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$\mathbf{C}^\infty$–Symmetries and Reduction of Equations Without Lie Point Symmetries

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Abstract. It is proved that several usual methods of reduction for ordinary differential equations, that do not come from the Lie theory, are derived from the existence of $\mathcal{C}^\infty$-symmetries. This kind of symmetries is also applied to obtain two successive reductions of an equation that lacks Lie point symmetries but is a reduced equation of another one with a three dimensional Lie algebra of point symmetries. Some relations between $\mathcal{C}^\infty$-symmetries and potential symmetries are also studied.

1. Introduction

Let us consider the nth-order ordinary differential equation

$$\Delta(x, u^{(n)}) = 0. \tag{1}$$

In the literature there appear several methods of reduction for (1). One of the most important is based on the existence of Lie point symmetries of the equation. However, there are also equations that lack Lie point symmetries but can be reduced. This is the case, for instance, when by means of

$$y = y(x, u), \quad v = f(x, u, u_x), \tag{2}$$

equation (1) transforms into

$$\Delta_1(y, v^{(n-1)}) = 0, \tag{3}$$

or when (1) can be written in the form

$$D_x(\Delta_2(x, u^{(n-1)})) = 0, \tag{4}$$

where D_x denotes the total derivative with respect to the independent variable x. There are also many examples of integrable equations that lack Lie point symmetries ([2, 8, 9, 10, 11, 17]). In this paper we extend the concept of $\mathcal{C}^\infty$-symmetry, that appears in [17], and we prove that these classes of reductions are particular

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cases of the algorithm of reduction derived from the existence of extended $\mathcal{C}^\infty$-symmetries of the equation. These $\mathcal{C}^\infty$-symmetries can be found by a well-defined algorithm, somewhat similar to the Lie algorithm.

It may also happen that (1) has no Lie point symmetries but, by means of a Bäcklund transformation

$$u = f(x, v, v_x), \tag{5}$$

equation (1) is transformed into an equation of the form

$$\widetilde{\Delta}(x, v^{(n+1)}) = 0 \tag{6}$$

that has a non-trivial Lie algebra of point symmetries $\mathcal{G}$. This case can happen when (1) is the reduced equation of (6) after using a generator of $\mathcal{G}$: it may occur that the unused generators are not inheritable to the reduced equation. In the literature, these lost symmetries are called type I hidden symmetries (the term type II hidden symmetries refers to the symmetries that are gained after an order reduction). It may be said that the origin of the theory of hidden symmetries is in the concept of exponential vector fields ([20]), that provides order reductions but are not local vector fields. Many recent studies about lost symmetries have been done (see [1]-[3],[10]-[12], [16] and references therein). In [13], the concept of solvable structure ([6]) is applied to study hidden symmetries of type II. In particular, the authors show how the hidden symmetries of type II appearing in [12] are related to a solvable structure for the unreduced equations.

Let us observe that when (5) does not depend on v then $X = \frac{\partial}{\partial v}$ is a Lie point symmetry of (6), (1) is the corresponding reduced equation and, if $n \geq 3$, the order of the original equation can be reduced by two. By using some results that appear in [18], in this paper we prove that if an equation of the form (6) admits a three dimensional Lie algebra of point symmetries then the order of (6) can successively be reduced by three: if any of the generators of $\mathcal{G}$ is used to reduce the order then the remaining generators are inheritable, at some stage of the reduction process, as $\mathcal{C}^\infty-$symmetries of the reduced equations. We also show, through an example, that these $\mathcal{C}^\infty-$symmetries (derived from type I hidden symmetries) can be used to construct a solvable structure of the reduced equations.

It may also happen that, for some function f, (6) can be written in the conserved form

$$D_x(\Delta_3(x, v^{(n)})) = 0, \tag{7}$$

for some function Δ_3. In this case we also have the trivial reduction

$$\Delta_3(x, v^{(n)}) = 0, \tag{8}$$

which, as we prove in this paper, corresponds to a $\mathcal{C}^\infty$-symmetry of (6). Then, we could use the symmetries of (8) to obtain solutions of the original equation (1). This is the way followed by Bluman ([7]): these symmetries are called potential symmetries of (1). Potential symmetries are not, in general, either contact or Lie-Bäcklund symmetries of (1) because v, as defined by (5), cannot be expressed in terms of x, u and derivatives of u with respect to x to some finite order. Since Lie point symmetries of (8) can be used to reduce its order, potential symmetries are useful to find solutions of equation (1), because if $v(x) = \phi(x)$ solves equation (8) then $u(x) = f(x, \phi(x), \phi_x(x))$ solves equation (1). Let us observe that Lie point symmetries of any reduced equation

$$\Delta_3(x, v^{(n)}) = C, \tag{9}$$

where $C \in \mathbb{R}$ is an arbitrary constant, lead to a similar process for equation (1). Therefore, in this paper we will understand a potential symmetry of equation (1) as a Lie point symmetry of equation (8), for some $C \in \mathbb{R}$. In practice, Bäcklund transformations (5) that let us write equation (1) in conserved form (7) are difficult to find, if there is one, because this form is too restrictive. In this paper we prove that some special potential symmetries of (1), that are here called super-potential symmetries, can be considered as $\mathcal{C}^\infty$-symmetries of (1) and, therefore, two procedures to obtain solutions of (1) are available. This is illustrated through an example and both methods are compared.

2. Notations and preliminary results

Let us consider an nth-order ordinary differential equation

$$\Delta(x, u^{(n)}) = 0, \tag{10}$$

with $(x, u) \in M$, for some open subset $M \subset X \times U \simeq \mathbb{R}^2$. We denote by $M^{(k)}$ the corresponding $k-$jet space $M^{(k)} \subset X \times U^{(k)}$, for $k \in \mathbb{N}$. Their elements are $(x, u^{(k)}) = (x, u, u_1, \cdots, u_k)$, where, for $1 \le i \le k$, u_i denotes the derivative of order i of u with respect to x. We assume that the implicit function theorem can be applied to equation (10), and, as a consequence, that this equation can locally be written in the explicit form

$$u_n = \Psi(x, u^{(n-1)}). \tag{11}$$

The vector field

$$A_{(x,u)} = \frac{\partial}{\partial x} + u_1 \frac{\partial}{\partial u} + \cdots + \Psi(x, u^{(n-1)}) \frac{\partial}{\partial u_{n-1}} \tag{12}$$

will be called the vector field associated with equation (11).

It is well-known ([23]) that a vector field X on M is a Lie point symmetry of equation (11) if and only if there exists a function $\rho \in C^\infty(M^{(1)})$ such that

$$[X^{(n-1)}, A_{(x,u)}] = \rho A_{(x,u)}, \tag{13}$$

where $X^{(n-1)}$ denotes the usual $(n-1)$th prolongation of the vector field X. The generalized Lie symmetries ([22]) are vector fields Y defined on $M^{(n-1)}$ that satisfy $[Y, A_{(x,u)}] = \rho A_{(x,u)}$, for some function $\rho \in C^\infty(M^{(j)})$.

A Lie point symmetry X can be used to reduce the order of the equation by one: we introduce a change of variables $\{y = y(x, u), \alpha = \alpha(x, u)\}$ such that the vector field X can be written as $X = \frac{\partial}{\partial \alpha}$, in some open set of variables (y, α), that will also be denoted by M. Since X is a Lie point symmetry of the equation, (11) can be written in terms of variables $(y,\alpha^{(n)})$ of $M^{(n)}$ in the form

$$\alpha_n = \Phi(y,\alpha_1,\alpha_2,\cdots,\alpha_{n-1}). \tag{14}$$

If we set $w=\alpha_1$ in (14) we obtain a reduced equation

$$w_{n-1} = \Phi(y,w,w_1,\cdots,w_{n-2}), \tag{15}$$

where (y,w) are in some open set $M_1 \subset \mathbb{R}^2$.

It can easily be checked that the vector field associated with equation (14), written in the new variables, is

$$A_{(y,\alpha)} = \frac{1}{D_x(y(x,u))} A_{(x,u)}. \tag{16}$$

The vector field associated with the reduced equation (15) can be constructed as follows. Let $\pi_X^{(k)} : M^{(k)} \to M_1^{(k-1)}$ be the projection $(y,\alpha,\alpha_1,\cdots,\alpha_k) \mapsto (y,w,\cdots,w_{k-1}) = (y,\alpha_1,\cdots,\alpha_k)$, for $k \in \mathbb{N}$. A vector field V on $M^{(k)}$ will be called $\pi_X^{(k)}-$projectable if

$$[X^{(k)},V] = fX^{(k)}, \tag{17}$$

for some function $f \in C^\infty(M^{(k)})$. This implies that V, in the variables $(y,\alpha^{(k)})$, must take the following form

$$V = \xi(y,\alpha_1,\cdots,\alpha_k)\frac{\partial}{\partial y} + \eta(y,\alpha,\alpha_1,\cdots,\alpha_k)\frac{\partial}{\partial \alpha} + \sum_{i=1}^{k} \eta_i(y,\alpha_1,\cdots,\alpha_k)\frac{\partial}{\partial \alpha_i}. \tag{18}$$

The $\pi_X^{(k)}-$projection of V on $M_1^{(k-1)}$ is the vector field

$$(\pi_X^{(k)})_*(V) = \xi(y,w,\cdots,w_{k-1})\frac{\partial}{\partial y} + \sum_{i=1}^{k} \eta_i(y,w,\cdots,w_{k-1})\frac{\partial}{\partial w_{i-1}}. \tag{19}$$

With this definition, it can be checked that the vector field $A_{(y,\alpha)}$ is $\pi_X^{(n-1)}-$projectable and its projection is the vector field $A_{(y,w)}$ associated with the reduced equation (15).

The concept of Lie point symmetry for an ordinary differential equation can be generalized in several ways: conditional symmetries, Lie-Bäcklund symmetries, etc. ([5],[4],[20], [21]). In [17], we have introduced the concept of $C^\infty-$symmetry. This concept is somewhat similar to the concept of Lie point symmetry, but it is based on a different way to prolong vector fields. The following prolongation method generalizes the method that appears in [17].

Definition 2.1. Generalized prolongation formula

Let $X = \xi(x,u)\frac{\partial}{\partial x} + \eta(x,u)\frac{\partial}{\partial u}$ be a vector field defined on M, and let $\lambda \in C^\infty(M^{(k)})$ be an arbitrary function. The *$\lambda-$prolongation of order n of X*, denoted by $X^{[\lambda,(n)]}$, is the vector field defined on $M^{(n+k-1)}$ by

$$X^{[\lambda,(n)]} = \xi(x,u)\frac{\partial}{\partial x} + \sum_{i=0}^{n} \eta^{[\lambda,(i)]}(x,u^{(i+k-1)})\frac{\partial}{\partial u_i} \tag{20}$$

where $\eta^{[\lambda,(0)]}(x,u)=\eta(x,u)$ and

$$\begin{array}{rl}\eta^{[\lambda,(i)]}(x,u^{(i+k-1)}) = & D_x\left(\eta^{[\lambda,(i-1)]}(x,u^{(i+k-2)})\right)-D_x(\xi(x,u))u_i \\ & +\lambda\left(\eta^{[\lambda,(i-1)]}(x,u^{(i+k-2)})-\xi(x,u)u_i\right),\end{array} \tag{21}$$

for $1\leq i\leq n$.

Let us observe that, if $\lambda=0$, the $\lambda-$prolongation of order n of X is the usual nth prolongation of X. If $Q=\eta(x,u)-\xi(x,u)u_1$ is the characteristic of $X=\xi(x,u)\frac{\partial}{\partial x}+\eta(x,u)\frac{\partial}{\partial u}$ then

$$X^{[\lambda,(n)]}=X_Q^{[\lambda,(n)]}+\xi(x,u)D_x, \tag{22}$$

where

$$X_Q^{[\lambda,(n)]}=\sum_{i=1}^{n}(D_x+\lambda)^i(Q)\frac{\partial}{\partial u_i}. \tag{23}$$

Definition 2.2. Let $\Delta(x,u^{(n)})=0$ be an nth-order ordinary differential equation. We will say that a vector field X, defined on M, is a $C^\infty(M^{(k)})-$*symmetry* of the equation, $1\leq k<n$, if there exists a function $\lambda\in C^\infty(M^{(k)})$ such that

$$X^{[\lambda,(n)]}(\Delta(x,u^{(n)}))=0, \quad \text{when} \quad \Delta(x,u^{(n)})=0. \tag{24}$$

In this case we will also say that X is a $\lambda-$symmetry or a C^∞-symmetry, if there is no place for confusion.

By a straightforward generalization of a result that appears in [17], it can be checked that a vector field X on M is a $C^\infty(M^{(k)})-$symmetry of the equation (11) if and only if there exist two functions, $\lambda,\rho\in C^\infty(M^{(k)})$, such that

$$[X^{[\lambda,(n-1)]},A_{(x,u)}]=\lambda X^{[\lambda,(n-1)]}+\rho A_{(x,u)}. \tag{25}$$

Let us observe that if X is a Lie point symmetry then, $[X^{(n-1)},A_{(x,u)}]=\rho A_{(x,u)}$ for some $\rho\in C^\infty(M^{(1)})$, and for any function $f\in C^\infty(M)$, $fX^{(n-1)}$ satisfies

$$[fX^{(n-1)},A_{(x,u)}]=-A_{(x,u)}(f)X^{(n-1)}+f\rho A_{(x,u)}. \tag{26}$$

Therefore, fX is a $C^\infty(M^{(1)})-$symmetry for $\lambda=-\frac{A_{(x,u)}(f)}{f}$.

Conversely, if X is a $\lambda-$symmetry, for $\lambda\in C^\infty(M^{(k)})$, and $f\in C^\infty(M^{(j)})$, then, by using (25), we have

$$[fX^{[\lambda,(n-1)]},A_{(x,u)}]=(f\lambda-A_{(x,u)}(f))X^{[\lambda,(n-1)]}+f\rho A_{(x,u)}. \tag{27}$$

If we choose f such that $A_{(x,u)}(f)=f\lambda$ then $fX^{[\lambda,(n-1)]}$ becomes a Lie symmetry in the generalized sense.

Example 2.3. It can be proved ([17]) that the second order equation:

$$u_{xx} + \frac{x^2}{4u^3} + u + \frac{1}{2u} = 0 \tag{28}$$

has no Lie point symmetries. The vector field $X = u\frac{\partial}{\partial u}$ is a $\lambda-$symmetry, for $\lambda = \frac{x}{u^2}$, of equation (28). In this case, $A_{(x,u)} = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} - \left(\frac{x^2}{4u^3} + u + \frac{1}{2u}\right)\frac{\partial}{\partial u_x}$ and, by (21), $X^{[\lambda,(1)]} = u\frac{\partial}{\partial u} + (u_x + \frac{x}{u})\frac{\partial}{\partial u_x}$. The vector field $Y = X^{[\lambda,(1)]}$ verifies formula (25), because

$$[Y, A_{(x,u)}] = [X^{[\lambda,(1)]}, A_{(x,u)}] = \frac{x}{u}\frac{\partial}{\partial u} + \left(\frac{xu_x}{u^2} + \frac{x^2}{u^3}\right)\frac{\partial}{\partial u_x} = \lambda Y. \tag{29}$$

However, there is no function ρ such that (13) is satisfied. Therefore, $[Y, A_{(x,u)}] \neq \rho A_{(x,u)}$ for any function ρ; i.e. Y is not a Lie symmetry, in the generalized sense, of equation (28). In order to find a function f such that fY is a generalized Lie symmetry, we must solve the following partial differential equation:

$$f_x + f_u u_x - f_{u_x}\left(\frac{x^2}{4u^3} + u + \frac{1}{2u}\right) = \frac{x}{u^2}f. \tag{30}$$

Therefore, it seems that it is easier to calculate $C^\infty-$symmetries than the associated generalized Lie symmetries.

An algorithm to determine the $C^\infty-$symmetries of an equation follows from (24): this equation generates a system of equations for the infinitesimals of the $C^\infty-$symmetry X, in which λ is also an unknown function. This gives, with respect to Lie method, a higher level of freedom in the resolution of these determining equations. In particular, it may happen that the determining equations for Lie point symmetries only admit the trivial solution but the corresponding equations for $C^\infty-$symmetries have non-trivial solutions, as in Example 2.3.

The $C^\infty(M^{(k)})$-symmetries can be used to obtain reduction processes. The corresponding method is described in Theorem 2.5. In order to prove this theorem, we need a preliminary result.

Theorem 2.4. *Let X be a vector field defined on $M \subset X \times U$ and let $\lambda \in C^\infty(M^{(k)})$. If $\alpha = \alpha(x, u^{(j)})$, $\beta = \beta(x, u^{(j)})$ are functions in $C^\infty(M^{(j)})$ such that*

$$X^{[\lambda,(j)]}(\alpha(x, u^{(j)})) = X^{[\lambda,(j)]}(\beta(x, u^{(j)})) = 0, \tag{31}$$

then

$$X^{[\lambda,(j+1)]}\left(\frac{D_x\alpha(x, u^{(j)})}{D_x\beta(x, u^{(j)})}\right) = 0. \tag{32}$$

Proof. It is clear that

$$[X^{[\lambda,(j+1)]}, D_x] = \lambda X^{[\lambda,(j+1)]} + \mu D_x, \tag{33}$$

where $\mu = -D_x(X(x)) - \lambda X(x) \in C^\infty(M^{(k)})$. Therefore,

$$X^{[\lambda,(j+1)]}\left(\frac{D_x\alpha}{D_x\beta}\right) = \frac{1}{(D_x\beta)^2}\left(D_x\beta\cdot X^{[\lambda,(j+1)]}(D_x\alpha) - D_x\alpha\cdot X^{[\lambda,(j+1)]}(D_x\beta)\right)$$

$$= \frac{1}{(D_x\beta)^2}\left(D_x\beta\cdot[X^{[\lambda,(j+1)]},D_x](\alpha) - D_x\alpha\cdot[X^{[\lambda,(j+1)]},D_x](\beta)\right) \tag{34}$$

$$= \frac{1}{(D_x\beta)^2}\left(D_x\beta\cdot(\mu\cdot D_x\alpha) - D_x\alpha\cdot(\mu\cdot D_x\beta)\right) = 0.$$

This proves the theorem. ∎

The following theorem, and its proof, gives a method to reduce an equation that admits a $\mathcal{C}^\infty$-symmetry.

Theorem 2.5. *Let X be a λ-symmetry, with $\lambda \in C^\infty(M^{(k)})$, of the equation $\Delta(x,u^{(n)}) = 0$. Let $y = y(x,u)$ and $w = w(x,u,u_1,\cdots,u_k)$ be two functionally independent invariants of $X^{[\lambda,(n)]}$. The general solution of the equation can be obtained by solving a reduced equation of the form $\Delta_r(y,w^{(n-k)}) = 0$ and an auxiliary kth-order equation $w = w(x,u,u_1,\cdots,u_k)$.*

Proof. Let $y = y(x,u)$ and $w = w(x,u,u_1,\cdots,u_k)$ be two functionally independent invariants of $X^{[\lambda,(k)]}$ such that w depends on u_k. By Theorem 2.4,

$$w_1 = \frac{D_x w(x,u,u_1,\cdots,u_k)}{D_x y(x,u)} \tag{35}$$

is an invariant for $X^{[\lambda,(k+1)]}$. The set $\{y,w,w_1\}$ is functionally independent, because w_1 depends on u_{k+1}. From w_1 and y we can obtain, by derivation, a $(k+2)$th-order invariant for $X^{[\lambda,(n)]}$ and so on. Therefore, the set

$$\{y,w,w_1(x,u,u_1,\cdots,u_{k+1}),\cdots,w_{n-k}((x,u,u_1,\cdots,u_n)\} \tag{36}$$

is a set of functionally independent invariants of $X^{[\lambda,(n)]}$. Since X is, by hypothesis, a $C^\infty(M^{(k)})$-symmetry of $\Delta(x,u^{(n)}) = 0$, it can be checked, by Definition 2.1, that this equation can be written in terms of (36). The resulting equation is a $(n-k)$th-order equation of the form

$$\Delta_r(y,w^{(n-k)}) = 0. \tag{37}$$

We can recover the general solution of $\Delta(x,u^{(n)}) = 0$ from the general solution of (37) and the corresponding kth-order auxiliary equation:

$$w = w(x,u,u_1,\cdots,u_k). \tag{38}$$

∎

3. $C^\infty-$Symmetries and order reductions

In this section we show that many of the known reduction processes for ordinary differential equations can be obtained, through the former method, as a consequence of the existence of $C^\infty-$symmetries of the given equations.

Theorem 3.1. *Let*

$$\Delta_1(x, u^{(n)}) = 0 \tag{39}$$

be an nth-order ordinary differential equation. Let us suppose that there exists a transformation

$$\left.\begin{array}{l} y = y(x,u), \\ w = w(x,u,u_1), \end{array}\right\} \tag{40}$$

where $w_{u_1} \neq 0$*, such that (39) can be written, in terms of variables* (y,w)*, in the form*

$$\Delta_2(y, w^{(n-1)}) = 0. \tag{41}$$

There exists a $C^\infty-$ *symmetry* X *of equation (39) such that (41) is the corresponding reduced equation.*

Proof. Let $\alpha \in C^\infty(M)$ be such that the functions y, α are functionally independent. We denote $\alpha_1 = \frac{d\alpha}{dy} = \frac{D_x(\alpha(x,u))}{D_x(y(x,u))} \in C^\infty(M^{(1)})$. We consider on $M^{(1)}$ the local coordinates (y, α, α_1). We determine a vector field of the form $X = \xi(y,\alpha)\frac{\partial}{\partial y} + \eta(y,\alpha)\frac{\partial}{\partial \alpha}$ and a function $\lambda(y,\alpha,\alpha_1) \in C^\infty(M^{(1)})$ such that X is a $\lambda-$symmetry of the equation and the functions y, w in (40) are invariants of $X^{[\lambda,(1)]}$.

We set $\xi = 0$ and $\eta = 1$; by Definition 2.1, $X^{[\lambda,(1)]} = \frac{\partial}{\partial \alpha} + \lambda \frac{\partial}{\partial \alpha_1}$. We determine λ with the condition $X^{[\lambda,(1)]}(w) = 0$ and we find that $\lambda = -\frac{w_\alpha}{w_{\alpha_1}}$.

1. Let us prove that the vector field $X = \frac{\partial}{\partial \alpha}$ is a $\lambda-$symmetry of the equation for the function $\lambda = -\frac{w_\alpha}{w_{\alpha_1}}$. We denote $w_i = \frac{d^{i)}w}{dy^{i)}}$, for $1 \leq i \leq n-1$. It is clear that the set $\{y, \alpha, w, \cdots, w_{n-1}\}$ is a system of coordinates in $M^{(n)}$. By the construction of X and λ, we have that $\{y, w, \cdots, w_{n-1}\}$ are invariants for the vector field $X^{[\lambda,(n)]}$. Therefore, in the new local coordinates, $X^{[\lambda,(n)]} = \frac{\partial}{\partial \alpha}$. Since, by hypothesis, equation (39) can be written in these local coordinates as equation (41), we obtain

$$X^{[\lambda,(n)]}(\Delta_2(y, w^{(n-1)})) = \frac{\partial}{\partial \alpha}(\Delta_2(y, w^{(n-1)})) = 0. \tag{42}$$

   This proves that X is a $\lambda-$symmetry of the equation.

2. In order to check that (41) is the reduced equation that, by Theorem 2.5, corresponds to the λ-symmetry, it is sufficient to observe that the reduced equation can be obtained by writing the equation in terms of the complete system $\{y, w, \cdots, w_{n-1}\}$ of invariants of $X^{[\lambda,(n)]}$.

This proves the theorem. ∎

Theorem 3.2. *Let*

$$D_x(\Delta(x, u^{(n-1)})) = 0, \tag{43}$$

be an nth-order ordinary differential equation such that Δ is an analytical function of its arguments. There exists a function $\lambda \in C^\infty(M^{(k)}), k \leq n-1$, such that the vector field $X = \frac{\partial}{\partial u}$ is a $\lambda-$symmetry of the equation. The trivial order reduction

$$\Delta(x, u^{(n-1)}) = C, \quad C \in \mathbb{R}, \tag{44}$$

admitted by the equation, can be obtained as the auxiliary equation that corresponds to the reduction process, by means of X, that appears in Theorem 2.5.

Proof. 1. We try to find $\lambda \in C^\infty(M^{(k)}), k \leq n-1$, with the condition

$$X^{[\lambda,(n-1)]}(\Delta(x, u^{(n-1)})) = 0 \text{ when } D_x(\Delta(x, u^{(n-1)})) = 0. \tag{45}$$

In terms of the characteristic $Q \equiv 1$ of X, we have

$$X^{[\lambda,(n-1)]} = \sum_{i=0}^{n-1} (D_x + \lambda)^i(1) \frac{\partial}{\partial u_i}. \tag{46}$$

Hence, the equation $X^{[\lambda,(n-1)]}(\Delta(x, u^{(n-1)})) = 0$ can be written as

$$(D_x + \lambda)^{n-1}(1) \frac{\partial \Delta}{\partial u_{n-1}} = -\sum_{i=0}^{n-2} (D_x + \lambda)^i(1) \frac{\partial \Delta}{\partial u_i}. \tag{47}$$

Let us observe that, since the set of analytical functions is closed under differentiation, if λ is an analytical function in $M^{(k)}$ then, for $1 \leq i \leq n-1$, the function defined by $(D_x + \lambda)^i(1)$ is analytical in $M^{(k+i-1)}$ and in the partial derivatives of λ with respect to all its arguments up to the order $i-1$.

Since the order of (43) is n, we have $\Delta_{u_{n-1}}(x, u^{(n-1)}) \neq 0$, in some open set of $M^{(n-1)}$. Therefore, the implicit function theorem for analytical functions ([15]) let us, locally, write (43) in the form $u_n = F(x, u^{(n-1)})$, where F is an analytical function on its arguments.

In (47), we must replace u_n by F and u_{n+h}, $h \geq 1$, by the corresponding derivatives. The resulting equation is defined by functions that depend analytically on their arguments. It is easy to see that, in (47), the derivative

$$\frac{\partial^{n-2)} \lambda}{\partial x^{n-2)}} \tag{48}$$

does only appear in the first member and its coefficient is $\Delta_{u_{n-1}} \neq 0$. Equation (47) can be solved for $\frac{\partial^{n-2)} \lambda}{\partial x^{n-2)}}$ and the resulting partial differential equation for λ can be written in the form

$$\frac{\partial^{n-2)} \lambda}{\partial x^{n-2)}} = G(x, u^{(n-1)}, \lambda^{(n-2)}), \tag{49}$$

where $\lambda^{(n-2)}$ denotes the partial derivatives of λ with respect to its arguments, of orders $\leq n-2$, and the function G is analytical on its arguments and does not depend on $\frac{\partial^{n-2)}\lambda}{\partial x^{n-2)}}$.

With these conditions, Cauchy-Kovalevsky Theorem ([15]) ensures the existence of analytical solutions, $\lambda(x, u^{(n-1)})$, to equation (49).

Next, we prove that, if λ is a solution of (49), the vector field $X = \frac{\partial}{\partial u}$ is a λ-symmetry of the equation.

By Definition 2.2, it can be checked, that

$$[X^{[\lambda,(n-1)]}, D_x] = \lambda X^{[\lambda,(n-1)]}. \tag{50}$$

By applying both members of this expression to $\Delta(x, u^{(n-1)})$ we get

$$\begin{aligned} X^{[\lambda,(n-1)]}(D_x(\Delta(x,u^{(n-1)}))) - D_x(X^{[\lambda,(n-1)]}(\Delta(x,u^{(n-1)}))) \\ = \lambda X^{[\lambda,(n-1)]}(\Delta(x,u^{(n-1)})). \end{aligned} \tag{51}$$

Since λ is such that $X^{[\lambda,(n-1)]}(\Delta(x,u^{(n-1)})) = 0$ when $D_x(\Delta(x,u^{(n-1)})) = 0$, we get

$$X^{[\lambda,(n-1)]}(D_x(\Delta(x,u^{(n-1)}))) = 0, \quad \text{when} \quad D_x(\Delta(x,u^{(n-1)})) = 0, \tag{52}$$

i.e. X is a λ-symmetry of the equation.

2. The algorithm to reduce the order of the equation, by using the λ-symmetry X, leads to the first order ordinary differential equation

$$w_y = 0, \tag{53}$$

where $y = x$ and $w = \Delta(x, u^{(n-1)})$ are two functionally independent invariants of the vector field $X^{[\lambda,(n-1)]}$. The general solution of equation $w_y = 0$ is $w = C$, $C \in \mathbb{R}$. Therefore, the general solution the original equation is obtained by solving the ordinary differential equation (44). ■

4. Reduction of equations without Lie point symmetries.

Let us suppose that the ordinary differential equation

$$\Delta(x, u^{(n)}) = 0 \tag{54}$$

has no Lie point symmetries. We will also suppose that by means of $u = f(x, v_x)$ equation (54) is transformed into the $(n+1)$th-order equation

$$\widetilde{\Delta}(x, v^{(n+1)}) = 0, \tag{55}$$

and that this equation has a three dimensional Lie algebra of point symmetries $\mathcal{G}$. By Theorem 3.1, the order reduction of equation (55) to equation (54) corresponds to the use of the Lie point symmetry $\frac{\partial}{\partial v}$. By the classification of three dimensional Lie algebras that appears in [14], the structure of the Lie algebra $\mathcal{G}$ generated by X_1, X_2 and X_3 corresponds, by means of some linear combination of the generators, to some of the types enumerated in the following table:

Let us analyze the reduction of (54) by using the three dimensional symmetry algebra $\mathcal{G}$ of (55). We denote $X = \frac{\partial}{\partial v}$, the vector field that lets reduce (55) to (54). Here X may be any of the generators X_i, $1 \leq i \leq 3$ and, depending on i, several ways of step by step reduction of (54) can be followed.

Table 1: Three-dimensional solvable algebras

I	II	III	IV
$[X_1, X_2] = 0$	$[X_1, X_2] = 0$	$[X_1, X_2] = X_3$	$[X_1, X_2] = 0$
$[X_1, X_3] = 0$	$[X_1, X_3] = X_3$	$[X_1, X_3] = 0$	$[X_1, X_3] = aX_1 + bX_2$
$[X_2, X_3] = 0$	$[X_2, X_3] = 0$	$[X_2, X_3] = 0$	$[X_2, X_3] = cX_1 + dX_2$

Table 2: Three-dimensional non-solvable algebras

V	VI
$[X_1, X_2] = 2X_3$	$[X_1, X_2] = X_3$
$[X_1, X_3] = X_1$	$[X_1, X_3] = -X_2$
$[X_2, X_3] = -X_2$	$[X_2, X_3] = X_1$

1. In cases I to III, the kernel $Z(\mathcal{G}) = \{X \in \mathcal{G} : [X, Y] = 0, Y \in \mathcal{G}\}$ is not trivial. If we use any of the generators to perform a first reduction then the corresponding reduced equation conserves at least one Lie point symmetry. Since, by hypothesis, (54) has no Lie point symmetries, $X = \frac{\partial}{\partial v}$ cannot be in $\mathcal{G}$. This contradiction proves that, with our hypotheses, these three cases cannot happen.

2. In case IV, the following chains of normal subalgebras in $\mathcal{G}$ hold:

$$< X_1 > \triangleright < X_1, X_2 > \triangleright < X_1, X_2, X_3 >,$$
$$< X_2 > \triangleright < X_1, X_2 > \triangleright < X_1, X_2, X_3 > .$$

If we first reduce with X_i, $1 \leq i \leq 2$, the reduced equation always inherits a Lie point symmetry. Since (54) has no Lie point symmetries, necessarily the first reduction is performed by using $X = X_3 = \frac{\partial}{\partial v}$. Let us study how the symmetries X_1 and X_2 can be used to reduce, successively, the order of (54) by two.

(a) We may assume that $b = 0$: if this is not the case, we can use a linear change of coordinates (possibly with complex coefficients) to get $b = 0$. Let $f_1 \in C^\infty(M)$ be a function such that $X_3(f_1) = af_1$. Then

$$[f_1 X_1^{(k)}, X_3^{(k)}] = f_1(aX_1^{(k)}) - X_3(f_1)X_1^{(k)} = 0 \quad (k \in \mathbb{N}). \qquad (56)$$

By (17), $f_1 X_1^{(k)}$ is a $\pi_{X_3}^{(k)}-$projectable vector field. Since $[X_1^{(k)}, D_x] = -D_x(X_1(x))D_x$, we get

$$[f_1 X_1^{(k)}, D_x] = -\frac{D_x(f_1)}{f_1} f_1 X_1^{(k)} - f_1 D_x(X_1(x))D_x, \qquad (57)$$

and, by taking (33) into account, it follows that

$$f_1 \cdot X_1^{(k)} = (f_1 X_1)^{[\lambda_1,(k)]}, \quad \text{for } \lambda_1 = -\frac{D_x(f_1)}{f_1}. \qquad (58)$$

Let us denote $Y_1 = (\pi_{X_3}^{(1)})_*(f_1 X_1^{(1)})$. The vector field Y_1 is a $C^\infty-$symmetry of equation (54), for the function λ_1 given, in coordinates (x, v), by:

$$\lambda_1 = -\frac{D_x(f_1)}{f_1}. \tag{59}$$

(b) By Theorem 2.5, we use the λ_1-symmetry Y_1 to reduce the order of equation (54). Let $\{y = y(x,u), w = w(x,u,u_1)\}$ be two functionally independent invariants of $Y_1^{[\lambda_1,(1)]}$. We denote by

$$\widehat{\Delta}(y, w^{(n-1)}) = 0 \tag{60}$$

the corresponding reduced equation. Let $\beta = \beta(x,u)$ be such that $Y_1(\beta) = 1$. We denote $\pi^{(k)} = \pi_{Y_1}^{(k-1)} \circ \varphi \circ \pi_{X_3}^{(k)}$, where φ stands for the change of variables $\{x, u^{(k)}\} \leftrightarrow \{y, \beta, w^{(k-1)}\}$. Let f_2 be a function such that $X_3(f_2) = df_2$ and $X_1(f_2) = 0$. Since, for $2 \le k \le n-1$,

(i) $[f_1X_1^{(k)}, f_2X_2^{(k)}] = f_1X_1^{(k)}(f_2)X_2^{(k)} - f_2X_2^{(k)}(f_1)X_1^{(k)} = f_{12}^1 f_1 X_1^{(k)}$, where $f_{12}^1 = -\frac{f_2}{f_1}X_2^{(k)}(f_1)$,

(ii) $[f_2X_2^{(k)}, X_3^{(k)}] = f_2(cX_1^{(k)} + dX_2^{(k)}) - X_3^{(k)}(f_2)X_2^{(k)} = f_{23}^2 f_1 X_1^{(k)}$, where $f_{13}^2 = \frac{f_1}{f_2}c$,

the vector field $f_2X_2^{(k)}$ is $\pi^{(k)}-$projectable.

Since, by hypothesis, the vector field X_2 is a Lie point symmetry of the equation (55), we can write

$$[f_2X_2^{(n-1)}, A_{(x,v)}] = \lambda_2 f_2 X_2^{(n-1)} + \mu A_{(x,v)}$$

for some functions $\lambda_2, \mu \in C^\infty(M^{(1)})$. By (i) and (ii), the Jacobi identity for the vector fields

$$\{f_1X_1^{(n-1)}, f_2X_2^{(n-1)}, \mu A_{(x,v)}\} \text{ and } \{f_2X_2^{(n-1)}, X_3^{(n-1)}, \mu A_{(x,v)}\}$$

let us prove that the functions λ_2 and μ are both $f_1X_1^{(n-1)}$-invariant and $X_3^{(n-1)}$-invariant. Hence,

$$[(\pi^{(n-2)})_*(f_2X_2^{(n-1)}), A_{(y,w)}] = \widetilde{\lambda_2}(\pi^{(n-2)})_*(f_2X_2^{(n-1)}) + \widetilde{\mu}A_{(y,w)}, \tag{61}$$

where $(\pi^{(n-2)})^*(\lambda_2) = \widetilde{\lambda_2}$ and $(\pi^{(n-2)})^*(\mu) = \widetilde{\mu}$. Let us denote $Z_2 = (\pi^{(2)})_*(f_2X_2^{(2)})$. Clearly, (61) shows that Z_2 is a C^∞-symmetry of the reduced equation, and Z_2 can be used to reduce again the order.

3. In case V we have

$$< X_1 > \triangleright < X_1, X_3 >, \quad < X_2 > \triangleright < X_2, X_3 > . \tag{62}$$

It can be assumed, as before, that vector field X is X_3. By proceeding as for Case IV, it can be checked ([18]) that if $f_1, f_2 \in C^\infty(M)$ are such that

$$X_3(f_1) = f_1, \quad X_3(f_2) = -f_2, \tag{63}$$

then the vector field $f_1X_1^{(1)}$ and $f_2X_2^{(1)}$ are $\pi_{X_3}^{(1)}-$projectable. The projections $Y_1 = (\pi_{X_3}^{(1)})_*(f_1X_1^{(1)})$ and $Y_2 = (\pi_{X_3}^{(1)})_*(f_2X_2^{(1)})$, are $C^\infty-$symmetries of equation (54). Any of these two $C^\infty-$symmetries can be used to reduce the order of equation (54). The unused $C^\infty-$symmetry can also be recovered as a $C^\infty-$symmetry of the corresponding reduced equation ([18]).

4. In case VI the vector field $\frac{\partial}{\partial v}$ can be any of the generators and the main results are also valid. We have developed an procedure that allows us to recover the unused generators for the algebra as $C^\infty-$symmetries for the reduced equations. Theoretical results about this situation have been developed in [19].

The former results prove that if (54) has no Lie point symmetries but (55) has a three dimensional Lie algebra of point symmetries $\mathcal{G}$ such that $\frac{\partial}{\partial v} \in \mathcal{G}$ then, by using the generators of $\mathcal{G}$, the order of (54) can successively be reduced by two.

In order to illustrate these ideas, let us consider the following second order differential equation:

$$\begin{aligned} &8(ux+1)u_{xx} - 24xu_x^2 - 2(u^2x^2 + 2ux + 24u + 1)u_x \\ &+x^3u^5 + (5x^2 + 8x)u^4 + (7x + 32)u^3 + 3u^2 = 0. \end{aligned} \tag{64}$$

In Appendix A we prove that this equation has no Lie point symmetries. We are going to transform (64) into a third order equation that admits a (non-solvable) three-dimensional algebra of symmetry. We show how this algebra allows us to recover two of its point symmetries as $C^\infty-$symmetries of equation (64). As a consequence, equation (64) can be solved through two first order equations: one of them is a Ricatti equation, and the other one can be solved by a quadrature, because it is a linear equation.

By means of the transformation $u = v_x$, equation (64) becomes the third order equation

$$\begin{aligned} &8(v_xx+1)v_{xxx} - 24xv_{xx}^2 - 2(v_x^2x^2 + 2v_xx + 24v_x + 1)v_{xx} \\ &+x^3v_x^5 + (5x^2 + 8x)v_x^4 + (7x + 32)v_x^3 + 3v_x^2 = 0. \end{aligned} \tag{65}$$

It can be checked that this equation admits a three-dimensional Lie algebra generated by

$$X_1 = e^{-v}\frac{\partial}{\partial x}, \quad X_2 = -e^v x^2 \frac{\partial}{\partial x} + 2e^v x \frac{\partial}{\partial v}, \quad X_3 = \frac{\partial}{\partial v}. \tag{66}$$

Since

$$[X_1, X_2] = 2X_3, \quad [X_1, X_3] = X_1, \quad [X_2, X_3] = -X_2, \tag{67}$$

the symmetry algebra of equation (65) is the non-solvable Lie algebra $sl(2,\mathbb{R})$ associated to the unimodular group, and corresponds to case V. Equation (64) can be considered as the reduced equation that corresponds to the reduction derived from the use of the Lie point symmetry X_3. Symmetries X_1 and X_2 are not inheritable, as Lie point symmetries, to the reduced equation (64). However, these lost symmetries (hidden symmetries of type I) can be recovered ([18]) as $C^\infty-$symmetries of equation (64). Let us choose $f_1 = e^v$ and $f_2 = e^{-v}$. The vector fields $f_1X_1^{(1)}$ and $f_2X_2^{(1)}$ are $\pi_{X_3}^{(1)}-$projectable and the projections

$$Y_1 = \tfrac{\partial}{\partial x} + u^2\tfrac{\partial}{\partial u}, \quad Y_2 = -x^2\tfrac{\partial}{\partial x} + (2 + 4xu + x^2u^2)\tfrac{\partial}{\partial u} \tag{68}$$

are $C^\infty-$symmetries of equation (64), for $\lambda_1 = -u$ and $\lambda_2 = u$, respectively (see (59)). Let us observe that the $C^\infty-$symmetries Y_1 and Y_2 could also be found by solving the determining equations for the $C^\infty-$symmetries of equation (64).

In terms of variables

$$\left\{ z = \frac{1}{u} + x, \beta = x, \mu = \frac{\beta_z}{\beta - z} = \frac{u^3}{u_x - u^2} \right\}, \tag{69}$$

the vector field $Y_1^{[\lambda_1,(1)]}$ is simply expressed as $Y_1^{[\lambda_1,(1)]} = \frac{\partial}{\partial \beta}$. The corresponding reduced equation is

$$24\mu + 8\mu_z z + \mu^3 z^3 - 2\mu^2 z(4+z) = 0. \tag{70}$$

This first order equation inherits Y_2 as the $C^\infty-$symmetry

$$Z_2 = -2z^2 \frac{\partial}{\partial z} - 2\mu z(-3+\mu z)\frac{\partial}{\partial \mu}, \quad \widetilde{\lambda}_2 = -\mu. \tag{71}$$

In the system of coordinates $\{s = \frac{\mu z - 2}{z^3 \mu}, r = \frac{1}{2z}\}$ the vector field Z_2 can be written as $Z_2 = \frac{\partial}{\partial r}$. Therefore, in these coordinates, equation (70) takes the form of the Ricatti equation

$$r_s = \frac{4r^2}{s} - 1. \tag{72}$$

When the general solution of equation (72) is expressed, in terms of $\{z, \mu\}$, as $\mu = H(z, C_1)$, the auxiliary first order equation that let us recover the solution of equation (70) is the linear equation $\beta_z = H(z, C_1)(\beta - z)$, which can be solved by quadrature.

Next we show how the $C^\infty-$symmetries (68) can be used to construct a solvable structure for equation (64). The $C^\infty-$symmetries Y_1 and Y_2 are in involution because $[Y_1^{[\lambda_1,(1)]}, Y_2^{[\lambda_2,(1)]}] = -2xY_1^{[\lambda_1,(1)]}$. Let g_1 be any function such that $A_{(x,u)}(g_1) = \lambda_1 g_1$. Then $g_1 Y_1^{[\lambda_1,(1)]}$ is a generalized Lie symmetry of equation (64), that is

$$[g_1 Y_1^{[\lambda_1,(1)]}, A_{(x,u)}] = \rho_1 A_{(x,u)}, \tag{73}$$

for some function ρ_1. Let g_2 be a function such that

$$A_{(x,u)}(g_2) = \lambda_2 g_2 \quad \text{and} \quad Y_1^{[\lambda_1,(1)]}(g_2) = 0. \tag{74}$$

A function g_2 can be found as follows: since $\{z, \mu\}$ in (69) are invariants of $Y_1^{[\lambda_1,(1)]} = \frac{\partial}{\partial \beta}$, we can choose any function $g_2 = g_2(z(x,u), \mu(x,u,u_x))$ such that $A_{(z,\mu)}(g_2) = \widetilde{\lambda}_2 g_2 = -\mu g_2$, where $A_{(z,\mu)}$ is the vector field associated to equation (70). If g_2 verifies (74), then

$$[g_2 Y_2^{[\lambda_2,(1)]}, A_{(x,u)}] = \rho_2 A_{(x,u)}, \quad \text{and} \quad [g_2 Y_2^{[\lambda_2,(1)]}, g_1 Y_1^{[\lambda_1,(1)]}] = g Y_1^{[\lambda_1,(1)]}, \tag{75}$$

for some function ρ_2 and $g = 2xg_1g_2 + g_2 Y_2^{[\lambda_2,(1)]}(g_1)$. By (73) and (75), we deduce that $\{A_{(x,u)}, g_1 Y_1^{[\lambda_1,(1)]}, g_2 Y_2^{[\lambda_2,(1)]}\}$ constitutes a solvable structure of equation (64).

The same construction can be done for any equation with symmetry algebra of case V, because the $C^\infty-$symmetries Y_1 and Y_2 are always in involution (see Theorem 4 in [18]).

5. $\mathcal{C}^\infty$-symmetries and potential symmetries.

In this section we study some relationships between potential symmetries, introduced by Bluman ([7]), and $\mathcal{C}^\infty$-symmetries. Although we only consider 2nd-order equations, the ideas included in this section may directly be generalized to equations of greater order.

Let us assume that equation

$$\Delta(x, u^{(2)}) = 0 \tag{76}$$

has no Lie point symmetries and that by means of the Bäcklund transformation $u = f(x, v_x)$ equation (76) can be written in conserved form

$$D_x(\tilde{\Delta}(x, v^{(2)})) = 0. \tag{77}$$

Equation (77) can be reduced to (76) by means of the Lie point symmetry $\frac{\partial}{\partial v}$. By other hand, (77) can trivially be reduced to equation

$$\tilde{\Delta}(x, v^{(2)}) = C, \tag{78}$$

where C is an arbitrary constant. By Theorem 3.2, this reduction is also associated to the existence of a $\mathcal{C}^\infty$-symmetry of (77); in general, this reduction does not come from the existence of a Lie point symmetry.

If, for some $C \in \mathbb{R}$, X is a Lie point symmetry of (78) then X is not necessarily a point symmetry of equation $\tilde{\Delta}(x, v^{(2)}) = C'$, for $C' \neq C$, and is not a point symmetry of (77). In this paper, any point symmetry of equation (78), for some $C \in \mathbb{R}$, will be called potential symmetry of equation (76). Let us recall that Bluman ([7]) considered the concept of potential symmetry only for $C = 0$ and, in this case, it may happen that the general solution of (76) cannot be obtain from one of the equations of type (78). This occurs in some trivial cases. For instance, if equation (76) is $u_{xx} = 0$ then, by writing $u = v_x$, equation (77) is $v_{xxx} = 0$ and equation (78) is $v_{xx} = 0$. The general solution of this equation takes the form $v = ax + b$, with $a, b \in \mathbb{R}$. Clearly $u = v_x = a$ does not give the general solution of $u_{xx} = 0$.

If X is a vector field, in variables (x, v), that is a point symmetry of every equation of the form (78) (and does not depend on C) then X will be called a super-potential symmetry of equation (76). We prove in this section that super-potential symmetries can be recovered as $\mathcal{C}^\infty$-symmetries of (76).

Let us assume that the vector field X is a Lie point symmetry of equation (78), for every $C \in \mathbb{R}$; i.e. X is a super-potential symmetry of equation (76). In this case $\tilde{\Delta}(x, v^{(2)})$ is a $X^{(2)}$-invariant function. If $\phi = \phi(x)$ is an arbitrary solution of (77) then $D_x(\tilde{\Delta}(x, \phi^{(2)}(x))) = 0$ and there exists a constant $C \in \mathbb{R}$ such that $\tilde{\Delta}(x, \phi^{(2)}(x)) = C$. Hence ϕ is a solution of equation (78) and X transforms ϕ into another solution of the same equation (78). This proves that the transformed solution is also a solution of (77). Therefore X is also a Lie point symmetry of (77). This fact can also be proved by another procedure: since $[X^{(3)}, D_x] = \mu D_x$, for some function μ, it follows that $\mu D_x \tilde{\Delta} = X^{(3)}(D_x \tilde{\Delta}) - D_x(X^{(3)} \tilde{\Delta}) = X^{(3)}(D_x \tilde{\Delta})$ and $X^{(3)}(D_x \tilde{\Delta}) = 0$ when $D_x \tilde{\Delta} = 0$.

If X_1, X_2 are super-potential symmetries of (76) and X_1, X_2, X_3, with $X_3 = \frac{\partial}{\partial v}$, are the generators of a Lie algebra, then we have proved, in section

4., that X_1, and X_2, can be recovered as $C^\infty-$symmetries of equation (76). Since $C^\infty-$symmetries can be calculated by a well-defined algorithm, and can be used to reduce the order, we could solve the original equation without knowing the associated conserved form (77), which is needed to calculate potential symmetries. In this case two methods of reduction can be used, that are illustrated through the following example.

Let us consider the following second order differential equation:

$$u^5 + e^{2\left(\frac{1}{u}+x\right)}\left(u^4 + u^5 - 3\,{u_1}^2 + u\,u_2\right) = 0. \tag{79}$$

It can be checked (see Appendix B) that this equation has no Lie point symmetries.

Method A. By means of the transformation $u = v_x = v_1$ equation (79) becomes the third order differential equation:

$$ {v_1}^5 + e^{2\left(\frac{1}{v_1}+x\right)}\left({v_1}^4 + {v_1}^5 - 3\,{v_2}^2 + v_1\,v_3\right) = 0. \tag{80}$$

Equation (80) can be expressed in conserved form as

$$D_x\left(e^{-2\,v}\left(e^{-2\left(\frac{1}{v_1}+x\right)} + \left(1 + \frac{1}{v_1} - \frac{v_2}{{v_1}^3}\right)^2\right)\right) = 0. \tag{81}$$

Let us consider any second order equation associated to equation (81) or (80):

$$e^{-2\left(\frac{1}{v_1}+x\right)} + \left(1 + \frac{1}{v_1} - \frac{v_2}{{v_1}^3}\right)^2 = Ce^{2\,v}, \tag{82}$$

where C is an arbitrary constant. It can be checked that equation (82) admits

$$X_1 = e^{-v}\frac{\partial}{\partial x},\quad X_2 = -e^{-v}(v + x + 1)\frac{\partial}{\partial x} + e^{-v}\frac{\partial}{\partial v} \tag{83}$$

as Lie point symmetries. Hence, X_1 and X_2 are super-potential symmetries of equation (79). Since $[X_1, X_2] = 0$, any of these two Lie point symmetries can be used to reduce the order of equation (82), in such a way that the reduced equation inherits the unused symmetry as a Lie point symmetry. As a consequence, equation (82) can be solved by quadratures.

Next, we use the Lie point symmetry X_1 to reduce the order of equation (82). Let us introduce coordinates $\{y = v, \alpha = e^v x\}$ in some open set $M \subset \mathbb{R}^2$. In variables $\{y, \alpha\}$, the vector field X_1 can be written as $\frac{\partial}{\partial \alpha}$. We consider the corresponding system of coordinates $\{y, \alpha^{(2)}\}$ in $M^{(2)}$, and the map $\pi^{(2)}_{X_1} : M^{(2)} \to M^{(1)}_1$ defined by $(y, \alpha^{(2)}) \mapsto (y, w, w_1)$, where $w = \alpha_1$. In terms of $\{y, w^{(1)}\}$ equation (82) takes the form of the first order reduced equation:

$$e^{-2\,(we^{-y}-y)} + (w_1 + e^y - w)^2 = e^{4\,y}\,C. \tag{84}$$

It can be checked that the vector field $(\pi^{(1)}_{X_1})_* X^{(1)}_2$, that will be denoted by $\widetilde{X_2}$, can be written in terms of $\{y, w\}$ as

$$\widetilde{X_2} = e^{-y}\frac{\partial}{\partial y} + (-1 + e^{-y}w)\frac{\partial}{\partial w}. \tag{85}$$

The vector field $\widetilde{X_2}$ is a Lie point symmetry of equation (84), and therefore, it can be used to integrate the equation. By means of the change of variables $\{z = e^{-y}w + y, \beta = e^y\}$, for which $\widetilde{X_2} = \frac{\partial}{\partial \beta}$, equation (84) takes the form

$$\beta_z = \pm \frac{e^z}{\sqrt{-1 + e^{2\,z}\,C}}. \tag{86}$$

The general solution of (86) can be obtained by a quadrature:

$$C_1 \beta = \pm \ln(C_1\, e^z + \sqrt{-1 + C_1^2\, e^{2\,z}}) + C_2, \tag{87}$$

where $C_1^2 = C$ and C_2 is an arbitrary constant. This solution can be written in the simple form $C_1 e^z = \cosh(C_1\beta - C_2)$. Since $z = e^{-y}w + y$ and $\beta = e^y$, the general solution of equation (84) can be expressed as

$$w = -e^y\, \left(C_2 + \ln(2C_1) - C_1\, e^y + y - \ln(1 + e^{2\,(C_2 - \,C_1\, e^y)})\right). \tag{88}$$

By integration with respect to y we get :

$$\alpha = e^y \left(\frac{C_1}{2}\, e^y - (-1 + C_2 + \ln(2C_1) + y)\right) - \frac{1}{2C_1} \int \frac{\ln(1+t)}{t} dt + C_3, \tag{89}$$

where $t = e^{2\,(C_2 - \,C_1\, e^y)}$ and C_3 is an arbitrary constant. The solution of equation (81) can be expressed as

$$x = \left(\frac{C_1}{2}\, e^v - (-1 + C_2 + \ln(2C_1) + v)\right) - e^{-v} \frac{1}{2C_1} \int \frac{\ln(1+t)}{t} dt + C_3 e^{-v}, \tag{90}$$

where $t = e^{2\,(C_2 - \,C_1\, e^v)}$. Since $u = v_x = \frac{e^y}{w - \alpha}$, the general solution of equation (79) is given by

$$u^{-1} = \ln(1 + e^{2\,(C_2 - \,C_1\, e^v)}) - 1 + \frac{C_1\, e^v}{2} + \frac{e^{-v}}{2C_1} \int \frac{\ln(1+t)}{t} dt - C_3 e^{-v}. \tag{91}$$

Method B. Let us observe that the Lie point symmetries X_1 and X_2 of equation (82), that are super-potential symmetries of equation (79), are also Lie point symmetries of equation (80). Equation (80) does also admit the vector field $X_3 = \frac{\partial}{\partial v}$ as Lie point symmetry. It can be checked that the following relations hold:

$$[X_1, X_2] \;=\; 0, \;\; [X_1, X_3] \;=\; X_1, \;\; [X_2, X_3] \;=\; X_1 + X_2. \tag{92}$$

Therefore, equation (80) admits a three-dimensional solvable algebra, $\mathcal{G}$ generated by $\{X_1, X_2, X_3\}$, that corresponds to case IV (with $a = 1, b = 0, c = d = 1$), and equation (79) is the X_3-reduced equation.

In what sequel, we show that the point symmetries X_1, and X_2, used above as super-potential symmetries, can be recovered as $C^\infty-$symmetries of equation (79). Since $C^\infty-$symmetries can be calculated by an algorithm, and can be used to reduce the order, we could solve the original equation without knowing the associated conserved form (81) needed to calculate potential symmetries.

A function f_1 such that $X_3(f_1) = f_1$ is given by $f_1 = e^v$. Clearly, the vector field $f_1 X_1^{(1)}$ is $\pi_{X_3}^{(1)}-$projectable and the vector field

$$Y_1 = (\pi_{X_3}^{(1)})_*(f_1 X_1^{(1)}) = \frac{\partial}{\partial x} + u^2 \frac{\partial}{\partial u} \tag{93}$$

is a $C^\infty-$symmetry of equation (79), for the function

$$\lambda_1 = -\frac{D_x(e^v)}{e^v} = -v_x = -u. \tag{94}$$

By Theorem 2.5, the order of equation (79) can be reduced by using the λ_1-symmetry Y_1. It can be checked that in terms of variables

$$\left\{ z = \frac{1}{u} + x, \beta = x, \mu = \frac{\beta - z}{\beta_z} = \frac{u_x}{u^3} - \frac{1}{u} \right\} \tag{95}$$

the vector field $Y_1^{[\lambda_1,(1)]}$ is simply expressed as $Y_1^{[\lambda_1,(1)]} = \frac{\partial}{\partial \beta}$ and the corresponding reduced equation is

$$1 - e^{2\,z}\,(-1 + \mu + \mu\,\mu_z) = 0. \tag{96}$$

The unused Lie point symmetry X_2 can also be recovered as a $C^\infty-$symmetry of equation (96). By taking $f_2 = e^v$, the vector field $f_2 X_2$ is projectable to the space of variables $\{z, \mu\}$; its projection is given by

$$Y_2 = -\frac{\partial}{\partial z} + (-1 + \mu)\frac{\partial}{\partial \mu}. \tag{97}$$

The vector field Y_2 is a $C^\infty-$symmetry of equation (96) for the function $\lambda_2 = -\frac{D_z(e^v)}{e^v} = -\frac{D_x(v)}{D_x(z)} = \frac{1}{\mu}$.

A function g such that gY_2 is a Lie point symmetry of (96) is given by

$$g = \frac{e^z}{\sqrt{1 + e^{2z}\,(1-\mu)^2}}. \tag{98}$$

In coordinates $\{s = (1-\mu)e^z, r = \sqrt{1 + e^{2\,z}\,(-1+\mu)^2}e^{-z}\}$ the vector field gY_2 can be written as $\frac{\partial}{\partial r}$ and equation (96) can be integrated by a quadrature:

$$r_s = \frac{1}{\sqrt{1+s^2}}. \tag{99}$$

Let us compare the two methods of reduction we have used. When method A is applied to equation (79), we have to construct the conserved form (81). The order of the integrated equation (82) is reduced by means of the potential symmetries (83). The solution (90) of equation (82) is obtained by quadratures. To recover solutions of equation (79), we must solve equation (90) for v and then u can be obtained by derivation ($u = v_x$).

When method B ($C^\infty-$symmetries) is applied, we construct directly the reduced equation (99). The solutions of equation (79) can be obtained as follows. We first solve equation (99) in the form $r = H(s, C_1)$ and obtain μ in the form $\mu = G(z, C_1)$. Since $\mu = \frac{\beta - z}{\beta_z}$, we must solve the linear equation $\beta = z + \beta_z G(z, C_1)$. From the solution $\beta = J(z, C_1, C_2)$ of this equation we directly obtain the solution of (79) in the implicit form $x = J(x + \frac{1}{u}, C_1, C_2, C_3)$.

6. Conclusions

The introduction of the concept of $C^\infty(M^{(1)})-$symmetry ([17]) was motivated by the existence of ordinary differential equations that can be reduced or integrated but lack Lie point symmetries. In this paper, that concept is extended to define the $C^\infty(M^{(k)})-$symmetries and it is proved that several classes of order reduction methods come from of the existence of $C^\infty(M^{(k)})-$symmetries but not from the existence of Lie point symmetries.

We have also studied another equations, without Lie point symmetries, that can be obtained by reduction of equations with a three-dimensional Lie algebra of point symmetries $\mathcal{G}$, that may be solvable or not. The elements of $\mathcal{G}$ are hidden symmetries of the original equation, can be recovered as $C^\infty-$symmetries of the reduced equation and can be used to get new order reductions. This is illustrated through an example, that corresponds to a non-solvable algebra. We have also shown how these $\mathcal{C}^\infty-$symmetries can be used to construct a solvable structure of the reduced equation.

We have also proved that a class of potential symmetries, that are here called super-potential symmetries, can be recovered as $C^\infty-$symmetries of the original equation. Since these $C^\infty-$symmetries can be calculated by a well-defined algorithm, we do not have to determine the Bäcklund transformations needed to write the equation in conserved form, that in practice are difficult to find.

7. Appendix A: Equation (64) has no Lie point symmetries.

If a vector field $X = p(x, u)\frac{\partial}{\partial x} + r(x, u)\frac{\partial}{\partial u}$ is a Lie point symmetry of equation (64), the infinitesimal $p(x, u)$ must satisfy the equation $(ux + 1)\frac{\partial^2 p}{\partial u^2} + 3x\frac{\partial p}{\partial u} = 0$. Therefore,

$$p(x, u) = \frac{p_1(x)u(ux + 2) + 2p_2(x)}{(u\,x + 1)^2}, \tag{100}$$

where p_1 and p_2 depend only on x. The infinitesimal $r(x, u)$ must satisfy

$$\begin{aligned}
&(ux + 1)^4 \tfrac{\partial^2 r}{\partial u^2} - 3x(ux + 1)^3 \tfrac{\partial r}{\partial u} + 3x^2(ux + 1)^2 r + 2\,p_2\,u^2\,x^3 \\
&-p_1\,u^2\,x^2 + 8\,p_2'\,u\,x^2 + 4\,p_2\,u\,x^2 - 3\,p_1\,u^2\,x + 32\,p_2\,u\,x \\
&-4\,p_1'\,u\,x - 2\,p_1\,u\,x + 8\,p_2'\,x + 2\,p_2\,x - 18\,p_1\,u + 2\,p_2 - 4\,p_1' - p_1 = 0.
\end{aligned} \tag{101}$$

It can be checked that the general solution of equation (101) is given by

$$\begin{aligned} r(x,u) = \; & \tfrac{1}{6x^3(ux+1)^2}\left(6r_1u^5x^5 + 30r_1u^4x^4 + 6x^3(r_2x^2+10r_1)u^3\right. \\ & -2x^2(2p_2x^2 - 9r_2x^2 - p_1x - 3p_1 - 30r_1)u^2 - (6p_2'x^2 \\ & +8p_2x^2 - 18r_2x^2 + 24xp_2 - 3p_1'x - 4p_1x - 21p_1 - 30r_1)xu \\ & \left.+6r_2x^2 - 6p_2'x^2 - 4p_2x^2 - 12p_2x + 3p_1'x + 2p_1x + 9p_1 + 6r_1\right), \end{aligned} \tag{102}$$

where r_1 and r_2 are arbitrary functions depending on x. If, in the determining equations, we cancel out the coefficients of powers of u then $r_1(x) = 0, r_2(x) = \frac{-2p_1'(x)}{3x}$, $p_2(x) = \frac{2p_1'(x)}{x^2} + \frac{(x^2-12)p_1(x)}{2x^3}$ and $p_1(x)$ must satisfy:

$$(x^2+36)xp_1'(x) - 108p_1(x) = 0. \tag{103}$$

By solving this equation, we get $p_1(x) = \frac{a\,x^3}{(x^2+36)^{3/2}}$, for $a \in \mathbb{R}$, and hence $p_2(x) = \frac{a\,x^4+24\,a\,x^2}{2(x^2+36)^{5/2}}$.

With these values, we evaluate again the determining equations and we find $a = 0$. Therefore $p_1 = p_2 = 0$. By (100), $p = 0$ and, by (102), $r = 0$. Equation (64) has no Lie point symmetries.

8. Appendix B: Equation (79) has no Lie point symmetries

Next, we prove that equation (79) has no Lie point symmetries. If a vector field $X = p(x,u)\frac{\partial}{\partial x} + r(x,u)\frac{\partial}{\partial u}$ is a Lie point symmetry of equation (79) the infinitesimals p and r must satisfy the following determining equations:

$$\begin{aligned} e1: \;& \tfrac{\partial^2 p}{\partial u^2}\,u + 3\,\tfrac{\partial p}{\partial u} = 0, \\ e2: \;& \left(\tfrac{\partial^2 r}{\partial u^2} - 2\,\tfrac{\partial^2 p}{\partial u \partial x}\right)u^2 - 3\,\tfrac{\partial r}{\partial u}\,u + 3\,r = 0, \\ e3: \;& 3\,\tfrac{\partial p}{\partial u}\,u^5\,e^{-2\,x-\frac{2}{u}} + 3\,\tfrac{\partial p}{\partial u}\,u^5 + 3\,\tfrac{\partial p}{\partial u}\,u^4 + 2\,\tfrac{\partial^2 r}{\partial u \partial x}\,u - \tfrac{\partial^2 p}{\partial x^2}\,u - 6\,\tfrac{\partial r}{\partial x} = 0, \\ e4: \;& \left(-\tfrac{\partial r}{\partial u}\,u^4 + 2\,\tfrac{\partial p}{\partial x}\,u^4 - 2\,p\,u^4 + 4\,r\,u^3 + 2\,r\,u^2\right)e^{-2\,x-\frac{2}{u}} \\ & -\tfrac{\partial r}{\partial u}\,u^4 + 2\,\tfrac{\partial p}{\partial x}\,u^4 - \tfrac{\partial r}{\partial u}\,u^3 + 4\,r\,u^3 + 2\,\tfrac{\partial p}{\partial x}\,u^3 + 3\,r\,u^2 + \tfrac{\partial^2 r}{\partial x^2} = 0. \end{aligned} \tag{104}$$

From equation $e1$, we get

$$p = \frac{p_2}{u^2} + \frac{p_1}{2} \tag{105}$$

where p_1, p_2 are arbitrary functions on x. By substituting this value into equation $e2$ we obtain:

$$\frac{\partial^2 r}{\partial u^2}\,u^3 - 3\,\frac{\partial r}{\partial u}\,u^2 + 3\,r\,u + 4\,\frac{dp_2}{dx} = 0. \tag{106}$$

Therefore

$$r = \; r_1\, u^3 + r_2\, u - \tfrac{1}{2\,u}\tfrac{dp_2}{dx}, \tag{107}$$

where r_1, r_2 are functions depending on x. Since equation $e3$ becomes

$$-12\, p_2\, u^2 (u e^{-2\,x-\frac{2}{u}} + u + 1) - 8\, \tfrac{dr_2}{dx} u^2 - \tfrac{d^2 p_1}{dx^2} u^2 + 6\, \tfrac{d^2 p_2}{dx^2} = 0, \tag{108}$$

it follows that $p_2 = 0$. Therefore, equation $e4$ can be written as

$$\begin{array}{l} \left(r_1\, u^6 + 2\, r_1\, u^5 + 3\, r_2\, u^4 + \tfrac{dp_1}{dx}\, u^4 - p_1\, u^4 + 2\, r_2\, u^3\right)\, e^{-2\,x-\frac{2}{u}} \\ +r_1\, u^6 + 3\, r_2\, u^4 + \tfrac{dp_1}{dx}\, u^4 + 2\, r_2\, u^3 + \tfrac{d^2 r_1}{dx^2}\, u^3 + \tfrac{dp_1}{dx}\, u^3 + \tfrac{d^2 r_2}{dx^2}\, u = 0. \end{array} \tag{109}$$

We finally deduce that $r_1 = 0, r_2 = 0$ and $p_1 = 0$. By (107), $r = 0$ and by (105), $p = 0$. Therefore, equation (79) has no Lie point symmetries.

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