homology groups computed as lists of Young diagrams. We computed the whole homology of the algebras $\mathcal{L}(III,2,)$, $\mathcal{L}(IV,2,)$, $\mathcal{L}(V,2)$ and $\mathcal{L}(III,3)$ and the groups $H_i(\mathcal{L}(III,r))$, $i=1,\dots,4$, for all r.

At the moment we cannot explain our results, but we are confident that by

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explaining these particular cases one will get a deeper understanding of the general problem.

1. Preliminaries

Fix a natural number $r \geq 2$, let $X = \{X_1, \dots, X_r\}$ and let V_r be the $\mathbb{C}$-vector space spanned by X. Consider the tensor algebra

$$T(V_r) = \mathbb{C} \oplus V_r \ \oplus V_r^{\otimes 2} \oplus \cdots \oplus V_r^{\otimes N} \oplus \cdots$$

as a Lie algebra via the usual bracket for associative algebras.

The complex free Lie algebra of rank r is the Lie subalgebra $\mathcal{L}(r)$ of $T(V_r)$ generated by V_r. $\mathcal{L}(r)$ inherits the grading from $T(V_r)$, so that

$$\mathcal{L}(r) = H_1(r) \oplus H_2(r) \oplus \cdots \oplus H_N(r) \oplus \cdots \tag{1}$$

The complex free N-step nilpotent Lie algebra of rank r is the Lie algebra

$$\mathcal{L}(N, r) = \frac{\mathcal{L}(r)}{\sum_{i \geq N+1} H_i(r)} .$$

By an abuse of notation we write this algebra as

$$\mathcal{L}(N, r) = H_1(r) \oplus H_2(r) \oplus \cdots \oplus H_N(r)$$

where the bracket of two homogeneous elements of degree i and j is 0 if $i + j > N$. The elements of $\mathcal{L}(r)$ are sometimes called Lie polynomials. Several characterizations can be found in [6]. The universal enveloping algebra of $\mathcal{L}(r)$ is the tensor algebra $T(V_r)$.

The subspace $H_i(r)$ in (1) is the subspace of homogeneous Lie polynomials of degree i, which is the sum of $H_{i_1, i_2, \dots, i_r}(r)$, the subspaces of homogeneous Lie polynomials of multidegree $(i_1, i_2, \dots, i_r)$, with $i_1 + \cdots + i_r = i$. If $M_r(i) = \dim H_i(r)$ and $M_r(i_1, i_2, \dots, i_r) = \dim H_{i_1, i_2, \dots, i_r}(r)$, then

$$M_r(i) \;=\; \frac{1}{i} \sum_{d \mid i} \mu(d) r^{i/d} \tag{2}$$

$$M_r(i_1, i_2, \dots, i_r) \;=\; \frac{1}{i} \sum_{d \mid i_k} \mu(d) \frac{\left(\frac{i}{d}\right)!}{\left(\frac{i_1}{d}\right)! \dots \left(\frac{i_r}{d}\right)!} \tag{3}$$

Explicit basis can be constructed for $H_i(r)$. One well known basis is the Hall basis $\mathcal{H}(i)$, which is recursively defined (c.f. [7]); $\mathcal{H}(1) = X$, $\mathcal{H}(2)$ is a subset of $[\mathcal{H}(1), \mathcal{H}(1)]$ and in general $\mathcal{H}(i)$ is a subset of $\cup_{j+k=i} [\mathcal{H}(j), \mathcal{H}(k)]$. We take $\mathcal{H} = \cup_{k=1}^{N} \mathcal{H}(k)$ as a basis for $\mathcal{L}(N, r)$.

Remark 1.1. All generators $\{X_1, \dots, X_r\}$ appear the same number of times, if counted with multiplicity, in all the different Lie polynomials in $\mathcal{H}(i)$. In fact, the number of times that X_k appears in $\mathcal{H}(i)$ is

$$T_k(i) = \sum_{j=1}^{i-1} j \sum_{i_1 + \cdots + i_{k-1} + i_{k+1} + \cdots + i_r = i-j} M(i_1, \dots, i_{k-1}, j, i_{k+1}, \dots, i_r).$$

It is straightforward to check, using (3), that $T_1(i) = \cdots = T_r(i)$. Since each element in $\mathcal{H}(i)$ is an i-bracket, it is clear that this common number is $T(i) = \frac{M_r(i)i}{r}$.

We include here some background for the representation theory of $GL(r, \mathbb{C})$, we fix some notation and make some conventions. We assume all representations to be of finite dimension. Proofs, as well as the general theory, can be found in any standard book, e.g. [1].

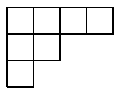
From now on, we fix a basis $\mathcal{B} = \{v_1, \dots, v_r\}$ of V_r and we denote by $\{E_{ij} : i, j = 1, \dots, r\}$ the canonical basis of $\mathrm{End}(V_r)$. Let $\mathfrak{gl}(V_r)$ be the Lie algebra of $GL(r, \mathbb{C})$ and fix the triangular decomposition $\mathfrak{gl}(V_r) = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, where $\mathfrak{n}^-$, $\mathfrak{h}$ and $\mathfrak{n}^+$ are the subalgebras consisting of endomorphisms whose matrices in the basis $\mathcal{B}$ are respectively strictly lower-triangular, diagonal and strictly upper-triangular. Now $\{E_{11}, \dots, E_{rr}\}$ is a basis of the Cartan subalgebra $\mathfrak{h}$. We will denote the corresponding dual basis by $\{\epsilon_1, \dots, \epsilon_r\}$. In particular $\{\epsilon_i - \epsilon_j : i < j\}$ is the set of *positive roots* corresponding to the triangular decomposition chosen above.

A linear functional λ on $\mathfrak{h}$ is called a *weight* if it takes integer values on the vectors $E_{ii} - E_{jj}$ for all $i < j$. A weight is said to be a *dominant weight* if $\lambda(E_{ii} - E_{jj}) \geq 0$. A partition of length r is an r-tuple of non-negative integers $\lambda = (\lambda_1, \dots, \lambda_r)$ such that $\lambda_1 \geq \cdots \geq \lambda_r$. Any partition of length r defines a dominant weight, which we will denote again by λ, given by $\sum \lambda_i \epsilon_i$.

By a polynomial representation we mean a finite dimensional representation of $GL(r, \mathbb{C})$ such that the matrix entries are given by polynomial functions. It is well known that every polynomial representation of $GL(r, \mathbb{C})$ can be decomposed as a sum of irreducible polynomial subrepresentations. In each irreducible polynomial representation W there is a unique (up to scalars) non-zero vector v such that $\mathfrak{n}^+.v = 0$ and $H.v = \lambda(H)v$ where λ is a dominant weight. Such a vector is called a *highest weight vector* of weight λ, and W is called an irreducible representation of highest weight λ. In addition, $W = \mathcal{U}(\mathfrak{n}^-).v$, where $\mathcal{U}(\mathfrak{n}^-)$ is the enveloping algebra of $\mathfrak{n}^-$.

The isomorphism classes of irreducible polynomial representations of $GL(V)$ are in one-to-one correspondence with the partitions of length r. Given λ, $\mathcal{W}_\lambda$ will denote an irreducible representation of highest weight λ.

A partition $\lambda = (\lambda_1, \dots, \lambda_r)$ is often represented by its Young diagram $Y(\lambda)$, a graphical arrangement of λ_i boxes in the i-th row starting in the first column. So, any polynomial representation of $GL(r, \mathbb{C})$ can be described (up to isomorphism) as a direct sum of Young diagrams with at most r rows. For example, to the partition $\lambda = (4, 2, 1, 0)$ it corresponds the Young diagram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2. General results

The group $GL(r, \mathbb{C})$ acts naturally on V_r and thus acts on $T(V_r)$, acting in each coordinate. This action is polynomial. Since

$$g[x, y] = g(x \otimes y - y \otimes x) = gx \otimes gy - gy \otimes gx = [gx, gy],$$

for any $x, y \in T(V_r)$ and $g \in GL(r,\mathbb{C})$, it follows that $GL(r,\mathbb{C})$ acts on $\mathcal{L}(r)$ preserving each one of the components in (1).

More generally, if W is a $GL(r,\mathbb{C})$-representation, then the exterior powers of W, $\Lambda^p W$, are also polynomial $GL(r,\mathbb{C})$-representations with $GL(r,\mathbb{C})$ acting in each coordinate. In this case the induced action of the Lie algebra $\mathfrak{gl}(r,\mathbb{C})$ on the exterior powers of W is given by

$$A(w_1 \wedge \ldots \wedge w_p) = \sum_{i=1}^{p} w_1 \wedge \ldots \wedge Aw_i \wedge \ldots \wedge w_p,$$

for $A \in \mathfrak{gl}(r,\mathbb{C})$ and $w_i \in W$.

Recall that the homology of a Lie algebra $\mathfrak{g}$, with trivial coefficients, is the homology of the complex

$$\cdots \longrightarrow \Lambda^p\mathfrak{g} \xrightarrow{\partial_p} \Lambda^{p-1}\mathfrak{g} \longrightarrow \cdots \longrightarrow \Lambda^2\mathfrak{g} \xrightarrow{\partial_2} \mathfrak{g} \longrightarrow \mathbb{C} \longrightarrow 0 \tag{4}$$

where

$$\partial_p(x_1 \wedge \ldots \wedge x_p) = \sum_{i<j} (-1)^{i+j+1} [x_i, x_j] \wedge x_1 \wedge \ldots \widehat{x_i} \ldots \widehat{x_j} \ldots \wedge x_p. \tag{5}$$

The exterior algebra complex, that computes the homology of $\mathcal{L}(N,r)$, is a $GL(r,\mathbb{C})$-module and it is easy to verify that the differential is a $GL(r,\mathbb{C})$-morphism. Therefore, the homology groups $H_p(\mathfrak{g}) = \ker \partial_p / \operatorname{Im} \partial_{p+1}$ inherit that $GL(r,\mathbb{C})$-module structure.

2.1. Poincaré duality.

All complex finite dimensional nilpotent Lie algebras $\mathfrak{g}$ enjoy Poincaré duality, that is $\dim H_i(\mathfrak{g}) = \dim H_{\dim \mathfrak{g} - i}(\mathfrak{g})$.

We prove here that for $\mathcal{L}(N,r)$ a stronger version of this duality holds. It is not true that $H_i(\mathcal{L}(N,r))$ and $H_{n-i}(\mathcal{L}(N,r))$ ($n = \dim \mathcal{L}(N,r)$) are $GL(r,\mathbb{C})$-isomorphic, but there is a simple formula relating these $GL(r,\mathbb{C})$-modules.

Theorem 2.1. *Let $\mathcal{L}(N,r)$ be the complex free N-step nilpotent Lie algebra of rank r and let n be its dimension. Let $M_r(i)$ be the dimension of the i-th homogeneous component of $\mathcal{L}(N,r)$ and set $T = \sum_{i=1}^{N} \frac{M_r(i)i}{r}$. Then*

$$H_{n-i}(\mathcal{L}(N,r)) \simeq H_i(\mathcal{L}(N,r))^* \otimes \det{}^T$$

as $GL(r,\mathbb{C})$-modules.

To prove this theorem we need some propositions.

From now on we set $\mathfrak{n} = \mathcal{L}(N,r)$, $n = \dim \mathcal{L}(N,r)$ and we fix a Hall basis for $\mathfrak{n}$, $\{e_1, \ldots, e_n\}$.

Consider the bilinear form B on $\Lambda\mathfrak{n}$ defined on homogeneous elements $v \in \Lambda^p\mathfrak{n}$ and $w \in \Lambda^q\mathfrak{n}$ by

$$B(v,w) = \begin{cases} 0, & \text{if } p+q \neq n; \\ \lambda, & \text{if } p+q = n; \end{cases}$$

here in the second case λ is such that $v \wedge w = \lambda(e_1 \wedge \ldots \wedge e_n)$. Let us take $\{v_{i_1 \ldots i_p} = e_{i_1} \wedge \ldots \wedge e_{i_p} : i_1 < \cdots < i_p,\ p = 1 \ldots n\} \cup \{v_\emptyset = 1\}$ as a basis of $\Lambda\mathfrak{n}$. The *Hodge map* $* : \Lambda\mathfrak{n} \longrightarrow \Lambda\mathfrak{n}$ is defined by $*v_{i_1 \ldots i_p} = cv_{\overline{i_1 \ldots i_p}}$ $(c = \pm 1)$, where $\overline{i_1 \ldots i_p}$ is the ordered complement of $\{i_1, \ldots, i_p\}$ in $\{1, \ldots, n\}$ and $v_{i_1 \ldots i_p \overline{i_1 \ldots i_p}} = cv_{1 \ldots n}$. Since $v \wedge *v = e_1 \wedge \ldots \wedge e_n$, $B(v, *v) = 1$ and B is non-degenerate.

Lemma 2.2. *If $v, w \in \Lambda\mathfrak{n}$, then $B(gv, gw) = (\det g)^T B(v, w)$ for any $g \in GL(r, \mathbb{C})$.*

Proof. We can assume that $v \in \Lambda^p\mathfrak{n}$ and $w \in \Lambda^q\mathfrak{n}$ with $p + q = n$. Let z be a generator of $\Lambda^n\mathfrak{n}$ and let $g = SU$ be the (multiplicative) Jordan decomposition of g; there exist $h \in GL(r, \mathbb{C})$ such that $P = hUh^{-1}$ is upper-triangular with all diagonal entries equal to 1. Since $Pz = z$, then also $Uz = z$. On the other hand, because of Remark 1.1

$$Sz = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_r \end{bmatrix} z = (\lambda_1^T \ldots \lambda_r^T) z.$$

Therefore $gz = (\det g)^T z$ and $B(gv, gw) = gv \wedge gw = g(v \wedge w) = g(\lambda z) = \lambda(\det g)^T z = (\det g)^T (v \wedge w) = (\det g)^T B(v, w)$. ∎

Proposition 2.3. $\Lambda^{n-i}\mathfrak{n} \simeq \left(\Lambda^i\mathfrak{n}\right)^* \otimes \det^T$ *as $GL(r, \mathbb{C})$-modules.*

Proof. For each $v \in \Lambda^{n-i}\mathfrak{n}$ let $\alpha_v \in \left(\Lambda^i\mathfrak{n}\right)^*$ be defined by $\alpha_v(w) = B(v, w)$. Let $\Psi : \Lambda^{n-i}\mathfrak{n} \longrightarrow \left(\Lambda^i\mathfrak{n}\right)^* \otimes \mathbb{C}$ be defined by $\Psi(v) = \alpha_v \otimes 1$, where $\mathbb{C}$ is the $GL(r, \mathbb{C})$-module $\det^T$. Since B is non-degenerate, Ψ is a $\mathbb{C}$-isomorphism. Moreover, Ψ is a $GL(r, \mathbb{C})$-isomorphism. In fact,

$$\begin{aligned} \Psi(gv)(w) &= (\alpha_{gv} \otimes 1)(w) = B(gv, w) = B(gv, gg^{-1}w) \\ &= (\det g)^T B(v, g^{-1}w) \end{aligned}$$

and

$$\begin{aligned} g\Psi(v)(w) &= g(\alpha_v \otimes 1)(w) = (g\alpha_v \otimes (\det g)^T)(w) \\ &= (\det g)^T B(v, g^{-1}w). \end{aligned}$$

∎

Lemma 2.4. *The square*

$$\begin{array}{ccc} \left(\Lambda^{n-i}\mathfrak{n}\right)^* \otimes \mathbb{C} & \xleftarrow{-(\partial^{n-i})^t \otimes \mathrm{id}} & \left(\Lambda^{n-(i+1)}\mathfrak{n}\right)^* \otimes \mathbb{C} \\ {\scriptstyle \Psi_i}\big\uparrow & & \big\uparrow{\scriptstyle \Psi_{i+1}} \\ \Lambda^i\mathfrak{n} & \xleftarrow[\partial^{i+1}]{} & \Lambda^{i+1}\mathfrak{n} \end{array}$$

is commutative if i is even and anti-commutative if i is odd.

Proof. Take $v \in \Lambda^{i+1}\mathfrak{n}$ and $w \in \Lambda^{n-i}\mathfrak{n}$; we may assume that $v = a_{j_1} \wedge \ldots \wedge a_{j_{i+1}}$ and $w = b_{l_1} \wedge \ldots \wedge b_{l_{n-i}}$ with $a_{j_p}, b_{l_q} \in \{e_1, \ldots, e_n\}$. To make formulas easier to read we will use the following notations

$$a(j_p, j_q) = a_{j_1} \wedge \ldots \wedge \widehat{a_{j_p}} \wedge \ldots \wedge \widehat{a_{j_q}} \wedge \ldots \wedge a_{j_{i+1}};$$
$$b(l_r, l_s) = b_{l_1} \wedge \ldots \wedge \widehat{b_{l_r}} \wedge \ldots \wedge \widehat{b_{l_s}} \wedge \ldots \wedge b_{l_{n-i}}.$$

By the definition of ∂ and Ψ we have

$$\partial^{i+1} v = \sum_{p<q} (-1)^{p+q+1} [a_{j_p}, a_{j_q}] \wedge a(j_p, j_q);$$
$$\Psi_i(\partial^{i+1} v) w = \sum_{p<q} (-1)^{p+q+1} [a_{j_p}, a_{j_q}] \wedge a(j_p, j_q) \wedge b_{l_1} \wedge \ldots \wedge b_{l_{n-i}}. \tag{6}$$

On the other hand

$$\begin{aligned} -(\partial^{n-i})^t (\Psi_{i+1} v) w &= -(\Psi_{i+1} v)(\partial^{n-i} w) = -v \wedge \partial^{n-i} w \\ &= \sum_{r<s} (-1)^{r+s} a_{j_1} \wedge \ldots \wedge a_{j_{i+1}} \wedge [b_{l_r}, b_{l_s}] \wedge b(l_r, l_s). \end{aligned} \tag{7}$$

We can identify v and w with the sets $\{a_{j_1}, \ldots, a_{j_{i+1}}\}$ and $\{b_{l_1}, \ldots, b_{l_{n-i}}\}$ respectively. Thus, since $\#v = i+1$ and $\#w = n-i$ it follows that $\#(v \cap w) \geq 1$.

If $\#(v \cap w) \geq 3$ then the sums in (6) and in (7) are both equal to 0.

If $\#(v \cap w) = 2$ we may assume that $a_{j_{p_o}} = b_{l_{r_o}} = x$ and $a_{j_{q_o}} = b_{l_{s_o}} = y$. Hence,

$$\begin{aligned} \Psi_i(\partial^{i+1} v) w &= (-1)^{p_0+q_0+1} [x, y] \wedge a(j_{p_0}, j_{q_0}) \wedge b_{l_1} \wedge \ldots \wedge b_{l_{n-i}} \\ &= (-1)^{p_0+q_0+r_0+s_0+2i-4} [x, y] \wedge b_{l_{r_0}} \wedge b_{l_{s_0}} \wedge a(j_{p_0}, j_{q_0}) \wedge b(l_{r_0}, l_{s_0}) \\ -(\partial^{n-i})^t (\Psi_{i+1} v) w &= (-1)^{r_0+s_0} a_{j_1} \wedge \ldots \wedge a_{j_{i+1}} \wedge [x, y] \wedge b(l_{r_0}, l_{s_0}) \\ &= (-1)^{r_0+s_0+p_0+q_0+i-2} [x, y] \wedge a_{j_{p_0}} \wedge a_{j_{q_0}} \wedge a(j_{p_0}, j_{q_0}) \wedge b(l_{r_0}, l_{q_0}) \end{aligned}$$

If $\#(v \cap w) = 1$ we may assume that $a_{j_{p_0}} = b_{l_{r_0}} = x$. It turns out that those summands in (6) with $p \neq p_0$ and $q \neq p_0$ are zero. Suppose $p = p_0$ in (6), then

$$\Psi_i(\partial^{i+1} v) w = \sum_{p_0<q} (-1)^{p_0+q+1} \underbrace{[a_{j_{p_0}}, a_{j_q}] \wedge a(j_{p_0}, j_q) \wedge b_{l_1} \wedge \ldots \wedge b_{l_{n-i}}}_{=c_q}$$

But $c_q \neq 0$ if and only if the coefficient of a_{j_q} in the expansion of $[a_{j_{p_o}}, a_{j_q}]$, as an element in $\mathfrak{n}$, is $\neq 0$. Since the homogeneous degree of $[a_{j_{p_o}}, a_{j_q}]$ is greater than that of a_{j_p} we conclude that $c_q = 0$ for all q and then the sum in (6) is equal to 0. The case when $q = p_0$ follows similarly.

In an analogous way we conclude that the sum in (7) is equal to 0. ∎

Proof. [Proof of Theorem 2.1] The homology of $\mathfrak{n}$ is the homology of the complex

$$\cdots \longrightarrow \Lambda^{i+1}\mathfrak{n} \xrightarrow{\partial^{i+1}} \Lambda^{i}\mathfrak{n} \xrightarrow{\partial^{i}} \Lambda^{i-1}\mathfrak{n} \longrightarrow \cdots$$

and its cohomology is the homology of the complex

$$\cdots \longleftarrow (\Lambda^{i+1}\mathfrak{n})^{*} \xleftarrow{-(\partial^{i+1})^{t}} (\Lambda^{i}\mathfrak{n})^{*} \xleftarrow{-(\partial^{i})^{t}} (\Lambda^{i-1}\mathfrak{n})^{*} \longleftarrow \cdots$$

Consider the following diagram of $GL(r,\mathbb{C})$-modules and $GL(r,\mathbb{C})$-morphisms with commutative or anti-commutative squares.

$$\begin{tikzcd}
\cdots \left(\Lambda^{i+1}\mathfrak{n}\right)^{*} \otimes \mathbb{C} & \left(\Lambda^{i}\mathfrak{n}\right)^{*} \otimes \mathbb{C} \arrow[l, "-(\partial^{i+1})^{t}\otimes \mathrm{id}"'] & \left(\Lambda^{i-1}\mathfrak{n}\right)^{*} \otimes \mathbb{C} \cdots \arrow[l, "-(\partial^{i})^{t}\otimes \mathrm{id}"'] \\
\cdots \Lambda^{n-(i+1)}\mathcal{L} \arrow[u, "\Psi"] & \Lambda^{n-i}\mathfrak{n} \arrow[l, "\partial^{n-i}"'] \arrow[u, "\Psi"] & \Lambda^{n-(i-1)}\mathfrak{n} \cdots \arrow[l, "\partial^{n-(i-1)}"'] \arrow[u, "\Psi"]
\end{tikzcd}$$

Since the upper complex computes the cohomology of $\mathfrak{n}$ it follows that Ψ induces a $GL(r,\mathbb{C})$-isomorphism

$$H_{n-i}(\mathfrak{n}) \simeq H^{i}(\mathfrak{n}) \otimes \det^{T}.$$

On the other hand the linear isomorphism $\alpha : H^{i}(\mathfrak{n}) \longrightarrow H_{i}(\mathfrak{n})^{*}$ defined by $\alpha([f]) = f_{*}$ for any $[f] \in H^{i}(\mathfrak{n})$, where $f_{*}([z]) = [f(z)]$ (recall that $\Lambda^{i}\mathfrak{n}^{*} \simeq (\Lambda^{i}\mathfrak{n})^{*}$), is a $GL(r,\mathbb{C})$-morphism. So we arrive to the desired $GL(r,\mathbb{C})$-isomorphism

$$H_{n-i}(\mathfrak{n}) \simeq H_{i}(\mathfrak{n})^{*} \otimes \det^{T}$$

■

Remark 2.5. It follows that as $GL(r,\mathbb{C})$-modules $H_{n} \simeq \det^{T}$, for every $r \geq 2$.

Remark 2.6. The previous isomorphism can be interpreted in terms of Young diagrams as follows.

For each diagram Y_{λ} that fits in the $r \times T$ rectangle let Y_{λ}^{c} be the diagram obtained by rotating 180 degrees the complementary arrangement of Y_{λ} in the $r \times T$ rectangle.

For example, if $r = 4$ we have $T = 6$, the diagram of the representation $\det^{T}$ is the 4×6 rectangle $\ydiagram{6,6,6,6}$ and for $Y_{\lambda} = \ydiagram{6,4,2,1}$ the corresponding diagram $Y_{\lambda}^{c} = \ydiagram{5,4,2}$.

Now, $Y_{\lambda} \in H_{n-i}$ if and only if $Y_{\lambda}^{c} \in H_{i}$. This follows from the fact that λ is a highest weight of a representation W of $GL(r,\mathbb{C})$ if and only if $(T,\ldots,T) - \lambda$ is a highest weight of $W^{*} \otimes \det^{T}$.

2.2. Homology stabilization.

The inclusion $\{x_1,\ldots,x_r\} \hookrightarrow \{x_1,\ldots,x_r,x_{r+1}\}$ induces a $\mathbb{C}$-monomorphism $T(V_r) \hookrightarrow T(V_{r+1})$, a Lie monomorphism $\mathcal{L}(N,r) \hookrightarrow \mathcal{L}(N,r+1)$ and a $\mathbb{C}$-monomorphism $\Lambda^{p}\mathcal{L}(N,r) \hookrightarrow \Lambda^{p}\mathcal{L}(N,r+1)$ for each $0 \leq p \leq n$. Moreover, by considering $GL(r,\mathbb{C})$ as the subgroup of $GL(r+1,\mathbb{C})$ consisting of matrices $\left(\begin{smallmatrix} A & 0 \\ 0 & 1 \end{smallmatrix}\right)$, with $A \in GL(r,\mathbb{C})$, all these maps are $GL(r,\mathbb{C})$-morphisms.

Let us denote $\mathfrak{n}' = \mathcal{L}(N,r+1)$ and consider the homology groups $H_p(\mathfrak{n})$ and $H_p(\mathfrak{n}')$, for $0 \leq p \leq n$, as $GL(r,\mathbb{C})$ and $GL(r+1,\mathbb{C})$-modules respectively.

Lemma 2.7. *If $0 \neq [v] \in H_p(\mathfrak{n})$ is a highest weight vector, then $0 \neq [v] \in H_p(\mathfrak{n}')$ is also a highest weight vector.*

Proof. Let $\lambda = (\lambda_1, \dots, \lambda_r)$ be the weight of $v \in \ker \partial_p$. It is clear that $v \in \ker \partial'_p$ and since $E_{r+1,r+1}v = 0$, then $\lambda' = (\lambda_1, \dots, \lambda_r, 0)$ is the weight of $v \in \ker \partial'_p$. In addition $E_{ij}v = 0$ for $1 \leq i < j = r+1$ and therefore v is a highest weight vector in $\ker \partial'_p$.

Suppose now that $v = \partial'_{p+1}(w)$ for some $w \in \Lambda^{p+1}\mathfrak{n}'$. We may assume that w is a weight vector since $w =$ weight vector + cycle. Moreover the weight of w is λ', because ∂'_{p+1} is a $GL(r+1, \mathbb{C})$-morphism, and thus $w \in \Lambda^{p+1}\mathfrak{n}$. Being $\partial_{p+1} = \partial'_{p+1}|_{\Lambda^{p+1}\mathfrak{n}}$, it follows that $v = \partial_{p+1}(w)$ or $0 = [v] \in H_p(\mathfrak{n})$. ■

Corollary 2.8. *Each Young diagram, counted with multiplicity, in the decomposition of $H_p(\mathfrak{n})$ is in the decomposition of $H_p(\mathfrak{n}')$.*

Proof. Let Y_λ be a Young diagram in the decomposition of $H_p(\mathfrak{n})$ and take v a highest weight vector of weight λ in $H_p(\mathfrak{n})$. By the previous Lemma v is a highest weight vector in $H_p(\mathfrak{n}')$ of weight $\lambda' + (\lambda_1, \dots, \lambda_r, 0)$, so that the corresponding Young diagram is Y_λ. ■

Theorem 2.9. *If $r \geq pN$, then the Young diagram decompositions of the homology groups $H_p(\mathcal{L}(N,r))$ and $H_p(\mathcal{L}(N,r+1))$ are identical.*

Proof. Let $Y_{\lambda'}$ be a Young diagram in the decomposition of $H_p(\mathcal{L}(N,r+1))$ and take v a corresponding highest weight vector with weight $\lambda' = (\lambda_1, \dots, \lambda_{r+1})$. Since $v \in \Lambda^p\mathcal{L}(N, r+)$, then $\lambda_1 + \dots + \lambda_{r+1} \leq pN$ and therefore $\lambda_{r+1} = 0$; this means that $v \in \Lambda^p\mathcal{L}(N,r)$ and finally $Y_{\lambda'} = Y_\lambda$, with $\lambda = (\lambda_1, \dots, \lambda_r)$ and Y_λ is in the Young decomposition of $H_p(\mathcal{L}(N,r))$. ■

Remark 2.10. The Young diagram decomposition of $H_p(\mathcal{L}(N,s))$, if $s \leq r$, can be read directly from that of $H_p(\mathcal{L}(N,r))$. In fact, $H_p(\mathcal{L}(N,s))$ decomposes as the sum of all those Young diagrams in $H_p(\mathcal{L}(N,r))$ with at most s rows.

Remark 2.11. Theorem 2.9 says that in order to compute $H_p(\mathcal{L}(N,r))$ for all r's it is enough to do it for $r = pN$. It turns out, in the cases treated here, that a smaller r is enough as well.

2.3. Minimal weights.

Let $0 \to \mathfrak{h} \to \mathcal{L} \to \mathfrak{g} \to 0$ be a free presentation of $\mathfrak{g}$, an arbitrary Lie algebra over $\mathbb{C}$. Let $\mathcal{F}$ be the augmentation ideal of $\mathcal{U}(\mathcal{L})$ (universal enveloping algebra of $\mathcal{L}$) and $\mathcal{R}$ be the ideal of $\mathcal{U}(\mathcal{L})$ generated by $i(\mathfrak{h})$, where $i : \mathcal{L} \to \mathcal{U}(\mathcal{L})$ is the canonical map. We have then a Gruenberg like formula for the homology of $\mathfrak{g}$.

Theorem 2.12. *Let $\mathfrak{g}$, $\mathcal{F}$ and $\mathcal{R}$ be as above. Then*

$$\begin{aligned} H_{2n+1}(\mathfrak{g}) &= \frac{\mathcal{F}\mathcal{R}^n \cap \mathcal{R}^n\mathcal{F}}{\mathcal{F}\mathcal{R}^n\mathcal{F} + \mathcal{R}^{n+1}}, \\ H_{2n}(\mathfrak{g}) &= \frac{\mathcal{R}^n \cap \mathcal{F}\mathcal{R}^{n-1}\mathcal{F}}{\mathcal{R}^n\mathcal{F} + \mathcal{F}\mathcal{R}^n}. \end{aligned}$$

These formulas for Lie algebra homology are the analogous of those for group homology given by Gruenberg in [2] and [3]. Since $\mathcal{U}(\mathcal{L})$ is a tensor algebra the ideals $\mathcal{F}$ and $\mathcal{R}$ are free as left ideals. The proof is then almost identical as Gruenberg's proof.

Definition 2.13. Given a Young diagram Y_λ we define its *total weight* as $|Y_\lambda| = \lambda_1 + \cdots + \lambda_r$, if $\lambda = (\lambda_1, \ldots, \lambda_r)$.

Theorem 2.14. *Let $\mathfrak{n}$ be a complex free N-step nilpotent Lie algebra of any rank. If Y_λ is a Young diagram in the decomposition of $H_i(\mathfrak{n})$, then*

$$\begin{aligned} nN + (n+1) \leq\ & |Y_\lambda|, \quad \text{if } i = 2n+1 \\ nN + n \leq\ & |Y_\lambda|, \quad \text{if } i = 2n \end{aligned}$$

Proof. In all cases $\mathcal{U}(\mathcal{L})$ is a tensor algebra, $\mathcal{F}$ is the ideal of polynomials without constant term and $\mathcal{R}$ is the ideal generated by the Lie polynomials of degree $\geq N+1$. ∎

3. Explicit computations

To compute explicitly the homology of $\mathcal{L}(N,r)$ we can proceed directly with the complex (4). The first step is the decomposition of the exterior powers of $\mathcal{L}(N,r)$ as a sum of irreducible $GL(r,\mathbb{C})$-representations. Then we compute the differential ∂ (see 5) on the highest weight vectors corresponding to the previous decomposition to finally determine the homology groups.

On the other hand we can use a Laplacian Δ, as in [5]. Recall that two linear operators ∂ and d on $\Lambda\mathfrak{n}$ such that $\partial^2 = d^2 = 0$ are disjoint if

1. $d\partial(x) = 0 \Longrightarrow \partial(x) = 0$;

2. $\partial d(x) = 0 \Longrightarrow d(x) = 0$.

In this case there is a canonical isomorphism from the kernel of the Laplacian $\Delta = d\partial + \partial d$ to the derived space of homology of $\Lambda\mathfrak{n}$, $\frac{\ker d}{\operatorname{Im} d}$. If $\partial = d^*$, the adjoint of d with respect to an inner product defined on $\Lambda\mathfrak{n}$, then d and ∂ are disjoint.

Proposition 3.1. *There is a $GL(r,\mathbb{C})$-morphism d in $\Lambda\mathfrak{n}$ which is disjoint to ∂.*

Proof. Consider the unitary group $U(r) \subseteq GL(r,\mathbb{C})$ and $\mathfrak{u}(r)$ its Lie algebra. There is on $\mathfrak{n}$ a $U(r)$-invariant inner product. (It is unique, up to scalars, in each irreducible component of $\mathfrak{n}$.) We extend this inner product to the exterior powers of $\mathfrak{n}$, via the determinant, remaining $U(r)$-invariant. Let $d = \partial^*$ be the adjoint of ∂. d is $U(r)$-equivariant and therefore also $\mathfrak{u}(r)$-equivariant. Since d is $\mathbb{C}$-linear and $\mathfrak{gl}(r,\mathbb{C})$ is the complexification of $\mathfrak{u}(r)$ it turns out to be a $\mathfrak{gl}(r,\mathbb{C})$-morphism and then a $GL(r,\mathbb{C})$-morphism. ∎

Let $d_p = d|_{\Lambda^p\mathfrak{n}}$ and let $\{f_1, \dots, f_n\}$ be an orthonormal basis of $\mathfrak{n}$. Then

$$d_p(x_1 \wedge \ldots \wedge x_p) = \sum_{i=1}^{p} (-1)^{i+1} d_1(x_i) \wedge x_1 \wedge \ldots \widehat{x_i} \ldots \wedge x_p,$$

where $d_1(f_k) = \sum_{i<j} c_{ij}^k f_i \wedge f_j$ and $[f_i, f_j] = \sum c_{ij}^k f_k$. It is straightforward to verify the equation $\langle \partial_{p+1} v, w \rangle = \langle v, d_p w \rangle$ on the induced orthonormal basis of $\Lambda\mathfrak{n}$ (recall the definition of ∂_p in (4)).

We used both methods in the cases of rank 2 algebras and $\mathcal{L}(III, 3)$ and the direct computations in the case of $\mathcal{L}(III, r)$.

We may notice that some of the spaces involved are very big. For example, the algebra $\mathcal{L}(III, 7)$ is of dimension 140, hence its fourth exterior power has dimension 15329615 and its fifth exterior power has dimension 416965528. These spaces are involved in the computation of $H_4(\mathcal{L}(III, 7))$.

3.1. Presentation of data.

In all cases the homology groups are displayed as lists of Young diagrams, that is as sums of irreducible $GL(r, \mathbb{C})$-representations. The dimension of each group and the total dimension are also given. We notice that in all cases $H_0 = \mathbb{C}$, so we omit it. For large algebras we give the corresponding number T and therefore, by virtue of Theorem 2.1 and Remark 2.6, we only show half of the homology groups. We do not present the highest weight cycles corresponding to each Young diagram because of the length of all this data. However, we can make them available to the interested reader.

For the cases were a Laplacian has been used we include more information. We write down Hall basis for these algebras and we give the $U(r)$-invariant inner products and the corresponding d_1 operators.

3.2. Rank 2 algebras.

Hall basis for the first 5 homogeneous components of the free Lie algebra $\mathcal{L}(2)$ are listed below.

$$\begin{aligned}
H_1 &= \langle x, y \rangle \\
&= \langle r_1, r_2 \rangle \\
H_2 &= \langle [xy] \rangle \\
&= \langle t_1 \rangle \\
H_3 &= \langle [x[xy]], [y[xy]] \rangle \\
&= \langle u_1, u_2 \rangle \\
H_4 &= \langle [x[x[xy]]], [y[x[xy]]], [y[y[xy]]] \rangle \\
&= \langle v_1, v_2, v_3 \rangle \\
H_5 &= \langle [x[x[x[xy]]]], [y[x[x[xy]]]], [y[y[x[xy]]]], [y[y[y[xy]]]], [[xy][x[xy]]], \\
&\quad [[xy][y[xy]]] \rangle \\
&= \langle w_1, w_2, w_3, w_4, w_5, w_6 \rangle
\end{aligned}$$

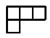
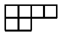
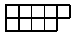
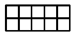
All the basis vectors are weight vectors. Since weight vectors of different weights are orthogonal, we only compute the norms and the non-zero products.

$$||r_1|| = 1,\ ||r_2|| = 1;$$
$$||t_1|| = 1;$$
$$||u_1|| = 1,\ ||u_2|| = 1;$$
$$||v_1|| = 1,\ ||v_2|| = \frac{1}{2},\ ||v_3|| = 1;$$
$$||w_1|| = 1,\ ||w_2|| = \frac{7}{9},\ ||w_3|| = \frac{4}{9},\ ||w_4|| = 1,\ ||w_5|| = 1,\ ||w_6|| = 1,$$
$$< w_2, w_5 >= -\frac{2}{3},\ < w_3, w_6 >= -\frac{1}{3}.$$

This inner product allows us to compute d_1 (and hence d). We have,

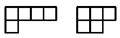
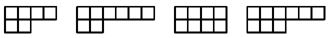
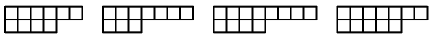
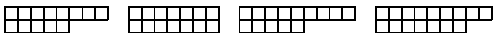
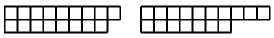
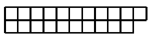
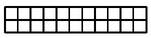
$$d_1(r_1) = 0,\ d_1 = (r_2) = 0;$$
$$d_1(t_1) = r_1 \wedge r_2;$$
$$d_1(u_1) = r_1 \wedge t_1,\ d_1(u_2) = r_2 \wedge t_1;$$
$$d_1(v_1) = r_1 \wedge u_1,\ d_1(v_2) = \tfrac{1}{2} r_1 \wedge u_2 + \tfrac{1}{2} r_2 \wedge u_1,\ d_1(v_3) = r_2 \wedge u_2;$$
$$d_1(w_1) = r_1 \wedge v_1,\ d_1(w_2) = \tfrac{2}{9} r_1 \wedge v_2 + \tfrac{7}{9} r_2 \wedge v_1 - \tfrac{2}{3} t_1 \wedge u_1,$$
$$d_1(w_3) = \tfrac{1}{9} r_1 \wedge v_3 + \tfrac{8}{9} r_2 \wedge v_2 - \tfrac{1}{3} t_1 \wedge u_2,\ d_1(w_4) = r_2 \wedge v_3;$$
$$d_1(w_5) = \tfrac{2}{3} r_1 \wedge v_2 + t_1 \wedge u_1 - \tfrac{2}{3} r_2 \wedge v_1,$$
$$d_1(w_6) = \tfrac{2}{3} r_1 \wedge v_3 + t_1 \wedge u_2 - \tfrac{2}{3} r_2 \wedge v_2.$$

3.3. The homology of $\mathcal{L}(III, 2)$.

Group	Young decomposition	Dimension
H_1		2
H_2		3
H_3		3
H_4		2
H_5		1

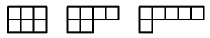
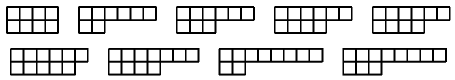
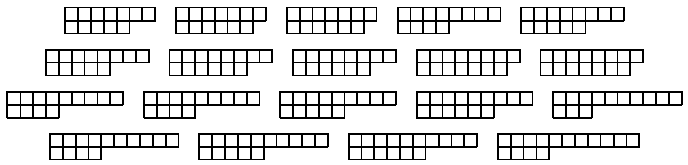
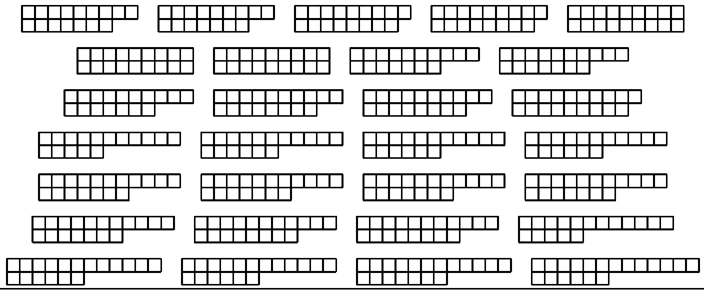
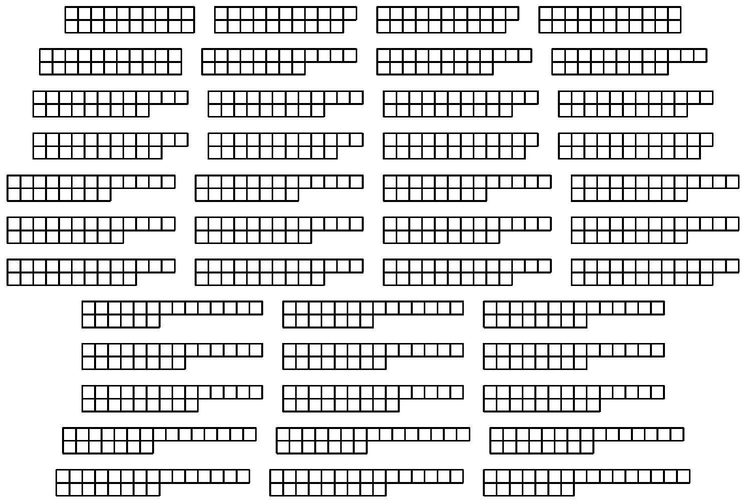
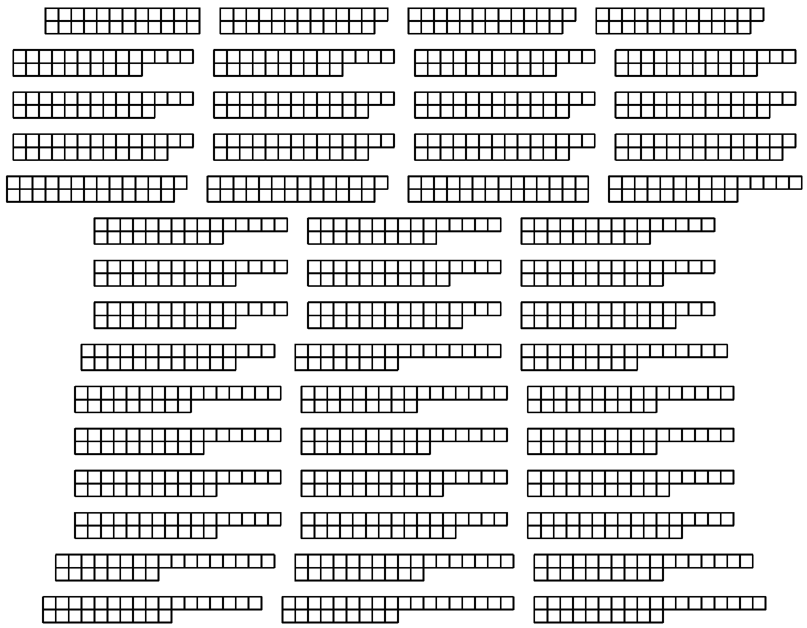
Total homology: 12

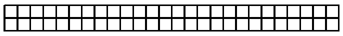
3.4. The homology of $\mathcal{L}(IV, 2)$.

Group	Young decomposition	Dimension
H_1		2
H_2		6
H_3		13
H_4		16
H_5		13
H_6		6
H_7		2
H_8		1

Total homology: 60

3.5. The homology of $\mathcal{L}(V, 2)$.

Group	Young decomposition	Dim
H_1		2
H_2		9
H_3		39
H_4		85
H_5		145
H_6		206
H_7		258

$\det^T =$  $(T = 26)$. Total homology: 1232

3.6. The homology of $\mathcal{L}(III, 3)$.

A Hall basis for $\mathcal{L}(III, 3)$ is given by

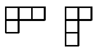
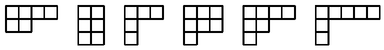
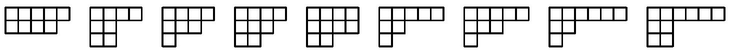
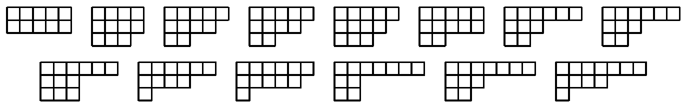
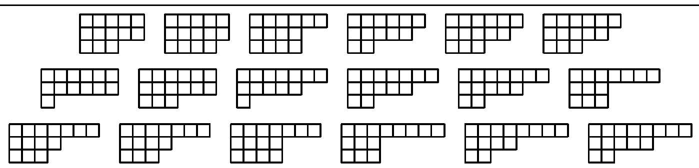
$$\begin{aligned} H_1 &= \langle x, y, z\rangle \\ &= \langle r_1, r_2, r_3\rangle \\ H_2 &= \langle [xy], [xz], [yz]\rangle \\ &= \langle t_1, t_2, t_3\rangle \\ H_3 &= \langle [x[xy]], [x[xz]], [y[xy]], [y[xz]], [y[yz]], [z[xy]], [z[xz]], [z[yz]]\rangle \\ &= \langle u_1, u_2, \dots, u_8\rangle \end{aligned}$$

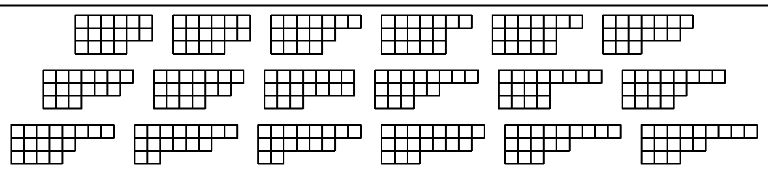
As in the case of rank 2 algebras all basis vectors are weight vectors and therefore we only compute the norms and the non-zero products (vectors of different weights are orthogonal).

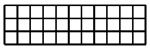
$$\begin{gathered} ||r_1|| = 1,\ ||r_2|| = 1,\ ||r_3|| = 1; \\ ||t_1|| = 1,\ ||t_2|| = 1,\ ||t_3|| = 1; \\ ||u_1|| = 1,\ ||u_2|| = 1,\ ||u_3|| = 1,\ ||u_5|| = 1,\ ||u_7|| = 1,\ ||u_8|| = 1, \\ ||u_4|| = \frac{2}{3},\ ||u_6|| = \frac{2}{3}; \\ < u_4, u_6 >= \frac{1}{3}. \end{gathered}$$

This inner product allow us to compute d_1 (and hence d). We have,

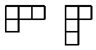
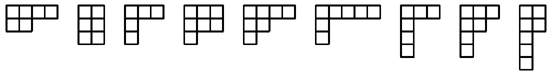
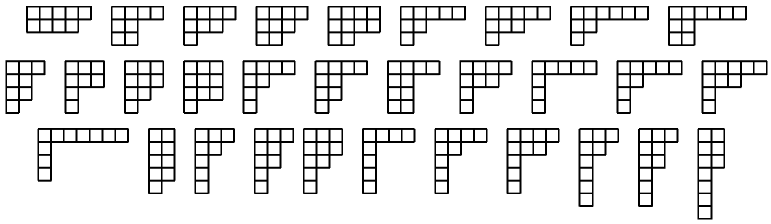
$$\begin{gathered} d_1(r_1) = 0,\ d_1 = (r_2) = 0,\ d_1(r_3); \\ d_1(t_1) = r_1 \wedge r_2,\ d_1(t_2) = r_1 \wedge r_3,\ d_1(t_3) = r_2 \wedge r_3; \\ d_1(u_1) = r_1 \wedge t_1,\ d_1(u_2) = r_1 \wedge t_2,\ d_1(u_3) = r_2 \wedge t_1; \\ d_1(u_5) = r_2 \wedge t_3,\ d_1(u_7) = r_3 \wedge t_2,\ d_1(u_8) = r_3 \wedge t_3; \\ d_1(u_4) = \tfrac{1}{3} r_1 \wedge t_3 + \tfrac{2}{3} r_2 \wedge t_2 + \tfrac{1}{3} r_3 \wedge t_1, \\ d_1(u_6) = -\tfrac{1}{3} r_1 \wedge t_3 + \tfrac{1}{3} r_2 \wedge t_2 + \tfrac{2}{3} r_3 \wedge t_1\ . \end{gathered}$$

Group	Young decomposition	Dim
H_1		3
H_2		18
H_3		70
H_4		171
H_5		327
H_6		462

H_7		504

$\det^T =$  $(T = 11)$. Total homology: 2608

3.7. The first homology groups of $\mathcal{L}(III, r)$.

Group	Young decomposition	Dim
H_1		$p_1(r)$
H_2		$p_2(r)$
H_3		$p_3(r)$
H_4		$p_4(r)$

The polynomials $p_i(r)$ are:

$$p_1(r) = r,$$

$$p_2(r) = \frac{1}{4}r^2(r+1)(r-1) \sim \frac{1}{4}r^4 \ ,$$

$$p_3(r) = \frac{1}{420}(10r^4 + 14r^3 + 31r^2 - 56r - 74)r(r+1)(r-1) \sim \frac{1}{42}r^7 \ ,$$

$$p_4(r) = \frac{1}{1209600}(1001r^7 + 3975r^6 + 18371r^5 - 4635r^4 - 54436r^3$$
$$-100500r^2 + 111744r + 120960)r(r+1)(r-1) \sim \frac{1001}{1209600}r^{10} \ .$$

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