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The Product Formula for the Spherical Functions on Symmetric Spaces of Noncompact Type

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Abstract. In this paper, we prove the existence of the product formula for the spherical functions on symmetric spaces of noncompact type. To this end, we study the analyticity properties of the Cartan decomposition and we find a limited Taylor expansion of the abelian factor in this decomposition.

1. Introduction

Let G be a semisimple noncompact connected Lie group with finite center and K a maximal compact subgroup of G and $X = G/K$ the corresponding Riemannian symmetric space of noncompact type. We have a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and we choose a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$. In what follows, Σ corresponds to the root system of $\mathfrak{g}$ and Σ^+ to the positive roots. This implies that we have chosen a set of simple positive roots $\alpha_1, \ldots, \alpha_r$ where $r = \dim \mathfrak{a}$ is the rank of the symmetric space. We have the root space decomposition $\mathfrak{g} = \mathfrak{g}_0 + \sum_{\alpha\in\Sigma} \mathfrak{g}_\alpha$. Recall that $\mathfrak{k}$, the Lie algebra of K, can be described as

$$\mathfrak{k} = \mathrm{span}\,\{X_\alpha + \theta(X_\alpha)\colon\ X_\alpha \in \mathfrak{g}_\alpha,\ \alpha \in \Sigma^+ \cup \{0\}\}$$

where θ is the Cartan automorphism. Let $\mathfrak{n} = \sum_{\alpha\in\Sigma^+} \mathfrak{g}_\alpha$ and $\bar{\mathfrak{n}} = \sum_{\alpha\in\Sigma^+} \mathfrak{g}_{-\alpha} = \sum_{\alpha\in\Sigma^+} \theta(\mathfrak{g}_\alpha)$. Denote the groups corresponding to the Lie algebras $\mathfrak{a}$, $\mathfrak{n}$ and $\bar{\mathfrak{n}}$ by A, N and $\bar{N}$ respectively. We have the Cartan decomposition $G = K\,A\,K$ and the Iwasawa decomposition $G = K\,A\,N$. Let $\mathfrak{a}^+ = \{H \in A\colon\ \alpha(H) > 0\ \forall\ \alpha \in \Sigma^+\}$ and $A^+ = \exp(\mathfrak{a}^+)$.

If λ is a complex-valued functional on $\mathfrak{a}$, the corresponding spherical function is

$$\phi_\lambda(e^H) = \int_K e^{(i\,\lambda-\rho)(\mathcal{H}(e^H\,k))}\,dk$$

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where $g = k\, e^{\mathcal{H}(g)}\, n \in KAN$ and $\rho = (1/2) \sum_{\alpha \in \Sigma^+} m_\alpha\, \alpha$ (m_α denotes the multiplicity of the root α). A spherical function, like any K-biinvariant function, can also be considered as a K-invariant function on the Riemannian symmetric space of noncompact type $X = G/K$. Naturally, such a function is completely determined by its values on A (or on A^+). The books [4, 7, 8] constitute a standard reference on these topics.

Suppose that the symmetric space G/K is of noncompact type. Suppose also that $X \in \mathfrak{a}^+$ and $Y \in \mathfrak{a}^+$. We will assume these hypotheses throughout the paper. At the end of the paper, we relax the last hypothesis.

Remark 1.1. The condition $X \in \mathfrak{a}^+$ and $Y \in \mathfrak{a}^+$ is for convenience. We could assume instead that $\alpha(X) \neq 0$ and $\alpha(Y) \neq 0$ for every $\alpha \in \Sigma^+$.

In [8, (32), page 480], Helgason shows that a Weyl-invariant measure $\mu_{X,Y}$ exists on the Lie algebra $\mathfrak{a}$ such that

$$\phi_\lambda(e^X)\, \phi_\lambda(e^Y) = \int_{\mathfrak{a}} \phi_\lambda(e^H)\, d\mu_{X,Y}(H) \tag{1}$$

(unlike us, Helgason states his results at the group level).

It is known [8] that

$$\phi_\lambda(e^X)\, \phi_\lambda(e^Y) = \int_K \phi_\lambda(e^X\, k\, e^Y)\, dk.$$

The measure $\mu_{X,Y}$ is then to satisfy

$$\int_K f(e^X\, k\, e^Y)\, dk = \int_{\mathfrak{a}} f(e^H)\, d\mu_{X,Y}(H)$$

for all continuous functions f which are biinvariant under the action of K.

The support of the measure $\mu_{X,Y}$ is shown to be included in $C(X) + C(Y)$ when $0 \notin W \cdot X + W \cdot Y$ where $C(H)$ is the convex hull of the orbit of H under the action of the Weyl group W. In fact, this is true for all X and Y using properties of weak convergence of probability measures.

The natural question is whether the measure $\mu_{X,Y}$ is absolutely continuous with respect to the Lebesgue measure on $\mathfrak{a}$ *i.e.* whether we have a "product formula"

$$\phi_\lambda(e^X)\, \phi_\lambda(e^Y) = \int_{\mathfrak{a}} \phi_\lambda(e^H)\, k(H, X, Y)\, dH \tag{2}$$

where $k(H, X, Y)$ is Weyl invariant in each of the variables. Helgason also discusses this measure and some partial results in [9].

Flensted-Jensen and Koornwinder give explicit formulae for the rank one case in [2]. In fact, they give a product formula for a larger class of special functions, namely the Jacobi functions. The formulae given can be derived using an addition formula which is not currently available in higher rank situations. The reader may wish to consult also [11]. In [6], the authors prove the existence of the product formula for the spherical functions in the complex case and we study properties of the integral kernel of this formula.

In this paper, we prove the existence of the product formula (2) for symmetric spaces of noncompact type. In [9, page 367], Helgason writes that "It is obviously an interesting problem to relate $\mu_{b,c}$ to the structure of G". The product formula is an important step in solving that problem.

In Section 2, we discuss our approach to prove the existence of the product formula. Our method is based on the approach Flensted-Jensen and Ragozin use in [3] to prove the existence of the kernel of the Abel transform (see also [8, Theorem 10.11, page 478]). It leads us to study analyticity properties of the Cartan decomposition and in particular the analyticity of the function $g \to a$ where $g = k_1 a k_2$ is the Cartan decomposition of g, a question interesting in itself. It is worthwhile mentioning that in order to prove the product formula, we do not use direct knowledge of the spherical functions unlike our result of [6].

In Section 3, a rank-one reduction is used. This section contains most of the computations.

In Section 4, we put everything together to prove the existence of the product formula. We conclude by the discussion of cases $X \in \partial\mathfrak{a}^+$ or $Y \in \partial\mathfrak{a}^+$ and by giving some interesting applications of the product formula (they were given in the complex case in [6]).

2. Product formula and analyticity of the Cartan decomposition

To prove the product formula (2), it is sufficient to show that there exists a kernel $k(H, X, Y)$ such that for every K-biinvariant continuous function f and for every X, $Y \in \mathfrak{a}$, we have

$$\int_K f(e^X\, k\, e^Y)\, dk = \int_{\mathfrak{a}} f(e^H)\, k(H, X, Y)\, dH. \tag{3}$$

Let $a\colon G \to \overline{\mathfrak{a}^+}$ be defined by $g = k_1\, e^{a(g)}\, k_2$. Equation (3) is equivalent to

$$\int_K f(e^{a(e^X\, k\, e^Y)})\, dk = \int_{\mathfrak{a}} f(e^H)\, k(H, X, Y)\, dH. \tag{4}$$

Let M be the centralizer of A in K. It is possible to write (4) as

$$\int_{K/M} f(e^{a(e^X\, k\, e^Y)})\, dk_M = \int_{\mathfrak{a}} f(e^H)\, k(H, X, Y)\, dH. \tag{5}$$

if the measure on the coset space K/M is properly normalized. Indeed, if $m \in M$, $a(e^X\, k\, m\, e^Y) = a(e^X\, k\, e^Y)$.

The question is therefore to show whether the measure on $\mathfrak{a}$ given by the left-hand side of (4) or (5) is absolutely continuous with respect to the Lebesgue measure on $\mathfrak{a}$. Our approach, embodied by Proposition 2.8 at the end of this section, is inspired by [3] and [8]. Since their approach relies on the analyticity of the Iwasawa decomposition, in order to adapt their method, we have to investigate the analyticity of the Cartan decomposition. Contrary to the Iwasawa decomposition, the function $g \to a(g)$ is not analytic on the whole group G although it is known to be smooth on $K\, A^+\, K$ by [7, Corollary 1.2, page 402]. We go a little further.

Proposition 2.1. *Suppose G/K is a symmetric space of noncompact type and let $g = k_1 e^{a(g)} k_2$, $a(g) \in \overline{\mathfrak{a}^+}$, be the Cartan decomposition of $g \in G$. Then the function $g \to a(g)$ is analytic on $K A^+ K$.*

Proof. Let $g = k \exp X$ where $k \in K$ and $X \in \mathfrak{p}$. The map $g \to (k, X) \in K \times \mathfrak{p}$ is analytic ([4, Prop. 2.1.11, p. 60]). Now, one represents $X \in \mathfrak{p}$ as $X = \mathrm{Ad}(k)H$ where $k \in K/M$ and $H \in \overline{\mathfrak{a}^+}$ and the Jacobian of the analytic map $(k\,M, H) \to \mathrm{Ad}(k)H$ is equal to $\prod_{\alpha\in\Sigma^+} \alpha(H)^{m_\alpha}$ ([8, p. 195]). It follows that the mapping $g \to X \to H = a(g)$ is analytic when $a(g) \in \mathfrak{a}^+$. ■

We now give a result which will prove useful several times.

Proposition 2.2. *Let H be a connected Lie group and f: $H \to \mathbf{R}$ an analytic function. If $f \neq 0$ then the zeros of f form a closed set C of zero Haar measure in H.*

Proof. This well known property of analytic functions follows for example from Lelong's theorem on semi-analytic sets (see for example [12]).

Let us give here a simple elementary proof of this property. Consider $h \in C$ and a coordinate map ϕ on a neighbourhood U of h such that $V = \phi(U)$ is a product of open intervals in $\mathbf{R}^m$, $m = \dim H$. The set $U \cap C$ is of Haar measure zero if and only if its image by ϕ is of zero Lebesgue measure in $\mathbf{R}^m$. We proceed by induction. If $m = 1$ then it is well known that if $F = f \circ \phi^{-1}$: $(a, b) \to \mathbf{R}$ is analytic then either $F = 0$ or the zeros of f are isolated and therefore of Lebesgue measure zero.

If V is a product of $m > 1$ intervals, we write it $V = I \times J$ with $I = (a, b) \subset \mathbf{R}$ and J a product of $m-1$ intervals. Consider $A = \{x \in I\colon F(x, y) = 0$ for all $y \in J\} \subset I$. Either A is of strictly positive Lebesgue measure in I or A is of zero measure. The first case implies that F is identically zero on V. In the second case, if $x \in I \setminus A$ then by induction hypothesis, the set of y's in J such that $F(x, y) = 0$ is of measure zero in $\mathbf{R}^{m-1}$. An application of Fubini's theorem allows us to conclude the proof. ■

Remark 2.3. Let μ be the Haar measure on H. The complement of a closed set of μ-measure zero is a dense open set.

Corollary 2.4. *Let H be a connected Lie group and N an analytic manifold. Let f: $H \to N$ an analytic map. Let $k = \min(\dim H, \dim N)$.*

If the differential df is of rank k at a point $h \in H$ then it is of rank k on an open set $H \setminus C \subset H$ where the Haar measure of C is zero.

Proof. The differential df is an analytic map on H. Let $s_k(h)$ be the sum of squares of all determinants of $df|_h$ of dimension k. The function s_k is non-negative analytic on H. According to the hypothesis, $s_k(h) > 0$ and we apply Proposition 2.2 to s_k. ■

We will also need the following algebraic result.

Theorem 2.5. (Hermite) *Let P be a polynomial of degree n with real coefficients. The number of distinct roots of P is equal to the rank of the matrix*

$$B = \begin{pmatrix} p_0 & p_1 & \dots & p_{n-1} \\ p_1 & p_2 & \dots & p_n \\ \vdots & \vdots & \ddots & \vdots \\ p_{n-1} & p_n & \dots & p_{2n-2} \end{pmatrix}$$

where $p_k = \sum_j b_j^k$ and $b_1, \dots, b_n$ are the roots of P. The Newton polynomials p_k are polynomials of the coefficients of P.

Proof. Refer to [5]. ■

Remark 2.6. The matrix B is sometimes called the Bezoutian of the the polynomial P.
We will be applying the above result to $P(t) = \det(t\,I - A)$ where A is a real symmetric matrix. In that case, the rank of the corresponding matrix B will denote the number of distinct real roots of P *i.e.* the number of distinct eigenvalues of A.

In the sequel we want to use Proposition 2.2 and Corollary 2.4 in order to study the absolute continuity of the measure $\mu_{X,Y}$ which is characterized by the equality

$$\int_K f(e^{a(e^X\,k\,e^Y)})\,dk = \int_{\mathfrak{a}} f(e^H)\,d\mu_{X,Y}(H). \tag{6}$$

for all K-biinvariant continuous functions on G. Since the application

$$F\colon\ K \to \overline{\mathfrak{a}^+}, \qquad F(k) = a(e^X\,k\,e^Y)$$

is only analytic on $K_0 = \{k \in K :\ a(e^X\,k\,e^Y) \in \mathfrak{a}^+\} \subset K$, we cannot apply the analytic continuation arguments of Proposition 2.2 and Corollary 2.4 because K_0 may not be connected. We will surmount this difficulty by working on a subset K' of K_0 introduced in the following lemma. We will write $k \in K'$ if $H = a(e^X k e^Y) \in \mathfrak{a}^+$ and $\alpha(H) \neq \beta(H)$ for different roots α and β.

Let us fix a basis of $\mathfrak{g}$ according to [7, Lemma 3.5, page 261]. This is a basis constructed as a union of bases of root spaces $\mathfrak{g}_0$ and $\mathfrak{g}_\alpha$. If we have $g = k_1\,e^H\,k_2$ (i.e. $H = a(g)$) then

$$\mathrm{Ad}(g) = \mathrm{Ad}(k_1)\,e^{\mathrm{ad}_H}\,\mathrm{Ad}(k_2)$$

and

$$\mathrm{Ad}(\theta(g)^{-1})\,\mathrm{Ad}(g) = \mathrm{Ad}(k_2)^{-1}\,e^{2\,\mathrm{ad}_H}\,\mathrm{Ad}(k_2).$$

In the above basis, the operator $e^{2\,\mathrm{ad}_H}$ is given by the matrix

$$\mathrm{diag}[\overbrace{1,\dots,1}^{\dim\mathfrak{g}_0},\ \overbrace{e^{2\,\alpha_1(H)},\dots\,e^{2\,\alpha_1(H)}}^{\dim\mathfrak{g}_{\alpha_1}},\dots,\ e^{-2\,\alpha_1(H)},\dots,e^{-2\,\alpha_{n_0}(H)}] \tag{7}$$

where $n_0 = |\Sigma^+|$ is the number of positive roots.

Lemma 2.7. *Let*

$$G' = \{g \in G\colon\ \mathrm{Ad}(\theta(g)^{-1})\,\mathrm{Ad}(g) \text{ has } 1 + 2\,|\Sigma^+| \text{ distinct eigenvalues}\}$$

and $K' = \{k \in K\colon\ e^X\,k\,e^Y \in G'\}$. *If* $K' \neq \emptyset$ *then* K' *is an open set in* K *whose complement is of measure zero.*

Proof. The map from G to the set of linear operators on $\mathfrak{g}$ defined by $g \to \mathrm{Ad}(\theta(g)^{-1})\,\mathrm{Ad}(g)$ is analytic.

For a given g, let $\Phi(g) = \det(t\,I - \mathrm{Ad}(\theta(g)^{-1})\,\mathrm{Ad}(g))$ be the characteristic polynomial of $\mathrm{Ad}(\theta(g)^{-1})\,\mathrm{Ad}(g)$ and let B_g be the matrix defined in Theorem 2.5 for the polynomial $\Phi(g)$. Let $m = 1 + 2\,|\Sigma^+|$ which is the largest possible number of distinct roots of $\Phi(g)$ (refer to (7)). According to Theorem 2.5, $\Phi(g)$ has m distinct roots if and only if $s_m(B_g) > 0$ where $s_m(B)$ is the sum of squares of all determinants of order m of the matrix B.

Consider the analytic map $k \to s_m(B_{e^X\,k\,e^Y})$. Assuming that there exists $k \in K'$, this map is not identically zero on K so the statement follows by Proposition 2.2. ∎

Proposition 2.8. *Let* $F(k) = a(e^X\,k\,e^Y)$. *In order to show that the measure* $\mu_{X,Y}$ *in (1) and (6) is absolutely continuous with respect to the Lebesgue measure on* $\mathfrak{a}$ *it suffices to show that the differential of the map* F *is surjective for at least one point* $k_0 \in K'$ *(refer to Lemma 2.7).*

Proof. Denote by $P_n[t]$ the space of real polynomials of degree $n = \dim\mathfrak{g}$ with the highest term t^n. Consider the map $\Phi\colon\ G \to P_n[t]$ defined as in Lemma 2.7. Clearly, Φ is an analytic map which verifies $\Phi(g) = \Phi(e^{a(g)})$. Denote by Φ^A the restriction of Φ to A. The map Φ^A is analytic on A. The map $\Psi = \Phi^A \circ e^F$ is analytic on K since $\Psi(k) = \Phi(e^X\,k\,e^Y)$.

The map Φ^A is of rank less than or equal to $r = \dim\mathfrak{a}$. We now show that Φ^A is of rank r on the set $A' = A \cap G'$ open in A. Denote $m_i = m_{\alpha_i}$. Then

$$\Phi^A(t) = (t-1)^{m_0} \prod_{i=1}^{n_0} ((t - e^{2\,\alpha_i})\,(t - e^{-2\,\alpha_i}))^{m_i}.$$

Suppose that $Q = \sum_{k=1}^r a_k\,\frac{\partial\Phi^A}{\partial\alpha_k} = 0$. If $1 \leq j \leq r$ then on A' we have

$$\begin{aligned}\left.\frac{d^{m_j-1}}{dt^{m_j-1}}Q\right|_{t=e^{2\,\alpha_j}} &= -\ 2\,a_j\,m_j!\,e^{2\,\alpha_j}\,(e^{2\,\alpha_j}-1)^{m_0}\,(e^{2\,\alpha_j}-e^{-2\,\alpha_j})^{m_j} \\ &\quad \times \prod_{i\neq j} ((e^{2\,\alpha_j} - e^{2\,\alpha_i})\,(e^{2\,\alpha_j} - e^{-2\,\alpha_i}))^{m_i} = 0\end{aligned}$$

so the a_k are all zero and $\frac{\partial\Phi^A}{\partial\alpha_k}$, $k = 1, \ldots, r$, are linearly independent. It follows that Φ^A is of rank r on A'.

From that we deduce that at a point $k \in K'$ where dF is surjective, the map $\Psi(k) = \Phi^A(e^{F(k)})$ is of full rank r.

By Corollary 2.4, we find an open set U of K such that $K\backslash U$ is of measure 0 and such that $d\Psi$ is of rank r everywhere on U. Let $V = K' \cap U$ where K' is as in Lemma 2.7. If $k \in V$, then $d\Psi$ is surjective at k and $d\Phi^A$ is surjective at $F(k)$. Since $d\Psi = d\Phi^A \circ d(e^F)$, we conclude that dF is surjective at k.

The proof of the fact that if S is of zero Lebesgue measure in $\mathfrak{a}^+$ than $\mu_{X,Y}(S) = 0$ is now identical as in [3] or [8, p. 479]. ■

Remark 2.9. The analyticity of the roots of a polynomial as functions of its coefficients was studied by Brillinger in [1] but these results are not precise enough to be used in the proof of the Proposition 2.8.

Remark 2.10. In the proofs of Lemma 2.7 and Proposition 2.8 one can use in an equivalent way the polynomial

$$\tilde{\Phi}(g) = \det(t\,I - \mathrm{ad}X)$$

where $g = k \exp X$ with $k \in K$ and $X \in \mathfrak{p}$ instead of the polynomial $\Phi(g) = \det(t\,I - \mathrm{Ad}(\theta(g)^{-1})\,\mathrm{Ad}(g))$. The map $g \to \tilde{\Phi}(g)$ is analytic and one verifies easily that $\tilde{\Phi}(g) = \tilde{\Phi}(e^{a(g)})$. The rest of the proofs remains identical.

The remainder of this paper consists mainly in showing that there exists an element $k \in K$ such that the hypotheses of Lemma 2.7 and Proposition 2.8 are fulfilled *i.e.* that there exists $k \in K$ such that $e^X\,k\,e^Y \in G'$ and $dF(k)$ is surjective. This will be done in Theorem 4.5 for an element of K of the form

$$k_0 = \exp(t_1\,(X_{\alpha_1} + \theta(X_{\alpha_1})) + \ldots + t_r\,(X_{\alpha_r} + \theta(X_{\alpha_r})))$$

where $X_{\alpha_1}, \ldots, X_{\alpha_r}$ are root vectors and the t_i's are not zero but small enough.

3. A rank one reduction

The main result of this section is Proposition 3.2 and its corollary. There we restrict the map F of Proposition 2.8 to elements of the form

$$k = \exp(t_1\,(X_{\alpha_1} + \theta(X_{\alpha_1})) + \ldots + t_r\,(X_{\alpha_r} + \theta(X_{\alpha_r})))$$

and compute a limited Taylor expansion about $k = e$ using a rank one reduction.

Lemma 3.1. *Let α be a positive root with associated root vector X_α and let $H_\alpha = [X_\alpha, \theta(X_\alpha)]$. Then $\alpha(H_\alpha) < 0$. If $\alpha_1, \ldots, \alpha_r$ are simple positive roots then the vectors $H_{\alpha_1}, \ldots, H_{\alpha_r}$ form a basis of $\mathfrak{a}$.*

Proof. Define $A_\alpha \in \mathfrak{a}$ by $\alpha(H) = B(H, A_\alpha)$ for all $H \in \mathfrak{a}$. Using [7, (7), page 407], we get $H_\alpha = B(X_\alpha, \theta(X_\alpha))\,A_\alpha$ and therefore,

$$\alpha(H_\alpha) = B(X_\alpha, \theta(X_\alpha))\,\alpha(A_\alpha) = B(X_\alpha, \theta(X_\alpha))\,B(A_\alpha, A_\alpha) < 0$$

since the quadratic form $X \to B(X, \theta(X))$ is strictly negative definite everywhere ([7, Proposition 7.4 page 184]) while the form $H \to B(H, H)$ is strictly positive definite on $\mathfrak{a}$ (the second statement is a consequence of the former since $\theta = -\mathrm{id}$ on $\mathfrak{a}$). ■

Proposition 3.2. *Assume $a<0$ and $b<0$. Let α be a positive root, X_α an associated root vector and let $H_\alpha=[X_\alpha,\theta(X_\alpha)]$. Let K_α be the connected subgroup of K with subalgebra $\mathfrak{k}_\alpha$ generated by $X_\alpha+\theta(X_\alpha)$. Then there exist $k_1(t)$, $k_2(t)\in K_\alpha$ and $S>0$ such that for t small enough, we have*

$$e^{a\,H_\alpha}\,e^{t\,(X_\alpha+\theta(X_\alpha))}\,e^{b\,H_\alpha}=k_1(t)\,\exp\Big((a+b)\,H_\alpha+S\,H_\alpha\,t^2+O(t^3)H_\alpha\Big)\,k_2(t). \qquad (8)$$

Proof. Let $c=\sqrt{-\alpha(H_\alpha)/2}$ (which is a positive real number by Lemma 3.1). Let $\tilde{\mathfrak{g}}=\operatorname{span}\{H_\alpha,X_\alpha,\theta(X_\alpha)\}$. It is easy to check that $\tilde{\mathfrak{g}}$ is a Lie algebra and that the map $\eta\colon\tilde{\mathfrak{g}}\to\mathfrak{sl}(2,\mathbf{R})$ defined by $\eta(b_1\;H_\alpha+b_2\,X_\alpha+b_3\,\theta(X_\alpha))=\begin{bmatrix}-c^2\,b_1 & c\,b_2\\ -c\,b_3 & c^2\,b_1\end{bmatrix}$ is a Lie algebra isomorphism.

For t small enough, the result depends only on the Lie algebra and we can use the correspondence given by η. We know that we have

$$e^{a\,H_\alpha}\,e^{t\,(X_\alpha+\theta(X_\alpha))}\,e^{b\,H_\alpha}=k_1\,\exp\Big((a+b)\,H_\alpha+e_1\,H_\alpha\,t+e_2\,H_\alpha\,t^2+O(t^3)\,H_\alpha\Big)\,k_2$$

with $k_1=k_1(t)$, $k_2=k_2(t)\in K_\alpha$ (the Cartan decomposition in the connected subgroup of G corresponding to $\tilde{\mathfrak{g}}$). Call the above equality $A=B$ and write $\theta(A)^{-1}\,A=\theta(B)^{-1}\,B$. This gives

$$\begin{aligned}&e^{b\,H_\alpha}\,e^{-t\,(X_\alpha+\theta(X_\alpha))}\,e^{2\,a\,H_\alpha}\,e^{t\,(X_\alpha+\theta(X_\alpha))}\,e^{b\,H_\alpha}\\ =\;&k_2^{-1}\,\exp\Big(2\,(a+b)\,H_\alpha+2\,e_1\,H_\alpha\,t+2\,e_2\,H_\alpha\,t^2+O(t^3)\,H_\alpha\Big)\,k_2.\end{aligned}$$

Using the isomorphism η, this becomes

$$e^{\begin{bmatrix}-b\,c^2 & 0\\ 0 & b\,c^2\end{bmatrix}}\,e^{\begin{bmatrix}0 & -c\,t\\ c\,t & 0\end{bmatrix}}\,e^{\begin{bmatrix}-2\,a\,c^2 & 0\\ 0 & 2\,a\,c^2\end{bmatrix}}\,e^{\begin{bmatrix}0 & c\,t\\ -c\,t & 0\end{bmatrix}}\,e^{\begin{bmatrix}-b\,c^2 & 0\\ 0 & b\,c^2\end{bmatrix}}=$$

$$\tilde{k}_2^{-1}\,e^{\begin{bmatrix}-2((a+b)+e_1\,t+e_2\,t^2+O(t^3))c^2 & 0\\ 0 & 2((a+b)+e_1\,t+e_2\,t^2+O(t^3))c^2\end{bmatrix}}\,\tilde{k}_2$$

where $\tilde{k}_2=\tilde{k}_2(t)$ corresponds to $k_2=k_2(t)$ under the isomorphism. The exponential map on a one dimensional matrix Lie algebra is the classical matrix exponential so the product of exponentials on the lefthand-side of the last formula is equal to $\begin{bmatrix}d_1 & d_2\\ d_3 & d_4\end{bmatrix}$ with

$$\begin{aligned}d_1&=\cos(c\,t)^2\,(e^{-2\,c^2\,(b+a)}-e^{2\,c^2\,(-b+a)})+e^{(2\,c^2\,(-b+a))},\\ d_2&=-\cos(c\,t)\,\sin(c\,t)\,(e^{2\,c^2\,a}-e^{-2\,c^2\,a}),\\ d_3&=-\cos(c\,t)\,\sin(c\,t)\,(e^{2\,c^2\,a}-e^{(-2\,c^2\,a)}),\\ d_4&=\cos(c\,t)^2\,(e^{2\,c^2\,(b+a)}-e^{2\,c^2\,(-b+a)})+e^{(2\,c^2\,(-b+a))}.\end{aligned}$$

Taking the trace on both sides, we find that

$$\begin{aligned}&\cos(c\,t)^2\,e^{-2\,c^2\,(b+a)}+e^{2\,c^2\,(-b+a)}-\cos(c\,t)^2\,e^{2\,c^2\,(-b+a)}\\ &\qquad+e^{-2\,c^2\,(-b+a)}-\cos(c\,t)^2\,e^{-2\,c^2\,(-b+a)}+\cos(c\,t)^2\,e^{2\,c^2\,(b+a)}\\ =\;&2\,\cosh(2\,((a+b)+e_1\,t+e_2\,t^2))+O(t^3).\end{aligned}$$

Comparing the limited Taylor expansions of the last equality, we find easily that $e_1 = 0$ and that

$$S = e_2 = \frac{\sinh(2\,a\,c^2)\,\left(e^{-2\,b\,c^2}\sinh(2\,(a+b)\,c^2) - \sinh(2\,a\,c^2)\right)}{-\sinh(2\,(a+b)\,c^2)e^{-2\,(a+b)\,c^2}}.$$

It is not difficult to see that the last expression is strictly positive when $a < 0$ and $b < 0$. ∎

Corollary 3.3. *Let α be a positive root, X_α an associated root vector and let $H_\alpha = [X_\alpha, \theta(X_\alpha)]$. Suppose $\alpha(X) > 0$ and $\alpha(Y) > 0$. Then there exists $S > 0$ such that for t small enough, we have*

$$a(e^X\, e^{t\,(X_\alpha+\theta(X_\alpha))}\, e^Y) = X + Y + S\,H_\alpha\,t^2 + O(t^3)H_\alpha.$$

Proof. Write

$$X = \frac{\alpha(X)}{\alpha(H_\alpha)}\,H_\alpha + \overbrace{(X - \frac{\alpha(X)}{\alpha(H_\alpha)}\,H_\alpha)}^{X'} \text{ and } Y = \frac{\alpha(Y)}{\alpha(H_\alpha)}\,H_\alpha + \overbrace{(Y - \frac{\alpha(Y)}{\alpha(H_\alpha)}\,H_\alpha)}^{Y'}.$$

Since $\alpha(X') = 0 = \alpha(Y')$ and X', $Y' \in \mathfrak{a}$, they both commute with the elements of $\tilde{\mathfrak{g}}$ (refer to the proof of Proposition 3.2). For the same reason $e^{X'}$, $e^{Y'}$ and $e^{X'+Y'}$ commute with the elements of K_α. We then have

$$\begin{aligned}
e^X\,e^{t\,(X_\alpha+\theta(X_\alpha))}\,e^Y &= e^{X'}\,e^{\frac{\alpha(X)}{\alpha(H_\alpha)}\,H_\alpha}\,e^{t\,(X_\alpha+\theta(X_\alpha))}\,e^{\frac{\alpha(Y)}{\alpha(H_\alpha)}\,H_\alpha}\,e^{Y'}\\
&= e^{X'}\,k_1(t)\,e^{\left(\frac{\alpha(X)}{\alpha(H_\alpha)}+\frac{\alpha(Y)}{\alpha(H_\alpha)}\right)H_\alpha+S\,H_\alpha\,t^2+O(t^3)H_\alpha}\,k_2(t)e^{Y'}\\
&= k_1(t)\,e^{X'+Y'+\left(\frac{\alpha(X)}{\alpha(H_\alpha)}+\frac{\alpha(Y)}{\alpha(H_\alpha)}\right)\,H_\alpha+S\,H_\alpha\,t^2+O(t^3)H_\alpha}\,k_2(t)\\
&= k_1(t)\,e^{X+Y+S\,H_\alpha\,t^2+O(t^3)\,H_\alpha}\,k_2(t)
\end{aligned}$$

($k_i(t) \in K_\alpha \subset K$). For t small enough, $X + Y + S\,H_\alpha\,t^2 + O(t^3)\,H_\alpha \in \mathfrak{a}^+$ which allows us to conclude. ∎

4. The existence of the product formula

Definition 4.1. Let α_1, …, α_r be the simple positive roots and choose associated root vectors X_{α_1}, …, X_{α_r}. Suppose X, $Y \in \mathfrak{a}^+$. Define g: $\mathbf{R}^r \to \mathfrak{a}$ by

$$g(t_1, \ldots, t_r) = a(e^X\,e^{t_1\,(X_{\alpha_1}+\theta(X_{\alpha_1}))+\ldots+t_r\,(X_{\alpha_r}+\theta(X_{\alpha_r}))}\,e^Y).$$

Remark 4.2. In order to prove that the differential of the map F defined in Proposition 2.8 is surjective at a point $k_0 \in K'$, it clearly suffices to show that there exists a point $\mathbf{t} = (t_1, \ldots, t_r)$ such that $g(\mathbf{t}) \in \mathfrak{a}^+$, $\alpha(g(\mathbf{t})) \neq \beta(g(\mathbf{t}))$ whenever $\alpha \neq \beta$, $\alpha, \beta \in \Sigma^+$ and such that the Jacobian of g in $\mathbf{t}$ is nonzero.

It will also be convenient to think of g as a map into $\mathbf{R}^r$ (using the basis H_{α_i}, $i = 1$, …, r, of $\mathfrak{a}$ where $H_{\alpha_i} = [X_{\alpha_i}, \theta(X_{\alpha_i})]$).

Lemma 4.3. *We have*

$$g(\pm t_1, \dots, \pm t_r) = g(t_1, \dots, t_r)$$

where the signs are independent.

Proof. Fix $\epsilon_i \in \{-1, 1\}$, $i = 1, \dots, r$. Recall that $\mathfrak{g} = \oplus_\alpha \mathfrak{g}_\alpha$ where the sum is taken over all the roots and that for any root α, we have $\alpha = \sum_{i=1}^r n_i \alpha_i$ where the n_i are integers that are all greater than or equal to 0 or all smaller than or equal to 0 and that the representation is unique. We define the map ψ: $\mathfrak{g} \to \mathfrak{g}$ by

$$\psi(X) = \epsilon_1^{n_1} \cdots \epsilon_r^{n_r} X \qquad \text{if } X \in \mathfrak{g}_\alpha,\ \alpha = \sum_{i=1}^r n_i \alpha_i$$

and we extend the map linearly to $\mathfrak{g}$. Note that $\psi(X) = X$ when $X \in \mathfrak{g}_0$. It is easy to check that ψ is a Lie algebra isomorphism (using the relation $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ where $\mathfrak{g}_{\alpha+\beta} = \{0\}$ if $\alpha + \beta$ is not a root).

Let $\tilde{G}$ be the simply connected covering of G. Since G is connected, we know that $G \simeq \tilde{G}/Z$ where Z is a subgroup of $Z(\tilde{G})$, the center of $\tilde{G}$ (see [13, page 26]). There exists a group automorphism Ψ on $\tilde{G}$ such that $\Psi(e^X) = e^{\psi(X)}$ for all $X \in \mathfrak{g}$. Now, $\Psi(Z(\tilde{G})) = Z(\tilde{G})$ so it is possible to define the group automorphism Ψ on $\tilde{G}/Z(\tilde{G})$. Now, $\tilde{G}/Z(\tilde{G}) \simeq G/Z(G)$ since they are both isomorphic to $\mathrm{Int}(\mathfrak{g})$ (see [7, Corollary 5.2, page 129]). Note that $G/Z(G)$ has Lie algebra $\mathfrak{g}$ and finite center (its center is $\{eZ(G)\}$ by [7, Corollary 5.3, page 129]). It is easy to see that $Z(G)$ is a subgroup of K. Indeed, let K' be any maximal compact subgroup containing the finite set $Z(G)$. Since all maximal compact subgroups of G are conjugate, there exists $g \in G$ such that $g\, K'\, g^{-1} = K$. But then $Z(G) = g\, Z(G)\, g^{-1} \subset K$. Hence, with some abuse of language, $G/Z(G) = (K/Z(G))\, A\, N$ and $G/Z(G) = (K/Z(G))\, A\, (K/Z(G))$.

All this implies that the function $g : \mathbf{R}^r \to \mathfrak{a} = \mathbf{R}^r$ is the same if we define it *via* the Iwasawa and Cartan decompositions of the group G or of the group $G/Z(G)$. On $G/Z(G)$, the automorphism Ψ is well defined. We will therefore work on $G/Z(G)$ or, in other words, work on G supposing that the center of G is $\{e\}$. Note also that the map Ψ sends the groups K, A, N and $\bar{N}$ onto themselves. In addition, the automorphism Ψ is the identity on A.

We have $a(\Psi(g)) = a(g)$ since $g = k_1\, e^{a(g)}\, k_2$ implies

$$\Psi(g) = \Psi(k_1)\, \Psi(e^{a(g)})\, \Psi(k_2) = \Psi(k_1)\, e^{\psi(a(g))}\, \Psi(k_2) = \Psi(k_1)\, e^{a(g)}\, \Psi(k_2).$$

Hence, writing $Z_i = X_{\alpha_i} + \theta(X_{\alpha_i})$,

$$\begin{aligned} g(\pm t_1, \dots, \pm t_r) &= a(e^X\, e^{\pm t_1 Z_1 + \dots + \pm t_r Z_r} e^Y) = a(e^X\, e^{\psi(t_1 Z_1 + \dots + t_r Z_r)} e^Y) \\ &= a(e^X\, \Psi(e^{t_1 Z_1 + \dots + t_r Z_r}) e^Y)) = a(\Psi(e^X\, e^{t_1 Z_1 + \dots + t_r Z_r} e^Y)) \\ &= g(t_1, \dots, t_r) \end{aligned}$$

which proves the lemma. ∎

Proposition 4.4. *There exist $S_1 > 0, \ldots, S_r > 0$ such that we have*

$$g(t_1,\ldots,t_r) = X + Y + \sum_{i=1}^{r} S_i\, H_{\alpha_i}\, t_i^2 + O(\|\mathbf{t}\|^3).$$

Proof. Taking $\alpha = \alpha_i$ and $X_\alpha = X_{\alpha_i}$ in Corollary 3.3, it is clear that

$$g(t_1,\ldots,t_r) = X + Y + \sum_{i=1}^{r} S_i\, H_{\alpha_i}\, t_i^2 + \sum_{i<j} c_{ij}\, t_i\, t_j\, H_{ij} + O(\|\mathbf{t}\|^3) \tag{9}$$

where $S_i > 0$ for each i. The invariance given in Lemma 4.3 implies that the coefficients c_{ij} of the terms $c_{ij}\, t_i\, t_j\, H_{ij}$ must be 0 (it would suffice to take $\epsilon_i = -1$ and $\epsilon_k = 1$ for $k \neq i$). ■

Theorem 4.5. *The product formula (2) exists for every symmetric space of noncompact type.*

Proof. We show the existence of a point $\mathbf{t} = (t_1,\ldots,t_r)$ with properties required in Remark 4.2.

The Jacobian of g is easily computed; for $\mathbf{s} = (s_1,\ldots,s_r)$ close to the origin, the Jacobian is $2^r\,(\prod_{i=1}^{r} s_i\, S_i)\det(H_{\alpha_1},\ldots,H_{\alpha_r}) + O(\|\mathbf{s}\|^2)$ which is clearly nonzero when all $s_i \neq 0$. Fix such an $\mathbf{s}$. It follows that g is a diffeomorphism of an open neighbourhood U of $\mathbf{s}$ onto an open neighbourhood V of $g(\mathbf{s})$. The Lebesgue measure of the set $C = \{v \in V \mid \alpha(v) = \beta(v) \text{ for some roots } \alpha \neq \beta\}$ is zero so there exists $\mathbf{t} \in U$ with nonzero coefficients such that $g(\mathbf{t}) \notin C$. ■

As announced in the introduction, we now discuss what happens without the hypothesis $X \in \mathfrak{a}^+$ and $Y \in \mathfrak{a}^+$.

Lemma 4.6. *Let Δ be an irreducible root system and Δ_0 be the set of simple positive roots. If $\beta_1 \in \Delta_0$ then one may order the elements of Δ_0 in such a way $\beta_2, \ldots, \beta_r$ ($r = |\Delta_0|$) that $\sum_{i=1}^{k} \beta_i \in \Delta$ for all $k = 1,\ldots,r$.*

Proof. We use induction on r. The result is trivial when $r = 1$.

Suppose that the statement is true for any root system with $r - 1$ simple positive roots.

Consider the Dynkin diagram D of a root system with $|\Delta_0| = r$. As a graph, this diagram is a finite tree so there exists a root $\alpha_r \in \Delta$ such that $\langle \alpha, \alpha_r \rangle$ for only one other root in Δ, say $\alpha = \alpha_{r-1}$ (geometrically, it means that the only vertex of D connected with α_r is α_{r-1}).

Let Δ' be the root system generated by the simple roots $\Delta_0 \setminus \{\alpha_r\}$ (the Dynkin diagram D' of Δ' is obtained from D by suppressing the vertex α_r and the edge $[\alpha_{r-1}, \alpha_r]$).

Let us order the elements of $\Delta_0 \setminus \{\alpha_r\}$ in such a way $\alpha_1, \ldots, \alpha_{r-1}$ that $\sum_{i=k}^{r-1} \alpha_i \in \Delta$ for all $k = 1, \ldots, r-1$. This is possible by the induction hypothesis.

Let $\beta = \alpha_1 + \ldots + \alpha_{r-1} \in \Delta$. As $\langle \alpha_p, \alpha_r \rangle = 0$ when $p < r-1$ and $\langle \alpha_{r-1}, \alpha_r \rangle < 0$ (all this follows from the fact that D contains the edge $[\alpha_{r-1}, \alpha_r]$

and no other edge with vertex α_r) we infer that $\langle\beta,\alpha_r\rangle < 0$. This implies [7, Lemma 2.18 page 291] that $\beta+\alpha_r = \alpha_1+\ldots+\alpha_r \in \Delta$. The statement of the lemma is then true for $\beta_1 = \alpha_i$, $i<r$. It is also true for $\beta_1 = \alpha_r$ since $\sum_{i=k}^{r-1}\alpha_i+\alpha_r \in \Delta$ for $k = 1, \ldots, r-1$ which follows from the same argument replacing β by $\alpha_k+\ldots+\alpha_{r-1}$.

The lemma could also be proven using the classification of root systems by a case by case examination. See for instance [7]. ■

Proposition 4.7. *Let $X\in\partial\mathfrak{a}^+$, $X\neq 0$ and $Y\in\mathfrak{a}^+$ (or vice-versa). Then $\mu_{X,Y}$ has a density.*

Proof. We refer to the notation of Lemma 4.6. Let $\beta_1\in\Delta_0$ be such that $\beta_1(X)\neq 0$. Let $\beta_2, \ldots, \beta_r$ be as in Lemma 4.6. We define $g(t_1,\ldots,t_r)$ as in Definition 4.1 using the roots $\alpha_k = \sum_{i=1}^k \beta_i$, $k=1, \ldots, r$, which are no longer simple. Then equation (9) still holds.

We define the map ψ: $\mathfrak{g}\to\mathfrak{g}$ by

$$\psi(X) = \epsilon_1^{n_1}\cdots\epsilon_r^{n_r}\,X \qquad \text{if } X\in\mathfrak{g}_\alpha,\ \alpha=\sum_{i=1}^r n_i\,\beta_i$$

and we extend the map linearly to $\mathfrak{g}$. If $i<j$, we choose $\epsilon_i=-1$ and all other ϵ_k's to be 1. We then have

$$g(0,\ldots,0,t_i,0,\ldots,0,-t_j,0,\ldots,0) = g(0,\ldots,0,t_i,0,\ldots,0,t_j,0,\ldots,0)$$

which shows that the term $t_i\,t_j$ does not appear in (9). ■

We now discuss some instructive examples. Let $G/K = \mathbf{SL}(3,\mathbf{R})/\mathbf{SO}(3)$. Let $K_1 = \{\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix} : \theta\in\mathbf{R}\}$, $K_2 = \{\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{bmatrix} : \theta\in\mathbf{R}\}$ and $K_3 = \{\begin{bmatrix}\cos\theta & 0 & -\sin\theta\\ 0 & 1 & 0\\ \sin\theta & 0 & \cos\theta\end{bmatrix} : \theta\in\mathbf{R}\}$. If $K=\mathbf{SO}(3)$ then $K = K_1\,K_3\,K_2$ (in [10], we have $K=K_1\,K_2\,K_3$ which is equivalent if we interchange the first 2 coordinates in $\mathbf{R}^3$) and $K=K_1\,K_2\,K_1$ (see [15]).

If $X = \begin{bmatrix}a & 0 & 0\\ 0 & a & 0\\ 0 & 0 & -2\,a\end{bmatrix}\in\partial\mathfrak{a}^+$, $a>0$, and $Y = \begin{bmatrix}b & 0 & 0\\ 0 & b & 0\\ 0 & 0 & -2\,b\end{bmatrix}\in\partial\mathfrak{a}^+$, $b>0$, then for $k = k_1\,k_2\,k_1'\in K = K_1\,K_2\,K_1$, we have with the help of Corollary 3.3

$$e^X\,k\,e^Y = e^X\,k_1\,k_2\,k_1'\,e^Y = k_1\,e^X\,k_2\,e^Y\,k_1' = k_1\,k_2'\,e^{X+Y+c\,H_\beta}\,k_2''\,k_1'$$

where $H_\beta = \begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix}$. The set $\{a(e^X\,k\,e^Y)\colon k\in K\}$ is contained in a segment and therefore $\mu_{X,Y}$ cannot have a density. Note that in this case, $X+Y\in\partial\mathfrak{a}^+$.

If $X = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & -2\,a \end{bmatrix} \in \partial\mathfrak{a}^+$, $a > 0$ and $Y = \begin{bmatrix} 2\,b & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & -b \end{bmatrix} \in \partial\mathfrak{a}^+$, $b > 0$, then for $k = k_1\,k_3\,k_2 \in K = K_1\,K_3\,K_2$, we have with the help of Corollary 3.3

$$e^X\,k\,e^Y = e^X\,k_1\,k_3\,k_2\,e^Y = k_1\,e^X\,k_3\,e^Y\,k_2 = k_1\,k_3'\,e^{X+Y+c\,H_\beta}\,k_3''\,k_2$$

where $H_\beta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. The set $\{a(e^X\,k\,e^Y)\colon\ k \in K\}$ is again contained in a segment and the measure $\mu_{X,Y}$ has not a density. Note that in this case, $X + Y \in \mathfrak{a}^+$.

Similar remarks can be made for the same examples in the case $G/K = \mathbf{SL}(3,\mathbf{C})/\mathbf{SU}(3)$ using [6, Proposition 4.4].

Let us take $G/K = \mathbf{SL}(4,\mathbf{R})/\mathbf{SO}(4)$. Suppose X, $Y \in \partial\mathfrak{a}^+$ with $\alpha_1(X) = 0$, $\alpha_2(X) > 0$, $\alpha_3(X) > 0$ and $\alpha_2(Y) = 0$, $\alpha_1(Y) > 0$, $\alpha_3(Y) > 0$. Then α_3, $\alpha_2 + \alpha_3$ and $\alpha_1 + \alpha_2 + \alpha_3$ are independent roots which are nonzero on X and Y (note that $X + Y \in \mathfrak{a}^+$). Then the reasoning used in the proof of Proposition 4.7 shows that $\mu_{X,Y}$ has a density.

The above examples show that the density of $\mu_{X,Y}$ in the case where X and $Y \in \partial\mathfrak{a}^+$ may exist or not.

In a further analysis of the kernel of the product formula the following observations should be useful. We define the map $k\colon\ G \to K$ by $g = k(g)\,e^{\mathcal{H}(g)}\,n$ (the Iwasawa decomposition of g). The map $\bar{n} \to k(\bar{n})\,M$ is a diffeomorphism of $\bar{N}$ into an open subset of K/M whose complement is of measure zero with respect to dk_M. Combining (5) with [8, Th.5.20, page 198] we get

$$\int_{\bar{N}} f(e^{a(e^X\,k(\bar{n})\,e^Y)})\,e^{-2\rho\,(\mathcal{H}(\bar{n}))}\,d\bar{n} = \int_{\mathfrak{a}} f(e^H)\,k(H,X,Y)\,dH.$$

where the measures are normalized to ensure that $\int_{K/M} dk_M = \int_{\bar{N}} e^{-2\rho\,(\mathcal{H}(\bar{n}))}\,d\bar{n}$. It will be interesting to study the application $\bar{n} \to a(e^X\,k(\bar{n})\,e^Y)$ on $\bar{N}$. Let us give here a Taylor expansion of this function.

Corollary 4.8.

(i) Let α be a positive root, X_α an associated root vector and let $H_\alpha = [X_\alpha, \theta(X_\alpha)]$. Suppose $\alpha(X) > 0$ and $\alpha(Y) > 0$. Then there exists $S > 0$ such that for t small enough, we have

$$a(e^X\,k(e^{t\,\theta(X_\alpha)})\,e^Y) = X + Y + S\,H_\alpha\,t^2 + O(t^3)H_\alpha.$$

(ii) Let $\alpha_1,\ldots,\alpha_r$ be the simple positive roots and $X_{\alpha_1},\ldots,X_{\alpha_r}$ be associated root vectors. Suppose X, $Y \in \mathfrak{a}^+$. Then

$$a(e^X\,k(e^{t_1\,\theta(X_{\alpha_1})+\ldots+t_r\,\theta(X_{\alpha_r})})\,e^Y) = X + Y + \sum_{i=1}^r S_i\,H_{\alpha_i}\,t_i^2 + O(\|\mathbf{t}\|^3).$$

Proof. The part (i) follows from Corollary 3.3 using Lemma 4.9 below. The part (ii) may be then deduced from (i) similarly as the Proposition 4.4 is deduced from Corollary 3.3. ∎

Lemma 4.9. *Let α be a positive root and let X_α be any associated root vector. Then $k(e^{t\,\theta(X_\alpha)}) = e^{t\,(X_\alpha+\theta(X_\alpha))+O(t^3)}$.*

Proof. This is straightforward when we use the rank one reduction of Section 3. This can also be shown using Campbell-Baker-Hausdorff formula [14, Theorem 2.15.4]. ■

The following results are straightforward applications of the product formula and its proof. The proof of Corollary 4.10 is the same as that given in [6] in the complex case.

Corollary 4.10. *Let G be a semisimple Lie group of noncompact type and let μ, ν be two K-biinvariant finite measures on G such that $\mu(K) = \nu(K) = 0$. Suppose that $\mu(K\partial A^+K) = 0$ or $\nu(K\partial A^+K) = 0$. Then the measure $\mu * \nu$ is absolutely continuous.*

Corollary 4.11.

(i) Let $X \in \overline{\mathfrak{a}^+}, Y \in \mathfrak{a}^+$. Then the support of $\mu_{X,Y}$ contains a nonempty open set.

(ii) Let G be a simple Lie group of noncompact type and let $g \in K\,A^+\,K$. Then the orbit $K\,g\,K$ generates G.

For some interesting applications of the product formula in the arithmetic of probability measures and statistics see also [6].

5. Conclusion

Naturally, we would like to have more information about the kernel of the product formula and its support $a(e^X\,K\,e^Y)$. All our result guarantees is that the kernel $H \to k(H, X, Y)$ is of class L^1 on $\mathfrak{a}$.

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