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Kazhdan Constants and Matrix Coefficients of $\mathrm{Sp}(n, \mathbf{R})$

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Abstract. An infinitesimal Kazhdan constant of $\mathrm{Sp}(2, \mathbf{R})$ is computed. The methods used to prove this can also be employed to determine a quantitative estimate of the asymptotics of the matrix coefficients of $\mathrm{Sp}(n, \mathbf{R})$ in an elementary manner. An application of the result gives explicit Kazhdan constants for $\mathrm{Sp}(n, \mathbf{R})$, $n \geq 2$.

1. Introduction

A locally compact group G has Kazhdan's property T, if for a compact subset $Q \subset G$ and an $\varepsilon > 0$, every unitary representation π which has a (Q, ε)-invariant vector, i. e. a vector $\xi \in H_\pi$ such that $\|\pi(g)\xi - \xi\| < \varepsilon \|\xi\|$ for all $g \in Q$, has in fact a nonzero invariant vector. If such (Q, ε) for a group exists it is called a Kazhdan pair. This group theoretic property introduced in [9] has remarkable applications, for an account see [5] and [12].

In this paper, by "representation" we shall always mean "unitary representation". Let G be a connected Lie group. If π is a representation of G, a vector $\xi \in H_\pi$ is called a C^∞-vector if $g \mapsto \langle \pi(g)\xi, \eta \rangle$ is a C^∞-function for all $\eta \in H_\pi$, cf. for example [14]. The space of C^∞-vectors is denoted by H_π^∞. Let K be a maximal compact subgroup of G. A vector $\xi \in H_\pi$ is K-finite if the linear span of $\pi(K)\xi$ is finite-dimensional. We denote by $H_{\pi,K}^\infty$ the space of K-finite, C^∞-vectors in H_π.

Let $X_1, \ldots, X_m$ be a basis of the Lie algebra of G, then $\Delta = -\sum_{k=1}^m X_k^2$ denotes the Laplacian. If π is a representation of G, let $d\pi$ denote the derived representation of the Lie algebra. It can be extended to the universal enveloping algebra.

In [1, Theorem 3.10], it was shown that property T for a connected Lie group G, is equivalent to the existence of an $\varepsilon > 0$ such that

$$\langle d\pi(\Delta)\xi, \xi \rangle \geq \varepsilon \|\xi\|^2$$

for every $\xi \in H_\pi^\infty$ and every π without nonzero fixed vector.

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In [2, page 94], it was shown that restriction to the space $H_{\pi,K}^{\infty}$ is possible, namely:

Theorem 1.1. *The connected Lie group G has property T if and only if there exists a constant $\varepsilon > 0$ such that*

$$\inf\left\{\langle d\pi\left(\Delta\right)\xi,\xi\rangle : \xi \in H_{\pi,K}^{\infty}, \|\xi\| = 1\right\} \geq \varepsilon$$

for any unitary representation π of G without nonzero fixed vector.

We define the infinitesimal Kazhdan constant as

$$\kappa_K\left(\Delta, G\right) = \inf\left\{\langle d\pi\left(\Delta\right)\xi,\xi\rangle : \xi \in H_{\pi,K}^{\infty}, \|\xi\| = 1, \pi \not> 1\right\}.$$

The symplectic group $\mathrm{Sp}\left(n,\mathbf{R}\right) \subset \mathrm{GL}\left(2n,\mathbf{R}\right)$ is the group of isometries of the skew symmetric bilinear form induced by

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

where I_n is the $n \times n$ identity matrix. So

$$\mathrm{Sp}\left(n,\mathbf{R}\right) = \left\{g \in \mathrm{GL}\left(2n,\mathbf{R}\right) : g^T J g = J\right\}.$$

In the following $G = \mathrm{Sp}\left(n,\mathbf{R}\right)$ and $K = \mathrm{Sp}\left(n,\mathbf{R}\right) \cap \mathrm{SO}\left(2n\right)$ is the standard maximal compact subgroup of G.

Let π be a strongly continuous representation of G on a Hilbert space H_π. A vector $\xi \in H_\pi$ is called K-finite if the linear span of the set $\pi\left(K\right)\xi$ in H_π is finite-dimensional. Denote this dimension by $\delta\left(\xi\right) = \dim\langle\pi\left(K\right)\xi\rangle$.

Theorem 1.2. *For every $\mathrm{Sp}\left(2,\mathbf{R}\right) \cap \mathrm{SO}\left(4\right)$-finite unit C^∞-vector η and every representation π of $\mathrm{Sp}\left(2,\mathbf{R}\right)$ without nonzero invariant vectors*

$$\langle d\pi\left(\Delta\right)\eta,\eta\rangle \geq \frac{1}{4\pi}\sup_{0<\vartheta<\pi/2}\frac{\left(\sin\left(2\vartheta\right)\right)^2}{\vartheta} > 0.11532$$

for a suitable Laplacian Δ on $\mathrm{Sp}\left(2,\mathbf{R}\right)$, described after Theorem 3.5.

By Theorem 1.1, this implies that $\mathrm{Sp}\left(2,\mathbf{R}\right)$ has Kazhdan's property T which was shown for any local field in [4] and [13] and with an elementary proof in [3].

The group G can be decomposed as $G = KAK$, where

$$A = \left\{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a = \mathrm{diag}\left(a_1,\ldots,a_n\right)\right\},$$

the subgroup of the diagonal matrices in G. In fact, the matrices in A in the decomposition can be chosen more specially as $G = KA^+K$, where

$$A^+ = \left\{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : \begin{array}{c} a = \mathrm{diag}\left(a_1,\ldots,a_n\right), \\ a_1 \geq a_2 \geq \ldots \geq a_n \geq 1 \end{array}\right\}.$$

This can be achieved by suitable conjugation of an element of A by permutation matrices contained in K.

The asymptotics of the matrix coefficients will be given for the dense subspace of K-finite vectors of a representation π.

The quantitative estimate of the asymptotic of matrix coefficients will be given in terms of the Harish-Chandra function Ξ defined by

$$\Xi \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \frac{1}{2\pi} a^{-1} \int_0^{2\pi} \left| a^{-4} (\cos \vartheta)^2 + (\sin \vartheta)^2 \right|^{-1/2} d\vartheta,$$

cf. for example [7, page 215].

Let $g \in G$ with the decomposition $g = k_1 h k_2$, $k_1, k_2 \in K$,

$$h = \operatorname{diag}\left(a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1}\right) \in A^+,$$

then define

$$\Psi(g) = \Xi \begin{pmatrix} \sqrt{a_1 a_2} & 0 \\ 0 & \sqrt{a_1 a_2}^{-1} \end{pmatrix}.$$

The next theorem gives a quantitative estimate for the asymptotics of matrix coefficients.

Theorem 1.3. *Let π be a strongly continuous representation of* $\mathrm{Sp}(n, \mathbf{R})$, *$n \geq 2$, without nonzero invariant vectors, then*

$$|\varphi_{\xi,\eta}(g)| \leq \|\xi\| \, \|\eta\| \sqrt{\delta(\xi)\, \delta(\eta)}\, \Psi(g)$$

for two K-finite vectors $\xi, \eta \in H_\pi$, where $\varphi_{\xi,\eta}(g) = \langle \pi(g)\xi, \eta\rangle$.

Here the main application of this theorem is the proof of Kazhdan's property T of $\mathrm{Sp}(n, \mathbf{R})$, $n \geq 2$, with an explicit Kazhdan pair.

Theorem 1.4. *Let $0 < \delta < 1$, $\varepsilon = 0.32 \times \sqrt{2\delta}$, and $Q = \Psi^{-1}([1-\delta, 1])$, then (Q, ε) is a Kazhdan pair of* $\mathrm{Sp}(n, \mathbf{R})$.

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2. Preliminaries

A set in the dual space of $S^2(\mathbf{R}^2)$ is determined, where $S^2(\mathbf{R}^2)$ is identified with the vector space of the symmetric 2×2-matrices. This set will be important for the computation of an explicit estimate of the infinitesimal Kazhdan constant of $\mathrm{Sp}(2, \mathbf{R})$ in Section 3 and for the determination of an explicit quantitative estimate of the asymptotics of matrix coefficients of $\mathrm{Sp}(n, \mathbf{R})$ in Section 5. The asymptotics will be employed in the last section to obtain a Kazhdan pair for $\mathrm{Sp}(n, \mathbf{R})$.

For $\mathrm{SL}(3, \mathbf{R})$ M. B. Bekka and M. Mayer in [2] have determined a lower bound of the infinitesimal Kazhdan constant associated with a Laplacian.

Let π be a representation of $\mathrm{Sp}(2, \mathbf{R})$ on H_π. The strategy for establishing the estimates consists in considering the restriction of π to $\mathrm{SL}(2, \mathbf{R}) \ltimes S^2(\mathbf{R}^2)$.

There is a spectral measure on the dual group $\widehat{N}$ corresponding to $\pi|_N$ where $N = S^2(\mathbf{R}^2)$ is an abelian subgroup. The main problem here will be to find a set $W \subset \widehat{N}$ of which the spectral measure can be computed and estimated under the action of a suitably defined one parameter subgroup.

The subgroup

$$P = \left\{ \begin{pmatrix} a & b \\ 0 & a^{T^{-1}} \end{pmatrix} : a \in \mathrm{SL}(2,\mathbf{R}), ab^T = ba^T \right\} \cong \mathrm{SL}(2,\mathbf{R}) \ltimes S^2\left(\mathbf{R}^2\right)$$

will be considered. If ξ is a vector fixed by the subgroup

$$N = \left\{ \begin{pmatrix} 1 & 0 & x & y \\ 0 & 1 & y & z \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbf{R} \right\} \cong \mathbf{R}^3,$$

then ξ is a fixed vector of $\mathrm{Sp}(2,\mathbf{R})$, cf. for example [10, page 88].

So it can be supposed that π has no nonzero N-invariant vector. Let E be the spectral measure of $\widehat{N}$. Then $\pi|_N = \int_{\widehat{N}} \chi \, dE(\chi)$ and $E(\{0\}) = 0$. For Borel sets $W \subset \widehat{N}$ we have $E(a \cdot W) = \pi(a) E(W) \pi(a)^{-1}$ for all $a \in \mathrm{SL}(2,\mathbf{R})$.

Let ρ denote the action of $\mathrm{SL}(2,\mathbf{R})$ on $S^2(\mathbf{R}^2)$ by $\rho(a) b = aba^T$. We have that $a^T = \omega a^{-1} \omega^{-1}$ for $\omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $a \in \mathrm{SL}(2,\mathbf{R})$. So the dual operation on $\widehat{N}$ is equivalent to the usual operation ρ since $\mathrm{tr}\left(ba^T ca\right) = \mathrm{tr}\left(aba^T c\right)$.

The following basis

$$s_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, s_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, s_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

of $S^2(\mathbf{R}^2)$ is chosen. The isomorphism

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto xs_1 + ys_2 + zs_3 = \begin{pmatrix} x+y & z \\ z & x-y \end{pmatrix}$$

yields an identification between $\widehat{N} \cong N \cong S^2(\mathbf{R}^2)$ and $\mathbf{R}^3$. The spectral measure E is now considered to be defined on $\mathbf{R}^3$.

For an angle $0 < \vartheta < \pi$ and $h \in \mathbf{R}$ define

$$S_h^+(\vartheta) = \left\{ \begin{pmatrix} x \\ hx + y\cos\beta \\ y \sin\beta \end{pmatrix} : x \in \mathbf{R}, y > 0, -\vartheta < \beta \leq \vartheta \right\}.$$

For $0 < \vartheta < \frac{\pi}{2}$ one has

$$S_h^+(\vartheta) = \left\{ \begin{pmatrix} x \\ hx + y \\ y \tan\beta \end{pmatrix} : x \in \mathbf{R}, y > 0, -\vartheta < \beta \leq \vartheta \right\}.$$

Let $g_0(\alpha) = \begin{pmatrix} \cos(\alpha/2) & -\sin(\alpha/2) \\ \sin(\alpha/2) & \cos(\alpha/2) \end{pmatrix}$. Then in the chosen basis $g_0(\alpha)$ acts on $\mathbf{R}^3$ by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}.$$

Hence

$$g_0(\alpha) \cdot S_0^+(\vartheta) = \left\{ \begin{pmatrix} x \\ y\cos\beta \\ y\sin\beta \end{pmatrix} : x \in \mathbf{R}, y > 0, -\vartheta + \alpha < \beta \leq \vartheta + \alpha \right\}.$$

This implies that $g_0(2\vartheta) \cdot S_0^+(\vartheta)$ and $S_0^+(\vartheta)$ are disjoint.

Let ξ be a unit eigenvector of the image $\pi(K)$, then $\pi(g_0(\alpha))\xi = e^{in\alpha/2}\xi$ for an $n \in \mathbf{Z}$. As $g_0(\alpha) \in \mathrm{SL}(2, \mathbf{R})$,

$$\begin{aligned} \pi(g_0(\alpha)) E\left(S_0^+(\vartheta)\right)\xi &= \pi(g_0(\alpha)) E\left(S_0^+(\vartheta)\right) \pi(g_0(\alpha))^{-1} \pi(g_0(\alpha))\xi \\ &= E\left(g_0(\alpha) \cdot S_0^+(\vartheta)\right) \pi(g_0(\alpha))\xi \\ &= e^{in\alpha/2} E\left(g_0(\alpha) \cdot S_0^+(\vartheta)\right)\xi \end{aligned}$$

and so

$$\begin{aligned} \left\| E\left(S_0^+(\vartheta)\right)\xi \right\| &= \left\| \pi(g_0(\alpha)) E\left(S_0^+(\vartheta)\right)\xi \right\| \\ &= \left\| E\left(g_0(\alpha) \cdot S_0^+(\vartheta)\right)\xi \right\|. \end{aligned}$$

On the other hand $S_0^+(\vartheta)$ and $g_0(\alpha) \cdot S_0^+(\vartheta)$ are disjoint for $2\vartheta \leq \alpha \leq 2\pi - 2\vartheta$. Hence $E\left(S_0^+(\vartheta)\right) E\left(g_0(\alpha) \cdot S_0^+(\vartheta)\right) = 0$ and $E\left(S_0^+(\vartheta)\right)\xi$ is orthogonal to $E\left(g_0(\alpha) \cdot S_0^+(\vartheta)\right)\xi$.

Let now $n \geq 2$, $\vartheta = \pi/n$, and $\alpha_j = 2\pi j/n$ for $0 \leq j \leq n-1$, then

$$\mathbf{R}^3 \setminus \{0\} = \bigcup_{j=0}^{n-1} g_0(\alpha_j) \cdot S_0^+(\vartheta)$$

where the union is disjoint. So $\sum_{j=0}^{n-1} E\left(g_0(\alpha_j) \cdot S_0^+(\vartheta)\right) = \mathrm{id}_{H_\pi}$. This way ξ can be decomposed into vectors of equal length $\xi = \sum_{j=0}^{n-1} E\left(g_0(\alpha_j) \cdot S_0^+(\vartheta)\right)\xi$. If ξ is a unit vector

$$\left\| E\left(S_0^+(\vartheta)\right)\xi \right\|^2 = 1/n = \vartheta/\pi.$$

This equality can be extended first to all $\vartheta = r\pi$ with $r \in \mathbf{Q} \cap]0,1[$ and then to all $r \in]0,1[$. This proves the following.

Lemma 2.1. *For $0 < \vartheta < \pi$:*

$$\left\| E\left(S_0^+(\vartheta)\right)\xi \right\|^2 = \vartheta/\pi.$$

Let now

$$\begin{aligned} S_h^-(\vartheta) &= \left\{ \begin{pmatrix} x \\ hx + y\cos\beta \\ y\sin\beta \end{pmatrix} : x \in \mathbf{R}, y < 0, -\vartheta < \beta \leq \vartheta \right\} \\ &= g_0(\pi) \cdot S_{-h}^+(\vartheta), \end{aligned}$$

then $\left\|E\left(S_0^-(\vartheta)\right)\xi\right\|^2 = \left\|E\left(S_0^+(\vartheta)\right)\xi\right\|^2 = \vartheta/\pi$.

Let $S_0(\vartheta) = S_0^+(\vartheta) \cup S_0^-(\vartheta)$, $W^+(\vartheta) = S_1^+(\vartheta) \cap S_0^+(2\vartheta)$, $W^-(\vartheta) = S_1^-(\vartheta) \cap S_0^-(2\vartheta)$, and $W(\vartheta) = W^-(\vartheta) \cup W^+(\vartheta)$ for $0 < \vartheta < \pi/2$.

The determination of the spectral measure of $W(\vartheta)$ is a more difficult task. This will be done in the next section.

3. Kazhdan constants associated with a Laplacian

The next proposition is an important step in the determination of the infinitesimal Kazhdan constant associated with Δ.

Proposition 3.1. *For $0 < \vartheta < \pi/2$ and a unit K-eigenvector ξ the spectral measure is $\|E(W(\vartheta))\xi\|^2 = 2\vartheta/\pi$.*

The proof is postponed to Appendix A.

Now it will be investigated how $W(\vartheta)$ behaves under the action of the one parameter group $g_1(t) = \begin{pmatrix} \exp(t/2) & 0 \\ 0 & \exp(-t/2) \end{pmatrix}$. Then $g_1(t)$ acts on $S^2(\mathbf{R}^2)$ with the above basis by

$$\begin{pmatrix} \cosh t & \sinh t & 0 \\ \sinh t & \cosh t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence

$$\begin{aligned} g_1(t) \cdot S_1^+(\vartheta) &= \left\{ \begin{pmatrix} x \\ x+y \\ ye^t \tan\beta \end{pmatrix} : x \in \mathbf{R}, y > 0, -\vartheta < \beta \leq \vartheta \right\} \\ &= S_1^+\left(\arctan\left(e^t \tan\vartheta\right)\right) \end{aligned}$$

and

$$g_1(t) \cdot W^{\pm}(\vartheta) = S_1^{\pm}\left(\arctan\left(e^t \tan\vartheta\right)\right) \cap \left(g_1(t) \cdot S_0^{\pm}(2\vartheta)\right).$$

Here

$$\begin{aligned} & g_1(t) \cdot S_0^{\pm}(\vartheta) \\ = & \left\{ \begin{pmatrix} x\cosh t + y\cos\beta\sinh t \\ x\sinh t + y\cos\beta\cosh t \\ y\sin\beta \end{pmatrix} : x \in \mathbf{R}, \pm y > 0, -\vartheta < \beta \leq \vartheta \right\}. \end{aligned}$$

Since $x \in \mathbf{R}$ is arbitrary, replace x by $\frac{x - y\cos\beta\sinh t}{\cosh t}$. Then the first coordinate becomes x and the second $(x - y\cos\beta\sinh t)\tanh t + y\cos\beta\cosh t = x\tanh t + \frac{y\cos\beta}{\cosh t}$. So

$$\begin{aligned} g_1(t) \cdot S_0^{\pm}(\vartheta) &= \left\{ \begin{pmatrix} x \\ x\tanh t + \frac{y\cos\beta}{\cosh t} \\ y\sin\beta \end{pmatrix} : x \in \mathbf{R}, \pm y > 0, -\vartheta < \beta \leq \vartheta \right\} \\ &= S_{\tanh t}^{\pm}\left(\arctan\left(\cosh t \tan\vartheta\right)\right). \end{aligned}$$

The next proposition determines a ϑ_t dependent of t and ϑ such that $W(\vartheta_t)$ is contained in $g_1(t) \cdot W(\vartheta)$ giving in the corollary below a lower bound for the spectral measure of $W(\vartheta_t)$ as an immediate consequence.

Proposition 3.2. *For* $0 < \vartheta < \pi/2$ *and* $t > 0$ *holds* $g_1(t) \cdot W(\vartheta) \supseteq W(\arctan(e^t \tan\vartheta))$.

The proof is postponed to Appendix B.

Corollary 3.3. *For* $0 < \vartheta < \pi/2$ *and* $t > 0$,

$$\|E(g_1(t) \cdot W(\vartheta))\xi\|^2 \geq \frac{2}{\pi}\arctan\left(e^t \tan\vartheta\right).$$

The purpose of all this is to obtain an estimate of $\|d\pi(Y_1)\xi\|$, where $Y_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, for a smooth $\mathrm{SO}(2)$-finite unit vector ξ. Observe that $g_1(t) = \exp(tY_1)$.

Proposition 3.4. *Let* π *be a representation of* $\mathrm{SL}(2,\mathbf{R}) \ltimes S^2(\mathbf{R}^2)$ *without nonzero* $S^2(\mathbf{R}^2)$*-invariant vectors, then*

$$\|d\pi(Y_1)\xi\| \geq \frac{1}{2\sqrt{2\pi}}\frac{\sin(2\vartheta)}{\sqrt{\vartheta}}$$

for every smooth $\mathrm{SO}(2)$*-eigenvector* ξ *with* $\|\xi\| = 1$.

Proof. For ξ smooth of norm 1:

$$\begin{aligned} \|E(g_1(t) \cdot W(\vartheta))\xi\| &= \|\pi(g_1(t))E(W(\vartheta))\pi(g_1(-t))\xi\| \\ &= \|E(W(\vartheta))\pi(g_1(-t))\xi\|. \end{aligned}$$

Differentiating at $t = 0$ yields

$$\begin{aligned} & \frac{d}{dt}\|E(g_1(t) \cdot W(\vartheta))\xi\|^2\Big|_{t=0} \\ = \; & \frac{d}{dt}\|E(W(\vartheta))\pi(g_1(-t))\xi\|^2\Big|_{t=0} \\ = \; & -\langle d\pi(Y_1)\xi, E(W(\vartheta))\xi\rangle - \langle E(W(\vartheta))\xi, d\pi(Y_1)\xi\rangle. \end{aligned}$$

If f is a real function differentiable at 0 with $f(0) = 0$ and $f(x) \geq 0$ for $x \geq 0$, then $f'(0) \geq 0$. Together with Corollary 3.3 this implies

$$\begin{aligned} & \frac{d}{dt}\|E(g_1(t) \cdot W(\vartheta))\xi\|^2\Big|_{t=0} \\ \geq \; & \frac{2}{\pi}\frac{d}{dt}\arctan\left(e^t\tan\vartheta\right)\Big|_{t=0} = \frac{2}{\pi}\frac{1}{1+(\tan\vartheta)^2}\tan\vartheta \\ = \; & \frac{2}{\pi}(\cos\vartheta)^2\tan\vartheta = \frac{2}{\pi}\cos\vartheta\sin\vartheta = \frac{1}{\pi}\sin(2\vartheta). \end{aligned}$$

Hence

$$\begin{aligned} 2\|d\pi(Y_1)\xi\|\sqrt{\frac{2\vartheta}{\pi}} &\geq -\langle d\pi(Y_1)\xi, E(W(\vartheta))\xi\rangle - \langle E(W(\vartheta))\xi, d\pi(Y_1)\xi\rangle \\ &= \frac{d}{dt}\|E(g_1(t) \cdot W(\vartheta))\xi\|^2\Big|_{t=0} \\ &\geq \frac{1}{\pi}\sin(2\vartheta) \end{aligned}$$

and

$$\|d\pi(Y_1)\xi\| \geq \frac{1}{2\sqrt{2\pi}}\frac{\sin(2\vartheta)}{\sqrt{\vartheta}}$$

for every smooth K-eigenvector ξ of norm 1. ■

For $Y_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, conjugate to Y_1, the same equality holds. Together with $Y_0 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, the three elements Y_0, Y_1, Y_2 form a basis of the Lie algebra of $\mathrm{SL}(2, \mathbf{R})$ orthogonal with respect to the Killing form. The corresponding Casimir operator is $C = \frac{1}{2}(Y_1^2 + Y_2^2 - Y_0^2)$ and the corresponding Laplacian is $\Delta = -Y_1^2 - Y_2^2 - Y_0^2 = -2C - 2Y_0^2$.

Theorem 3.5. *Let π be a representation of $\mathrm{SL}(2, \mathbf{R}) \ltimes S^2(\mathbf{R}^2)$ without nonzero $S^2(\mathbf{R}^2)$-invariant vectors, then*

$$\langle d\pi(\Delta)\eta, \eta\rangle \geq \frac{1}{4\pi}\sup_{0<\vartheta<\pi/2}\frac{(\sin(2\vartheta))^2}{\vartheta}$$

for every smooth $\mathrm{SO}(2)$-finite unit vector η.

Proof. Let $\eta = \sum_{k=1}^{r}\xi_k$ be the orthogonal decomposition of η into $d\pi(Y_0)$-eigenvectors, then the observation that C commutes with Y_0 implies

$$\begin{aligned}
& \langle d\pi(\Delta)\eta, \eta\rangle \\
= & \left\langle d\pi\left(-2Y_0^2\right)\eta, \eta\right\rangle + \langle d\pi(-2C)\eta, \eta\rangle \\
= & \sum_{k=1}^{r}\left\langle d\pi\left(-2Y_0^2\right)\xi_k, \xi_k\right\rangle + \langle d\pi(-2C)\xi_k, \xi_k\rangle = \sum_{k=1}^{r}\langle d\pi(\Delta)\xi_k, \xi_k\rangle \\
\geq & \sum_{k=1}^{r} 2\left(\frac{1}{2\sqrt{2\pi}}\frac{\sin(2\vartheta)}{\sqrt{\vartheta}}\right)^2\|\xi_k\|^2 = \frac{1}{4\pi}\frac{\sin^2(2\vartheta)}{\vartheta}\|\eta\|^2.
\end{aligned}$$

■

The following basis of the Lie algebra $\mathrm{sp}(2, \mathbf{R})$ will be considered which contains elements corresponding to Y_1 and Y_2. The Lie algebra $\mathrm{sp}(2, \mathbf{R})$ admits a Cartan decomposition into $\mathrm{sp}(2, \mathbf{R}) = k \oplus p$ where $k = \mathrm{so}(4, \mathbf{R}) \cap \mathrm{sp}(2, \mathbf{R})$ and $p = S^2(\mathbf{R}^2) \cap \mathrm{sp}(2, \mathbf{R})$. With

$$\begin{aligned}
X_0 &= \frac{1}{2}\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, X_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
X_2 &= \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, X_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}
\end{aligned}$$

and

$$X_4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, X_5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$X_6 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, X_7 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$X_8 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, X_9 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

the Casimir operator satisfies

$$C = 2\left(X_1^2 + X_2^2 + X_3^2 + X_4^2 + X_5^2 + X_6^2 - \left(X_7^2 + X_8^2 + X_9^2 + X_0^2\right)\right).$$

The elements X_0, X_7, X_8, X_9 form a basis of k and $X_1, X_2, X_3, X_4, X_5, X_6$ form a basis of p. Let η be a smooth $\mathrm{Sp}\,(2, \mathbf{R}) \cap \mathrm{SO}\,(4)$-finite unit vector, then

$$\langle d\pi\,(\Delta)\,\eta, \eta\rangle \geq \langle d\pi\,(\Delta_1)\,\eta, \eta\rangle\,,$$

with $\Delta_1 = -X_0^2 - X_1^2 - X_2^2 = -2X_0^2 - 2C_1$ where $C_1 = \frac{1}{2}\left(-X_0^2 + X_1^2 + X_2^2\right)$ is the Casimir operator of a Lie subalgebra isomorphic to $\mathrm{sl}\,(2, \mathbf{R})$. By Theorem 3.5 this shows that $\langle d\pi\,(\Delta)\,\eta, \eta\rangle \geq (4\pi)^{-1}\,(\sin\,(2\vartheta))^2\,/\vartheta$ for every smooth $\mathrm{Sp}\,(2, \mathbf{R}) \cap \mathrm{SO}\,(4)$-finite unit vector η of a representation π without nonzero $S^2\,(\mathbf{R}^2)$-invariant vectors.

To conclude the proof of Theorem 1.2 let π be a representation of $\mathrm{Sp}\,(2, \mathbf{R})$ without nonzero invariant vector. If the restriction to $S^2\,(\mathbf{R}^2)$ would have a nonzero invariant vector this would imply the contradiction that $\mathrm{Sp}\,(2, \mathbf{R})$ would have a nonzero invariant vector by an argument similar to the one for $\mathrm{SL}\,(2, \mathbf{R})$ in [10, page 88]. For more details see also the proof of Theorem 4.3. In the notation used there a nonzero $S^2\,(\mathbf{R}^2)$-invariant vector would imply a nonzero vector invariant under $G_{1,1}$, $G_{1,2}$, and $G_{2,2}$ (see next section). But these three subgroups together generate $\mathrm{Sp}\,(2, \mathbf{R})$.

By Theorem 3.5 now only the maximum of the function $\vartheta \mapsto (\sin\,(2\vartheta))^2\,/\vartheta$ has to be considered which is obtained at approximately $\vartheta \approx 0.582781$ so

$$\frac{1}{4\pi} \sup_{0<\vartheta<\pi/2} \frac{(\sin\,(2\vartheta))^2}{\vartheta} \approx 0.115325 > 0.11532.$$

4. Vanishing of matrix coefficients

In this section the qualitative behavior of the matrix coefficients of $\mathrm{Sp}\,(n, \mathbf{R})$ will be analyzed in an elementary manner. The case $\mathrm{SL}\,(n, \mathbf{R})$ was done in [7].

The following notion will be used.

Let X be a Hausdorff topological space. A complex valued function f is said to vanish at infinity if for every $\varepsilon > 0$ there exists a compact set $C \subset X$ such that $|f(x)| < \varepsilon$ for all $x \in X \setminus C$.

A sequence goes to ∞ in X if it has no limit point in X. If X is second countable a complex valued function f vanishes at ∞ if $\lim_{m\to\infty} f(x) = 0$ for every sequence $(x_m)_{m\in\mathbf{N}}$ in X going to ∞. This will be used for $\mathrm{Sp}(n,\mathbf{R})$.

The following is easily deduced from the fact that $\mathrm{Sp}(n,\mathbf{R}) = KA^+K$ and $\pi(K)$ is compact.

Lemma 4.1. *Let π be a representation of $\mathrm{Sp}(n,\mathbf{R})$ on H_π such that the matrix coefficients do not vanish at infinity; then there are $\xi, \eta \in H_\pi$ and a sequence $(g_m)_{m\in\mathbf{N}}$ with $g_m \in A^+$ and $g_m \to \infty$ such that $(\langle \pi(g_m)\xi, \eta\rangle)_{m\in\mathbf{N}}$ does not converge to 0.*

The subgroup

$$N_1 = \left\{ \begin{pmatrix} 1 & 0 & x & y^T \\ 0 & I & y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & I \end{pmatrix} : x \in \mathbf{R}, y \in \mathbf{R}^{n-1} \right\}$$

of $\mathrm{Sp}(n,\mathbf{R})$ will be important. The following proposition shows that a representation of $\mathrm{Sp}(n,\mathbf{R})$ which has a matrix coefficient that does not vanish at ∞ has in fact a nonzero vector which is N_1-invariant. The next theorem will show this vector is in fact invariant by proving that some specific subgroups generate $\mathrm{Sp}(n,\mathbf{R})$.

Proposition 4.2. *Let π be a strongly continuous unitary representation of $\mathrm{Sp}(n,\mathbf{R})$ on H_π and suppose that a matrix coefficient of π does not vanish at ∞, then there is a nonzero N_1-invariant vector.*

Proof. By Lemma 4.1 there is a sequence $(g_m)_{m\in\mathbf{N}}$ which goes to infinity with $g_m \in A^+$ and a $\xi \in H_\pi$ such that the sequence $(\pi(g_m)\xi)_{m\in\mathbf{N}}$ does not converge weakly to 0. After passing to a subsequence it can be assumed that $(\pi(g_m)\xi)_{m\in\mathbf{N}}$ converges in the weak topology to $\eta \neq 0$ since $\pi(g_m)$ is unitary and the unit ball is compact in the weak topology.

Let $g_m = \begin{pmatrix} a_m & 0 \\ 0 & a_m^{-1} \end{pmatrix}$ with

$$a_m = \mathrm{diag}(a_{m,1}, \ldots, a_{m,n}), a_{m,1} \geq \ldots \geq a_{m,n} \geq 1.$$

As $a_m \to \infty$, we have $a_{m,1}^{-1} \to 0$. The elements $g_m^{-1} h g_m$ converge to the identity I_n for $m \to \infty$ and $h \in N_1$ as

$$\begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} I_n & b \\ 0 & I_n \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} I_n & a^{-1}ba^{-1} \\ 0 & I_n \end{pmatrix}$$

with a diagonal and $b \in S^2(\mathbf{R}^n)$,

$$\begin{pmatrix} a_{m,1}^{-1} & 0 \\ 0 & d_m^{-1} \end{pmatrix} \begin{pmatrix} x & y^T \\ y & 0 \end{pmatrix} \begin{pmatrix} a_{m,1}^{-1} & 0 \\ 0 & d_m^{-1} \end{pmatrix} = \begin{pmatrix} a_{m,1}^{-2}x & a_{m,1}^{-1}y^T d_m^{-1} \\ a_{m,1}^{-1} d_m^{-1} y & 0 \end{pmatrix}$$

with $d_m = \operatorname{diag}(a_{m,2}, \ldots, a_{m,n})$ and so $a_{m,1}^{-2} x \to 0$ and $a_{m,1}^{-1} d_m^{-1} y \to 0$ because $a_{m,j} \geq 1$ for all j.

Next it is proven that $\eta \in H_\pi$ is N_1-invariant. Let $h \in N_1$ with $h = \begin{pmatrix} I & b \\ 0 & I \end{pmatrix}$, then

$$\begin{aligned}
|\langle \pi(h)\eta - \eta, \zeta\rangle| &= \lim_{m\to\infty} |\langle \pi(h)\pi(g_m)\xi - \pi(g_m)\xi, \zeta\rangle| \\
&= \lim_{m\to\infty} \left|\left\langle \pi(g_m)\left(\pi\left(g_m^{-1} h g_m\right)\xi - \xi\right), \zeta\right\rangle\right| \\
&\leq \lim_{m\to\infty} \left\|\pi(g_m)\left(\pi\left(g_m^{-1} h g_m\right)\xi - \xi\right)\right\| \|\zeta\| \\
&= \lim_{m\to\infty} \left\|\pi\left(g_m^{-1} h g_m\right)\xi - \xi\right\| \|\zeta\| = 0
\end{aligned}$$

for all $\zeta \in H_\pi$ because of the strong continuity of π. So $\pi(h)\eta = \eta$. ■

With the help of the last proposition the following yields an elementary proof that the matrix coefficients of $\mathrm{Sp}(n, \mathbf{R})$ vanish at infinity.

Let $E_{j,k} \in \mathbf{R}^{n\times n}$ be the matrix which is zero in every entry except for the one at (j,k) which is 1. Let $\rho_{j,k} : \mathrm{SL}(2,\mathbf{R}) \to \mathrm{Sp}(n,\mathbf{R})$ be the homomorphisms

$$\rho_{j,k}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} I_n + (a-1)(E_{j,j} + E_{k,k}) & b(E_{j,k} + E_{k,j}) \\ c(E_{j,k} + E_{k,j}) & I_n + (d-1)(E_{j,j} + E_{k,k}) \end{pmatrix}$$

for $j,k = 1, \ldots, n$, $j \neq k$,

$$\rho_{k,k}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} I_n + (a-1)E_{k,k} & bE_{k,k} \\ cE_{k,k} & I_n + (d-1)E_{k,k} \end{pmatrix}$$

for $k = 1, \ldots, n$, and $\tilde{\rho}_{j,k} : \mathrm{SL}(2,\mathbf{R}) \to \mathrm{SL}(n,\mathbf{R})$ the homomorphisms

$$\tilde{\rho}_{j,k}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = I_n + (a-1)E_{j,j} + bE_{j,k} + cE_{k,j} + (d-1)E_{k,k}$$

for $j,k = 1, \ldots, n$. Let

$$\begin{aligned}
G_{j,k} &= \rho_{j,k}(\mathrm{SL}(2,\mathbf{R})), \\
\tilde{G}_{j,k} &= \left\{ \begin{pmatrix} \tilde{\rho}_{j,k}(g) & 0 \\ 0 & \left(\tilde{\rho}_{j,k}(g)^T\right)^{-1} \end{pmatrix} : g \in \mathrm{SL}(2,\mathbf{R}) \right\}
\end{aligned}$$

for $j,k = 1, \ldots, n$ be the corresponding subgroups.

The first proof of the following was given in [6].

Theorem 4.3. *Let π be a unitary representation of $\mathrm{Sp}(n,\mathbf{R})$ which does not contain the trivial representation, then the matrix coefficients of π vanish at infinity.*

Proof. Assume by contradiction that at least one coefficient of π does not vanish at infinity.

For $n = 1$ a vector which is N_1-invariant is also invariant for $\mathrm{Sp}(n,\mathbf{R}) = \mathrm{SL}(2,\mathbf{R})$, see for example [10, page 88].

Now suppose $n \geq 2$, then by Lemma 4.2 there is an N_1-invariant ξ. The case $n = 1$ implies that this vector is also $G_{1,k}$-invariant for $k = 1, \ldots, n$. Let G be the subgroup of $\mathrm{Sp}(n, \mathbf{R})$ generated by these subgroups. It will be shown that $G = \mathrm{Sp}(n, \mathbf{R})$.

Let

$$\omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \omega_{j,k} = \rho_{j,k}(\omega), \tilde{\omega}_{j,k} = \tilde{\rho}_{j,k}(\omega)$$

for $j, k = 1, \ldots, n$. Then $\omega_{1,k}\rho_{1,1}(g)\,\omega_{1,k}^{-1} = \left(\rho_{k,k}(g)^T\right)^{-1}$ for $k = 2, \ldots, n$. This implies $G_{k,k} \subset G$. Since $\omega_{1,1}\rho_{1,k}(g)\,\omega_{1,1}^{-1} = \left(\tilde{\rho}_{1,k}(g)^T\right)^{-1}$ for $k = 2, \ldots, n$, we have $\tilde{G}_{1,k} \subset G$. Also $\tilde{\omega}_{1,j}\tilde{\rho}_{1,k}(g)\,\tilde{\omega}_{1,j}^{-1} = \tilde{\rho}_{j,k}(g)$ for $j, k = 2, \ldots, n$, $j \neq k$ which gives $\tilde{G}_{j,k} \subset G$. Finally $\tilde{\omega}_{1,j}\rho_{1,k}(g)\,\tilde{\omega}_{1,j}^{-1} = \rho_{j,k}(g)$ for $j, k = 2, \ldots, n$, $j \neq k$ and $G_{j,k} \subset G$.

This implies $G = \mathrm{Sp}(n, \mathbf{R})$, see [8, Section 6.9]. So ξ is G-invariant. ■

5. An estimate for the decay of the matrix coefficients

Before studying the decay of the matrix coefficients of $\mathrm{Sp}(n, \mathbf{R})$ the matrix coefficients of the semi-direct product $\mathrm{SL}(2, \mathbf{R}) \ltimes S^2(\mathbf{R}^2)$ are considered. A set in the unitary dual $\widehat{S^2(\mathbf{R}^2)}$ of the additive group of $S^2(\mathbf{R}^2)$ will help to determine an estimate for the matrix coefficients of the representations of $\mathrm{SL}(2, \mathbf{R}) \ltimes S^2(\mathbf{R}^2)$ without nonzero $S^2(\mathbf{R}^2)$-invariant vectors.

Theorem 5.1. *Let π be a representation of $\mathrm{SL}(2, \mathbf{R}) \ltimes S^2(\mathbf{R}^2)$ on H_π without nonzero $S^2(\mathbf{R}^2)$-invariant vectors, then*

$$|\varphi_{\xi,\eta}(g_0(\alpha)\,g_1(t)\,g_0(\beta))| = |\langle \pi(g_0(\alpha)\,g_1(t)\,g_0(\beta))\,\xi, \eta\rangle| \leq c_{\xi,\eta}e^{-t/2}$$

for $\xi, \eta \in H_{\pi,K}$ and $c_{\xi,\eta}$ is a constant depending only on ξ and η.

Proof. Let $\Phi : \mathbf{R}^3 \to \widehat{S^2(\mathbf{R}^2)}$ be the isomorphism

$$(\Phi(x, y, z))(u) = \exp\left(i\,\mathrm{tr}\left(\begin{pmatrix} x+z & y \\ y & z \end{pmatrix} u\right)\right)$$

for $u \in S^2(\mathbf{R}^2)$. We identify $\mathbf{R}^3$ with $\widehat{S^2(\mathbf{R}^2)}$ via Φ. Let $s > 1$ and

$$X_s = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbf{R}^3 : s^{-2} < y^2 + z^2 < s^2 \right\},$$

then $\bigcup_{s>1} X_s = \mathbf{R}^3 \backslash \{0\}$. As π has no nonzero $S^2(\mathbf{R}^2)$-invariant vectors, $E(X_s)\,\eta$ converges to η for $\eta \in H_\pi$ where E is the spectral measure associated to $\pi|_{S^2(\mathbf{R}^2)}$. So it is enough to prove the statement for eigenvectors $\xi, \eta \in E(X_s)\,H_\pi$ of $\pi(K)$ as the matrix coefficients are sesquilinear in ξ and η.

Let $t > 2\ln s$, then

$$\begin{aligned}
\varphi_{\xi,\eta}(g_1(t)) &= \langle \pi(g_1(t))\,\xi, \eta\rangle = \langle \pi(g_1(t))\,E(X_s)\,\xi, E(X_s)\,\eta\rangle \\
&= \langle E(g_1(t)\cdot X_s)\,\pi(g_1(t))\,\xi, E(X_s)\,\eta\rangle \\
&= \langle \pi(g_1(t))\,\xi, E((g_1(t)\cdot X_s)\cap X_s)\,\eta\rangle .
\end{aligned}$$

By the Cauchy–Schwarz inequality:

$$|\varphi_{\xi,\eta}(g_1(t))| \le \|\xi\|\,\|E((g_1(t)\cdot X_s)\cap X_s)\,\eta\| .$$

The one-parameter subgroup generated by $g_1(t)$ operates in the following way on $\mathbf{R}^3$. Then

$$\begin{aligned}
g_1(-t)\begin{pmatrix} x+z & y \\ y & z \end{pmatrix} g_1(-t) &= \begin{pmatrix} e^{-t}(x+z) & y \\ y & e^t z \end{pmatrix} \\
&= \begin{pmatrix} e^{-t}x - 2z\sinh t + e^t z & y \\ y & e^t z \end{pmatrix}
\end{aligned}$$

so by the isomorphism Φ

$$g_1(t)\cdot\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} e^{-t}x - 2z\sinh t \\ y \\ e^t z \end{pmatrix}.$$

Hence

$$g_1(t)\cdot X_s = \left\{\begin{pmatrix} x \\ y \\ z \end{pmatrix} : s^{-2} < y^2 + e^{-2t}z^2 < s^2\right\} \subset \left\{\begin{pmatrix} x \\ y \\ z \end{pmatrix} : |z| < e^t s\right\}.$$

As

$$(g_1(t)\cdot X_s)\cap X_s \subset \left\{\begin{pmatrix} x \\ y \\ z \end{pmatrix} : y^2+z^2 > s^{-2}, |z| < e^t s\right\}$$

we have $|z|\left(\sqrt{y^2+z^2}\right)^{-1} < e^t s\,(s^{-1})^{-1} = e^t s^2$. Now $z = r\cos\beta$ with $r = \sqrt{y^2+z^2}$ where $|\cos\beta| < e^t s^2$. Let $\vartheta = \arccos(e^t s^2)$, then $-\pi < \beta < \pi$ if and only if $-\pi < \beta < -\vartheta$ or $\vartheta < \beta < \pi$. By definition of $S_h(\vartheta)$ and $W(\vartheta)$, cf. Section 3,

$$\begin{aligned}
|\varphi_{\xi,\eta}(g_1(t))| &\le \sqrt{1 - \|E(S_1(\vartheta))\,\xi\|^2} \le \sqrt{1 - \|E(W(\vartheta))\,\xi\|^2} \\
&= \sqrt{1 - \frac{2}{\pi}\arccos(e^{-t}s^2)} = \sqrt{\frac{2}{\pi}\arcsin(e^{-t}s^2)} \\
&\le s e^{-t/2}.
\end{aligned}$$

Finally for $t \le 2\ln s$, $|\varphi_{\xi,\eta}(g_1(t))| \le 1 \le s e^{-t/2}$ holds. ∎

There is an estimate for the matrix coefficients of the regular representation of $\mathrm{SL}(2,\mathbf{R})$ which depends on the Harish-Chandra Ξ function, cf. for example [7, page 217]. For $t \in \mathbf{R}$:

$$\Xi\left(g_1(t)\right) = (2\pi)^{-1} e^{-t/2} \int_0^{2\pi} \left| e^{-2t} (\cos\vartheta)^2 + (\sin\vartheta)^2 \right|^{-1/2} d\vartheta.$$

Theorem 5.2. *Let π be a representation of $\mathrm{SL}(2,\mathbf{R}) \ltimes S^2(\mathbf{R}^2)$ without nonzero $S^2(\mathbf{R}^2)$-invariant vectors, then for the matrix coefficient of any two vectors $\xi, \eta \in H_\pi$ there is the pointwise estimate*

$$\left|\varphi_{\xi,\eta}\left(g_1(t)\right)\right| \leq \|\xi\| \, \|\eta\| \sqrt{\dim\langle\pi(K)\xi\rangle \dim\langle\pi(K)\eta\rangle}\,\Xi\left(g_1(t)\right),$$

where $\langle\pi(K)\xi\rangle$ is the subspace spanned by the orbit $\pi(K)\xi$.

The proof can be copied word by word form [7, page 226] replacing the corresponding statement by Theorem 5.1.

Hence it is possible to prove Theorem 1.3, which describes the asymptotics of matrix coefficients of $\mathrm{Sp}(n,\mathbf{R})$.

Proof of Theorem 1.3. Consider the subgroups

$$\begin{aligned} \tilde{G}_{1,2} &= \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & {a^T}^{-1} & 0 \\ 0 & 0 & 0 & I \end{pmatrix} : a \in \mathrm{SL}(2,\mathbf{R}) \right\}, \\ P_{1,2} &= \left\{ \begin{pmatrix} a & 0 & b & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & {a^T}^{-1} & 0 \\ 0 & 0 & 0 & I \end{pmatrix} : a \in \mathrm{SL}(2,\mathbf{R}),\, ab^T = ba^T \right\} \end{aligned}$$

isomorphic to $\mathrm{SL}(2,\mathbf{R})$ and $\mathrm{SL}(2,\mathbf{R}) \ltimes S^2(\mathbf{R}^2)$ respectively.

Let π be a representation of $\mathrm{Sp}(n,\mathbf{R})$ without nonzero invariant vectors, then the representation $\pi_{1,2} = \pi|_{P_{1,2}}$ also has no nonzero $S^2(\mathbf{R}^2)$-invariant vectors, as the matrix coefficients of π and hence the ones of $\pi_{1,2}$ vanish at ∞, as shown in Theorem 4.3.

To $\tilde{G}_{1,2} \subset \mathrm{Sp}(n,\mathbf{R})$ the estimate of Theorem 5.2 is applied. Let

$$K_{1,2} = \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & I \end{pmatrix} : a \in \mathrm{SO}(2,\mathbf{R}) \right\}$$

be a maximal compact subgroup of $\tilde{G}_{1,2}$.

Let $\omega = \begin{pmatrix} I - E_{2,2} & -E_{2,2} \\ E_{2,2} & I - E_{2,2} \end{pmatrix} \in K$, then

$$\omega g \omega^{-1} = \operatorname{diag}\left(a_1, a_2^{-1}, a_3, \ldots, a_n, a_1^{-1}, a_2, a_3^{-1}, \ldots, a_n^{-1}\right).$$

Now

$$\begin{pmatrix} \sqrt{a_1 a_2} & 0 \\ 0 & \sqrt{a_1 a_2}^{-1} \end{pmatrix} \begin{pmatrix} \sqrt{a_1/a_2} & 0 \\ 0 & \sqrt{a_1/a_2} \end{pmatrix} = \begin{pmatrix} a_1 & 0 \\ 0 & a_2^{-1} \end{pmatrix}.$$

Let

$$\begin{aligned}\tilde{g} &= \operatorname{diag}\left(\sqrt{a_1 a_2}, \sqrt{a_1 a_2}^{-1}, 1, \dots, 1, \sqrt{a_1 a_2}^{-1}, \sqrt{a_1 a_2}, 1, \dots, 1\right),\\ h &= \operatorname{diag}\left(\sqrt{a_1/a_2}, \sqrt{a_1/a_2}, a_3, \dots, a_n, \sqrt{a_2/a_1}, \sqrt{a_2/a_1}, a_3^{-1}, \dots, a_n^{-1}\right),\end{aligned}$$

then

$$\begin{aligned}\left|\varphi_{\xi,\eta}(g)\right| &= \left|\langle \pi(g)\xi, \eta\rangle\right| = \left|\langle \pi(\tilde{g})\pi(h\omega)\xi, \pi(\omega)\eta\rangle\right|\\ &\leq \|\xi\|\,\|\eta\|\sqrt{\delta(\xi)\delta(\eta)}\,\Xi\begin{pmatrix}\sqrt{a_1 a_2} & 0\\ 0 & \sqrt{a_1 a_2}^{-1}\end{pmatrix}\end{aligned}$$

by Theorem 5.2, as π is unitary,

$$\begin{aligned}\dim\langle \pi(K_{1,2})\pi(h\omega)\xi\rangle &= \dim\langle \pi(h)\pi(K_{1,2})\pi(\omega)\xi\rangle\\ &= \dim\langle \pi(K_{1,2})\pi(\omega)\xi\rangle,\end{aligned}$$

and $K_{1,2}\omega \subset K$. ■

6. Kazhdan pairs

For $g \in \mathrm{Sp}(n, \mathbf{R})$ there are $k_1, k_2 \in K$ and

$$h = \operatorname{diag}\left(a_1, a_2, \dots, a_n, a_1^{-1}, a_2^{-1}, \dots a_n^{-1}\right) \in A^+$$

such that $g = k_1 h k_2$. This implies

$$\begin{aligned}\left|\varphi_{\xi,\eta}(g)\right| &= \left|\left\langle \pi(h)\pi(k_2)\xi, \pi(k_1)^{-1}\eta\right\rangle\right|\\ &\leq \|\xi\|\,\|\eta\|\sqrt{\delta(\xi)\delta(\eta)}\,\Xi\left(g_1\left(\ln\left(a_1 a_2\right)\right)\right).\end{aligned}$$

Let Ψ be defined by $\Psi(g) = \Xi\left(g_1\left(\ln\left(a_1 a_2\right)\right)\right)$.

Theorem 6.1. *Let $0 < \varepsilon < 1$ and $\delta = \left(4\sin\left(\frac{\arcsin\varepsilon}{2}\right) + \varepsilon\right)^2/2 < 1$, then $\left(\Psi^{-1}\left([1-\delta, 1]\right), \varepsilon\right)$ is a Kazhdan pair of $\mathrm{Sp}(n, \mathbf{R})$.*

The proof can again be copied word by word from [7, page 230–231] replacing $\mathrm{SL}(n, \mathbf{R})$ by $\mathrm{Sp}(n, \mathbf{R})$, δ by $1-\delta$, and the corresponding statement by Theorem 5.2.

For given δ the ε in the last theorem can be estimated. We can now prove Theorem 1.4.

Proof of Theorem 1.4. Let at first $0 < \varepsilon < 1$ be arbitrary. The Taylor expansion of $x \mapsto \sqrt{1+x}$ at 0 shows

$$\begin{aligned}4\sin\left(\left(\arcsin\varepsilon\right)/2\right) &= 2\sqrt{2}\sqrt{1-\sqrt{1-\varepsilon^2}}\\ &\geq 2\sqrt{2}\sqrt{1-\left(1-\varepsilon^2/2\right)} = 2\varepsilon\end{aligned}$$

for $0 < \varepsilon < 1$, hence $4\sin\left(\left(\arcsin\varepsilon\right)/2\right) + \varepsilon \geq 3\varepsilon \geq \sqrt{2}$ for $\varepsilon \geq \sqrt{2}/3$. So $\varepsilon < \sqrt{2}/3$ can be assumed. Again with the above mentioned Taylor expansion we have $\sqrt{1+x} \leq 1 + x/2$ for $x \geq -1$ and so

$$\sqrt{1+x} = \sqrt{1-\frac{x}{1+x}}^{-1} \geq \left(1-\frac{x}{2+2x}\right)^{-1} = 1 + \frac{x}{2+x}.$$

By letting $x = -\varepsilon^2$ this yields

$$\begin{aligned}
& \sqrt{1-\sqrt{1-\varepsilon^2}} \\
\leq\ & \sqrt{1-\left(1-\frac{\varepsilon^2}{2-\varepsilon^2}\right)} = \frac{\varepsilon}{\sqrt{2}}\sqrt{\frac{1}{1-\varepsilon^2/2}} = \frac{\varepsilon}{\sqrt{2}}\sqrt{1+\frac{\varepsilon^2/2}{1-\varepsilon^2/2}} \\
\leq\ & \frac{\varepsilon}{\sqrt{2}}\left(1+\frac{\varepsilon^2/2}{2-\varepsilon^2}\right) = \frac{\varepsilon}{\sqrt{2}}\left(1+\frac{1}{4\varepsilon^{-2}-2}\right) < \frac{\varepsilon}{\sqrt{2}}\frac{17}{16}
\end{aligned}$$

for $0 < \varepsilon < \sqrt{2}/3$ and hence $4\sin\left(\left(\arcsin\varepsilon\right)/2\right) + \varepsilon < \left(17/8+1\right)\varepsilon = \left(25/8\right)\varepsilon$.

Now let $0 < \varepsilon = \left(8/25\right)\sqrt{2\delta} < \sqrt{2}/3$, then $4\sin\left(\left(\arcsin\varepsilon\right)/2\right) + \varepsilon < \sqrt{2\delta}$ and the last theorem shows that $\left(Q,\varepsilon\right)$ is a Kazhdan pair. ■

A Proof of Proposition 3.1

The idea is to decompose $W\left(\vartheta\right)$ suitably such that it can be rearranged to $S_0\left(\vartheta\right)$ using only rotations $g_0\left(\alpha\right)$.

The union $W\left(\vartheta\right) = W^-\left(\vartheta\right) \cup W^+\left(\vartheta\right)$ is disjoint and

$$W\left(\vartheta\right) \cap S_0\left(\vartheta\right) = \left(W^+\left(\vartheta\right) \cap S_0^+\left(\vartheta\right)\right) \cup \left(W^-\left(\vartheta\right) \cap S_0^-\left(\vartheta\right)\right)$$

and

$$\begin{aligned}
W\left(\vartheta\right) \setminus S_0\left(\vartheta\right) &= \left(W^+\left(\vartheta\right) \cup W^-\left(\vartheta\right)\right) \setminus S_0\left(\vartheta\right) \\
&= \left(W^+\left(\vartheta\right) \setminus S_0\left(\vartheta\right)\right) \cup \left(W^-\left(\vartheta\right) \setminus S_0\left(\vartheta\right)\right) \\
&= \left(W^+\left(\vartheta\right) \setminus S_0^+\left(\vartheta\right)\right) \cup \left(W^-\left(\vartheta\right) \setminus S_0^-\left(\vartheta\right)\right).
\end{aligned}$$

Hence $\left\|E\left(W\left(\vartheta\right)\right)\xi\right\|^2 = \left\|E\left(W\left(\vartheta\right) \cap S_0\left(\vartheta\right)\right)\xi\right\|^2 + \left\|E\left(W\left(\vartheta\right) \setminus S_0\left(\vartheta\right)\right)\xi\right\|^2$,

$$\begin{aligned}
& \left\|E\left(W\left(\vartheta\right) \cap S_0\left(\vartheta\right)\right)\xi\right\|^2 \\
=\ & \left\|E\left(W^+\left(\vartheta\right) \cap S_0^+\left(\vartheta\right)\right)\xi\right\|^2 + \left\|E\left(W^-\left(\vartheta\right) \cap S_0^-\left(\vartheta\right)\right)\xi\right\|^2,
\end{aligned}$$

and

$$\begin{aligned}
& \left\|E\left(W\left(\vartheta\right) \setminus S_0\left(\vartheta\right)\right)\xi\right\|^2 \\
=\ & \left\|E\left(W^+\left(\vartheta\right) \setminus S_0^+\left(\vartheta\right)\right)\xi\right\|^2 + \left\|E\left(W^-\left(\vartheta\right) \setminus S_0^-\left(\vartheta\right)\right)\xi\right\|^2.
\end{aligned}$$

Then

$$\begin{aligned}
W^\pm\left(\vartheta\right) \setminus S_0^\pm\left(\vartheta\right) &= \left(S_1^\pm\left(\vartheta\right) \cap S_0^\pm\left(2\vartheta\right)\right) \setminus S_0^\pm\left(\vartheta\right) \\
&= S_1^\pm\left(\vartheta\right) \cap \left(S_0^\pm\left(2\vartheta\right) \setminus S_0^\pm\left(\vartheta\right)\right)
\end{aligned}$$

where the sign is either everywhere $+$ or everywhere $-$ and

$$S_0^\pm\left(2\vartheta\right) \setminus S_0^\pm\left(\vartheta\right) = g_0\left(\frac{3\vartheta}{2}\right) \cdot S_0^\pm\left(\frac{\vartheta}{2}\right) \cup g_0\left(-\frac{3\vartheta}{2}\right) \cdot S_0^\pm\left(\frac{\vartheta}{2}\right)$$

and the union is again disjoint.

We have

$$\begin{aligned} S_0^+(\vartheta) &= \left(W^+(\vartheta)\cap S_0^+(\vartheta)\right)\\ &\quad \cup g_0(\pi+2\vartheta)\cdot\left(S_1^-(\vartheta)\cap g_0\left(-\frac{3\vartheta}{2}\right)\cdot S_0^-\left(\frac{\vartheta}{2}\right)\right)\\ &\quad \cup g_0(\pi-2\vartheta)\cdot\left(S_1^-(\vartheta)\cap g_0\left(\frac{3\vartheta}{2}\right)\cdot S_0^-\left(\frac{\vartheta}{2}\right)\right)\end{aligned}$$

where the union is again disjoint. The validity of this equality for $S_0^+(\vartheta)$ can be deduced from the following equalities for the three sets. It can be shown that

$$\begin{aligned} W^+(\vartheta)\cap S_0^+(\vartheta) &= \left\{\begin{pmatrix} x\\ x+y\\ y\tan\beta\end{pmatrix} : x>0, y>0, -\vartheta<\beta\le\vartheta\right\}\\ &\quad \cup\left\{\begin{pmatrix} x\\ y\\ y\tan\beta\end{pmatrix} : x\le 0, y>0, -\vartheta<\beta\le\vartheta\right\},\end{aligned}$$

$$\begin{aligned} &g_0(\pi+2\vartheta)\cdot\left(S_1^-(\vartheta)\cap g_0\left(-\frac{3\vartheta}{2}\right)\cdot S_0^-\left(\frac{\vartheta}{2}\right)\right)\\ &= \left\{\begin{pmatrix} x\\ hx+y\\ y\tan\vartheta\end{pmatrix} : x>0, 0\le h<1, y>0\right\}\end{aligned}$$

and

$$\begin{aligned} &g_0(\pi-2\vartheta)\cdot\left(S_1^-(\vartheta)\cap g_0\left(\frac{3\vartheta}{2}\right)\cdot S_0^-\left(\frac{\vartheta}{2}\right)\right)\\ &= \left\{\begin{pmatrix} x\\ hx+y\\ -y\tan\vartheta\end{pmatrix} : x>0, 0<h\le 1, y>0\right\}.\end{aligned}$$

An analogous statement holds for $S_0^-(\vartheta)$.

Now $W(\vartheta)$ will be decomposed accordingly and put together again from rotated pieces to $S_0(\vartheta)$. With the above

$$\begin{aligned} \|E(W(\vartheta))\xi\|^2 &= \left\|E\left(W^+(\vartheta)\cap S_0^+(\vartheta)\right)\xi\right\|^2\\ &\quad+\left\|E\left(W^+(\vartheta)\cap g_0\left(\frac{3\vartheta}{2}\right)\cdot S_0^+\left(\frac{\vartheta}{2}\right)\right)\xi\right\|^2\\ &\quad+\left\|E\left(W^+(\vartheta)\cap g_0\left(-\frac{3\vartheta}{2}\right)\cdot S_0^+\left(\frac{\vartheta}{2}\right)\right)\xi\right\|^2\\ &\quad+\left\|E\left(W^-(\vartheta)\cap S_0^-(\vartheta)\right)\xi\right\|^2\\ &\quad+\left\|E\left(W^-(\vartheta)\cap g_0\left(\frac{3\vartheta}{2}\right)\cdot S_0^-\left(\frac{\vartheta}{2}\right)\right)\xi\right\|^2\\ &\quad+\left\|E\left(W^-(\vartheta)\cap g_0\left(-\frac{3\vartheta}{2}\right)\cdot S_0^-\left(\frac{\vartheta}{2}\right)\right)\xi\right\|^2\end{aligned}$$

and by the K-invariance

$$\begin{aligned}
& \left\| E\left(W^{+}(\vartheta) \cap g_0\left(\frac{3\vartheta}{2}\right) \cdot S_0^{+}\left(\frac{\vartheta}{2}\right)\right)\xi\right\|^2 \\
= & \left\| E\left(g_0(\pi - 2\vartheta) \cdot \left(S_1^{+}(\vartheta) \cap g_0\left(\frac{3\vartheta}{2}\right) \cdot S_0^{+}\left(\frac{\vartheta}{2}\right)\right)\right)\xi\right\|^2 \\
= & \left\| E\left(g_0(\pi - 2\vartheta) \cdot S_1^{+}(\vartheta) \cap g_0\left(-\frac{\vartheta}{2}\right) \cdot S_0^{-}\left(\frac{\vartheta}{2}\right)\right)\xi\right\|^2
\end{aligned}$$

and analogously

$$\begin{aligned}
& \left\| E\left(W^{+}(\vartheta) \cap g_0\left(-\frac{3\vartheta}{2}\right) \cdot S_0^{+}\left(\frac{\vartheta}{2}\right)\right)\xi\right\|^2 \\
= & \left\| E\left(g_0(\pi + 2\vartheta) \cdot S_1^{+}(\vartheta) \cap g_0\left(\frac{\vartheta}{2}\right) \cdot S_0^{-}\left(\frac{\vartheta}{2}\right)\right)\xi\right\|^2, \\
& \left\| E\left(W^{-}(\vartheta) \cap g_0\left(\frac{3\vartheta}{2}\right) \cdot S_0^{-}\left(\frac{\vartheta}{2}\right)\right)\xi\right\|^2 \\
= & \left\| E\left(g_0(\pi - 2\vartheta) \cdot S_1^{-}(\vartheta) \cap g_0\left(-\frac{\vartheta}{2}\right) \cdot S_0^{+}\left(\frac{\vartheta}{2}\right)\right)\xi\right\|^2, \\
& \left\| E\left(W^{-}(\vartheta) \cap g_0\left(-\frac{3\vartheta}{2}\right) \cdot S_0^{-}\left(\frac{\vartheta}{2}\right)\right)\xi\right\|^2 \\
= & \left\| E\left(g_0(\pi + 2\vartheta) \cdot S_1^{-}(\vartheta) \cap g_0\left(\frac{\vartheta}{2}\right) \cdot S_0^{+}\left(\frac{\vartheta}{2}\right)\right)\xi\right\|^2.
\end{aligned}$$

This yields

$$\begin{aligned}
\|E(W(\vartheta))\xi\|^2 &= \left\|E\left(S_0^{+}(\vartheta)\right)\xi\right\|^2 + \left\|E\left(S_0^{-}(\vartheta)\right)\xi\right\|^2 \\
&= 2\left\|E\left(S_0^{+}(\vartheta)\right)\xi\right\|^2 = \frac{2\vartheta}{\pi}.
\end{aligned}$$

A more detailed proof can be found in [11, page 59–68].

B Proof of Proposition 3.2

It is enough to prove that

$$W^{\pm}(\vartheta) \supseteq g_1(-t) \cdot W^{\pm}\left(\arctan\left(e^t \tan\vartheta\right)\right)$$

where either both signs are $+$ or both $-$. Therefore it has to be shown that

$$S_1^{\pm}(\vartheta) \cap S_0^{\pm}(2\vartheta) \supseteq S_1^{\pm}(\vartheta) \cap S^{\pm}_{-\tanh t}\left(\arctan\left(\cosh t \tan\left(2\arctan\left(e^t \tan\vartheta\right)\right)\right)\right).$$

So let

$$\begin{pmatrix} x \\ x+y \\ y\tan\beta \end{pmatrix} \in S_1^{+}(\vartheta) \cap S^{+}_{-\tanh t}\left(\arctan\left(\cosh t \tan\left(2\arctan\left(e^t \tan\vartheta\right)\right)\right)\right)$$

with $x \in \mathbf{R}$, $y > 0$ and $-\vartheta < \beta \leq \vartheta$. Then there is $z > 0$ and α with

$$\begin{aligned} & -\arctan\left(\cosh t \tan\left(2\arctan\left(e^t \tan\vartheta\right)\right)\right) \\ < \; & \alpha \leq \arctan\left(\cosh t \tan\left(2\arctan\left(e^t \tan\vartheta\right)\right)\right) \end{aligned}$$

such that $x + y = -x\tanh t + z$ and $y\tan\beta = z\tan\alpha$. If $x \geq 0$, then $x + y > 0$, since $y > 0$. Hence $0 < \frac{y\tan\beta}{x+y} \leq \tan\beta \leq \tan\vartheta$ for $0 < \beta \leq \vartheta$ and $0 \geq \frac{y\tan\beta}{x+y} \geq \tan\beta > -\tan\vartheta$ for $-\vartheta < \beta \leq 0$. So $\begin{pmatrix} x \\ x+y \\ y\tan\beta \end{pmatrix} \in S_0^+(2\vartheta)$. If $x < 0$, then $x + y = -x\tanh t + z > 0$. If $0 < \beta \leq \vartheta$ and $y \geq -2x(\cos\vartheta)^2$, then

$$\begin{aligned} 0 \; &< \; \frac{y\tan\beta}{x+y} = \left(1 + \frac{-x}{x+y}\right)\tan\beta \\ &\leq \; \left(1 + \frac{1}{-1 + 2(\cos\vartheta)^2}\right)\tan\beta = \frac{2(\cos\vartheta)^2}{\cos(2\vartheta)}\tan\beta \\ &\leq \; \frac{2(\cos\vartheta)^2}{\cos(2\vartheta)}\tan\vartheta = \tan(2\vartheta). \end{aligned}$$

If $-\vartheta < \beta \leq 0$, holds analogously

$$\begin{aligned} 0 \; &\geq \; \frac{y\tan\beta}{x+y} = \left(1 + \frac{-x}{x+y}\right)\tan\beta \\ &\geq \; \left(1 + \frac{1}{-1 + 2(\cos\vartheta)^2}\right)\tan\beta = \frac{2(\cos\vartheta)^2}{\cos(2\vartheta)}\tan\beta \\ &> \; -\frac{2(\cos\vartheta)^2}{\cos(2\vartheta)}\tan\vartheta = -\tan(2\vartheta). \end{aligned}$$

For $z \leq -x\frac{\tanh t}{\cosh t\tan(2\arctan(e^t\tan\vartheta)) - \tan(2\vartheta)}\tan(2\vartheta)$ and

$$0 < \alpha \leq \arctan\left(\cosh t \tan\left(2\arctan\left(e^t \tan\vartheta\right)\right)\right)$$

holds

$$\begin{aligned} 0 \; &< \; \frac{z\tan\alpha}{-x\tanh t + z} = \left(1 - \frac{-x\tanh t}{-x\tanh t + z}\right)\tan\alpha \\ &\leq \; \left(1 - \frac{1}{1 + \frac{1}{\cosh t\tan(2\arctan(e^t\tan\vartheta)) - \tan(2\vartheta)}\tan(2\vartheta)}\right)\tan\alpha \\ &= \; \frac{\tan(2\vartheta)}{\cosh t\tan(2\arctan(e^t\tan\vartheta))}\tan\alpha \leq \tan(2\vartheta). \end{aligned}$$

For $-\arctan(\cosh t\tan(2\arctan(e^t\tan\vartheta))) < \alpha \leq 0$ analogously

$$\begin{aligned} 0 \; &\geq \; \frac{z\tan\alpha}{-x\tanh t + z} = \left(1 - \frac{-x\tanh t}{-x\tanh t + z}\right)\tan\alpha \\ &\geq \; \left(1 - \frac{1}{1 + \frac{1}{\cosh t\tan(2\arctan(e^t\tan\vartheta)) - \tan(2\vartheta)}\tan(2\vartheta)}\right)\tan\alpha \\ &= \; \frac{\tan(2\vartheta)}{\cosh t\tan(2\arctan(e^t\tan\vartheta))}\tan\alpha > -\tan(2\vartheta). \end{aligned}$$

But,

$$\begin{aligned}
& (\cosh t)\tan\left(2\arctan\left(e^t\tan\vartheta\right)\right)-\tan(2\vartheta)\\
= & (\cosh t)\frac{2e^t\tan\vartheta}{1-\left(e^t\tan\vartheta\right)^2}-\tan(2\vartheta)\\
= & (\cosh t)\frac{e^t\tan(2\vartheta)}{1-\left(e^t\tan\vartheta\right)^2}\left(1-(\tan\vartheta)^2\right)-\tan(2\vartheta)\\
= & \left((\cosh t)\frac{e^t}{1-\left(e^t\tan\vartheta\right)^2}\left(1-(\tan\vartheta)^2\right)-1\right)\tan(2\vartheta)
\end{aligned}$$

and hence

$$\begin{aligned}
& \frac{\tanh t}{\cosh t\tan\left(2\arctan\left(e^t\tan\vartheta\right)\right)-\tan(2\vartheta)}\tan(2\vartheta)\\
= & \frac{\tanh t}{(\cosh t)\frac{e^t}{1-(e^t\tan\vartheta)^2}\left(1-(\tan\vartheta)^2\right)-1}\\
= & \frac{\left(1-\left(e^t\tan\vartheta\right)^2\right)\tanh t}{(\cosh t)\,e^t\left(1-(\tan\vartheta)^2\right)-1+\left(e^t\tan\vartheta\right)^2}.
\end{aligned}$$

Now

$$\begin{aligned}
& (\cosh t)\,e^t\left(1-(\tan\vartheta)^2\right)-1+\left(e^t\tan\vartheta\right)^2\\
= & (\cosh t)\,e^t-1+e^t(\tan\vartheta)^2\left(-(\cosh t)+e^t\right)\\
= & \frac{e^{2t}-1}{2}+e^t(\tan\vartheta)^2\frac{e^t-e^{-t}}{2}\\
= & e^t\sinh t+e^t(\tan\vartheta)^2\sinh t=e^t\frac{\sinh t}{\cos^2\vartheta}.
\end{aligned}$$

This implies

$$\begin{aligned}
& \frac{\tanh t}{\cosh t\tan\left(2\arctan\left(e^t\tan\vartheta\right)\right)-\tan(2\vartheta)}\tan(2\vartheta)\\
= & \frac{\left(1-\left(e^t\tan\vartheta\right)^2\right)\tanh t}{e^t\frac{\sinh t}{\cos^2\vartheta}}=\frac{\left(1-\left(e^t\tan\vartheta\right)^2\right)\cos^2\vartheta}{e^t\cosh t}\\
= & \frac{e^{-t}\cos^2\vartheta-e^t\sin^2\vartheta}{\cosh t}.
\end{aligned}$$

Since $y\geq-2x(\cos\vartheta)^2$, one has $x+y\geq-x\left(-1+2(\cos\vartheta)^2\right)=-x\cos(2\vartheta)$ and

$$\begin{aligned}
z & \leq -x\frac{\tanh t}{\cosh t\tan\left(2\arctan\left(e^t\tan\vartheta\right)\right)-\tan(2\vartheta)}\tan(2\vartheta)\\
& = -x\frac{e^{-t}\cos^2\vartheta-e^t\sin^2\vartheta}{\cosh t},
\end{aligned}$$

so

$$-x\tanh t+z\leq-x\left(\tanh t+\frac{e^{-t}\cos^2\vartheta-e^t\sin^2\vartheta}{\cosh t}\right).$$

Also

$$\begin{aligned}
& \tanh t + \frac{e^{-t}\cos^2\vartheta - e^t \sin^2\vartheta}{\cosh t} \\
= \quad & \frac{\sinh t + e^{-t}\cos^2\vartheta - e^t\sin^2\vartheta}{\cosh t} \\
= \quad & \frac{(1 - 2\cos^2\vartheta)\sinh t + e^t\cos^2\vartheta - e^t\sin^2\vartheta}{\cosh t} \\
= \quad & \frac{-\cos(2\vartheta)\sinh t + e^t\cos(2\vartheta)}{\cosh t} = \frac{\cos(2\vartheta)\,(-\sinh t + e^t)}{\cosh t} \\
= \quad & \cos(2\vartheta)\,.
\end{aligned}$$

So $-x\tanh t + z \le -x\cos(2\vartheta)$. Hence $\begin{pmatrix} x \\ x+y \\ y\tan\beta \end{pmatrix} = \begin{pmatrix} x \\ -x\tanh t + z \\ z\tan\alpha \end{pmatrix} \in S_0^+(2\vartheta)$.

The inclusion $g_1(t)\cdot W^-(\vartheta) \supseteq W^-(\arctan(e^t\tan\vartheta))$ holds analogously. Therefore $g_1(t)\cdot W(\vartheta) \supseteq W(\arctan(e^t\tan\vartheta))$.

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