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Relative and Absolute Differential Invariants for Conformal Curves

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Abstract. In this paper we classify all vector relative differential invariants with Jacobian weight for the conformal action of $\mathrm{O}(n+1,1)$ on parametrized curves in $\mathbb{R}^n$. We then write a generating set of independent conformal differential invariants, for both parametrized and unparametrized curves, as simple combinations of the relative invariants. We also find an invariant frame for unparametrized curves via a Gram-Schmidt procedure. The invariants of unparametrized curves correspond to the ones found in [6]. As a corollary, we obtain the most general formula for evolutions of curves in $\mathbb{R}^n$ invariant under the conformal action of the group.

1. Introduction.

The basic theory of Invariance is due to Lie [14] and Tresse [24] and the introduction of their infinitesimal methods brought with it a rapid development of geometry which had its high point at the turn of the 19th century. The traditional approach of Lie was followed by the modern one of Cartan, whose moving frame method was developed during the first half of the 20th century. He showed how his method could produce differential invariants for several groups, including curves in the Euclidean, affine and projective planes [2], [3]. Invariants of parametrized projective curves were found early in the 20th century by Wilczynski in [25]. Those of unparametrized conformal curves were found by Fialkow in [6]. By differential invariants of parametrized curves we mean functions depending on the curve and its derivatives which are invariant under the action of the group, but not necessarily under reparametrizations. The invariants of unparametrized curves are, additionally, invariant under reparametrization. The theory of invariance is recently undergoing a revival and new methods have been developed, most notably the regularized moving frame method of Fels and Olver [4,5]. The recent work of Fels and Olver has led to the discovery of new invariants, among them differential invariants of parametrized objects. In [18] the authors used Fels and Olver's regularized moving frame method to classify all differential invariants of parametrized projective surfaces. In [16] the author found all invariants of parametrized planar curves under the action of $O(3,1)$, using the same

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approach. The method of Fels and Olver bypasses many of the complications inherent in the traditional method by avoiding completely the normalization steps that are necessary there. Some applications to image processing have also been found recently in [22].

The main result of this paper is the classification and the finding of explicit formulas for all vector relative differential invariants for parametrized curves in conformal n-space. They produce a moving frame, in the traditional sense, invariant under the group action (although not under reparametrization.) After classifying explicitly relative invariants the classification of differential invariants becomes immediate, for both parametrized and unparametrized curves. Their formulas in terms of relative invariants are strikingly simple. They involve only inner products and determinants. These relative invariants can also be used to generate a moving frame in the unparametrized case (in both traditional and Cartan's sense) thus connecting the relative invariants to the group theoretical definition of Frenet frame. (See Sharpe's book [23] for an excellent explanation of Cartan's formulation and the generation of conformal Frenet equations in Cartan's sense.) The relation with the Cartan frame will appear in a forthcoming paper. Our results for the unparametrized case coincide with those obtained by Fialkow in [6], although our construction and formulas are notably simpler.

Geometers have been traditionally interested in invariants and invariant evolutions of unparametrized submanifolds. These are used to solve the problem of classification. But both cases, parametrized and unparametrized, are highly relevant in the study of the relation between the geometry of curves and infinite dimensional Poisson brackets. This paper is motivated by the investigation of relationships between invariant evolutions of curves and surfaces, on one hand, and Hamiltonian structures of PDE's on the other. The idea behind this relationship is rather simple: if a curve or surface evolves invariantly in a certain homogeneous space, the evolution naturally induced on the differential invariants is sometimes a Hamiltonian system of PDEs. More interestingly, the Poisson structure is defined exclusively from the geometry of the curves or surfaces. For example, the so–called Adler–Gel'fand–Dikii bracket can be defined merely with the use of a projective frame and projective differential invariants of parametrized curves in $\mathbb{R}P^n$ ([15]). The presence of the projective group is essential for the understanding of the Poisson geometry of these brackets (see [26]). The same situation appears in, for example, 2 and 3-dimensional Riemannian manifolds with constant curvature in the case of unparametrized curves ([20], [12],[11], [19]). On the other hand, for this example it does not seem to hold in the parametrized case. Other examples are curves in CP^1 and reparametrizations of $\mathbb{R}P^2$ for the parametrized case ([16],[17]). The bibliography on integrable systems associated to these brackets is very extensive: see for example [10], [12], [27], [19] and references within. The study of Poisson tensors as related to the geometry of conformal curves and surfaces, and other homogeneous spaces, is still open.

In section two we introduce some of the definitions and concepts related to invariant theory and conformal geometry, as well as other concepts and results that will be needed in the paper. In section 3 we find a formula for the most general vector relative differential invariant with Jacobian weight for conformal

n–space, a formula which can be found in theorem 3.3. In section 4 we combine the vector invariants found in section 3 in order to classify differential invariants of parametrized curves. These differential invariants behave in many aspects like the differential invariants in Euclidean geometry, as is shown in theorem 4.1, formula (4.4). We select a family of independent generators formed by *homogeneous polynomials* on certain quotients of dot products of the curve and its derivatives. Their explicit expression can be found in theorem 4.6. These invariants are differential invariants at the infinitesimal level, that is, invariant under the action of the connected component of the group containing the identity. In section 4 we additionally show that the invariants found are also invariant under two chosen discrete symmetries of the group, symmetries which connect the different connected components. Our differential invariants will therefore be invariant under the complete conformal group. The last corollary in that section (corollary 4.11) states the most general form of an evolution of curves in $\mathbb{R}^n$ invariant under the conformal group. In section 5 we finally find a generating system of independent differential invariants which are also invariant under reparametrizations, with the mere use of inner products and determinants of vector relative invariants, as shown in theorem 5.1. These correspond to the invariants classified by Fialkow in [6]. We also find an invariant frame which is also invariant under reparametrizations. This frame will be obtained via a Gram-Schmidt process applied to the frame found in section 3. This result is given in Theorem 5.2. The last section is devoted to conclusions, and to the relation and the implications for infinite dimensional Hamiltonian systems of PDEs.

I would like to thank Peter Olver for his very useful comments on this paper and for so patiently explaining to me the regularized moving frame method and its details. The understanding of his method has given me a deep insight into the structure of differential invariants. I would also like to thank Jan Sanders for multiple conversations. Finally, my gratitude goes to the department of mathematics at the University of Leiden for its hospitality while this paper was written.

2. Notation and definitions.

2.1. *Preliminaries.*

Let $u : I \to \mathbb{R}^n$ be a parametrized curve, where I is some interval of $\mathbb{R}$. Let x be the parameter. Denote the components of u by $u(x) = (u^1, u^2, \dots, u^n)(x)$ and let u_s denote the s^{th} derivative of the curve u.

Let's denote by $p_{i,j}$ the following expression

$$p_{i,j} = \frac{u_i \cdot u_j}{u_1 \cdot u_1} \tag{2.1}$$

where we denote by $\cdot$ the usual dot product in $\mathbb{R}^n$.

Definition2.1. We say that *the degree* of $p_{i,j}$ is $i+j-2$ and we denote it by $d(p_{i,j}) = i+j-2$. Using the natural condition

$$d(p_{i,j}p_{r,s}) = d(p_{i,j}) + d(p_{r,s})$$

we can extend the concept of degree to products of $p_{i,j}$'s.

Let P be a polynomial in $p_{i,j}$, $i, j \geq 1$. We say that P is *homogeneous of degree* k if each one of the terms in P has degree k. For example, the polynomial $P = p_{1,2}^2 + p_{1,3}$ is a homogeneous polynomial of degree 2. The following properties are all quite obvious:

(a) Let P and Q be homogeneous polynomials of degree r and s. Then PQ is a homogeneous polynomial of degree $r + s$.

(b) If P is a homogeneous polynomial of degree k, then $\frac{dP}{dx}$ is a homogeneous polynomial of degree $k + 1$.

(c) The following formula holds

$$\frac{dp_{i,j}}{dx} = p_{i+1,j} + p_{i,j+1} - 2p_{1,2}p_{i,j}. \tag{2.2}$$

2.2. The conformal action of $\mathrm{O}(n+1,1)$ on $\mathbb{R}^n$.

Let $\mathrm{O}(n+1,1)$ be the set of $n+2 \times n+2$ matrices such that $M \in \mathrm{O}(n+1,1)$ if, and only if $MCM^T = C$ where C is given by

$$C = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \tag{2.3}$$

and where ^{T} denotes transposition.

We call *Minkowski length* the length defined by the matrix (2.3), that is $|x| = x^T C x$, and *Minkowski space* the space $\mathbb{R}^{n+2}$ (or its projectivisation) endowed with Minkowski length (we are of course abusing the lenguage here since the "length" of a nonzero vector can be zero). Let $\mathbb{RP}_0^{n+1}$ be the lightcone in Minkowski space, that is, points in $\mathbb{RP}^{n+1}$ with zero Minkowski length. We can also think of them as lines in $\mathbb{R}^{n+2}$ such that $xCx^T = 0$ whenever x is on the line.

$\mathrm{O}(n+1,1)$ acts naturally on $\mathbb{R}^{n+2}$ using the usual multiplication. Given that $\mathrm{O}(n+1,1)$ preserves the metric, it also acts on $\mathbb{RP}_0^{n+1}$. If $U \subset \mathbb{RP}^{n+1}$ is a coordinate neighborhood, the immersion of $\mathbb{RP}^{n+1}$ into $\mathbb{R}^{n+2}$ will take locally the form

$$\begin{aligned} \eta : U &\to \mathbb{R}^{n+2} \\ y &\to (y, 1). \end{aligned}$$

Now, $\mathbb{R}^n$ can be identified locally with $\mathbb{RP}_0^{n+1}$ using the map

$$\begin{aligned} \rho : \mathbb{R}^n &\to \mathbb{RP}_0^{n+1} \\ u &\to (q, u) \end{aligned}$$

where q is uniquely determined through the relationship $2q+(u^1)^2+\ldots+(u^n)^2 = 0$ which is necessary upon imposing the zero length condition. Let π be the projection of $\mathbb{R}^{n+2}-\{0\}$ on $\mathbb{RP}^{n+1}$.

The action of $O(n+1,1)$ on $\mathbb{R}^n$ is given by the map

$$O(n+1,1)\times\mathbb{R}^n \to \mathbb{R}^n$$

$$(N,u)\to N\cdot u = \rho^{-1}\left(\pi N(\eta(\rho(u)))\right),$$

that is, we lift u to an unique element on the lightcone, lift the line to $\mathbb{R}^{n+2}$, multiply by N and project back into $\mathbb{RP}^{n+1}_0$ and into $\mathbb{R}^n$. This procedure is the usual interpretation of conformal geometry as the geometry induced by the action of $O(n+1,1)$ on $\mathbb{R}^n$, for $n>2$ (see [8] for more details). The equivalence of the geometry defined by this action (in the Klein sense) and the usual conformal geometry can be found in some classical differential geometry books, but it is perhaps better explained in [13].

This action has, of course, its infinitesimal version, the representation of the subalgebra $o(n+1,1)$ as infinitesimal generators. This representation is generated by the following vectors fields of $\mathbb{R}^n$ (the case $n=2$ is listed in [21])

$$v_i = \frac{\partial}{\partial u^i},\quad i=1,\ldots n, \qquad v_{ij} = u^i\frac{\partial}{\partial u^j} - u^j\frac{\partial}{\partial u^i},\quad i<j,\ i,j=1,\ldots,n,$$
$$v=\sum_{i=1}^n u^i\frac{\partial}{\partial u^i}, \qquad w_i=\sum_{j\neq i}2u^iu^j\frac{\partial}{\partial u^j}+((u^i)^2-\sum_{i\neq j}(u^j)^2)\frac{\partial}{\partial u^i},\ i=1,\ldots n. \tag{2.4}$$

We will abuse the notation and denote the algebra of infinitesimal generators also by $o(n+1,1)$.

We recall that the group $O(n+1,1)$ has a total of four connected components: if $M\in O(n+1,1)$ is given by

$$M=\begin{pmatrix} a_1 & v_1^T & a_2\\ v_3 & A & v_4\\ a_3 & v_2^T & a_4\end{pmatrix}$$

with $a_i\in\mathbb{R}$, $v_i\in\mathbb{R}^n$ and A a $n\times n$ matrix, condition $MCM^T=C$ implies the equation

$$A\left[\frac{1}{(a_1a_3-a_2a_4)^2}\begin{pmatrix} v_1 & v_2\end{pmatrix}\begin{pmatrix} v_2^Tv_2 & 1-v_1^Tv_2\\ 1-v_1^Tv_2 & v_1^Tv_1\end{pmatrix}\begin{pmatrix} v_1^T\\ v_2^T\end{pmatrix}+\mathrm{Id}\right]A^T=\mathrm{Id}$$

where Id represents the identity matrix. Therefore, $M\in O(n+1,1)$ implies that the determinant of both M and A should be nonzero. Four connected components are found for each choice of sign in both determinants. Notice that multiplication by C itself changes the sign of the determinant of M, but not that of the inner matrix A. Multiplication by, for example

$$\begin{pmatrix} 1 & 0 & 0 & \ldots & 0\\ 0 & -1 & 0 & \ldots & 0\\ 0 & 0 & 1 & \ldots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \ldots & 1\end{pmatrix} \tag{2.5}$$

will change the sign of both determinants. These two multiplications define two discrete symmetries of the group and they are sufficient to connect the four connected components.

2.3. The Theory of differential invariance.

Let $\mathrm{J}^m = \mathrm{J}^m(\mathbb{R}, \mathbb{R}^n)$ denote the m^{th} order jet bundle consisting of equivalence classes of parametrized curves modulo m^{th} order contact. We introduce local coordinates x on $\mathbb{R}$, and u on $\mathbb{R}^n$. The induced local coordinates on J^m are denoted by u_k, with components u_k^α, where $u_k^\alpha = \frac{d^k u^\alpha}{dx^k}$, $0 \leq k \leq m$, $\alpha = 1, \ldots, n$, represent the derivatives of the dependent variables – u^α – with respect to the independent variable – x –.

Since $\mathrm{O}(n+1,1)$ preserves the order of contact between submanifolds, there is an induced action of $\mathrm{O}(n+1,1)$ on the jet bundle J^m known as its m^{th} *prolongation*, and denoted by $\mathrm{O}(n+1,1)^{(m)}$ (the underlying group being identical to $\mathrm{O}(n+1,1)$). Since we are studying the action on parametrized curves, $\mathrm{O}(n+1,1)$ does not act on x, and the prolonged action becomes quite simple, namely the action is given by

$$\begin{gathered} \mathrm{O}(n+1,1)^{(m)} \times \mathrm{J}^m \to \mathrm{J}^m \\ (g, u_k) \to (g \cdot u)_k. \end{gathered}$$

The prolonged action has also its infinitesimal counterpart, the *infinitesimal prolonged action* of $\mathrm{o}(n+1,1)^{(m)}$ on the tangent space to J^m. The infinitesimal generators of this action are the so-called *prolongations* of the infinitesimal generators in (2.4) above. In our special case of parametrized curves the prolongations of a vector $w \in \mathrm{o}(n+1,1)$, with w given as $w = \sum_{i=1}^n \xi^i \frac{\partial}{\partial u^i}$, are defined as $\mathrm{pr}^{(m)} w \in \mathrm{o}(n+1,1)^{(m)}$

$$\mathrm{pr}^{(m)} w = \sum_{i=1}^n \sum_{k \leq m} \xi_k^i \frac{\partial}{\partial u_k^i} \tag{2.6}$$

where again $\xi_k^i = \frac{d^k \xi^i}{dx^k}$.

Definition2.2. A map $F : \mathrm{J}^m \to \mathbb{R}^n$ is called a (infinitesimal) *relative vector differential invariant* of the conformal action of $\mathrm{O}(n+1,1)$ with *Jacobian weight* if for any $w \in \mathrm{o}(n+1,1)$, defined as $w = \sum_{i=1}^n \xi^i \frac{\partial}{\partial u^i}$, F is a solution of the system

$$\mathrm{pr}^{(m)} w(F) = \frac{\partial \xi}{\partial u} F, \tag{2.7}$$

where $\frac{\partial \xi}{\partial u}$ is the Jacobian, a matrix with (i,j) entries given by $\dfrac{\partial \xi^i}{\partial u^j}$, and where $\mathrm{pr}^{(m)} w(F)$ represents the application of the vector field $\mathrm{pr}^{(m)} w$ to each one of the entries in F.

A map $I : \mathrm{J}^m \to \mathbb{R}$ is called a m^{th} order *differential invariant* for the conformal action of $\mathrm{O}(n+1,1)$ on $\mathbb{R}^n$ if it is invariant under the prolonged action of $\mathrm{O}(n+1,1)^{(m)}$. The infinitesimal description of differential invariants is well known. I is an infinitesimal m$^{\text{th}}$ order differential invariant for the conformal action if and only if

$$\mathrm{pr}^{(m)}w(I) = 0$$

for any $w \in \mathrm{o}(n+1,1)$. In this case I is only invariant under the action of the connected component of $\mathrm{O}(n+1,1)$ containing the identity.

A set $\{I_1, \dots, I_k\}$ of differential invariants is called a *generating set* if any other differential invariant can be written as an analytic function of $I_1, \dots, I_k$ and their derivatives with respect to the parameter. They are called *independent (or differentially independent)* if no invariant in the set can be written as an analytic function of the other invariants and their derivatives with respect to the parameter x.

3. Classification of relative differential invariants.

This is the main section in the paper. Here we will give explicitly the formula for any relative vector differential invariant with Jacobian weight, which amounts to finding all possible solutions of (2.7). It is known ([21]),[7]) that these vectors can be used to write a general formula for evolutions of parametrized curves in $\mathbb{R}^n$ which are invariant under the action of $\mathrm{O}(n+1,1)$. By an invariant evolution we mean an evolution for which $\mathrm{O}(n+1,1)$ takes solutions to solutions. We show this in section 4. Furthermore, these vectors also determine the invariants of both parametrized and unparametrized conformal curves. In subsequent sections we will see that they play the analogous role in conformal geometry to that of the curve derivatives in Euclidean geometry, not only because they form an invariant frame, but because they are in fact the building blocks of the invariants. These are written in terms of the relative invariants much as Euclidean invariants are written in terms of u_k.

The following result is known and can be found for example in [7].

Proposition 3.1. *Let ν be a nondegenerate matrix of vector relative differential invariants with common Jacobian weight. Then, any other vector relative differential invariant with the same weight is given as*

$$F = \nu \mathcal{I}$$

where $\mathcal{I}$ is a vector of differential invariants.

One can rephrase this proposition as: given n independent solutions of (2.7), any additional solution can be written as a combination of them with invariant coefficients. The solution of (2.7) is a n–dimensional module over the ring of differential invariants. ¿From here, the classification of relative differential invariants is reduced to finding a nondegenerate matrix of vector relative differential invariants with Jacobian weight, ν, and to classify the differential invariants. We will solve the first part in this section and the second part in section 4 and 5. The knowledge of ν (almost) suffices for the complete classification of both vector relative differential invariants and absolute differential invariants.

The projective case was worked out in [7] and a nondegenerate matrix of relative invariants was found there. The classification of differential invariants in the projective case had been already found in the early 20th century in [25]. In the projective case the matrix of relative invariants had a factorization into a Wronskian matrix and an upper triangular matrix with ones down the diagonal. In the conformal case things are different and in some sense more complicated. The Wronskian matrix does not suffice to get a factorization of the matrix we search for, as we need to use derivatives up to order $n+1$. This need will be clear in chapter four when we recall the order of generating invariants, as given in the work of Green [9]. First of all, let us prove a simple, but fundamental lemma that we will need along the paper.

Lemma 3.2. *Assume u is such that the vectors $u_1, \dots, u_n$ are independent. Then, the functions $p_{i,j}$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, k$, $i \le j$, $k \ge n$, are functionally independent.*

In particular, for such a choice of u, the matrix

$$P = \begin{pmatrix} 1 & p_{1,2} & p_{1,3} & \cdots & p_{1,n} \\ p_{1,2} & p_{2,2} & p_{2,3} & \cdots & p_{2,n} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ p_{1,n} & p_{2,n} & p_{3,n} & \cdots & p_{n,n} \end{pmatrix} \tag{3.1}$$

is invertible.

Proof. Consider the map

$$\begin{array}{ccc} L : J^k & \to & \mathcal{P} \\ (x, u, u_1, \dots, u_k) & \to & (p_{1,2}, \dots, p_{1,k}, p_{2,2}, \dots, p_{2,k}, \dots, p_{n,n}, \dots, p_{n,k}) \end{array}$$

where $\mathcal{P} = \mathbb{R}^m$ and where m is the number of $p_{i,j}$'s. The proof of the lemma is equivalent to showing that L is a submersion for any u for which $u_1, \dots, u_n$ are independent. Since the dimension of the manifold $\mathcal{P}$ equals the number of different $p_{i,j}$'s, that is, equals $\frac{n(n+1)}{2} - 1 + n(k-n)$, we need to show that the rank of L is $\frac{n(n+1)}{2} - 1 + n(k-n)$. Define $\hat{L}(u)$ to be given by the matrix

$$\hat{L}(u) = \begin{pmatrix} 1 & p_{1,2} & \cdots & p_{1,n} & p_{1,n+1} & \cdots & p_{1,k} \\ p_{1,2} & p_{2,2} & \cdots & p_{2,n} & p_{2,n+1} & \cdots & p_{2,k} \\ \vdots & \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ p_{1,n} & p_{2,n} & \cdots & p_{n,n} & p_{n,n+1} & \cdots & p_{n,k} \end{pmatrix}.$$

Since each $\frac{\partial \hat{L}(u)}{\partial u_i^\alpha}$ is a matrix representation of a vector column in the Jacobian matrix of L (the one associated to $\frac{\partial L(u)}{\partial u_i^\alpha}$) we conclude that the rank of L at u equals the dimension of the linear subspace of matrices $n \times k$ generated by the set

$$\{\frac{\partial \hat{L}(u)}{\partial u_i^\alpha},$$

Indeed, both rank of L and dimension of (3.2) are equal to the dimension of the row space of the Jacobian matrix of L. A simple inspection reveals

$$\frac{\partial \hat{L}(u)}{\partial u_i^\alpha} = \frac{1}{u_1 \cdot u_1} \begin{pmatrix} 0 & \dots & 0 & u_1^\alpha & 0 & \dots & \dots & \dots & 0 \\ 0 & \dots & 0 & u_2^\alpha & 0 & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \vdots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & u_{i-1}^\alpha & 0 & \dots & \dots & \dots & 0 \\ u_1^\alpha & u_2^\alpha & \dots & 2u_i^\alpha & \dots & u_n^\alpha & u_{n+1}^\alpha & \dots & u_k^\alpha \\ 0 & \dots & 0 & u_{i+1}^\alpha & 0 & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \vdots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & u_n^\alpha & 0 & \dots & \dots & \dots & 0 \end{pmatrix} - 2\frac{u_1^\alpha}{u_1 \cdot u_1}\delta_i^1 \hat{L}(u), \tag{3.2}$$

for any $i \leq n$ and $\alpha = 1, \dots, n$ (δ is the Delta of Kronecker,) and

$$\frac{\partial \hat{L}(u)}{\partial u_i^\alpha} = \frac{1}{u_1 \cdot u_1} \begin{pmatrix} 0 & \dots & 0 & u_1^\alpha & 0 & \dots & 0 \\ \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & \dots & 0 & u_n^\alpha & 0 & \dots & 0 \end{pmatrix} \tag{3.3}$$

for any $i > n$, $\alpha = 1, \dots, n$, where the nonzero column is in place i. Since, by hypothesis, $u_1, \dots, u_n$ are independent, the matrices in (3.4) generate a subspace of dimension $n(k-n)$ for the different choices $\alpha = 1, \dots, n$. They generate matrices with zero left $n \times n$ block.

For the purpose of dimension counting, we can now assume that the right $n \times (k-n)$ block in matrices (3.3) are zero. We then see that for $i = n$, the different choices $\alpha = 1, \dots, n$ in (3.3) will generate the matrices $E_{r,n} + E_{n,r}$, $r = 1, \dots, n$. Here $E_{i,j}$ denotes the matrix having 1 in the (i,j) entry and zeroes elsewhere. For $i = n-1$, in that same group of matrices, $n-1$ appropiate combinations of the choices $\alpha = 1, \dots, n$ will generate the matrices $E_{r,n-1} + E_{n-1,r}$, $r = 1, \dots, n-1$. In general, for a given $1 < s \leq n$, appropiate choices of combinations of the matrices corresponding to $\alpha = 1, \dots, n$ will generate $E_{r,s} + E_{s,r}$ for $r = 1, \dots, s$. Obviously $E_{1,1}$ can never be generated by any of these matrices.

The dimension of the subspace (3.2) is thus $n + n - 1 + \ldots + 2 + n(n-k)$ and the lemma is proved.

■

The following is the main theorem in this chapter.

Theorem 3.3. *Let u be a parametrized curve in $\mathbb{R}^n$. Define D to be the $n \times n+1$ matrix given by*

$$D = \begin{pmatrix} u_1^1 & u_2^1 & \dots & u_{n+1}^1 \\ u_1^2 & u_2^2 & \dots & u_{n+1}^2 \\ \vdots & \vdots & \dots & \vdots \\ u_1^n & u_2^n & \dots & u_{n+1}^n \end{pmatrix} \tag{3.4}$$

where $u^i_j = \frac{d^j u^i}{dx^j}$. *Then, there exists a* $n+1 \times n$ *matrix* Q *of the form*

$$Q = \begin{pmatrix} 1 & g_1^3 & g_1^4 & \dots & g_1^{n+1} \\ 0 & g_2^3 & g_2^4 & \dots & g_2^{n+1} \\ 0 & 1 & g_3^4 & \dots & g_3^{n+1} \\ 0 & 0 & 1 & \dots & g_4^{n+1} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}, \tag{3.5}$$

whose entries are homogeneous polynomials in $p_{i,j}$ *as in (2.1), and such that* $\nu = DQ$ *is a matrix of relative vector differential invariants with Jacobian weight. For a curve* u *such that* $u_1, \dots, u_n$ *are independent vectors, the matrix* ν *is nondegenerate.*

The entries g^i_j will be defined in lemma 3.4 for any $i = 3, \dots, n+1$, $j = 1, \dots, i$.

Proof. (Proof of the theorem.) Let F_i be the columns of the matrix $\nu = DQ$. First of all, it is fundamentally trivial to show that $F_1 = (F^j_1) = (u^j_1)$ is a relative vector differential invariant. The reader can check the validity of this below. We will thus focus on the other columns $F_{i-1} = (F^j_{i-1})$ for $i \geq 3$, where

$$F^j_{i-1} = u^j_1 g^i_1 + u^j_2 g^i_2 + \dots + u^j_i g^i_i, \tag{3.6}$$

and where g^i_j will be defined in lemma 3.4.

First of all, notice that if v, v_i and v_{ij} are the infinitesimal generators of $\mathrm{o}(n+1,1)$ given as in (2.4), and if $p_{r,s}$ are defined as in (2.1), then, using definition (2.6) of prolonged vector field it is straightforward to show that

$$\mathrm{pr}^{(m)} v_i(p_{r,s}) = \mathrm{pr}^{(m)} v(p_{r,s}) = \mathrm{pr}^{(m)} v_{ij}(p_{r,s}) = 0,$$

for any $i, j = 1, \dots, n$, $i < j$, and where m is always chosen as high as necessary. Furthermore,

$$\mathrm{pr}^{(m)} v_i(u^r_s) = 0, \quad \mathrm{pr}^{(m)} v(u^r_s) = u^r_s \quad \text{and} \quad \mathrm{pr}^{(m)} v_{ij}(u^r_s) = \delta^r_j u^i_s - \delta^r_i u^j_s.$$

So, if we assume that our functions g^i_j are polynomials on $p_{i,j}$, we obtain the following conditions on F_r,

$$\mathrm{pr}^{(m)} v_i(F^s_r) = 0, \quad \mathrm{pr}^{(m)} v(F^s_r) = F^s_r$$

$$\mathrm{pr}^{(m)} v_{ij}(F^s_r) = \begin{cases} 0 & \text{if } s \neq i, j \\ F^i_r & \text{if } s = j \\ -F^j_r & \text{if } s = i. \end{cases}$$

Now, the Jacobian $\frac{\partial \xi}{\partial u}$ as in (2.7) is zero for the vector fields v_i, $i = 1, \dots, n$, $\frac{\partial \xi}{\partial u} = \mathrm{Id}$, for the vector field v, (where Id represents the identity matrix,) and $\frac{\partial \xi}{\partial u} = E_{ji} - E_{ij}$ for the vector fields v_{ij}, $i, j = 1, \dots, n$, $i < j$. We readily see that F_r satisfies equations (2.7) whenever the vector field w is one of the vector fields v, v_i or v_{ij}. (Notice that F_1 given as in the beginning of the proof would be a special case where $g_1^i = 1$ and $g_r^i = 0$ for $r > 1$. Hence F_1 will also satisfy these equations.) Thus, the main conditions and troubles will come from trying to find solutions of the form (3.7) to equations (2.7) with vector fields $w = w_i$, $i = 1, \ldots, n$.

If $i \neq j$ the following relationship is straightforward

$$\mathrm{pr}^{(m)} w_i(u_k^j) = 2 \sum_{p=0}^{k} \binom{k}{p} u_p^i u_{k-p}^j,$$

and if $i = j$, we obtain

$$\mathrm{pr}^{(m)} w_i(u_k^i) = \sum_{p=0}^{k} \binom{k}{p} \left(2u_p^i u_{k-p}^i - u_p \cdot u_{k-p}\right),$$

where $\cdot$ denotes the usual dot product in $\mathbb{R}^n$. Now notice that, in the case $w = w_i$, the Jacobian matrix $\frac{\partial \xi}{\partial u}$ in (2.7) is given by a matrix having (j, i) entry equals $2u^j$, $j = 1, \ldots, n$, having (i, j) entry equals $-2u^j$, $j = 1, \ldots, n$, $j \neq i$, and having (j, j) entry equals $2u^i$, $j = 1, \ldots, n$. Thus

$$\frac{\partial \xi}{\partial u} = \begin{pmatrix} 2u^i & 0 & \ldots & 2u^1 & \ldots & 0 \\ 0 & 2u^i & \ldots & 2u^2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ldots & \vdots \\ -2u^1 & -2u^2 & \ldots & 2u^i & \ldots & -2u^n \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 2u^n & \ldots & 2u^i \end{pmatrix} \tag{3.7}$$

Equation (2.7) in this case becomes

$$\mathrm{pr}^{(m)} w_i(F_{r-1}^j) = \begin{cases} 2u^i F_{r-1}^j + 2u^j F_{r-1}^i & \text{if } i \neq j \\ 4u^i F_{r-1}^i - \displaystyle\sum_{k=1}^{n} 2u^k F_{r-1}^k & \text{if } i = j, \end{cases} \tag{3.8}$$

$r \geq 3$. (Notice that if $F_1 = (u_1^j)$, then $\mathrm{pr}^{(m)} w_i(u_1^j) = \begin{cases} 2u^i u_1^j + 2u_1^i u^j & \text{if } i \neq j \\ 4u^i u_1^i - 2u \cdot u_1 & \text{if } i = j \end{cases}$ so F_1 will trivially hold these equations also.) If we are looking for solutions of the form (3.7) to these PDEs, equations (3.9) become the following equations for the entries g_s^r

$$\sum_{k=1}^{r} u_k^j \mathrm{pr}^{(m)} w_i(g_k^r) + 2 \sum_{k=2}^{r} \sum_{p=1}^{k-1} \binom{k}{p} u_p^i u_{k-p}^j g_k^r = 0 \tag{3.9}$$

for any $i \neq j$ and any $r \geq 3$, and

$$\sum_{k=1}^{r} u_k^i \mathrm{pr}^{(m)} w_i(g_k^r) + \sum_{k=2}^{r} \sum_{p=1}^{k-1} \binom{k}{p} \left[2u_p^i u_{k-p}^i g_k^r - \sum_{j=1}^{n} u_p^j u_{k-p}^j g_k^r\right] = 0 \tag{3.10}$$

for any $i = 1, \dots, n$, and any $r \geq 3$. Thus, we will prove the theorem once we find g_s^r polynomials homogeneous in $p_{i,j}$ and solutions of the system of PDEs given by (3.10) and (3.11). This is an overdetermined system. The first step to solving the system is to realize that, since u_k^j are independent variables in the jet space, the subsystem (3.10) will be solved whenever we solve the simpler system

$$\mathrm{pr}^{(m)} w_i(g_r^r) = 0$$

$$\mathrm{pr}^{(m)} w_i(g_k^r) + 2\sum_{p=1}^{r-k} \binom{k+p}{p} u_p^i g_{k+p}^r = 0 \tag{3.11}$$

for any $i = 1, \dots, n$. This system is given by the coefficients of u_k^j in (3.10), after a short rewriting of the equations. The first equation in (3.12) is immediately satisfied if we normalize assuming that $g_r^r = 1$ (in fact any choice of constant would do).

Additional information about our polynomials g_s^r can be found from the system (3.12). We can extend the 2.1 definition of degree to products of the form $u_k^\alpha p_{i,j}$ so that

$$d(u_k^\alpha p_{i,j}) = k + d(p_{i,j}).$$

Now, given that none of the vectors w_i in (2.4) involve derivatives of u in their coefficients, if g_s^r is a homogeneous polynomial

$$d(\mathrm{pr}^{(m)} w_i(g_s^r)) = d(g_s^r).$$

Therefore, *if homogeneous polynomials g_s^r are to be found satisfying (3.12), the degree of g_s^r needs to be $r - s$.*

Next, if g_s^r satisfy (3.12), then the following equation is also satisfied

$$\sum_{k=1}^{r} u_k^i \mathrm{pr}^{(m)} w_i(g_k^r) + 2\sum_{k=2}^{r}\sum_{p=1}^{k-1} \binom{k}{p} u_p^i u_{k-p}^i g_k^r = 0 \tag{3.12}$$

since this is just a combination of equations in (3.12). Substituting this relationship in (3.11) we obtain that, in order to additionally satisfy (3.8) (and therefore the complete system), some solutions of (3.12) must also satisfy

$$2\sum_{k=2}^{r}\sum_{p=1}^{k-1} \binom{k}{p} u_{k-p}^i u_p^i g_k^r = 2\sum_{k=2}^{r}\sum_{p=1}^{k-1} \binom{k}{p} u_{k-p}^i u_p^i g_k^r - \sum_{k=2}^{r}\sum_{p=1}^{k-1}\sum_{j=1}^{n} \binom{k}{p} u_{k-p}^j u_p^j g_k^r,$$

which can be rewritten as

$$\sum_{k=2}^{r}\sum_{p=1}^{k-1} \binom{k}{p} p_{p,k-p} g_k^r = 0, \tag{3.13}$$

for any $r \geq 3$. Summarizing, *we will have proved the theorem once we find homogeneous polynomial g_s^r solving (3.12) and having the additional condition (3.14).*

Althought this rewriting has considerably simplified our task, it is still nontrivial to find the solutions to this simplified system. We are lucky enough to have the following fundamental recursion formula:

Lemma 3.4. *Assume that g_s^r is defined via the following recursion formula*

$$g_k^{r+1} = -p_{1,2} g_k^r + g_{k-1}^r + (g_k^r)' \tag{3.14}$$

for $k \geq 2$ and any $r \geq 3$, and where $'$ represents the derivative with respect to x. Assume $g_r^r = 1$ for all r and $g_1^2 = -2p_{1,2}$, by definition. Assume also that, at each step r, g_1^r is determined by the relationship

$$g_1^r = -\sum_{k=2}^{r} p_{1,k} g_k^r. \tag{3.15}$$

Then the set $\{g_s^r\}$ obtained this way defines a solution of the system (3.12) and satisfies the additional condition (3.14).

Proof. (Proof of the lemma.) First of all we will describe the recursion. The procedure defines our set of homogeneous polynomials the following way: using the defined values $g_2^2 = 1$ and $g_1^2 = -2p_{1,2}$ we obtain g_2^3 using (3.15). We then fix $g_3^3 = 1$ and g_1^3 determined by (3.16). We can then find g_2^4 and g_3^4 using the recursion formula. We fix $g_4^4 = 1$ and g_1^4 determined by (3.16); we find g_2^5, g_3^5, g_4^5 using the recursion formula, and so it goes. Using g_k^r, $k = 1, \ldots, r$ we can find, using the recursion formula, the values for g_k^{r+1}, $k = 2, \ldots, r$. We then fix $g_{r+1}^{r+1} = 1$ and g_1^{r+1} determined by (3.16). Thus, we build the following diagram from top to bottom by filling in the central entries with the use of the row immediately above and the induction, and then finding the beginning and the end using (3.16).

$$\begin{array}{ccccccccc}
 & & & g_2^2 & & g_1^2 & & & \\
 & & g_3^3 & & g_2^3 & & g_1^3 & & \\
 & g_4^4 & & g_3^4 & & g_2^4 & & g_1^4 & \\
g_5^5 & & g_4^5 & & g_3^5 & & g_2^5 & & g_1^5 \\
 & \cdots & & \cdots & & \cdots & & \cdots &
\end{array}$$

We will obviously prove the lemma by induction on r. The lemma holds true for $r = 3$. For this value $g_3^3 = 1$ and (3.15), (3.16) gives us the values $g_2^3 = -3p_{1,2}$ and $g_1^3 = 3(p_{1,2})^2 - p_{1,3}$. If we use the fact that $\mathrm{pr}^{(m)} w_i(p_{1,3}) = 6u_2^i$ and $\mathrm{pr}^{(m)} w_i(p_{1,2}) = 2u_1^i$ and we substitute these values in (3.12) we obtain the desired results. As we pointed out before $g_3^3 = 1$ satisfies trivially the first equation in (3.12) and condition (3.14) is also trivially satisfied in this case.

Assume the lemma holds true for all g_j^s, $s \leq r$, $j = 1, \ldots, r$. We need to prove that g_{k+1}^{r+1} defined by induction in (3.12) satisfies

$$\mathrm{pr}^{(m)} w_i(g_{k+1}^{r+1}) = -2 \sum_{p=1}^{r-k} \binom{k+p+1}{p} u_p^i g_{k+p+1}^{r+1} \tag{3.16}$$

for any $k < r$ (since again the case $k = r$ is trivial). Using the induction hypothesis we have on one hand

$$\mathrm{pr}^{(m)} w_i(-p_{1,2} g_{k+1}^r) = -2u_1^i g_{k+1}^r + 2p_{1,2} \sum_{p=1}^{r-k-1} \binom{k+1+p}{p} u_p^i g_{k+p+1}^r,$$

$$\mathrm{pr}^{(m)} w_i(g_k^r) = -2 \sum_{p=1}^{r-k} \binom{k+p}{p} u_p^i g_{k+p}^r$$

and on the other hand

$$\begin{aligned}\mathrm{pr}^{(m)} w_i((g_{k+1}^r)') &= (\mathrm{pr}^{(m)} w(g_{k+1}^r))' \\ &= -2\sum_{p=2}^{r-k} \binom{k+p}{p-1} u_p^i g_{k+p}^r - 2\sum_{p=1}^{r-k-1} \binom{k+p+1}{p} u_p^i (g_{k+p+1}^r)',\end{aligned}$$

so the coefficient of u_p^i, $1 < p < r-k$ on the LHS of (3.17) is given by

$$\begin{aligned}2\binom{k+1+p}{p} p_{1,2} g_{k+p+1}^r - 2\binom{k+p}{p} g_{k+p}^r - 2\binom{k+p}{p-1} g_{k+p}^r \\ -2\binom{k+p+1}{p} (g_{k+p+1}^r)'.\end{aligned}$$

Using $\binom{k+p}{p} + \binom{k+p}{p-1} = \binom{k+p+1}{p}$ (a relationship which is also used later on) we have the above to be equal to

$$-2\binom{k+p+1}{p}\left(-p_{1,2} g_{k+p+1}^r + g_{k+p}^r + (g_{k+p+1}^r)'\right) = -2\binom{k+p+1}{p} g_{k+p+1}^{r+1}$$

which is the coefficient of u_p^i, $1 < p < r-k$, in the RHS of (3.17). The coefficient of u_{r-k}^i on the LHS is given by

$$-2\binom{r}{r-k} - 2\binom{r}{r-k-1} = -2\binom{r+1}{r-k}$$

which is also the coefficient of u_{r-k}^i in the RHS. Finally, the coefficient of u_1^i in the LHS is given by

$$\begin{aligned}-2g_{k+1}^r + 2p_{1,2}\binom{k+2}{1} g_{k+2}^r - 2\binom{k+1}{1} g_{k+1}^r - 2\binom{k+2}{1} (g_{k+2}^r)' \\ = -2\binom{k+2}{1} g_{k+2}^{r+1}\end{aligned}$$

which is equal to the coefficient of u_1^i in the RHS.

The last part is to prove condition (3.14) to be true, also by induction on r. Assume that

$$\sum_{k=2}^{r}\sum_{p=1}^{k-1} \binom{k}{p} p_{p,k-p} g_k^r = 0. \tag{3.17}$$

Differentiating (3.18) with respect to x, making use of (2.2), (3.18) and performing minor simplifications we obtain the following equation

$$\sum_{k=2}^{r}\sum_{p=2}^{k-1} \binom{k+1}{p} p_{p,k+1-p} g_k^r + \sum_{k=2}^{r} 2k\ p_{1,k} g_k^r + \sum_{k=2}^{r}\sum_{p=1}^{k-1} \binom{k}{p} p_{p,k-p} (g_k^r)' = 0.$$

Now, given that g_1^r is defined by (3.16) we have that the equation above can be rewritten, after some calculations, as

$$\sum_{k=2}^{r+1}\sum_{p=1}^{k-1}\binom{k}{p}p_{p,k-p}g_{k-1}^r+\sum_{k=2}^{r}\sum_{p=1}^{k-1}\binom{k}{p}p_{p,k-p}(g_k^r)'$$
$$=\sum_{k=2}^{r+1}\sum_{p=1}^{k-1}\binom{k}{p}p_{p,k-p}g_k^{r+1}=0$$

completing the last induction and the proof of the lemma. ∎

This is the end of the proof of the theorem. From the lemma we deduce that the matrix ν has columns which are relative differential invariants with Jacobian weight. We will finally show that, whenever $u_1,\dots,u_n$ are independent vectors, this matrix is nondegenerate. Indeed, the columns of ν are of the form (3.7) except for the first one which is given by the first derivative of the curve. First of all, notice that the differential orders of all g_j^i in the matrix Q are less or equal to $i-j+1$ so the highest order in each column of Q is that of g_1^i which has order i. Now, making use of the first column in ν, $F_1=u_1$, we can simplify the other columns of ν so that u_1 will not appear in the expression of F_i, $i>1$. Therefore, without losing generality, we can assume that $g_1^i=0$ for all $i=3,\dots,n+1$.

Now, if the first n derivatives of the curve u are linearly independent, then any column in the matrix ν, say F_i, must be independent from the previous columns, $F_j, 1\le j<i$, with perhaps the exception of the last column. Hence, we will conclude the proof of the theorem once we prove that the last column of ν cannot be a combination of the previous columns. In fact, we always have that u_{n+1} is a combination of u_i, $i=1,\dots,n$, so assume

$$u_{n+1}=\sum_{i=1}^n\alpha_iu_i.$$

The coefficients α_i are homogeneous rational functions of $p_{i,j}$, $i=1,\dots,n$, $j=1,\dots,n+1$ since they are the solution of the system

$$\begin{pmatrix}1 & p_{1,2} & \cdots & p_{1,n}\\ p_{1,2} & p_{2,2} & \cdots & p_{2,n}\\ \vdots & \vdots & \cdots & \vdots\\ p_{1,n} & p_{2,n} & \cdots & p_{n,n}\end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_n\end{pmatrix}=\begin{pmatrix}p_{1,n+1}\\ p_{2,n+1}\\ \vdots\\ p_{n,n+1}\end{pmatrix}\tag{3.18}$$

which is unique from lemma 3.2. Next, assume that F_n is a combination of the previous columns, that is

$$F_n=\sum_{i=1}^{n-1}\beta_iF_i.\tag{3.19}$$

Substituting definition (3.7) in (3.20) and equating the coefficients of u_i, $i=1,\dots.n$, we obtain the following equations relating α and β coefficients

$$\begin{pmatrix}1 & g_3^4 & \cdots & g_3^n\\ \vdots & \ddots & \ddots & \vdots\\ 0 & \dots & 1 & g_{n-1}^n\\ 0 & \dots & 0 & 1\end{pmatrix}\begin{pmatrix}\beta_2\\ \vdots\\ \beta_{n-1}\end{pmatrix}=\begin{pmatrix}g_3^{n+1}\\ \vdots\\ g_n^{n+1}\end{pmatrix}+\begin{pmatrix}\alpha_3\\ \vdots\\ \alpha_n\end{pmatrix}\tag{3.20}$$

$$\sum_{k=2}^{n-1} \beta_k g_2^{k+1} = \alpha_2 + g_2^{n+1} \tag{3.21}$$

$$\beta_1 + \sum_{k=2}^{n-1} \beta_k g_1^{k+1} = \alpha_1 + g_1^{n+1}. \tag{3.22}$$

¿From (3.21) we can solve for $\beta_2, \dots, \beta_{n-1}$ in terms of $\alpha_3, \dots, \alpha_n$ and g_j^i, and from (3.23) we can solve for β_1 in terms of $\alpha_1, \alpha_3, \dots, \alpha_n$ and g_j^i. Therefore, equation (3.22) represents a relationship between the coefficients α and g_j^i. We will see that this relationship cannot exist.

Equation (3.22) is an equality between rational functions on the variables $p_{i,j}$, $1 \le i \le n$, $1 \le i \le j \le n+1$. The functions α_k's are the only part of the equation which are not polynomials and their denominators equal the determinant of the matrix P. If we multiply equations (3.21), (3.22) by $\det(P)$, these equations become equations on $\hat\beta_k = \det(P)\beta_k$ and $\hat\alpha_k = \det(P)\alpha_k$ and they are now equalities between polynomials in $p_{i,j}$. The functions $\hat\alpha_k$ are given by

$$\hat\alpha_k = \det \begin{pmatrix} 1 & p_{1,2} & \cdots & p_{1,n+1} & \cdots & p_{1,n} \\ p_{1,2} & p_{2,2} & \cdots & p_{2,n+1} & \cdots & p_{2,n} \\ \vdots & \dots & \dots & \vdots & \dots & \vdots \\ p_{1,n} & p_{2,n} & \cdots & p_{n,n+1} & \cdots & p_{n,n} \end{pmatrix}$$

where the $p_{i,n+1}$ column is located in place k, and (3.21), (3.22) become

$$\begin{pmatrix} 1 & g_3^4 & \cdots & g_3^n \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 1 & g_{n-1}^n \\ 0 & \dots & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat\beta_2 \\ \vdots \\ \hat\beta_{n-1} \end{pmatrix} = \det(P) \begin{pmatrix} g_3^{n+1} \\ \vdots \\ g_n^{n+1} \end{pmatrix} + \begin{pmatrix} \hat\alpha_3 \\ \vdots \\ \hat\alpha_n \end{pmatrix} \tag{3.23}$$

$$\sum_{k=2}^{n-1} \hat\beta_k g_2^{k+1} = \hat\alpha_2 + \det(P) g_2^{n+1}. \tag{3.24}$$

Lemma 3.2 states that both sides of equation (3.25) must be equal term by term. But such is not the case. Let $C_{r,s}$ be the cofactor of the entry (r,s) in P. The right hand side of equation (3.25) contains a unique term of the form $p_{2,n+1}p_{3,3}\cdots p_{n,n}$, a term with $n-1$ factors appearing in $\hat\alpha_2$. On the other hand, on the left hand side of (3.25) $p_{2,n+1}$ appears in each $\hat\beta_k$. In fact, from (3.24) it is straightforward to check that in each $\hat\beta_k$ the factor $p_{2,n+1}$ is multiplied by $C_{2,k+1}$ plus a combination of other cofactors with coefficients depending on g_s^r, $r \ne s$. Thus, all the terms with minimum number of factors appear in $C_{2,k+1}p_{2,n+1}$, $k = 2, \dots, n-1$, as part of $\hat\beta_k$. These terms have already a minimum number of factors equal $n-1$. But in equation (3.25) they are multiplied by g_2^{k+1} $k = 2, \dots, n-1$. Hence, they must have at least n factors each. This is a contradiction to lemma 3.2, since the term $p_{2,n+1}p_{3,3}\cdots p_{n,n}$, with $n-1$ factors, can never appear in the left hand side of (3.25) . ∎

Example. In the case $n=3$ the g_j^i polynomials needed for the definition of F_i are g_i^3, $i=1,2,3$ and g_i^4, $i=1,2,3,4$. We can find these using relation (3.15). From lemma 3.4 we have

$$\begin{aligned}
&g_3^3 = 1,\\
&g_2^3 = -p_{1,2}g_2^2 + g_1^2 + (g_2^2)' = -3p_{1,2},\\
&g_1^3 = -p_{1,2}g_2^3 - p_{1,3}g_3^3 = -p_{1,3} + 3p_{1,2}^2,
\end{aligned}$$

since $g_1^2 = -2p_{1,2}$ by definition, and $g_i^i = 1$ for all i. We can now find g_i^4. Indeed

$$\begin{aligned}
&g_4^4 = 1,\\
&g_3^4 = -p_{1,2}g_3^3 + g_2^3 + (g_3^3)' = -4p_{1,2},\\
&g_2^4 = -p_{1,2}g_2^3 + g_1^3 + (g_2^3)' = -4p_{1,3} - 3p_{2,2} + 12p_{1,2}^2,\\
&g_1^4 = -p_{1,2}g_2^4 - p_{1,3}g_3^4 - p_{1,4}g_4^4 = -p_{1,4} + 8p_{1,2}p_{1,3} + 3p_{1,2}p_{2,2} - 12p_{1,2}^3.
\end{aligned}$$

With these values for g_j^i, the relative invariants are given by

$$\begin{aligned}
&F_1 = u_1,\\
&F_2 = u_3 - 3p_{1,2}u_2 + (3p_{1,2}^2 - p_{1,3})u_1,\\
&F_3 = u_4 - 4p_{1,2}u_3 + (-4p_{1,3} - 3p_{2,2} + 12p_{1,2}^2)u_2 +\\
&\quad (-p_{1,4} + 8p_{1,2}p_{1,3} + 3p_{2,2}p_{1,2} - 12p_{1,2}^2)u_1.
\end{aligned} \tag{3.25}$$

In the next sections we will make use of $\{F^r\}$ in order to find differential invariants of both parametrized and unparametrized curves and an invariant frame for unparametrized curves. The procedure is very close to that of Euclidean Geometry, with $\widetilde{F}^r = \frac{F_r}{(u_1\cdot u_1)^{\frac{1}{2}}}$ taking the role of u_r.

4. Independent set of differential invariants for parametrized curves.

As we indicated in the introduction, the knowledge of the relative invariants with Jacobian weight will suffice to obtain any differential invariant, with the exception of the invariant of lowest degree. Theorem 4.1 below shows to what extent $\widetilde{F}^r$ is taking the Euclidean Geometry role reserved for u_r. Perhaps we should recall that any Euclidean differential invariant for parametrized curves can be written as a function of the basic Euclidean invariants $u_r\cdot u_s$.

The simplest differential invariant has degree 2 and order 3. We will call it I_1 and it is given by

$$I_1 = p_{1,3} + \frac{3}{2}p_{2,2} - 3p_{1,2}^2. \tag{4.1}$$

There are many ways to finding I_1. For example, one can choose a general homogeneous polynomial of order 2, which would be given by

$$ap_{1,3} + bp_{2,2} + c(p_{1,2})^2$$

with $a, b, c \in \mathbb{R}$. We can then apply the prolongation of the vectors in (2.4) and then solve for a, b and c.

The rest of the generating invariants are found in terms of the vector relative invariants, as shown in the next theorem. Formula (4.4) below is quite striking, particularly since $\frac{d\widetilde{F}^{r-1}}{dx}$ is never equal to $\widetilde{F}^r$, as shown in (4.6) below.

4.4. Differential invariants: infinitesimal version.

The following theorem describes differential invariants at the infinitesimal level, that is, invariant under the action of the connected component of the group which contains the identity element. We will later prove that these invariants are indeed invariant under the action of the entire group.

Theorem 4.1. *Let G_i, $i = 1, 2, 3, 4$ be any four relative vector differential invariant with Jacobian weight. Assume that the following expression is a function of $p_{i,j}$ as in (2.1)*

$$G = \frac{G_1 \cdot G_2}{G_3 \cdot G_4} \tag{4.2}$$

where $\cdot$ denotes the usual inner product in $\mathbb{R}^n$. Then G satisfies

$$\mathrm{pr}^{(m)} w(G) = 0$$

for any $w \in \mathrm{o}(n+1, 1)$.

In particular, let F_k $k = 1, 2, 3, \ldots, n$ be given as in (3.7). Then,

$$V_{r,s} = \frac{F_r \cdot F_s}{F_1 \cdot F_1} = \widetilde{F}^r \cdot \widetilde{F}^s, \tag{4.3}$$

are all differential invariants (at the infinitesimal level), for any $r, s = 2, 3, \ldots, n$. Furthermore, the following surprising relationship holds for any $r, s = 2, \ldots, n-1$

$$\frac{dV_{r,s}}{dx} = V_{r+1,s} + V_{r,s+1}. \tag{4.4}$$

Notice that $V_{1,1} = 1$ and, as shown in (4.7) below, $V_{1,s} = 0$ for $s = 2, \ldots, n$.

Proof. First of all observe that if G is a function of $p_{i,j}$, then the prolongations of the vectors v_i, $i = 1, \ldots, n$, v and v_{ij}, $i, j = 1, \ldots, n$, $i < j$, as in (2.4) will immediately vanish on it since they vanish on each $p_{i,j}$. We thus need to check only w_i, $i = 1, 2, \ldots, n$. But it is rather simple to check that, if we denote by $\frac{\partial \xi}{\partial u}$ the Jacobian of w_i as given in (2.7), then, for any vector $V = (V_j)$ the following holds

$$\frac{\partial \xi}{\partial u} V = 2u^i V + 2V_i u - 2u \cdot V e_i$$

where e_i is the vector in $\mathbb{R}^n$ with all entries equal zero except for a 1 in place i. Therefore, if $G_i = (G_i^j)$ are all relative differential invariants with Jacobian weight

$$\mathrm{pr}^{(m)} w_i G = 2u_i G + 2G_1^i \frac{u \cdot G_2}{G_3 \cdot G_4} - 2G_2^i \frac{u \cdot G_1}{G_3 \cdot G_4} + 2u_i G + 2G_2^i \frac{u \cdot G_1}{G_3 \cdot G_4} - 2G_1^i \frac{u \cdot G_2}{G_3 \cdot G_4}$$

$$-G\left(2u_i + 2G_3^i \frac{u \cdot G_4}{G_3 \cdot G_4} - 2G_4^i \frac{u \cdot G_3}{G_3 \cdot G_4} + 2u_i + 2G_4^i \frac{u \cdot G_3}{G_3 \cdot G_4} - 2G_3^i \frac{u \cdot G_4}{G_3 \cdot G_4}\right) = 0.$$

Next, observe that $V_{r,s}$ are all in fact homogeneous polynomials on $p_{i,j}$ as in (2.1), well defined as far as $u_1 \cdot u_1$. Indeed, they can be rewritten as

$$V_{r,s} = \sum_{l=1}^{r+1} \sum_{k=1}^{s+1} p_{l,k} g_l^{r+1} g_k^{s+1}. \tag{4.5}$$

Hence, all $V_{r,s}$ are differential invariants of the action, at the infinitesimal level. To finish the proof of the theorem, we only need to prove formula (4.4). Indeed, one obtains first

$$\frac{dg_1^r}{dx} = -\sum_{k=2}^{r} \left[p_{2,k} + p_{1,k+1} - 2p_{1,2}p_{1,k}\right] g_k^r - \sum_{k=2}^{r} p_{1,k}\left[g_k^{r+1} + p_{1,2}g_k^r - g_{k-1}^r\right]$$

$$= g_1^{r+1} + 2p_{1,2}g_1^r - \sum_{k=2}^{r} p_{2,k}g_k^r.$$

So that we get

$$\frac{dF_{r-1}}{dx} = \sum_{k=1}^{r} \frac{d(u_k g_k^r)}{dx} = \sum_{k=1}^{r} u_{k+1} g_k^r + \sum_{k=2}^{r} u_k\left[g_k^{r+1} + p_{1,2}g_k^r - g_{k-1}^r\right] + u_1 \frac{dg_1^r}{dx}$$

$$= F_r + p_{1,2}F_{r-1} - u_1 \sum_{k=1}^{r} p_{2,k} g_k^r, \tag{4.6}$$

after using (3.16) and the recursion formula. Finally one has that, for any $s = 2, 3, \ldots, n$

$$\frac{u_1 \cdot F_s}{F_1 \cdot F_1} = \sum_{k=1}^{s+1} p_{1,k} g_k^{s+1} = 0 \tag{4.7}$$

from (3.16). Using (4.6) and (4.7) we get the desired formula. ∎

4.5. Discrete symmetries and differential invariants of $\mathrm{O}(n+1,1)$.

As I explained in section 2, $\mathrm{O}(n+1,1)$ has four connected components, connected via multiplication by either matrix (2.3), or matrix (2.5). Next theorem shows that $V_{r,s}$ are all invariant under these discrete symmetries and, hence, that they are invariant under the action of the entire group.

Theorem 4.2. *$V_{r,s}$ are all invariant under the prolonged action of multiplication by C in (2.3) and multiplication by (2.5).*

Notice that the prolonged action of multiplication by (2.5) trivially leaves $V_{r,s}$ invariant. The result of this action is simply to change the sign of the first component of the curve, and so $p_{i,j}$ are already been preserved. Now, the action of C is more involved. In fact, using the original definition given in section 2 of the $\mathrm{O}(n+1,1)$ action on a curve $u=(u^i)$, it is trivial to see that

$$Cu = \frac{u}{q}$$

where q is defined by the relation $2q+\sum_{i=1}^n (u^i)^2 = 0$. Hence, the prolonged action on derivatives will be defined by

$$Cu_i = \left(\frac{u}{q}\right)^{(i)}.$$

Let's denote by a hat the result under the action of C. That is, if G is a function on the jet space, $G = G(x,u,u_1,u_2,\ldots)$, then $\widehat{G}(x,u,u_1,u_2,\ldots) = G(x,\frac{u}{q},\left(\frac{u}{q}\right)^{(1)},\ldots)$. Before we prove the theorem we need to prove some technical lemmas.

Lemma 4.3. *Let g^i_j be defined by the recursion (3.15), and let q be determined by $2q+\sum_{i=1}^n (u^i)^2 = 0$ as above. Then*

$$q\sum_{s=k}^{r+1}\binom{s}{k}\left(\frac{1}{q}\right)^{(s-k)}\widehat{g}^{r+1}_s = g^{r+1}_k \tag{4.8}$$

for any $k=1,2,\ldots,r+1$, $r=2,\ldots,n$.

Proof. For $r=2$, cases $k=1,2,3$ are proved straightforwardly using the two transformations

$$\widehat{p}_{1,2} = p_{1,2}-\frac{q'}{q}, \qquad \widehat{p}_{1,3} = p_{1,3}-3\left(\frac{q''}{q}-\frac{q'^2}{q^2}\right). \tag{4.9}$$

Assume that

$$q\sum_{s=k}^{r}\binom{s}{k}\left(\frac{1}{q}\right)^{(s-k)}\widehat{g}^{r}_s = g^{r}_k \tag{4.10}$$

for any $k=1,2,\ldots,r$.

First case : if $k>1$ we can make use of recursion (3.15) to obtain

$$\begin{aligned} &q\sum_{s=k}^{r+1}\binom{s}{k}\left(\frac{1}{q}\right)^{(s-k)}\widehat{g}^{r+1}_s \\ &\quad= q\sum_{s=k}^{r}\binom{s}{k}\left(\frac{1}{q}\right)^{(s-k)}(\widehat{g}^{r}_s)' - \widehat{p}_{1,2}q\sum_{s=k}^{r}\binom{s}{k}\left(\frac{1}{q}\right)^{(s-k)}\widehat{g}^{r}_s \\ &\quad+ q\sum_{s=k}^{r}\binom{s}{k}\left(\frac{1}{q}\right)^{(s-k)}\widehat{g}^{r}_{s-1} + \binom{r+1}{k}q\left(\frac{1}{q}\right)^{(r-k+1)}. \end{aligned} \tag{4.11}$$

With the use of (4.10), its derivative and $\widehat{p}_{1,2} = p_{1,2} - \frac{q'}{q}$ we have that, after some straightforward calculations, (4.11) equals

$$-p_{1,2} g_k^r + (g_k^r)' + \sum_{s=k-1}^{r} \binom{s}{k-1} q \left(\frac{1}{q}\right)^{(s-k+1)} \widehat{g}_s^r$$

$$= (g_k^r)' - p_{1,2} g_k^r + g_{k-1}^r = g_k^{r+1}.$$

2nd Case: $k = 1$. Using cases $k > 1$ above, multiplying (4.8) by $p_{1,k}$ and applying (3.16) we have

$$g_1^{r+1} = -q \sum_{s=2}^{r+1} \left[\sum_{k=2}^{s} \binom{s}{k} \left(\frac{1}{q}\right)^{(s-k)} p_{1,k} \right] \widehat{g}_s^{r+1}.$$

Therefore, in order to prove this case, we need to show that the following equality holds

$$\sum_{s=2}^{r+1} \left[\sum_{k=2}^{s} \binom{s}{k} \left(\frac{1}{q}\right)^{(s-k)} p_{1,k} \right] \widehat{g}_s^{r+1} = \sum_{s=2}^{r+1} \left[\frac{\widehat{p}_{1,s}}{q} - s \left(\frac{1}{q}\right)^{(s-1)} \right] \widehat{g}_s^{r+1}. \qquad (4.12)$$

Now, (4.12) can be transformed using the equality

$$\sum_{k=2}^{s} \binom{s}{k} \left(\frac{1}{q}\right)^{(s-k)} p_{1,k} = \frac{1}{q} \widehat{p}_{1,s} - \left(\frac{1}{q}\right)^{(s)} p_{0,1} - s \left(\frac{1}{q}\right)^{(s-1)} + \frac{q'}{q^2} \widehat{p}_{0,s}$$

which is obtained straighforward from the definition. We thus need to show

$$\sum_{s=2}^{r+1} \left[\left(\frac{1}{q}\right)^{(s)} + \frac{u_1 \cdot u_1}{q^2} \widehat{p}_{0,s} \right] \widehat{g}_s^{r+1} = 0. \qquad (4.13)$$

But we can further transform (4.13) using the fact that

$$\left(\frac{1}{q}\right)^{(s)} = -\frac{u_1 \cdot u_1}{2q^2} \sum_{k=0}^{s} \binom{s}{k} \widehat{p}_{k,s-k} \qquad (4.14)$$

which is obtained explanding $\left(\frac{u}{q} \cdot \frac{u}{q}\right)^{(s)} = -2 \left(\frac{1}{q}\right)^{(s)}$ with the use of the binomial formula. If we use this relationship, we see that (4.13) equals

$$-\frac{u_1 \cdot u_1}{2q^2} \sum_{s=2}^{r+1} \sum_{k=1}^{s-1} \binom{s}{k} \widehat{p}_{k,s-k} \widehat{g}_s^{r+1} = 0$$

which is true since it is the transformed of (3.14). This concludes the proof of the first lemma.

■

Lemma 4.4.

$$q^2 \sum_{s=1}^{r+1} \left(\frac{1}{q}\right)^{(s)} \widehat{g}_s^{r+1} = \sum_{s=1}^{r+1} u \cdot u_s g_s^{r+1} = u \cdot F_r \tag{4.15}$$

Proof. Using the result of the previous lemma, we obtain that relation (4.15) can be rewritten as

$$q \sum_{s=1}^{r+1} \left(\frac{1}{q}\right)^{(s)} \widehat{g}_s^{r+1} = \sum_{s=1}^{r+1} \sum_{k-1}^{s} u \cdot u_k \binom{s}{k} \left(\frac{1}{q}\right)^{(s-k)} \widehat{g}_s^{r+1}$$

which itself can be written as

$$\sum_{s=1}^{r+1} \left(\frac{1}{q}\right)^{(s)} \widehat{g}_s^{r+1} = \sum_{s=1}^{r+1} \left[\widehat{u} \cdot \widehat{u}_s + 2\left(\frac{1}{q}\right)^{(s)}\right] \widehat{g}_s^{r+1}. \tag{4.16}$$

Equation (4.16) is equivalent to

$$\sum_{s=1}^{r+1} \left[\widehat{u} \cdot \widehat{u}_s + \left(\frac{1}{q}\right)^{(s)}\right] \widehat{g}_s^{r+1} = 0$$

which is simply a rewriting of (4.13) in the previous lemma, and which has been proved to be true. ∎

Theorem 4.5.

$$V_{r,s} = \widehat{V}_{r,s}.$$

Proof. From the definition of F_r in (3.7) and using the binomial formula we obtain

$$\begin{aligned}\widehat{F}_r &= \sum_{s=1}^{r+1} \left(\frac{u}{q}\right)^{(s)} \widehat{g}_s^{r+1} = \sum_{s=1}^{r+1} \sum_{k=0}^{s} \binom{s}{k} \left(\frac{1}{q}\right)^{(s-k)} u_k \widehat{g}_s^{r+1} \\ &= \sum_{k=1}^{r+1} \sum_{s=k}^{r+1} \binom{s}{k} \left(\frac{1}{q}\right)^{(s-k)} \widehat{g}_s^{r+1} u_k + \sum_{s=1}^{r+1} \left(\frac{1}{q}\right)^{(s)} \widehat{g}_s^{r+1} u.\end{aligned} \tag{4.17}$$

But, from lemmas 4.3 and 4.4 we have that

$$\widehat{F}_r = \frac{1}{q} F_r + \frac{u}{q^2} F_r \cdot u.$$

Using the fact that $u \cdot u = -2q$ we finally have

$$\widehat{F}_r \cdot \widehat{F}_s = \frac{1}{q^2} F_r \cdot F_s$$

which proves the invariance of $V_{r,s}$. ∎

Next theorem will finally select which $V_{r,s}$ form an independent system of differential conformal invariants. We will prove in the next subsection that they are also generators.

Theorem 4.6. *Let I_1 be given as in (4.1) and let I_k be defined as*

$$I_k = V_{k,k}, \tag{4.18}$$

$k = 2, 3, \dots, n$. Then $\{I_1, \dots, I_n\}$ form a set of (differentially) independent differential invariants.

Proof. Notice that, since F_r, $r = 1, \dots, n+1$, are all independent and we have relation (4.3), the independency of $I_2, \dots, I_n$ can be proved following a reasoning similar to the one in lemma 3.2. It is the appearance of I_1 in the set that will require a bit more care.

Recall that the highest derivative involved in the definition of I_k is the one appearing in g_1^{k+1}, that is, $k+1$. Hence I_1 has order 3 and I_k order $k+1$, $k = 2, \dots, n$. Also, each I_k has homogeneous degree $2k$, for $k = 1, 2, \dots, n$.

Assume that there exists I_k such that $I_1, \dots, I_{k-1}$ are independent and such that

$$I_k = G(I_1, I_2, \dots, I_{k-1}, I_1', I_2', \dots, I_{k-1}', \dots) \tag{4.19}$$

for some analytic function G depending on $I_1, \dots, I_{k-1}$ and their derivatives up to a certain order o_k.

Assume that $k \neq n$. We expand G analytically in I's and their derivatives. Assume

$$G = G_1 + G_2 + \dots \tag{4.20}$$

with G_1 linear, G_2 quadratic, etc. By homogeneity the linear part of G must necessarily be of the form

$$G_1 = \gamma_{k-1}\frac{d^2 I_{k-1}}{dx^2} + \gamma_{k-2}\frac{d^4 I_{k-2}}{dx^4} + \dots + \gamma_1 \frac{d^{2(k-1)} I_1}{dx^{2(k-1)}} \tag{4.21}$$

for some $\gamma_i \in \mathbb{R}$, $i = 1, \dots, k-1$. Indeed, the linear part of both sides in (4.19) must be equal. Since I_k is a homogeneous polynomial of degree $2k$ so must be G_1. Expression (4.21) is a combination of the only possible linear terms (in I's and derivatives) with that degree of homogeneity. (In fact, since in this case the LHS of (4.19) is a polynomial on $p_{i,j}$, $i \leq n$, $i \leq j \leq n$, so must be (4.20).) ¿From lemma 3.2 both linear parts must be equal term by term.

We now analyze the $p_{i,j}$–linear part of both sides of (4.19). The linear part of I_i, $i > 1$ equals $p_{i+1,i+1}$ and so the linear part of $\frac{d^{2(k-i)} I_i}{dx^{2(k-i)}}$ equals

$$\sum_{s=0}^{2(k-i)} \binom{2(k-i)}{s} p_{i+s+1,\ 2k-i-s+1}.$$

The $p_{i,j}$–linear part of I_k is $p_{k+1,k+1}$ and $p_{k+1,k+1}$ appears also in each term of (4.21), with coefficient $\binom{2(k-i)}{k-i}$ for any $i > 1$. Analogously $p_{k+1,k+1}$ appears in the term corresponding to I_1 with some nonzero coefficient α. Thus, from the comparison of $p_{k+1,k+1}$ coefficients we obtain the condition

$$\sum_{i=1}^{k-1} \gamma_i \binom{2(k-i)}{k-i} + \alpha\gamma_1 = 1. \tag{4.22}$$

On the other hand, $p_{1,2k+1}$ appears only in the derivative of I_1 and nowhere else in (4.19). Therefore $\gamma_1 = 0$. The term $p_{3,2k-1}$ appears only in the derivative of I_2 and nowhere else in (4.19), and so $\gamma_2 = 0$. And so on, if $\gamma_i = 0$ for $i = 1, \dots, r-1$, using the fact that $p_{r+1,2k-r+1}$ appears only in the derivative of I_r we obtain $\gamma_r = 0$. This contradicts (4.22).

Finally, *assume* $k = n$. Recall that

$$u_{n+1} = \sum_{k=1}^{n} \alpha_k u_k$$

with α_k solutions of (3.19). Recall that we denote by C_{rs} the (r,s) cofactor of P as in (3.1). Then, we have

$$\alpha_i = \sum_{r=1}^{n} \frac{C_{ri}}{\det P}\, p_{r,n+1}$$

and so

$$p_{n+1,n+1} = \sum_{i,j=1}^{n} \sum_{r,s=1}^{n} \frac{C_{ri}C_{sj}}{(\det P)^2}\, p_{i,j} p_{r,n+1} p_{s,n+1} = \frac{1}{\det P} \sum_{r,s=1}^{n} C_{sr} p_{r,n+1} p_{s,n+1}, \tag{4.23}$$

since $\sum_{i=1}^{n} C_{ri} p_{i,j} = \delta_r^j \det P$. Relation (4.19) for $k = n$ can thus be rewritten as

$$\sum_{r,s=1}^{n} C_{sr} p_{r,n+1} p_{s,n+1} + \det P \sum_{i=1}^{n} \sum_{j=1}^{n+1} p_{i,j} g_i^{n+1} g_j^{n+1} + \det P \sum_{i=1}^{n} p_{i,n+1} g_i^{n+1} = \det P [G_1 + G_2 + \dots] \tag{4.24}$$

and both sides of (4.24) are now polynomials on $p_{i,j}$, $i \leq n$, $i \leq j$. Thus, we can apply lemma 3.2 and compare term by term depending, for example, in number of factors, differential order, etc.

The terms with a minimum number of factors in the RHS of (4.24) have n factors. They correspond to, for example, $p_{2,2} \dots p_{n,n}$ or other terms in the determinant with $n-1$ factors, times terms in G_1 with only one $p_{i,j}$ factor.

The terms with a minimum number of factors in the LHS of (4.24) will appear in the cofactor expansion since the other two sums have terms with at least $n+1$ factors. But, for example, $p_{2,n+1}^2 p_{3,3} \cdots p_{n,n}$ appears only once in the LHS of (4.24), in the cofactor expansion, and it can never appear in the RHS of (4.24). The term $p_{2,n+1}^2 p_{3,3} \cdots p_{n,n}$ has two factors of order $n+1$ and the terms in the RHS have at most one factor with that order, the factor that comes from G_1. We found a contradiction to lemma 3.2.

This concludes the proof of the theorem. ■

4.6. A generating set of differential invariants.

In this subsection we will show that $\{I_1, \dots, I_n\}$ as found in the previous section, form a generating set of differential conformal invariants. That is, any other conformal differential invariant can be written as a function of them and their derivatives with respect to the parameter x. This result will be a direct consequence of the work of Green ([9]) and Fialkow ([6]). We will proceede to give some definitions and notations. Some of what follows is as found in [21].

Let $\mathrm{o}(n+1,1)^{(m)}(p)$ be the vector subspace of the tangent space to J^m spanned by the m^{th} prolongation of the infinitesimal generators of the action of $\mathrm{o}(n+1,1)$ on $\mathbb{R}^n$ at the point p. Denote by $s_m(p)$ the dimension of the generic m^{th} prolonged orbit at p, that is, the dimension of the orbit of the action of $\mathrm{O}(n+1,1)^{(m)}$ on J^m, going through p. (We will avoid here any comments about singular orbits. For more information see [21].) It is known that $s_m(p)$ coincides with the dimension of $\mathrm{o}(n+1,1)^{(m)}(p)$ (again, see [21]). From now on we won't make any references to the point p, assuming we are always working at a given point.

Let i_m be the dimension of J^m. That is, in our case $i_m = n(m+1)+1$ since we have n variables in $\mathbb{R}^n$ and the parameter x. The following proposition is a standard result in invariant theory ([21]).

Proposition 4.7. *There are exactly $i_m - s_m$ (differentially) independent differential invariants of order at most m. The number*

$$j_m = i_m - s_m - (i_{m-1} - s_{m-1}) \tag{4.25}$$

denotes the number of independent differential invariants of order equal m.

The next theorem describes the values of j_m for conformal parametrized and unparametrized curves, and concludes that our system is generating. In the parametrized case, the theorem is a corollary of the study on Cartan's polygons done by Green in [9]. He gives the precise numbers in example 7. In the unparametrized case we can obtain the numbers from the study done by Fialkow in [6].

Theorem 4.8. *In the case of parametrized curves on $\mathbb{R}^n$ under the conformal action of $\mathrm{O}(n+1,1)$ the values of j_m are*

$$j_0 = 1, \quad j_1 = j_2 = 0, \quad \textit{and} \quad j_k = k-1 \quad \textit{for any} \quad k = 3,4,5,\dots,n+1.$$

In the case of unparametrized curves in $\mathbb{R}^n$ the values are

$$j_0 = 0, \quad j_1 = j_2 = 0, \quad \textit{and} \quad j_k = k-1 \quad \textit{for any} \quad k = 3,4,5,\dots,n+1.$$

Corollary4.9. *$\{I_1, I_2, \dots, I_n\}$ form a generating set of independent differential invariants for the conformal action of $\mathrm{O}(n+1,1)$ on $\mathbb{R}^n$. That is, any other* *differential invariant of the action can be written as a function of* $I_1, I_2, \dots, I_n$ *and their derivatives with respect to the parameter.*

As a final corollary to this section, we will write a formula for the most general evolution of parametrized curves in $\mathbb{R}^n$ of the form

$$u_t = F(u, u_1, u_2, \dots, u_r, \dots) \tag{4.26}$$

invariant under the action of $O(n+1,1)$. That is, evolutions for which the group $O(n+1,1)$ takes solutions to solutions. The following can be found, for example in [7].

Proposition 4.10. *An evolution of the form (4.27) is invariant under the action of* $O(n+1,1)$ *on* $\mathbb{R}^n$ *if, and only if* F *is a relative vector differential invariant with Jacobian weight.*

This result, together with proposition 3.1, theorem 3.3 and theorem 4.6, gives us the desired formula.

Corollary4.11. *The most general evolution of the form (4.27) invariant under the conformal action of* $O(n+1,1)$ *on* $\mathbb{R}^n$ *is given by*

$$u_t = DQJ \tag{4.27}$$

where D *and* Q *are given as in (3.5) and (3.6), and where* J *is a general differential invariant vector, that is a vector whose entries are functions of* I_k, $k = 1, \dots, n$, *as in (4.18) and their derivatives with respect to the parameter* x.

That is, any invariant evolution of the form (4.27) can be written as (4.28) for some invariant vector J.

Example4.12. In the case $n = 3$ the differential invariants are generated by

$$\begin{aligned} I_1 &= p_{1,3} + \frac{3}{2}p_{2,2} - 3p_{1,2}^2 \\ I_2 &= \frac{F_2 \cdot F_2}{u_1 \cdot u_1} = p_{3,3} - 6p_{1,2}p_{2,3} - p_{1,3}^2 + 9p_{1,2}^2 p_{2,2} + 6p_{1,2}^2 p_{1,3} - 9p_{1,2}^4 \\ I_3 &= \frac{F_3 \cdot F_3}{u_1 \cdot u_1}, \ F_3 \text{ as in (3.26).} \end{aligned} \tag{4.28}$$

Any invariant evolution will be of the form

$$u_t = h_1 F_1 + h_2 F_2 + h_3 F_3$$

where h_i are arbitrary functions of I_1, I_2, I_3 and their derivatives, and where F_i are as in (3.26), $i = 1, 2, 3$.

5. Differential Invariants of unparametrized curves

In this section we write a complete system of independent and generating differential invariants for conformal unparametrized curves. They correspond to the invariants found in [6]. The first step is to identify an element of arc-length. Let's denote by $\phi^\dagger I$ the reparametrization of I by $x=\phi(y)$. That is, if $I=I(u)$ and $u=u(x)$, then $\phi^\dagger I = I(u\circ\phi)$. A straightforward calculation reveals that

$$\phi^\dagger I_1 = \phi'^2 I_1\circ\phi + S(\phi),$$

where $S(\phi)=\frac{\phi'''}{\phi'}-\frac{3}{2}\frac{\phi''}{(\phi')^2}$ is the *Schwarzian derivative of* ϕ. On the other hand

$$\phi^\dagger I_2 = \phi'^4 I_2\circ\phi.$$

Therefore, we can choose $ds = I_2^{\frac{1}{4}}dx$, I_2 being the lowest order invariant that can define an element of arc–length. I_2 is given explicitly in (4.29). As usual, the arc–length is well-defined for those curves for which $I_2>0$.

We define, very much in the Euclidean fashion, the following symmetric determinants

$$\Delta_k = \frac{1}{I_2^{m_k}}\begin{vmatrix} V_{2,2} & V_{2,3} & \dots & V_{2,k+1}\\ V_{3,2} & V_{3,3} & \dots & V_{3,k+1}\\ \vdots & \vdots & \dots & \vdots\\ V_{k+1,2} & V_{k+1,3} & \dots & V_{k+1,k+1}\end{vmatrix} \tag{5.1}$$

for $k=2,\dots,n-1$ and where $m_k=\frac{1}{4}k(k+3)$.

Theorem 5.1. *The expressions* Δ_k, $k=2,\dots,n$ *together with*

$$\Delta_1 = 4\frac{I_1}{I_2^{\frac{1}{2}}}+\frac{9}{8}\frac{(I_2')^2}{I_2^{\frac{5}{2}}}-\frac{I_2''}{I_2^{\frac{3}{2}}} \tag{5.2}$$

form a complete set of independent and generating differential invariants for conformal unparametrized curves. i.e., they are also invariant under reparametrizations and any other differential invariant which is invariant under reparametrization is a function of Δ_k, $k=1,\dots,n$ *and their derivatives with respect to the invariant differentiation* $\frac{1}{I_2^{1/4}}D_x$.

Proof. We will prove that they are invariant under reparametrizations. The argument is almost identical to the one that shows that the analogous determinants in Euclidean geometry, that is with u_{k-1} instead of F_k, are a generating system of differential invariants for unparametrized curves in $\mathbb{R}^n$.

Since $\phi^\dagger F_k$ is also a relative invariant of the conformal action, it must be a linear combination of F_s $s=1,2\dots,k$, with invariant coefficients. Indeed, the order of $\phi^\dagger F_k$ equals that of F_k so it must be an invariant combination of relative invariants with equal or less order. We claim that, for any $k\ge 2$, $\phi^\dagger F_k$ is a linear combination which *does not contain* F_1. This will suffice to show that Δ_k are invariant under reparametrizations. Indeed, if we assume that our claim is correct, for any $2\le k\le n$ we can write

$$\phi^\dagger F_k = \sum_{s=2}^{k} A_s^k (F_s\circ\phi). \tag{5.3}$$

Clearly, since $\phi^\dagger u_k = (\phi')^k u_k + \ldots$, where the dots represent lower order terms, one has that $A_k^k = (\phi')^{k+1}$ for any $2 \le k \le n$. Δ_k can be rewritten as

$$\Delta_k = \frac{1}{I_2^m} \det \begin{pmatrix} \tilde{F}_2^T \\ \vdots \\ \tilde{F}_k^T \end{pmatrix} \begin{pmatrix} \tilde{F}_2 & \ldots & \tilde{F}_k \end{pmatrix} = \frac{1}{I_2^m} \det \begin{pmatrix} \tilde{F}_2 & \ldots & \tilde{F}_k \end{pmatrix}^2,$$

where again $\tilde{F}_i = \frac{1}{(u_1 \cdot u_1)^{1/2}} F_i$. But, if our claim is correct,

$$\phi^\dagger \Delta_k = \frac{(\phi')^{-4m}}{I_2^m \circ \phi} \det \begin{pmatrix} \frac{A_2^2}{\phi'} \tilde{F}_2 \circ \phi & \frac{A_3^3}{\phi'} \tilde{F}_3 \circ \phi & \ldots & \frac{A_k^k}{\phi'} \tilde{F}_k \circ \phi \end{pmatrix}^2$$

using column reduction in the determinant. Factoring $\frac{A_i^i}{\phi'} = (\phi')^i$ from each F_i column, we have the invariance of Δ_k.

We will prove our claim by induction. We can make use of formula (4.6) and the fact that

$$\sum_{k=1}^{r} p_{2,k} g_k^r = \frac{u_2}{u_1 \cdot u_1} \cdot F_{r-1}$$

to obtain the following relationship

$$\phi^\dagger F_r = \phi' [\phi^\dagger F_{r-1}]' - \phi^\dagger p_{1,2} \phi^\dagger F_{r-1} + \phi' \left(\phi^\dagger \frac{u_2}{u_1 \cdot u_1} \cdot \phi^\dagger F_{r-1} \right) u_1. \tag{5.4}$$

Now, in order to show that $\phi^\dagger F_r$ does not contain a term on $F_1 = u_1$, it suffices to show that the coefficient of u_1 above is zero. A quick calculation reveals that, if

$$\phi^\dagger F_{r-1} = \sum_{s=2}^{k} A_s^{r-1} (F_s \circ \phi),$$

then the coefficient of u_1 in (5.4) equals

$$\phi' \left(\phi^\dagger u_2 \cdot \phi^\dagger F_{r-1} - \sum_{k=2}^{r-1} \sum_{s=1}^{k+1} A_{r-1}^k (p_{2,s} g_s^{k+1}) \circ \phi \right). \tag{5.5}$$

But

$$\begin{aligned} &\sum_{k=2}^{r-1} \sum_{s=1}^{k+1} A_{r-1}^k (p_{2,s} g_s^{k+1}) \circ \phi \\ &\qquad = \sum_{k=2}^{r-1} A_{r-1}^k \left[\frac{u_2}{u_1 \cdot u_1} \circ \phi \right] \cdot [F_k \circ \phi] = \left[\frac{u_2}{u_1 \cdot u_1} \circ \phi \right] \cdot \left[\phi^\dagger F_{r-1} \right], \end{aligned}$$

and also,

$$\phi^\dagger \frac{u_2}{u_1 \cdot u_1} = \frac{u_2}{u_1 \cdot u_1} \circ \phi + \frac{\phi''}{(\phi')^2} \frac{u_1}{u_1 \cdot u_1}.$$

Since $u_1 \cdot F_k = 0$ for any $k = 2, \ldots, n$, we obtain that (5.5) equals zero and the induction is now complete.

The last part of the proof of the Theorem is to show that these are indeed generators for the set of differential invariants of unparametrized curves (their independence is quite obvious from the definition). This is an immediate consequence of Theorem 4.8, the reasoning being identical to that of the parametrized case. ∎

Our last theorem of the paper describes an invariant frame for $n-$ conformal unparametrized curves.

Theorem 5.2. *Let F_i, $i = 1, \dots, n$ be the invariant frame defined in theorem 3.3. Consider $G_1 = I_2^{-1/4} F_1$ and let G_i be the frame obtained from F_i, $i = 2, \dots, n$ via a Gram–Schmidt orthonormalization process using the product $\langle F_i, F_j \rangle = I_2^{1/2} \tilde{F}_i \cdot \tilde{F}_j = I_2^{1/2} V_{i,j}$. Then $\{G_i\}$, $i = 1, \dots, n$ is an invariant conformal frame for unparametrized n-curves.*

Proof. The proof of this theorem is very simple. Obviously G_1 is invariant under reparametrizations and G_1 is orthogonal to G_i for $i = 2, \dots, n$, since F_1 was orthogonal to the other F_i's (recall that $V_{1,r} = 0$ for any r.) Also, Gram–Schmidt process as applied to F_i, $i = 2, \dots, n$ with the product $\frac{I_2^{1/2}}{F_1 \cdot F_1} \langle , \rangle$ will produce a system of relative invariants since the result is a combination of the F_i with invariant coefficients. Thus $\phi^\dagger G_i$ will also be a relative invariant. The same reasoning as in the previous theorem tell us that $\phi^\dagger G_i = \sum_{r=2}^i B_r^i G_r \circ \phi$, for some invariant coefficients B_r^i. But $\phi^\dagger$ also preserves the orthonormality conditions. That is, $\phi^\dagger G_2 = \phi^\dagger \left(\frac{1}{I_2^{3/2}} F_2 \right) = G_2 \circ \phi$. Also, $\phi^\dagger G_3$ must be orthonormal to $G_2 \circ \phi$ and it must have order 4. The only choice will be $G_3 \circ \phi$. Etc. This procedure results on $\phi^\dagger G_i = G_i \circ \phi$, for any $i = 1, \dots, n$, as stated by the theorem . ∎

Example5.3. In the case $n = 3$ the differential invariants for unparametrized curves are given by

$$\Delta_1 = 4 \frac{I_1}{I_2^{\frac{1}{2}}} + \frac{9}{8} \frac{(I_2')^2}{I_2^{\frac{5}{2}}} - \frac{I_2''}{I_2^{\frac{3}{2}}}$$

$$\Delta_2 = \frac{1}{I_2^{10/4}} \det \begin{pmatrix} V_{2,2} & V_{2,3} \\ V_{2,3} & V_{3,3} \end{pmatrix} = \frac{I_3}{I_2^{3/2}} - \frac{1}{4} \frac{(I_2')^2}{I_2^{5/2}},$$

and the invariant frame is given by

$$G_1 = \frac{1}{I_2^{1/4}} F_1$$

$$G_2 = \frac{1}{I_2^{3/2}} F_2$$

$$G_3 = \frac{1}{\Delta_2^{1/2} I_2^{3/4}} \left[F_3 - \frac{1}{2} \frac{I_2'}{I_2} F_2 \right].$$

6. Conclusions and further questions.

In this paper we have found a system of n vector relative differential invariants of parametrized conformal curves in $\mathbb{R}^n$, for any $n \geq 3$. With the use of these vectors we classify all differential invariants of both parametrized and unparametrized curves, and we also find an invariant frame in the unparametrized case. Furthermore, the invariants of parametrized curves are generated by a system of independent homogeneous polynomials on expressions of the form $\frac{u_i \cdot u_j}{u_1 \cdot u_1}$ where the degree of this term is defined to be $i+j-2$. The differential order of the generators are $3, 3, 4, 5, \ldots, n+1$ and the homogeneity degrees are $2k$, $k = 1, \ldots, n$, respectively. The fact that they are homogeneous allows one to find any other homogeneous invariant in a rather simple way.

As I said in the introduction, the inspiration for the approach adopted here was found in the study of the relationship between differential invariants and Hamiltonian structures of PDE's. One of the simplest connection in the parametrized case is that of two very well known evolutions, the Schwarzian KdV and the KdV equation. The Schwarzian KdV is given by

$$\phi_s = \phi' S(\phi)$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is a nondegenerate real map, and where

$$S(\phi) = \frac{\phi'\phi''' - \frac{3}{2}(\phi'')^2}{(\phi')^2}$$

is the *Schwarzian derivative* of the function ϕ. It is well–known that whenever ϕ is a solution of the Schwarzian KdV, its Schwarzian derivative satisfies KdV itself, that is, if $v = S(\phi)$, then

$$v_s = v''' + 3vv'.$$

In fact the relation is closer. If ϕ is a solution of

$$\phi_s = \phi' H \tag{6.1}$$

where H is a function of $S(\phi)$ and its derivatives with respect to x, then $v = S(\phi)$ satisfies

$$v_s = H''' + 2vH' + v'H. \tag{6.2}$$

It is well known that $S(\phi)$ is the unique generating differential invariant for the projective action of $\mathrm{SL}(2)$ on maps $\phi : \mathbb{R} \to \mathbb{RP}^1$. That is, the unique differential invariant of fractional transformations acting on ϕ. It is also trivial to check that evolution (6.1) is the formula for the most general evolution of maps $\phi : \mathbb{R} \to \mathbb{RP}^1$ invariant under the projective action of $\mathrm{SL}(2)$ (ϕ' has the role of ν in section 3 and H is a general invariant). Hence, if ϕ is a solution of (6.1), its generating invariant is a solution of (6.2). Equation (6.2) is the well known Hamiltonian evolution associated to the second KdV Hamiltonian bracket. It has a compatible symplectic structure, the derivative, and it produces completely integrable PDEs, namely KdV itself. As seen in this simple example, the Poisson structure $D_x^3 + 2vD_x + v'$ can be found only with the knowledge of the relative invariant ϕ' and the invariant $S(\phi)$.

This relationship holds true for higher dimensions: if ϕ is a curve $\phi : \mathbb{R} \to \mathbb{RP}^{n-1}$ and we consider the projective action of $\mathrm{SL}(n)$ on $\mathbb{RP}^{n-1}$, then, whenever ϕ evolves invariantly, a generating set of differential invariants evolves following the so called Adler–Gel'fand–Dikii Hamiltonian evolution. This result can be found in [15]. The Adler–Gel'fand–Dikii Hamiltonian structure has a symplectic companion and it also produces completely integrable systems, the so called generalized KdV systems.

In the unparametrized case a similar situation can be found, for example, in Euclidean geometry ([11]). If $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$ is a planar flow of curves satisfying evolution

$$\phi_s = h_1 T + h_2 N = \begin{pmatrix} T & N \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

where h_1 and h_2 are functions of the curvature κ and where T and N are tangent and normal, and if the evolution is arc–length preserving (i.e., if $h_2 = \frac{h_1'}{\kappa}$), then the evolution induced on the curvature is given by

$$\kappa_s = D_x(D_x \frac{1}{\kappa} D_x + D_x \kappa) h_1$$

and the Hamiltonian structure D_x can be defined with the use of the invariant frame $\{T, N\}$ and the invariant κ. The same situation appears again in 3-dimensional Riemannian manifolds with constant curvature where 4 compatible Poisson tensors can be found this way ([19]).

It is thus natural to investigate for which groups do we obtain these kind of structures, and whether or not one could obtain new integrable systems as a byproduct. A fundamental first step of independent importance is, of course, the classification of relative and absolute differential invariants.

Finally, I will write a short comment on still another use of these systems of PDE's, analogous to (6.2). The same way we can obtain these PDE's by classifying differential invariants and invariant evolutions, one can also go the other way around and search for help finding differential invariants among the coefficients of these PDE's. Namely, if we have the general matrix ν of relative invariants and the first invariant I_1, which is always the simplest one, one can write the evolution induced on I_1 by the invariant evolution of curves. The coefficients of the resulting equation will all be differential invariants themselves. In all cases of group actions that we have studied either a complete set of generators, or a significant partial subset, appeared already among these coefficients, which can be generated *only with the knowledge of* ν *and* I_1. It is a very simple procedure that produces invariants in simple form. In the case presented here part of the generators were originally found this way, and most of their properties were learned from these coefficients. For example, in the case $n = 3$, if a general invariant evolution for a curve u is given by

$$u_s = h_1 F_1 + h_2 F_2 + h_3 F_3$$

straightforward calculations show that the induced evolution on I_1 is given by

$$(I_1)_t = [D_x^3 + 2I_1 D_x + 2(I_1)']h_1 - [2I_2 D_x + \frac{3}{2}(I_2)']h_2 - [(I_2)' D_x + (I_2)'' - I_3]h_3.$$

This evolution already shows in its coefficients the rest of the generating invariants, I_2 and I_3. The appropiate way of combining the relative invariants was found while studying the coefficients of the evolution of the first invariant and, even though it is very simple, it was not guessed.

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