Journal of Lie Theory
Volume **13** (2003) 7–12

On the Group of Isometries on a Locally Compact Metric Space

Antonios Manoussos and Polychronis Strantzalos*

Communicated by W. A. F. Ruppert

Abstract. In the present paper we study conditions under which the group of isometries on a locally compact metric space is locally compact, or acts properly.

Keywords: Ellis semigroup, isometry, pointwise convergence, proper action.
Mathematics Subject Classification: Primary: 54H20, 37B; Secondary: 54E15, 54H15.

1. Introduction

It is long known from the work of van Dantzig and van der Waerden ([1], cf. also [2, Ch.I, Th.4.7]) that if (X, d) is a connected locally compact metric space then its group of isometries $I(X, d)$, when endowed with the topology of pointwise convergence, is always locally compact and acts properly on X. More recently it was shown by one of the authors ([6]) that the pointwise closure of $I(X, d)$ is locally compact if the space $\Sigma(X)$ of the connected components of X is quasicompact (compact but not necessarily Hausdorff) with respect to the quotient topology. The question whether $I(X, d)$ is closed in $C(X, X)$ (the space of all continuous selfmaps of X endowed with the topology of pointwise convergence) remained open. In this note we fill this gap (cf. also [4]), i.e., we show that if $\Sigma(X)$ is quasicompact then $I(X, d)$ coincides with its Ellis' semigroup, completing the proof of the following:

Theorem. *Let (X, d) be a locally compact metric space. Denote by $I(X, d)$ its group of isometries, with the topology of pointwise convergence, and by $\Sigma(X)$ the space of the connected components of X, endowed with the quotient topology. Then*

1. *If $\Sigma(X)$ is not quasicompact, then $I(X, d)$ need not be locally compact, nor act properly on X.*
2. *If $\Sigma(X)$ is quasicompact then*

* The authors want to thank the referee for his useful and constructive comments.

ISSN 0949–5932 / \$2.50

(a) *$I(X,d)$ is locally compact,*
(b) *the action $(I(X,d),X)$ is not always proper, and*
(c) *the action $(I(X,d),X)$ is proper if X is connected.*

For the sake of completeness, we give short and slightly improved proofs of some of the previously published partial results of the authors, these are crucial for a unified proof of the above theorem. Our treatment is based on the sets $(x,V_x)=\{g\in I(X,d): g(x)\in V_x\}$, where V_x is a neighborhood of $x\in X$. These sets form a neighborhood subbasis at the identity with respect to the topology of pointwise convergence, the natural topology of $I(X,d)$.

2. Generalities

2.1. The following simple examples establish 1 and 2(b) of the above theorem.

Example. Let $X=\mathbb{Z}$ with the discrete metric. Obviously $\Sigma(X)$ is not quasi-compact. It can be easily seen that $I(X,d)$ is the group of all bijections of $\mathbb{Z}$, which is not locally compact with respect to the topology of pointwise convergence, therefore it cannot act properly on a locally compact space.

Example. Let $X=Y\cup\{(1,0)\}\subset\mathbb{R}^2$ where $Y=\{(0,y): \ y\in\mathbb{R}\}$, and $d=\min\{1,\delta\}$, where δ denotes the Euclidean metric. As we shall see in §3, by Theorem 3.7, $I(X,d)$ is locally compact; however the action $(I(X,d),X)$ is not proper, because the isotropy group of $(1,0)$ is not compact, since it contains the translations of Y. So, the action of $I(X,d)$ on X is not proper, even if X has two components.

Since the sets (x,V_x) as above form a neighborhood subbasis at the identity in $I(X,d)$, the following condition is necessary for the local compactness of $I(X,d)$:

(a) There exist $x_i\in X$, $i=1,\dots,m$ such that $\bigcap_{i=1}^m(x_i,V_{x_i})$ is relatively compact in $C(X,X)$.

This condition becomes also sufficient if, in addition, the following condition is satisfied:

(b) $I(X,d)$ is closed in $C(X,X)$.

So, to prove that $I(X,d)$ is locally compact, we have to ensure that both of the above conditions are satisfied.

3. The local compactness of $I(X,d)$

The following is crucial for the investigation of the conditions 2.1(a) and (b):

3.1. Lemma. *Let (X,d) be a locally compact metric space, $F\subseteq I(X,d)$, and*

$$K(F)=\{x\in X: F(x)=\{f(x): f\in F\} \text{ is relatively compact}\}.$$

Then $K(F)$ is an open and closed subset of X.

Proof. Since F is an equicontinuous family of selfmaps of X we see that $K(F)$ is open. It remains to prove that $K(F)$ is closed.

We write $S(x,\eta)=\{y\in X \mid d(x,y)<\eta\}$ for any $x\in X$ and $\eta>0$, and $S(M,\eta)=\bigcup\{S(x,\eta)\mid x\in M\}$ for subsets $M\subseteq X$. Let x be a cluster point of $K(F)$ and let η be a positive real such that $S(x,5\eta)$ is relatively compact. Choose a point $k\in K(F)\cap S(x,\eta)$. Then $\overline{F(k)}\subseteq S(F(S(x,\eta)),\eta)=F(S(x,2\eta))$, and by the compactness of $\overline{F(k)}$ we can find a finite subset $L\subseteq F$ such that $\overline{F(k)}\subseteq L(S(x,2\eta))$. We show that $F(x)$ is contained in the relatively compact set $L(S(x,5\eta))$. To see this, pick $f\in F$ and let $g\in L$ such that $f(k)\in g(S(x,2\eta))$. Then

$$\begin{aligned} d(f(x),g(k)) &\leq d(f(x),f(k))+d(f(k),g(x))+d(g(x),g(k))\\ &= d(x,k)+d(f(k),g(x))+d(x,k)\leq 4\eta \end{aligned}$$

and therefore

$$f(x)\in S(g(k),4\eta)=g(S(k,4\eta))\subseteq g(S(x,5\eta))\subseteq L(S(x,5\eta)).$$

Thus $x\in K(F)$ and the proof is finished.

3.2. Remark. In the sequel we assume that $\Sigma(X)$ *is quasicompact* in the quotient topology via the natural map $q:X\to\Sigma(X)$. Note that $\Sigma(X)$ is a T_1–space, and need not be Hausdorff. Nevertheless

X is separable, hence second countable; so sequences are adequate in C(X,X).

The proof is similar to the lengthy one in [5] (see also [2, Appendix 2]).

3.3. Lemma. *Let (X,d) be a locally compact metric space with a quasicompact space of connected components $\Sigma(X)$. Then condition 2.1(a) is satisfied.*

Proof. Let V_x be a relatively compact neighborhood of $x\in X$. Then

$$(x,V_x)=\{g\in I(X,d):\ g(x)\in V_x\}$$

is a neighborhood of the identity in $I(X,d)$. Since $x\in K((x,V_x))$, $K((x,V_x))$ is not empty, and by Lemma 3.1 is open and contains entire components of X. Therefore $q(K((x,V_x)))$ is an open subset of $\Sigma(X)$. Since $\Sigma(X)$ is quasicompact, there are x_i, $i=1,\ldots,m$, such that the corresponding $q(K((x_i,V_{x_i})))$' s cover $\Sigma(X)$. This means that $X=\bigcup_{i=1}^m K((x_i,V_{x_i}))$, i.e., the neighborhood $F=\bigcap_{i=1}^m (x_i,V_{x_i})$ of the identity has the property: for every $x\in X$ the set $F(x)$ is relatively compact in X. Therefore, by Ascoli' s theorem, F is relatively compact in $C(X,X)$.

3.4. Now we prove that if $\Sigma(X)$ is quasicompact then $I(X,d)$ is a closed subspace of $C(X,X)$. Because of Remark 3.2, the elements f of the boundary of $I(X,d)$ in $C(X,X)$ are limits of sequences $\{f_n\in I(X,d), n\in\mathbb{N}\}$. Obviously, such an f preserves d; so the question is whether f is surjective. If $\Sigma(X)$ is not quasicompact then this is not always true:

Example. Let $X = \mathbb{Z}$ with the discrete metric. If $f_n(z) = z$ for $-n < z < 0$, $f_n(-n) = 0$, and $f_n(z) = z+1$ otherwise, then $f_n \to f$, where $f(z) = z$ for $z < 0$, and $f(z) = z+1$ for $z \geq 0$. Hence each f_n is an isometry, but f is not surjective since $0 \notin f(\mathbb{Z})$.

3.5. Lemma. *If $\Sigma(X)$ is quasicompact and $\{(f_n) : f_n \in I(X,d)\}$ is a sequence such that $f_n \to f$ for some selfmap f of X with respect to the topology of pointwise convergence, then $f(X)$ is open and closed in X.*

Proof. By Lemma 3.1, it suffices to show that $f(X) = K(F)$, where $F = \{f_n^{-1},\, n \in \mathbb{N}\}$. Indeed, since $d(f_n(x), f(x)) = d(x, f_n^{-1}(f(x))$, we have $f_n^{-1}(f(x)) \to x$, so (since X is locally compact) $f(x) \in K(F)$, for every $x \in X$. Now, if $y \in K(F)$, we may assume $f_{n_k}^{-1}(y) \to x$ for some $x \in X$, because $F(y)$ is relatively compact in X, hence $f(x) = y$.

3.6. Proposition. *If (X,d) is a locally compact metric space, and $\Sigma(X)$ is quasicompact, then $I(X,d)$ is closed in $C(X,X)$.*

Proof. Let $\{(f_n) : f_n \in I(X,d)\}$ be a sequence such that $f_n \to f$ for some selfmap f of X with respect to the topology of pointwise convergence. We prove that f is surjective. Let $y \in X$. We denote by S_x the connected component containing $x \in X$, and by S_n the component of $f_n^{-1}(y)$. If $\{S_n, n \in \mathbb{N}\}$ has a constant subnet $\{S_{n_i},\, i \in I\}$, then $S_{n_i} = S_0$, for some $S_0 \in \Sigma(X)$. Hence $S_{f_{n_i}^{-1}(y)} = S_0$, so $f_{n_i}(S_0) = S_y$, for every $i \in I$. Pick an $x \in S_0$, then $f_{n_i}(x) \in S_y$. But $f_{n_i}(x) \to f(x)$, so $f(x) \in S_y$. By Lemma 3.5 $S_y \subseteq f(X)$, hence $y \in f(X)$.

Suppose that $\{S_n, n \in \mathbb{N}\}$ has no constant subnet. By the quasicompactness of $\Sigma(X)$, there exists a subnet $\{S_{n_i},\, i \in I\}$ of $\{S_n, n \in \mathbb{N}\}$ such that $S_{n_i} \to S$, for some $S \in \Sigma(X)$. With the above notation, the following is true:

Claim. *There exists a subsequence $\{S_k,\, k \in \mathbb{N}\}$ of $\{S_n,\, n \in \mathbb{N}\}$ such that there are $x_k \in S_k$ with $x_k \to x_0$, for some $x_0 \in X$.*

Proof. If not, $R = (\bigcup_{n=1}^{\infty} S_n) \setminus S$ is closed in X. Indeed, let $\{(y_m) : y_m \in R\}$ be a sequence such that $y_m \to y \in X$. If $y_m \in (\bigcup_{n=1}^{n_0} S_n) \setminus S$ for $m > m_0$, then a subsequence of $\{y_m,\, m \in \mathbb{N}\}$ is contained in some S_i for some $i \in \{1, \ldots, n_0\}$, therefore $y \in S_i \subseteq R$, as required. If this is not the case, we construct a subsequence $\{y_{m_p},\, p \in \mathbb{N}\}$ of $\{y_m,\, m \in \mathbb{N}\}$ in the following way: For S_1 we choose a point $y_{m_1} \in S_{n_1}$ with $n_1 > 1$ and $d(y_{m_1}, y) < 1$, for $(\bigcup_{n=1}^{n_1} S_n) \setminus S$ a point $y_{m_2} \in S_{n_2}$ with $n_2 > n_1$ and $d(y_{m_2}, y) < \frac{1}{2}$, and so on. Obviously, $y_{m_p} \in S_{n_p}$ and $y_{m_p} \to y$, a contradiction.

Since S does not meet R, then $S \subseteq X \setminus R$. On the other hand $X \setminus R$ is open (since R is closed in X) and contains entire components (recall that R is a union of components), so $S_{n_i} \subseteq X \setminus R$, eventually. Therefore $S_{n_i} = S$, a contradiction, since we have assumed that $\{S_n, n \in \mathbb{N}\}$ has no constant subnet.

According to the Claim, there exists a sequence $\{(x_k) : x_k \in S_k\}$ such that $x_k \to x_0 \in X$, where $S_k = S_{f_k^{-1}(y)} = f_k^{-1}(S_y)$, from which follows $x_k = f_k^{-1}(y_k)$ for some $y_k \in S_y$. Then

$$\begin{aligned} d(y_k, f(x_0)) &\leq d(y_k, f_k(x_0)) + d(f_k(x_0), f(x_0)) \\ &= d(f_k^{-1}(y_k), x_0) + d(f_k(x_0), f(x_0)) \to 0, \end{aligned}$$

therefore $f(x_0) \in S_y$, which means that $S_y \cap f(X) \neq \emptyset$ and, by Lemma 3.5, $S_y \subseteq f(X)$, hence $y \in f(X)$, and f is surjective.

3.7. Theorem. *If $\Sigma(X)$ is quasicompact, then $I(X, d)$ is locally compact.*

Proof. This assertion follows from Lemma 3.3 and Proposition 3.6, since both conditions 2.1(a) and (b) are satisfied.

4. The properness of the action $(I(X, d), X)$

In this short section, applying the methods used previously, we give a complete proof of the following:

Proposition. *If (X, d) is locally compact and connected, then $I(X, d)$ is locally compact and the action $(I(X, d), X)$ is proper.*

Proof. Since X is connected $G = I(X, d)$ is locally compact by Theorem 3.7. So, we have to show that, for every x, $y \in X$, there are neighborhoods U_x, U_y of x and y respectively such that

$$(U_x, U_y) := \{g \in G : (gU_x) \cap U_y \neq \emptyset\}$$

is relatively compact in G. Let $U_x = S(x, \varepsilon)$ and $U_y = S(y, \varepsilon)$ be such that $S(y, 2\varepsilon)$ is relatively compact. Then, for $g \in (U_x, U_y)$ and $z \in U_x$ with $g(z) \in U_y$, we have

$$d(g(x), (y)) \leq d(g(x), g(z)) + d(g(z), y) = d(x, z) + d(g(z), y) < 2\varepsilon,$$

therefore $g \in F = \{g \in G : g(x) \in S(y, 2\varepsilon)\}$. Then $x \in K(F)$, and, according to Lemma 3.1, $K(F)$ coincides with the connected space X. From this and Ascoli's theorem it follows that F is relatively compact in $C(X, X)$. So $(U_x, U_y) \subseteq F$ is relatively compact in $C(X, X)$, hence in G, because G is closed (cf. Proposition 3.6).

This proves the Proposition and completes the proof of the Theorem in the Introduction.

5. Final Remark

Using the same arguments we can prove that if X is a locally compact metrizable space, then $I(X, d)$ is locally compact for all admissible metrics d, provided that the space $Q(X)$ of the quasicomponents of X is compact with respect to the quotient topology (note that $Q(X)$ is always Hausdorff) (cf. [3]). Recall that the quasicomponent of a point is the intersection of all open and closed sets which contain it. Our exposition is given via $\Sigma(X)$ because we regard the condition "$\Sigma(X)$ is quasicompact" as a topologically more natural condition than "$Q(X)$ is compact", although it is more restrictive: There are locally compact metric spaces with compact $Q(X)$ and non quasicompact $\Sigma(X)$ as the following example shows:

Example. The space of all connected components of the locally compact space

$$X = \left(\bigcup_{n=1}^{\infty}\{(\frac{1}{n}, y) : y \in [-1,1]\}\right) \cup \{(0,y) : y \in [-1,0)\} \cup \left(\bigcup_{k=1}^{\infty} I_k\right) \subseteq \mathbb{R}^2,$$

where

$$I_k = \{(0,y) : y \in (\frac{1}{k+1}, \frac{1}{k})\}, \quad k \in \mathbb{N}^*,$$

is not quasicompact, because the sequence $\{I_k\} \subseteq \Sigma(X)$ does not have a convergent subsequence in $\Sigma(X)$. On the contrary, $Q(X)$ is compact, because the quasicomponent of the point $(0,-1)$ consists of the set $\{(0,y) : y \in [-1,0)\}$ and the intervals I_k, $k \in \mathbb{N}^*$.

So the quasicompactness of $\Sigma(X)$ is not necessary for the local compactness of $I(X,d)$.

References

[1] van Dantzig, D., and B. L. van der Waerden, *Über metrisch homogene Räume*, Abh. Math. Sem. Hamburg **6** (1928), 367–376.

[2] Kobayashi, S., and K. Nomizu, "Foundations of Differential Geometry, Vol. I," Interscience, New York, 1963.

[3] Manoussos, A., *Contribution to the study of D–stable actions*, Ph.D. Thesis, University of Athens, Greece (1993) (in Greek).

[4] Manoussos, A., *The Ellis semigroup of the full group of isometries of a locally compact metric space*, Preprint.

[5] Sierpinsky, W., *Sur les espaces métriques localement séparables*, Fund. Math. **21** (1933), 107–113.

[6] Strantzalos, P., *Actions by Isometries*, Lecture Notes in Math. **1375** (1989), 319– 325.

Antonios Manoussos
123 Sapfous Street
176 75 Kallithea
Athens
Greece
amanou@cc.uoa.gr

Polychronis Strantzalos
Department of Mathematics
University of Athens
Athens, GR – 157 84
Greece
pstrantz@math.uoa.gr

Received August 25, 2000
and in final form March 9, 2002