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The Lie Algebra of a Nuclear Group

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Abstract. The Lie algebra of a nuclear group is a locally convex nuclear vector space.
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1. The Lie Algebra of Abelian Topological Groups

Let G be an abelian topological group. The set of all one–parameter–subgroups

$$\mathcal{L}(G) := \{\lambda : \mathbb{R} \to G : \ \lambda \text{ is a continuous homomorphism}\}$$

is called the *Lie algebra* of G. We define addition pointwise and scalar multiplication by the following formula

$$\mathbb{R} \times \mathcal{L}(G) \to \mathcal{L}(G),\ (s, \lambda) \mapsto (s \cdot \lambda : t \mapsto \lambda(st)).$$

It is topologized with the compact–open topology. We use the notation $P(K, U) := \{f : X \to Y : \ f(K) \subseteq U\}$ for sets X, Y and subsets $K \subseteq X$ and $U \subseteq Y$. If no confusion can arise, we will use the same notation $P(\cdot, \cdot)$ for sets of continuous functions, respectively, continuous homomorphisms. The system of neighbourhoods of the neutral element 0 of an abelian topological group G is denoted by $\mathcal{U}_G(0)$. We set $G^* := \{\chi : G \to \mathbb{T} : \ \chi \text{ is a continuous homomorphism}\}$ where $\mathbb{T}$ is the compact group of complex numbers of modulus 1. With multiplication defined pointwise and endowed with the compact–open topology, G^* is an abelian topological group, named *dual group* of G. We introduce the canonical homomorphism

$$\alpha_G : G \to G^{**},\ x \mapsto (\chi \mapsto \chi(x)).$$

The famous Pontryagin van–Kampen duality theorem states that if G is a locally compact abelian (LCA) group then α_G is a topological isomorphism.
[A proof can be found in [3], p. 351 or [4], p.378 or [7], p.84.]

Lemma 1.1. *α_G is continuous if and only if every compact subset of G^* is equicontinuous.*

If G is metrizable (more generally, a k–space) then α_G is continuous.

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[A proof of these well known facts can be found in [1] (5.10) and (5.12).]

If $\varphi : G \to H$ is a continuous homomorphism between abelian topological groups, we define the *dual homomorphism* φ^* in the usual way:
$\varphi^* : H^* \to G^*,\ \chi \mapsto \chi \circ \varphi$. The *polar* of a subset $A \subseteq G$ (G an abelian topological group) is defined by $A^0 := \{\chi \in G^* |\ \forall x \in A :\ \mathrm{Re}\chi(x) \geq 0\ \}$. The set A is called **quasi–convex** if for every $x \in G \setminus A$ there exists $\chi \in A^0$ such that $\mathrm{Re}\chi(x) < 0$. The group G is named *locally quasi–convex* if the neutral element has a neighbourhood basis consisting of quasi–convex sets. This generalizes the setting of local convexity; more precisely, a Hausdorff topological vector space is locally convex if and only if it is a Hausdorff locally quasi–convex group. (See e.g. (2.4) in [2].)

It is easy to verify that α_G is injective for every locally quasi–convex Hausdorff group G.

Proposition 1.2. (i) *For every abelian topological group G the associated Lie algebra $\mathcal{L}(G)$ is a topological vector space.*
(ii) *If G is a Hausdorff space, so is $\mathcal{L}(G)$.*
(iii) *If G is a locally quasi–convex Hausdorff group then $\mathcal{L}(G)$ is a locally convex vector space.*

Proof. i) See e.g. [3] Proposition (7.36), p. 334.

ii) For $\lambda_1 \neq \lambda_2 \in \mathcal{L}(G)$ there exist $t \in \mathbb{R}$ such that $\lambda_1(t) \neq \lambda_2(t)$ and disjoint open neighbourhoods $U_i \in \mathcal{U}(\lambda_i(t))$ of $\lambda_i(t)$ $(i = 1, 2)$. Hence the sets $P(\{t\}, U_i)$ form disjoint neighbourhoods of the λ_i.

iii) According to Proposition (2.4) in [2], it is sufficient to show that $\mathcal{L}(G)$ is a locally quasi–convex group. Therefore, we will prove that the neighbourhood basis $\{P(K, U) :\ K \text{ compact, } U \in \mathcal{U}_G(0) \text{ quasi–convex}\}$ consists of quasi–convex sets. To this end, take $\lambda_0 \notin P(K, U)$ and pick $t_0 \in K$ such that $\lambda_0(t_0) \notin U$. By the quasi-convexity of U, there exists $\chi \in U^0$ such that $\mathrm{Re}\chi(\lambda_0(t_0)) < 0$.

Since one–point sets are compact, the point evaluations $\mathcal{L}(G) \to G,\ \lambda \to \lambda(t)$ are continuous and hence $X : \mathcal{L}(G) \to \mathbb{T},\ \lambda \mapsto \chi(\lambda(t_0))$ is a character of $\mathcal{L}(G)$. It is obvious that $X \in P(K, U)^0$ and that $\mathrm{Re}X(\lambda_0) < 0$. ∎

Proposition 1.3. *Let $\varphi : H \to G$ be a continuous homomorphism between abelian topological groups. Then the induced mapping $\varphi_* : \mathcal{L}(H) \to \mathcal{L}(G),\ \lambda \mapsto \varphi \circ \lambda$ is a continuous linear mapping. Moreover, if φ is an embedding, so is φ_*.*

Proof. A straightforward calculation shows that φ_* is a linear mapping. The assertion follows from the fact that $\varphi_*(P(K, \varphi^{-1}(U)) = P(K, U) \cap \mathrm{im}\varphi_*$ (for $K \subseteq \mathbb{R}$ compact and $U \in \mathcal{U}_G(0)$). ∎

Proposition 1.4. *Let H be an abelian locally quasi–convex Hausdorff group such that α_H is continuous. Then*

$$\Phi : \mathcal{L}(H) \to \mathrm{Hom}_{co}(H^*, \mathbb{R}^*),\ \lambda \mapsto \lambda^*$$

is an embedding.

Proof. It is easily verified that Φ is a group homomorphism. Since H is a locally quasi–convex Hausdorff group, α_H is injective, and from this fact it is straightforward that Φ is also injective. For every compact subset $K \subseteq \mathbb{R}$ and every quasi–convex neighbourhood $U \in \mathcal{U}_H(0)$ we have

$$\Phi(P(K,U)) = \mathrm{im}(\Phi) \cap P(U^0, K^0). \qquad (*)$$

[Proof of $\subseteq$: Let $\lambda \in P(K,U), \chi \in U^0$, and $x \in K$. Then we obtain $\Phi(\lambda)(\chi)(x) = \lambda^*(\chi)(x) = \chi(\lambda(x))$. Since $\mathrm{Re}\chi(\lambda(x)) \geq 0$ for every $\chi \in U^0$, we get $\lambda(x) \in U$.
Proof of $\supseteq$: Let $\lambda^* \in P(U^0, K^0)$, $x \in K$, and $\chi \in U^0$. It follows from $\lambda^*(\chi) \in K^0$ that $\mathrm{Re}\chi(\lambda(x)) = \mathrm{Re}\lambda^*(\chi)(x) \geq 0$. Since U is quasi–convex, this yields $\lambda(x) \in U$.]

By assumption, α_H is continuous, hence the sets $\{U^0 : U \in \mathcal{U}_H(e)\}$ form a cobasis for the compact subsets of H^* (1.1). So the assertion follows from $(*)$. ∎

2. The Lie algebra of a nuclear group

The class of nuclear groups was introduced by Banaszczyk in [2]. It is a Hausdorff variety consisting of abelian groups which contains all LCA groups and all locally convex nuclear vector spaces and which is closed with respect to forming arbitrary products, subgroups and Hausdorff quotient groups. (See e.g. section 7 in [2].)

Here we use a technical description making use of the Kolmogoroff–diameters whose original setting is in the context of locally convex spaces.

Let V be a vector space. For symmetric and convex subsets $X, Y \subseteq V$ we define

$$\mathrm{d}_k(X,Y) := \inf\{c > 0 | \ \exists L_c \leq V : \ \dim L_c < k \text{ and } X \subseteq c \cdot Y + L_c\}.$$

As usual, $\inf \emptyset = \infty$. The number $\mathrm{d}_k(X,Y)$ is called the k–th *Kolmogoroff–diameter* of X with respect to Y.

A locally convex vector space V is called a *nuclear vector space* if for every symmetric and convex neighbourhood U of 0 there exists W, another symmetric and convex neighbourhood of 0, such that $\mathrm{d}_k(W,U) \leq k^{-1}$ for all $k \in \mathbb{N}$. Roughly speaking, W is considerably smaller than U.

Next, we wish to define the Kolmogoroff–diameters for groups.

Let W, U be subsets of a group G. We set $(\mathrm{d}_k(W,U)) \leq (c_k)$ ((c_k) a sequence of real numbers) if there exist: a vector space V, symmetric and convex subsets $X, Y \subseteq V$ which satisfy $\mathrm{d}_k(X,Y) \leq c_k$ for all $k \in \mathbb{N}$, a subgroup H of V, and a homomorphism $\varphi : H \to G$ which fulfills $W \subseteq \varphi(X \cap H)$ and $\varphi(H \cap Y) \subseteq U$.

An abelian Hausdorff topological group G is called a *nuclear group* if for every neighbourhood U of the neutral element 0, for every $c > 0$, and every $m \in \mathbb{N}$, there exists a neighbourhood W of 0 which satisfies $(\mathrm{d}_k(W,U)) \leq (c \cdot k^{-m})$.

Lemma 2.1. *Let G be a nuclear group such that α_G is continuous. Then for every compact subset $K \subseteq G^*$, every $c > 0$, and every $m \in \mathbb{N}$ there exists a compact subset $S \subseteq G^*$ which satisfies $(\mathrm{d}_k(K,S))_{k \in \mathbb{N}} \leq (c \cdot k^{-m})_{k \in \mathbb{N}}$.*

Proof. Fix a compact subset $K \subseteq G^*$, $c > 0$, and $m \in \mathbb{N}$.

Since α_G is continuous, there exists $U \in \mathcal{U}_G(0)$ such that $K \subseteq U^0$ (1.1). By assumption, G is a nuclear group, hence we can find a neighbourhood $W \in \mathcal{U}_G(e)$ such that $(\mathrm{d}_k(W, U)) \leq \widetilde{c} k^{-\widetilde{m}}$ for $\widetilde{c} = \frac{c}{\gamma_{\widetilde{m}}}$ and $\widetilde{m} = m + 5$. (As usual, $\gamma_{\widetilde{m}}$ denotes the universal constant defined in (2.14) in [2].) According to (16.4) in [2], we get $(\mathrm{d}_k(U^0, W^0)) \leq (\widetilde{c}\gamma_{\widetilde{m}} \cdot k^{-\widetilde{m}+5}) = (c \cdot k^{-m})$.

Since $S := W^0$ is compact ((3.5) in [1]) and $K \subseteq U^0$ we get $(\mathrm{d}_k(K, S)) \leq (c \cdot k^{-m})$ and hence the assertion follows. ■

Theorem 2.2. *Let G be a nuclear group such that α_G is continuous. Then $\mathrm{Hom}_{co}(G^*, \mathbb{R})$ is a nuclear group, too.*

Proof. For $K \subseteq G^*$ compact, $c > 0$, and $m \in \mathbb{N}$ we put:

$$\widetilde{m} := m + 4 \text{ und } \widetilde{c} < \min\left(\frac{c}{2\gamma_{m+4}\gamma_{m+2}}, \frac{\sqrt{6}}{2\pi\gamma_{m+4}}\right).$$

According to the Lemma 2.1, there exists a compact subset $S \subseteq G^*$ which satisfies $(\mathrm{d}_k(K, S)) \leq (\widetilde{c} k^{-\widetilde{m}})$. Thus there exist a vector space V, a subgroup $H \leq V$, two symmetric and convex subsets $X, Y \subseteq V$ which satisfy $\mathrm{d}_k(X, Y) \leq \widetilde{c} k^{-\widetilde{m}}$ for all $k \in \mathbb{N}$, and a homomorphism $\varphi : H \to G^*$ such that $K \subseteq \varphi(X \cap H)$ and $\varphi(Y \cap H) \subseteq S$.

According to (2.14) in [2], there exist pHsns (pre-Hilbert seminorms) p, q defined on $\langle X \rangle_{\mathbb{R}}$ such that their closed unit balls B_p, respectively B_q, satisfy $X \subset B_p$, $B_q \subseteq Y$ and $\mathrm{d}_k(B_p, B_q) \leq \gamma_{\widetilde{m}} \widetilde{c} k^{-\widetilde{m}+2}$. Without loss of generality ((2.13) in [2]) we may assume that $V = \langle X \rangle_{\mathbb{R}}$. In particular, we obtain:

$$K \subseteq \varphi(H \cap B_p) \quad \text{and} \quad \varphi(H \cap B_q) \subseteq S. \tag{1}$$

By $L(H)$, respectively $A(H)$, we denote the free vector space, respectively, the free abelian group generated by $\eta(H)$ where $\eta : H \mapsto \eta(H)$ is a bijection; further, we introduce the following mappings:

$$\pi : L(H) \to V, \ \sum \lambda_x \eta(x) \mapsto \sum \lambda_x x$$

and

$$\pi' : A(H) \to H, \ \sum k_x \eta(x) \mapsto \sum k_x x.$$

The compositions $\widetilde{p} := p \circ \pi$ and $\widetilde{q} := q \circ \pi$ are pHsns on $L(H)$ whose closed unit balls satisfy $B_{\widetilde{p}} = \pi^{-1}(B_p)$ and $B_{\widetilde{q}} = \pi^{-1}(B_q)$. Since π is surjective, we obtain ((2.8) b) in [2]):

$$\mathrm{d}_k(B_{\widetilde{p}}, B_{\widetilde{q}}) = \mathrm{d}_k(B_p, B_q) \leq \gamma_{\widetilde{m}} \widetilde{c} k^{-\widetilde{m}+2} \tag{2}$$

and hence $\sum_{k \in \mathbb{N}} \mathrm{d}_k(B_{\widetilde{p}}, B_{\widetilde{q}})^2 \leq (\gamma_{\widetilde{m}} \widetilde{c})^2 \sum_{k \in \mathbb{N}} k^{-2(\widetilde{m}-2)} < \frac{1}{4}$. Let

$$\widetilde{X} := \mathrm{conv}(A(H) \cap B_{\widetilde{p}}) \quad \text{and} \quad \widetilde{Y} := \mathrm{conv}(A(H) \cap B_{\widetilde{q}}). \tag{3}$$

Hence (3.20) in [2] implies $\mathrm{d}_k(\widetilde{X},\widetilde{Y}) \le 2\mathrm{d}_k(B_{\widetilde{p}},B_{\widetilde{q}}) \le 2\gamma_{\widetilde{m}}\widetilde{c}k^{-\widetilde{m}+2}$. Again by (2.14) in [2], there exist pHsns p',q' on $\langle\widetilde{X}\rangle_{\mathbb{R}} \le L(H)$ which satisfy

$$\widetilde{X} \subseteq B_{p'},\ B_{q'} \subseteq \widetilde{Y}, \text{ and } \mathrm{d}_k(B_{p'},B_{q'}) \le 2\gamma_{\widetilde{m}}\gamma_{\widetilde{m}-2}\widetilde{c}k^{-\widetilde{m}+4} \le c\cdot k^{-m}. \tag{4}$$

$$\begin{tikzcd} L(H) & & & \\ A(H) \arrow[u] \arrow[r, "\pi'"'] & H \arrow[r, "\varphi"'] & G^* \arrow[r, "\lambda"'] & \mathbb{R} \end{tikzcd}$$

We define

$$\Psi : \mathrm{aHom}(G^*,\mathbb{R}) \to \mathrm{aHom}(L(H),\mathbb{R}),\ \lambda \mapsto \widetilde{\lambda\circ\varphi}, \tag{5}$$

where $\widetilde{\lambda\circ\varphi}$ denotes the unique linear extension on $L(H)$ of the mapping $\lambda\circ\varphi : H \to \mathbb{R}$ and aHom all (not necessarily continuous) homomorphisms. Observe that the spaces $\mathrm{aHom}(\cdot)$ are vector spaces and that Ψ is a linear mapping. As a consequence of $\pi'(A(H)\cap B_{\widetilde{p}}) = H\cap B_p$, $\pi'(A(H)\cap B_{\widetilde{q}}) = H\cap B_q$, and $\widetilde{\lambda\circ\varphi}|_{A(H)} = \lambda\circ\varphi\circ\pi'$, we obtain

$$\begin{aligned}
\Psi^{-1}(P(B_{p'},[-1,1])) &= \{\lambda\in\mathrm{aHom}(G^*,\mathbb{R}) :\ \widetilde{\lambda\circ\varphi}(B_{p'}) \subseteq [-1,1]\} \\
&\overset{(4)}{\subseteq} \{\lambda\in\mathrm{aHom}(G^*,\mathbb{R}) :\ \widetilde{\lambda\circ\varphi}(\widetilde{X}) \subseteq [-1,1]\} \\
&\overset{(3)}{=} \{\lambda\in\mathrm{aHom}(G^*,\mathbb{R}) :\ \widetilde{\lambda\circ\varphi}(A(H)\cap B_{\widetilde{p}}) \subseteq [-1,1]\} \\
&= \{\lambda\in\mathrm{aHom}(G^*,\mathbb{R}) :\ \lambda\circ\varphi\circ\pi'(A(H)\cap B_{\widetilde{p}}) \subseteq [-1,1]\} \\
&= \{\lambda\in\mathrm{aHom}(G^*,\mathbb{R}) :\ \lambda\circ\varphi(H\cap B_p) \subseteq [-1,1]\} \\
&\overset{(1)}{\subseteq} \{\lambda\in\mathrm{aHom}(G^*,\mathbb{R}) :\ \lambda(K) \subseteq [-1,1]\}, \text{ and}
\end{aligned}$$

$$\begin{aligned}
\Psi^{-1}(P(B_{q'},[-1,1])) &= \{\lambda\in\mathrm{aHom}(G^*,\mathbb{R}) :\ \widetilde{\lambda\circ\varphi}(B_{q'}) \subseteq [-1,1]\} \\
&\overset{(4)}{\supseteq} \{\lambda\in\mathrm{aHom}(G^*,\mathbb{R}) :\ \widetilde{\lambda\circ\varphi}(\widetilde{Y}) \subseteq [-1,1]\} \\
&\overset{(3)}{=} \{\lambda\in\mathrm{aHom}(G^*,\mathbb{R}) :\ \widetilde{\lambda\circ\varphi}(A(H)\cap B_{\widetilde{q}}) \subseteq [-1,1]\} \\
&= \{\lambda\in\mathrm{aHom}(G^*,\mathbb{R}) :\ \lambda\circ\varphi\circ\pi'(A(H)\cap B_{\widetilde{q}}) \subseteq [-1,1]\} \\
&= \{\lambda\in\mathrm{aHom}(G^*,\mathbb{R}) :\ \lambda\circ\varphi(H\cap B_q) \subseteq [-1,1]\}
\end{aligned}$$

$$\overset{(1)}{\supseteq} \{\lambda \in \mathrm{aHom}(G^*, \mathbb{R}) : \lambda(S) \subseteq [-1,1]\}.$$

Trivially,

$$\{\lambda \in \mathrm{Hom}(G^*, \mathbb{R}) : \lambda(S) \subseteq [-1,1]\} \subseteq \Psi^{-1}(P(B_{q'}, [-1,1])) \cap \mathrm{Hom}(G^*, \mathbb{R})$$

and

$$\{\lambda \in \mathrm{Hom}(G^*, \mathbb{R}) : \lambda(K) \subseteq [-1,1]\} \supseteq \Psi^{-1}(P(B_{p'}, [-1,1])) \cap \mathrm{Hom}(G^*, \mathbb{R}).$$

Since these sets are symmetric and convex, we obtain:

$$\mathrm{d}_k(\{\lambda \in \mathrm{Hom}(G^*, \mathbb{R}) : \lambda(S) \subseteq [-1,1]\}, \{\lambda \in \mathrm{Hom}(G^*, \mathbb{R}) : \lambda(K) \subseteq [-1,1]\})$$

$$\leq \mathrm{d}_k(\Psi^{-1}(P(B_{q'}, [-1,1])) \cap \mathrm{Hom}(G^*, \mathbb{R}), \Psi^{-1}(P(B_{p'}, [-1,1])) \cap \mathrm{Hom}(G^*, \mathbb{R}))$$

$$\overset{\diamond}{\leq} \mathrm{d}_k(\Psi^{-1}(P(B_{q'}, [-1,1])), \Psi^{-1}(P(B_{p'}, [-1,1])))$$

$$\overset{*}{\leq} \mathrm{d}_k(P(B_{q'}, [-1,1]) \cap \mathrm{im}(\Psi), P(B_{p'}, [-1,1]) \cap \mathrm{im}(\Psi))$$

$$\overset{\bullet}{\leq} \mathrm{d}_k(P(B_{q'}, [-1,1]), P(B_{p'}, [-1,1]))$$

$$\overset{\circ}{=} \mathrm{d}_k(B_{p'}, B_{q'}) \overset{(4)}{\leq} c \cdot k^{-m}.$$

$\diamond$: Observe that polars of and inverse images under linear mappings of closed unit balls of pHsns are closed unit balls of pHsns; hence we may apply (2.13) in [2].
$*$ is a consequence of (18.5) in [1],
$\bullet$ follows from (2.13) in [2],
$\circ$ is (2.16) in [2]. ■

Corollary 2.3. *Let G be a nuclear group such that α_G is continuous. Then $\mathrm{Hom}_{co}(G^*, \mathbb{R})$ is a locally convex nuclear vector space.*

Proof. According to the above theorem, $\mathrm{Hom}_{co}(G^*, \mathbb{R})$ is a nuclear group and, obviously, a topological vector space. Now (8.9) in [2] implies that it is a nuclear locally convex vector space. ■

Remark 2.4. A similar result has been shown in (20.25) in [1], namely: If G is a nuclear group and α_G is continuous then $\mathrm{Hom}_{co}(G^*, \mathbb{T})$ is a nuclear group.

This is a rather trivial consequence of the estimate proved in (16.4) in [2].

The problem which prevents us to copy that proof is the following difference between characters (i.e. continuous homomorphism into $\mathbb{T}$) and real characters (i.e. continuous homomorphisms into $\mathbb{R}$):

Let H be a subgroup of a nuclear group G. Then every character $\chi : H \to \mathbb{T}$ can be extended to a character $\chi_G : G \to \mathbb{T}$ ((8.3) in [2]).

The analogous assertion for real characters is not true as the following example will show:

It is well known that the space of rapidly decreasing sequences $\mathbf{s} := \{(x_n)_{\in\mathbb{N}} : (\forall m \in \mathbb{N})\ p_m((x_n)_{n\in\mathbb{N}}) < \infty\}$ endowed with the topology induced by the norms $p_m((x_n)_{n\in\mathbb{N}}) := \sup\{n^m|x_n| : \ n \in \mathbb{N}\}$ is a locally convex nuclear vector space (see e.g.(6.1.3) in [8]). The subgroup $H := \langle e_k : \ k \in \mathbb{N}\rangle$ where $e_k = (\delta_{n,k})_{n\in\mathbb{N}}$ is discrete and free. Hence every real sequence (x_n) gives rise to a real character $H \to \mathbb{R}$, $\sum k_n e_n \mapsto \sum k_n x_n$.

For a continuous linear form φ of $\mathbf{s}$, there exists there exists $m \in \mathbb{N}$ such that $\varphi(B_{p_m})$ is bounded; this implies that $(\varphi(e_n)) = O(n^m)$.

We wish to use the result (2.2) in order to prove that the Lie algebra of a nuclear group is a nuclear vector space.

Proposition 2.5. *Let $G = \prod_{i\in I} G_i$, then $\mathcal{L}(G)$ is topologically isomorphic to $\prod_{i\in I} \mathcal{L}(G_i)$.*

Proof. See (7.38) in [3]. ∎

Theorem 2.6. *If G is a nuclear group then $\mathcal{L}(G)$ is a locally convex nuclear vector space.*

Proof. Since we already know that $\mathcal{L}(G)$ is a Hausdorff topological vector space ((1.2) i)), because of (8.9) in [2], it is sufficient to show that $\mathcal{L}(G)$ is a nuclear group.

According to (21.3) in [1], G can be embedded into a product of metrizable nuclear groups G_i. Hence $\mathcal{L}(G)$ is topologically isomorphic to a subspace of $\prod_{i\in I} \mathcal{L}(G_i)$ (1.3 and 2.5). Since the class of nuclear groups forms a variety, we may assume that G is metrizable.

By (1.1), the mapping α_G is continuous so (1.4) implies that $\mathcal{L}(G)$ can be identified with a subgroup of $\mathrm{Hom}_{co}(G^*, \mathbb{R}^*)$. According to (23.27) (e) in [4], $\mathbb{R}^*$ is topologically isomorphic to $\mathbb{R}$, hence the groups $\mathrm{Hom}_{co}(G^*, \mathbb{R}^*)$ and $\mathrm{Hom}_{co}(G^*, \mathbb{R})$ are topologically isomorphic. Now 2.2 implies that this group is nuclear and hence the assertion follows. ∎

Note: The question underlying this paper was posed by Helge Glöckner at the conference "Nuclear Groups and Lie Groups" in September 1999 in Madrid.

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