Journal of Lie Theory
Volume **13** (2003) 193–212

Homomorphisms and Extensions of Principal Series Representations

Catharina Stroppel

Communicated by B. Ørsted

Abstract. In this article we describe homomorphisms and extensions of principal series representations. Principal series are certain representations of a semisimple complex Lie algebra $\mathfrak{g}$ and are objects of the Bernstein-Gelfand-Gelfand-category $\mathcal{O}$. Verma modules and their duals are examples of such principal series representations. Via the equivalence of categories of [3] the principal series representations correspond to Harish-Chandra modules for $\mathfrak{g} \times \mathfrak{g}$ which arise by induction from a minimal parabolic subalgebra of $\mathfrak{g} \times \mathfrak{g}$. We show that all principal series have one-dimensional endomorphism rings and trivial self-extensions. We also give an explicit example of a higher dimensional homomorphism space between principal series. As an application of these results we prove the existence of character formulae for "twisted tilting modules". The twisted tilting modules are some indecomposable objects of $\mathcal{O}$ having a flag whose subquotients are principal series modules and for which a certain Ext-vanishing condition holds.

1. Introduction

For a finite dimensional semisimple complex Lie algebra $\mathfrak{g}$ with Borel and Cartan subalgebras $\mathfrak{b}$ and $\mathfrak{h}$ respectively, we consider the category $\mathcal{O}$ (originally defined by [4]). It is a certain subcategory of the category of all finitely generated modules over the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$. This category decomposes into direct summands $\mathcal{O}_\lambda$, indexed by dominant weights, where each direct summand has as objects certain $\mathfrak{g}$-modules with a fixed general central character.
For any weight μ there is a universal object $\Delta(\mu)$, the *Verma module* with μ as highest weight. It is an object of the category $\mathcal{O}_\lambda$ for the dominant weight λ in the same Weyl group orbit as μ.
The motivation to consider category $\mathcal{O}$ comes from the representation theory of complex semisimple Lie groups. In this context the Harish-Chandra modules over $\mathcal{U}(\mathfrak{g} \times \mathfrak{g})$ play a crucial role (see [23], [17], [24], [25], [7]).
Via a choice of an isomorphism of algebras

$$\mathcal{U}(\mathfrak{g} \times \mathfrak{g}) \cong \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})^{opp}$$

ISSN 0949–5932 / \$2.50

the Harish-Chandra modules over $\mathcal{U}(\mathfrak{g}\times\mathfrak{g})$ correspond to objects of the category of Harish-Chandra bimodules as in [3], [13] and [19].
The relationship of this category to the category $\mathcal{O}$ is the following. The functor $\bullet\otimes_{\mathcal{U}(\mathfrak{g})}\Delta(0)$ defines an equivalence of categories from the category $\mathcal{H}_o^1$ of Harish-Chandra bimodules with trivial central character from the right and a certain full subcategory of $\mathcal{O}$ ([3, II]). We call this functor the *Bernstein-Gelfand-equivalence* in the following. On the other hand given two objects M and N in $\mathcal{O}$ we can define an object $\mathcal{L}(M,N)\in\mathcal{H}$ by taking the locally $\mathfrak{k}$-finite vectors of the extremely large space of all complex linear maps from M to N.
Combining these functorial constructions, for each element x of the Weyl group, A. Joseph defined in [15] a completion functor $\mathbf{C}_x$ on the trivial block of $\mathcal{O}$ as follows

$$\mathbf{C}_x:\mathcal{O}_o\quad\longrightarrow\quad\mathcal{O}_o,\qquad M\longmapsto\mathcal{L}(\Delta(x^{-1}\cdot 0),M)\otimes_{\mathcal{U}(\mathfrak{g})}\Delta(0).$$

The principal series representations from the title are just the modules $\mathbf{C}_xM$, where M is a dual Verma module. They can also be described as the (co-)induced representations from some minimal parabolic subalgebra ([7, 9.3 and 9.6]). The character of such principal series modules is well-known, since Frobenius reciprocity yields the equality

$$[\mathbf{C}_x(\Delta(y\cdot 0)^\star)]=[\Delta(x^{-1}y\cdot 0)]$$

in the Grothendieck group. If we take x equal to the identity, we get all dual Verma modules as principal series representations; if x is the longest element in the Weyl group, we obtain all Verma modules in $\mathcal{O}_o$. Therefore, principal series representations can be thought of as "twisted" Verma modules (see [1]). R. Irving uses the term "shuffled" Verma modules for principal series modules, since they can be constructed by a shuffling process using translation functors, which is described in [11]. Although our results are based on these shuffling properties, we do not define shuffling functors explicitly.
In the following we describe homomorphisms and extensions of such principal series. As the main result, it turns out (just as with Verma modules) the principal series modules always have one-dimensional endomorphism rings and trivial self-extensions; but the homomorphism spaces between two principal series representations can have higher dimension. We can give some conditions, which have to be fulfilled for the existence of homomorphisms and extensions. This makes it possible to define generalized tilting modules.
The motivation to look at homomorphism spaces of principal series modules comes from the study of primitive ideals of $\mathcal{U}(\mathfrak{g})$. For L a simple $\mathfrak{g}$-module the corresponding primitive quotient $\mathcal{U}(\mathfrak{g})/\operatorname{Ann}_{\mathcal{U}(\mathfrak{g})}L$ is a Harish-Chandra bimodule. By a theorem of Duflo ([8, Proposition 10]) this quotient is the image of a certain homomorphism between a projective Verma module and some principal series. A corollary of the results of this article is that the intertwining maps occurring in the Duflo-Zhelobenko four-step exact sequence ([15, corollary 4.7]) are unique up to a scalar; therefore they describe the Duflo-map mentioned above. This gives some more insight into composition factors of the quotient of the universal enveloping algebra of $\mathfrak{g}$ by some primitive ideal. Details on how the results of this paper are related to primitive ideals can be found in [22].

Acknowledgment: I would like to thank H. H. Andersen and my adviser W. Soergel for some interesting and helpful comments related to this work.

2. The Category $\mathcal{O}$ and Harish-Chandra bimodules

Let $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$ be a semisimple complex Lie algebra with a fixed Borel and Cartan subalgebras. Let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{b} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}$ be the corresponding Cartan decomposition. The corresponding universal enveloping algebras are denoted by $\mathcal{U}(\mathfrak{g}), \mathcal{U}(\mathfrak{b})$ etc.
We consider the category $\mathcal{O}$ defined as

$$\mathcal{O} := \left\{ M \in \mathfrak{g}-\mathrm{mod} \;\middle|\; \begin{array}{l} M \text{ is finitely generated as a } \mathcal{U}(\mathfrak{g})\text{-module} \\ M \text{ is locally finite for } \mathfrak{n} \\ \mathfrak{h} \text{ acts diagonally on } M \end{array} \right\}$$

where the second condition means that $\dim_{\mathbb{C}} \mathcal{U}(\mathfrak{n}) \cdot m < \infty$ for all $m \in M$ and the last says that $M = \oplus_{\mu \in \mathfrak{h}^*} M_\mu$, where $M_\mu = \{m \in M \mid h \cdot m = \mu(h)m \text{ for all } h \in \mathfrak{h}\}$ is the μ-weight space of M.
Many results about this category can be found for example in [4], [12], [13]. We want to list a few of these properties needed in the sequel without giving proofs.

The category $\mathcal{O}$ decomposes into a direct sum of full subcategories $\mathcal{O}_\chi$, indexed by central characters χ of $\mathcal{U} = \mathcal{U}(\mathfrak{g})$. Let $S = S(\mathfrak{h}) = \mathcal{U}(\mathfrak{h})$ be the symmetric algebra over $\mathfrak{h}$ considered as regular functions on $\mathfrak{h}^*$. The Weyl group W acts on $\mathfrak{h}^*$ via the 'dot-action' $w \cdot \lambda = w(\lambda + \rho) - \rho$ for $\lambda \in \mathfrak{h}^*$, where ρ is the half-sum of positive roots. Let $\mathcal{Z} = \mathcal{Z}(\mathcal{U})$ be the center of $\mathcal{U}$. Using the so-called Harish-Chandra isomorphism (see e.g. [12, Satz 1.5], [7, Theorem 7.4.5]) $\mathcal{Z} \to S^{W\cdot}$ and the fact that S is integral over $S^{W\cdot}$ ([7, Theorem 7.4.8]) we get an isomorphism $\xi : \mathfrak{h}^*/(W\cdot) \to \operatorname{Max} \mathcal{Z}$. Here $\operatorname{Max} \mathcal{Z}$ denotes the set of maximal ideals in $\mathcal{Z}$. This yields the following decomposition

$$\mathcal{O} = \bigoplus_{\chi \in Max \mathcal{Z}} \mathcal{O}_\chi = \bigoplus_{\lambda \in \mathfrak{h}^*/(W\cdot)} \mathcal{O}_\lambda, \tag{1}$$

where $\mathcal{O}_\chi$ denotes the subcategory of $\mathcal{O}$ consisting of all objects annihilated by some power of χ. If $\xi(\lambda) = \chi$, then $\mathcal{O}_\lambda = \mathcal{O}_\chi$.
By definition $\mathcal{O}_\lambda$ is a *regular* summand of the category $\mathcal{O}$ if λ is regular; that is, if $\lambda - \rho$ is not zero on any coroot $\check{\alpha}$ belonging to $\mathfrak{b}$. Let $W_\lambda = \{w \in W \mid w \cdot \lambda = \lambda\}$ be the stabilizer of λ in W.

We consider $\mathbb{C}_\lambda$, the irreducible $\mathfrak{h}$-module with weight λ, as a $\mathfrak{b}$-module by trivially extended action to the whole of $\mathfrak{b}$. For all $\lambda \in \mathfrak{h}^*$ we have a standard module, the Verma module $\Delta(\lambda) = \mathcal{U} \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_\lambda$. The Verma module $\Delta(\lambda)$ is a highest weight module of highest weight λ and has central character $\xi(\lambda)$. We denote by $L(\lambda)$ the unique irreducible quotient of $\Delta(\lambda)$. We fix a system of Chevalley generators $\{x_\alpha, h_\alpha\}_{\alpha \in R}$ of $\mathfrak{g}$; i.e $x_\alpha \in \mathfrak{g}_\alpha$, $h_\alpha \in \mathfrak{h}$ with $[x_\alpha, x_{-\alpha}] = h_\alpha$ and $\alpha(h_\alpha) = 2$ and denote by τ the Chevalley antiautomorphism of $\mathfrak{g}$ defined by $x_\alpha \mapsto x_{-\alpha}$ and $h_\alpha \mapsto h_\alpha$. Let $\star$ denote the duality on $\mathcal{O}$; i.e. $M^\star$ is the maximal $\mathfrak{h}$-semisimple submodule of the representation M^* with the action twisted by τ,

i.e. $(x.f)(m) = f(\tau(x)m)$ for $f \in M^*$, $x \in \mathfrak{g}$ and $m \in M$. We denote by $\nabla(\lambda)$ the dual Verma module $\Delta(\lambda)^\star$.
For a $\mathcal{U}$-bimodule M *the adjoint action* of $\mathfrak{g}$ on M is defined by $x \cdot m := xm - mx$, where $x \in \mathfrak{g}$, $m \in M$. A bimodule M is *locally-$\mathfrak{g}$-finite* if each $m \in M$ lies in a finite dimensional subspace of M, which is invariant under the adjoint action of $\mathfrak{g}$. The category $\mathcal{H}$ of *Harish-Chandra bimodules* is defined as the full subcategory of the category of all $\mathcal{U}$-bimodules whose objects are

1. finitely generated and
2. locally $\mathfrak{g}$-finite.

The Chevalley antiautomorphism τ of $\mathfrak{g}$ can be extended to an isomorphism $\mathcal{U} \cong \mathcal{U}^{opp}$. We choose the isomorphism $\mathcal{U}(\mathfrak{g} \times \mathfrak{g}) \to \mathcal{U} \otimes \mathcal{U}$ to be the unique homomorphism induced by the map $\mathfrak{g} \times \mathfrak{g} \to \mathcal{U} \otimes \mathcal{U}$ given by $(x, y) \mapsto x \otimes 1 + 1 \otimes y$. Hence, there is an equivalence of categories

$$\mathcal{U} - \text{mod} - \mathcal{U} \cong \mathcal{U} \otimes \mathcal{U}^{opp} - \text{mod} \cong \mathcal{U} \otimes \mathcal{U} - \text{mod} \cong \mathcal{U}(\mathfrak{g} \times \mathfrak{g}) - \text{mod} \cong \mathfrak{g} \times \mathfrak{g} - \text{mod}.$$

Via the whole equivalence, the adjoint action of $\mathfrak{g}$ corresponds to the action of $\mathfrak{k} := \{(x, -\tau(x))\}$. Since $\mathfrak{g}$ is semisimple, so is $\mathfrak{k}$. Hence, *locally $\mathfrak{g}$-finite* corresponds to *locally $\mathfrak{k}$-finite* under the equivalence of categories; and is therefore the same as *semisimple as $\mathfrak{k}$-module.*
For a finitely generated $\mathcal{U}$-bimodule X the set of locally $\mathfrak{g}$-finite vectors for the adjoint action forms a subbimodule ([7, 1.7.9]); we denote it by X^{adf}.
For M, $N \in \mathcal{O}$ the vector space $\mathrm{Hom}_{\mathbb{C}}(M, N)$ becomes a $\mathcal{U}$-bimodule by setting for $x \in \mathfrak{g}$, $f \in \mathrm{Hom}_{\mathbb{C}}(M, N)$ and $m \in M$

$$(x.f)(m) = x.(f(m)) \quad \text{and} \quad (f.x)(m) = f(x.m),$$

where on the right hand side of each equality the dot "." stands for the $\mathfrak{g}$-module structure of M and N, respectively. The largest locally-$\mathfrak{g}$-finite submodule $\mathrm{Hom}_{\mathbb{C}}(M, N)^{adf}$ of $\mathrm{Hom}_{\mathbb{C}}(M, N)$ is denoted by $\mathcal{L}(M, N)$ and it is an object of $\mathcal{H}$.
Given two elements x and y of the Weyl group W we denote by $\mathcal{P}_{(x,y)}$ the *principal series* module

$$\mathcal{P}_{(x,y)} = \mathcal{L}\big(\Delta(x \cdot 0), \nabla(y \cdot 0)\big).$$

If $y = x$ we also write $\mathcal{P}_x$ instead of $\mathcal{P}_{(x,x)}$. (In the notation of [13] the bimodule $\mathcal{P}_{(x,y)}$ is $\mathcal{D}(\Delta(y \cdot 0), \nabla(x \cdot 0))$ which corresponds to $M(x^{-1}, yw_o)$ in [11]).
There is a functor $\eta : \mathcal{H} \longrightarrow \mathcal{H}$, which interchanges the left and the right bimodule structure. As vector spaces $\eta(X) = X$ and $(u, v) \in \mathfrak{g} \times \mathfrak{g}$ acts on $\eta(X)$ as (v, u) on X. We often write X^η instead of $\eta(X)$.
The action of the center of $\mathcal{U}(\mathfrak{g} \times \mathfrak{g})$ gives a decomposition of the category $\mathcal{H}$. With analogous notations to those in (1) we have

$$\mathcal{H} = \bigoplus_{(\zeta,\chi) \in \mathrm{Max}\, \mathcal{Z} \times \mathrm{Max}\, \mathcal{Z}} {}_\zeta\mathcal{H}_\chi = \bigoplus_{\lambda, \mu \in \mathfrak{h}^*/(W\cdot)} {}_\lambda\mathcal{H}_\mu,$$

where ${}_\lambda\mathcal{H}_\mu$ consists of all Harish-Chandra bimodules having generalized central character λ from the left and μ from the right.

2.1. Translation functors. Let λ, μ, λ', μ' be dominant integral weights. We denote by $\mathrm{pr}_{(\mu,\mu')}$ the projection onto the direct summand ${}_{\mu}\mathcal{H}_{\mu'}$. The *translation functors* are defined as follows

$$\begin{array}{rccl} \theta_{(\lambda,\lambda')}^{(\mu,\mu')}: & {}_{\lambda}\mathcal{H}_{\lambda'} & \longrightarrow & {}_{\mu}\mathcal{H}_{\mu'} \\ & X & \mapsto & \mathrm{pr}_{(\mu,\mu')}\big(X \otimes E(\mu-\lambda)^{l} \otimes E(\mu'-\lambda')^{r}\big), \end{array}$$

where $E(\mu-\lambda)$ denotes the finite dimensional simple $\mathfrak{g}$-module having extremal weight $(\mu-\lambda)$. The upper index l means we consider E as a bimodule with trivial right action and E^r denotes the bimodule $\eta(E^l)$ having a trivial left action. Let θ_s and θ_s^r be translation *through* the s-wall from the left and from the right hand side respectively. More precisely, given weights λ and μ we choose two other weights λ' and μ' such that $\lambda-\lambda'$ and $\mu-\mu'$ are integral and where their stabilizers satisfy $W_{\lambda'}=W_{\mu'}=\{1,s\}$. The translations through the wall are then defined as follows:

$$\begin{array}{rcl} \theta_s := \theta_{(\lambda',\mu)}^{(\lambda,\mu)} \circ \theta_{(\lambda,\mu)}^{(\lambda',\mu)}: & {}_{\lambda}\mathcal{H}_{\mu} \longrightarrow {}_{\lambda}\mathcal{H}_{\mu} & \text{and} \\ \theta_s^r := \theta_{(\lambda,\mu')}^{(\lambda,\mu)} \circ \theta_{(\lambda,\mu)}^{(\lambda,\mu')}: & {}_{\lambda}\mathcal{H}_{\mu} \longrightarrow {}_{\lambda}\mathcal{H}_{\mu}. & \end{array}$$

(Up to natural equivalence these functors are independent of the special choice of λ' and μ'.) The translation functors for category $\mathcal{O}$ are defined in an analogous way. So the translation from $\mathcal{O}_\lambda$ to $\mathcal{O}_\mu$ is given by the functor

$$\begin{array}{rccl} \theta_\lambda^\mu: & \mathcal{O}_\lambda & \longrightarrow & \mathcal{O}_\mu \\ & M & \mapsto & \mathrm{p}_\mu(M \otimes E(\mu-\lambda)), \end{array}$$

where p_μ is the projection onto $\mathcal{O}_\mu$. For λ and λ' as above, the translation through the s-wall is the functor $\theta_s = \theta_{\lambda'}^{\lambda} \circ \theta_{\lambda}^{\lambda'}$. Under the Bernstein-Gelfand-equivalence the two functors θ_s correspond. The isomorphisms of vector spaces $\mathrm{Hom}_{\mathbb{C}}(M\otimes E, N) \cong \mathrm{Hom}_{\mathbb{C}}(M, E^*\otimes N) \cong \mathrm{Hom}_{\mathbb{C}}(M,N)\otimes E^*$ are compatible with the $\mathfrak{g}$-bimodule structures and induce a canonical isomorphism

$$\theta_{(\lambda,\lambda')}^{(\mu,\mu')}\mathcal{L}(M,N) \cong \mathcal{L}(\theta_\mu^{\mu'}M, \theta_\lambda^{\lambda'}N). \tag{2}$$

The duality on $\mathcal{O}$ gives rise to a duality on the Harish-Chandra bimodules with trivial central character from the right. We denote it also by $\star$. For $X \in \mathcal{H}$ with trivial central character from the right $X^\star$ can also defined as the largest locally $\mathfrak{k}$-finite submodule of X^*, with the action twisted by τ (see [15, 2.7]).

2.2. Principal series and Joseph's Completion Functor. In this section we recall the definition of Joseph's completion functor and some of its properties, which are needed in the following section. All this can be found in [16] and [15].

Definition 2.1. Let $x \in W$. Joseph's completion functor $\mathbf{C}_x$ on $\mathcal{O}_0$ is defined as

$$\mathbf{C}_x(M) := \mathcal{L}(\Delta(x^{-1}\cdot 0), M) \otimes_{\mathcal{U}} \Delta(0).$$

Instead of $\mathbf{C}_{s_\alpha}$ we will often write $\mathbf{C}_\alpha$.
For M a dual Verma module we also call $\mathbf{C}_x(M) \in \mathcal{O}$ a *principal series*. This is compatible with the term used for Harish-Chandra bimodules in the sense of property (P4) in the next section.

2.3. Properties of the completion functors.

(P1) ([15, 2.2]) The functor $\mathbf{C}_x$ is covariant and left exact.

(P2) ([16, 2.9]) There is a natural equivalence of functors $\mathbf{C}_x \cong \mathbf{C}_{s_1} \cdots \mathbf{C}_{s_r}$, where $x = s_1 \cdot \ldots \cdot s_r$ is a reduced expression for x.

(P3) ([16, Lemma 2.10]) Concerning dual Verma modules, $\mathbf{C}_x \nabla(0) \cong \nabla(x \cdot 0)$ holds for all $x \in W$.

(P4) By definition $\mathbf{C}_{x^{-1}} \nabla(y \cdot 0)$ corresponds to $\mathcal{L}(\Delta(x \cdot 0), \nabla(y \cdot 0)) = \mathcal{P}_{(x,y)}$ via the equivalence of categories in [3]. In the Grothendieck group of $\mathcal{O}$ the equality

$$[\mathbf{C}_x \nabla(y \cdot 0)] = [\Delta(xy \cdot 0)]$$

holds for all x, $y \in W$ (see [16, 3.1]). A proof of this can be found in [7, 9.6.2].

(P5) ([15, Lemma 2.5]) For a simple root α we have

$$\mathbf{C}_\alpha \Delta(x \cdot 0) \cong \begin{cases} \Delta(s_\alpha x \cdot 0) & \text{if } s_\alpha x < x \\ \Delta(x \cdot 0) & \text{otherwise.} \end{cases}$$

Therefore, for Verma modules the completion in the sense of Joseph is therefore the same thing as completion in the sense of Enright ([9]). The Verma modules belong to the principal series: for $y \in W$ there is an isomorphism

$$\mathcal{L}\big(\Delta(w_o \cdot 0), \nabla(y \cdot 0)\big) \otimes_{\mathcal{U}} \Delta(0) \cong \Delta(w_o y \cdot 0). \tag{3}$$

(To see this let $a = w_o y^{-1}$ and $b = w_o a^{-1}$. By definition of the completion functors and their properties we get $\mathcal{L}\big(\Delta(w_o \cdot 0), \nabla(y \cdot 0)\big) \otimes_{\mathcal{U}} \Delta(0) = \mathbf{C}_{w_o} \nabla(y \cdot 0) \overset{(P2)}{\cong} \mathbf{C}_b \mathbf{C}_a \nabla(y \cdot 0) \overset{(P3)}{\cong} \mathbf{C}_b \mathbf{C}_a \mathbf{C}_y \nabla(0) \overset{(P2)}{\cong} \mathbf{C}_b \mathbf{C}_{w_o} \nabla(0) \overset{(P3)}{\cong} \mathbf{C}_b \nabla(w_o \cdot 0) \cong \mathbf{C}_b \Delta(w_o \cdot 0) \overset{(P5)}{\cong} \Delta(b w_o \cdot 0) = \Delta(w_o a^{-1} a y \cdot 0) = \Delta(w_o y \cdot 0)$.)

(P6) For each simple root α and all modules M in $\mathcal{O}_0$, the inclusion $\Delta(s_\alpha \cdot 0) \hookrightarrow \Delta(0)$ induces a canonical morphism $\phi_M^\alpha : M \longrightarrow \mathbf{C}_{s_\alpha} M$.

- We denote by $\mathbf{D}_\alpha^- M$ the image of this induced map. The kernel of ϕ_M^α is the largest α-finite submodule of M ([15, Lemma 2.4]); i.e. the largest submodule, whose composition factors are all of the form $L(x \cdot 0)$ with $\langle x \cdot 0, \check{\alpha} \rangle > 0$. A module M is called *α-free*, if ϕ_M^α is injective. In particular, every Verma module is α-free. (Note, that this definition does not agree with the one in [13].)
- Dually, we say that a module M is *α-cofree* if $M^\star$ is α-free and we define $\mathbf{D}_\alpha^+ M := (\mathbf{D}_\alpha^-(M^\star))^\star$. In particular every dual Verma module is α-cofree.

(P7) The isomorphism of vector spaces $\mathrm{Hom}_{\mathbb{C}}(M, N^*) \cong (N \otimes M)^*$ induces (see [13, 6.9 (3)]) an isomorphism $\mathcal{L}(M, N)^\eta \cong \mathcal{L}(N^\star, M^\star)$ for all objects M and N in $\mathcal{O}$.

Now we are ready to prove some results concerning the principal series modules.

3. Principal series and their duals

The following theorem was proved independently by Andersen and Lauritzen [1] and the author. We state the result and also give a proof here to show the connections with the main theorem which comes later.

Theorem 3.1. Dual Principal Series
For all $x \in W$ there is an isomorphism of bimodules $\mathcal{P}^\star_{(x,y)} \tilde{\longrightarrow} \mathcal{P}_{(w_o x, w_o y)}$.

Proof. The proof is by induction on the length of y. For $y = e$ the property (P3) of the completion functors gives $(\mathcal{P}_{(x,e)} \otimes_{\mathcal{U}} \Delta(0))^\star = (\mathbf{C}_{x^{-1}}\nabla(0))^\star \cong (\nabla(x^{-1} \cdot 0))^\star \cong \Delta(x^{-1} \cdot 0)$. On the other hand we have $\mathcal{P}_{(w_o x, w_o)} \otimes_{\mathcal{U}} \Delta(0) = \mathbf{C}_{(w_o x)^{-1}}\nabla(w_o \cdot 0) \cong \mathbf{C}_{(w_o x)^{-1}}\Delta(w_o \cdot 0) \cong \Delta(x^{-1} \cdot 0)$ by the properties (P2) and (P5). This is the starting point of the induction.
Consider for a simple reflection s such that $ys > y$ the exact sequence

$$\nabla(ys \cdot 0) \hookrightarrow \theta_s \nabla(y \cdot 0) \twoheadrightarrow \nabla(y \cdot 0).$$

Since $\mathcal{L}(\Delta(x \cdot 0), \bullet)$ is left exact, the character formula in (P4) gives for all $y \in W$ an exact sequence of the form

$$0 \to \underbrace{\mathcal{L}\big(\Delta(x \cdot 0), \nabla(ys \cdot 0)\big)}_{=\mathcal{P}_{(x,ys)}} \longrightarrow \underbrace{\theta_s \mathcal{L}\big(\Delta(x \cdot 0), \nabla(y \cdot 0)\big)}_{=:B} \xrightarrow{\text{can}} \underbrace{\mathcal{L}\big(\Delta(x \cdot 0), \nabla(y \cdot 0)\big)}_{=:A} \to 0. \quad (4)$$

(For the middle terms see (2)in the previous section.) On the other hand the exact sequence

$$\nabla(w_o y \cdot 0) \hookrightarrow \theta_s \nabla(w_o y \cdot 0) \twoheadrightarrow \nabla(w_o ys \cdot 0)$$

gives rise to a short exact sequence

$$\underbrace{\mathcal{L}\big(\Delta(w_o x \cdot 0), \nabla(w_o y \cdot 0)\big)}_{=:C} \xrightarrow{\text{can}} \underbrace{\theta_s \mathcal{L}\big(\Delta(w_o x \cdot 0), \nabla(w_o y \cdot 0)\big)}_{=:D} \longrightarrow \underbrace{\mathcal{L}\big(\Delta(w_o x \cdot 0), \nabla(w_o ys \cdot 0)\big)}_{\mathcal{P}_{(w_o x, w_o ys)}}.$$

By assumption there is an isomorphism $\psi : C^\star \tilde{\to} A$. The translation functors commute with the duality (see [13, 4.12 (9)]); hence we can choose an isomorphism $\beta : D^\star = (\theta_s C)^\star \cong \theta_s C^\star$. This implies the existence of an isomorphism $\tilde{\psi} = \theta_s \psi \circ \beta : D^\star \longrightarrow \theta_s A = B$ and gives the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{P}_{(x,ys)} & \longrightarrow & B & \xrightarrow{\text{can}} & A & \longrightarrow & 0 \\ & & & & \uparrow{\scriptstyle\tilde{\psi}} & & \uparrow{\scriptstyle\psi} & & \\ 0 & \longrightarrow & \mathcal{P}^\star_{(w_o x, w_o ys)} & \longrightarrow & D^\star & \xrightarrow{\text{can}^\star} & C^\star & \longrightarrow & 0 \end{array} \quad (5)$$

To prove the theorem, it is sufficient to observe that the modules on the left hand side are both kernels of the canonical map and therefore isomorphic. ■

Remark 3.1. In the next section (Theorem 4.1 b.) we prove independently of Theorem 3.1 that the diagram (5) commutes (up to a scalar), since the homomorphism space from $D^\star$ to A is one-dimensional.

The following lemma can be considered as a corollary of the previous theorem, but it is also the key lemma for the Endomorphism Theorem.

Lemma 3.2. *Let x, $y \in W$ and $M := \mathcal{L}(\Delta(x\cdot 0), \nabla(y\cdot 0)) \otimes_{\mathcal{U}} \Delta(0) \in \mathcal{O}_0$ be the corresponding principal series. The module M is α-cofree for a simple reflection $s = s_\alpha$ such that $xs > x$.*

Proof. Consider the dual module $M^\star \cong \mathcal{L}(\Delta(w_o x\cdot 0), \nabla(w_o y\cdot 0))$. We assume that $x < xs$. This implies $w_o x > w_o xs$. Therefore, there exists a reduced expression $w_o x = s_r \cdot \ldots \cdot s_1$ where $s_i = s_{\alpha_i}$ with $s_1 = s$. By definition we have $M^\star = \mathbf{C}_{\alpha_1}\cdots\mathbf{C}_{\alpha_r}\nabla(w_o y\cdot 0)$. In particular (see [16, 3.2]), $M^\star$ is α_1-free; hence M itself is α_1-cofree. ∎

Remark 3.3. a.) The previous Lemma can also be proved by the combinatorics of [16, 2.2] using the character formulas of the principal series modules: with the notations of [16] and defining $M := \mathbf{C}_{x^{-1}}\nabla(y\cdot 0)$ the following equalities hold:

$$\begin{aligned}
[\mathbf{D}_\alpha^+ M] &= -[\mathbf{C}_\alpha M] + [M] + s[M]\\
&= -[\Delta(sx^{-1}y\cdot 0)] + [\Delta(x^{-1}y\cdot 0)] + [\Delta(s(x^{-1}y)\cdot 0)]\\
&= [\Delta(x^{-1}y\cdot 0)] = [M].
\end{aligned}$$

By the definition of $\mathbf{D}_\alpha^+$, the module M is therefore α-cofree. This is the statement of the lemma.

b.) The statement of the lemma can be reformulated as follows: for all x, $y \in W$ and all simple reflections $s = s_\alpha \in W$ such that $ys_\alpha > y$, the module $\mathbf{C}_{y^{-1}}\nabla(x\cdot 0)$ is α-cofree, i.e. $\mathbf{D}_\alpha^+ C_{y^{-1}}\nabla(x\cdot 0) = C_{y^{-1}}\nabla(x\cdot 0)$. Therefore the exact sequence in [15, Proposition 3.2] (with $M = C_{y^{-1}}\nabla(x\cdot 0)$) turns out to be of the form

$$0 \to \mathcal{L}\big(\Delta(ys\cdot 0), \nabla(x\cdot 0)\big) \longrightarrow \theta_s^r\mathcal{L}\big(\Delta(y\cdot 0), \nabla(x\cdot 0)\big) \tag{6}$$
$$\xrightarrow{\text{can}} \mathcal{L}\big(\Delta(y\cdot 0), \nabla(x\cdot 0)\big) \to 0. \tag{7}$$

Applying the functor η gives (by property (P7)) just the exact sequence (4):

$$0 \to \mathcal{L}\big(\Delta(x\cdot 0), \nabla(ys\cdot 0)\big) \longrightarrow \theta_s\mathcal{L}\big(\Delta(x\cdot 0), \nabla(y\cdot 0)\big) \tag{8}$$
$$\xrightarrow{\text{can}} \mathcal{L}\big(\Delta(x\cdot 0), \nabla(y\cdot 0)\big) \to 0. \tag{9}$$

c.) Let $x \in W$ and let s be a simple reflection such that $xs > x$. Given an explicit isomorphism from $\mathcal{P}_{w_o xs, w_o z}$ into the dual of $\mathcal{P}_{xs,z}$, it is possible to construct an isomorphism from $\mathcal{P}_{w_o x, w_o z}$ into $\mathcal{P}_{x,z}$. The definitions give isomorphisms

$$\begin{aligned}
\Big(\mathcal{L}\big(\Delta(x\cdot 0), \nabla(z\cdot 0)\big)\Big)^\star &\cong \Big(\mathcal{L}\big(\Delta(0), \mathbf{C}_{x^{-1}}\nabla(z\cdot 0)\big)\Big)^\star\\
&\cong \mathcal{L}\Big(\Delta(0), \big(\mathbf{C}_{x^{-1}}\nabla(z\cdot 0)\big)^\star\Big).
\end{aligned}$$

With $y = w_o x$ this yields by [16, 2.6] and Lemma 3.2

$$\mathbf{C}_{x^{-1}}\nabla(z\cdot 0) \cong \big(\mathbf{C}_s(\mathbf{C}_s\mathbf{C}_{x^{-1}}\nabla(z\cdot 0))^\star\big)^\star;$$

hence,

$$\begin{aligned}\mathcal{L}\Big(\Delta(0),\big(\mathbf{C}_{x^{-1}}\nabla(z\cdot 0)\big)^{\star}\Big) &\cong \mathcal{L}\Big(\Delta(0),\mathbf{C}_s\big(\mathbf{C}_{xs^{-1}}\nabla(z\cdot 0)\big)^{\star}\Big)\\ &\cong \mathcal{L}\Big(\Delta(s\cdot 0),\mathbf{C}_{(w_oxs)^{-1}}\nabla(w_oz\cdot 0)\Big)\\ &\cong \mathcal{L}\Big(\Delta(w_ox\cdot 0),\nabla(w_oz\cdot 0)\Big).\end{aligned}$$

d.) The proof of Theorem 3.1 indicates that the statement is not based on the definition of principal series we gave, but on the existence of an exact sequence of the form (4); the abstract context for this is described in [1]. This enabled H. H. Andersen and N. Lauritzen to characterize principal series modules as geometric objects, i.e. as local cohomology bundles on the flag variety, or as semi-induced modules (see [1]). Using the results of [5] one may also consider principal series modules as certain $\mathcal{D}$-modules.

e.) The restriction to regular weights is not necessary; rather, it avoids some interfering indices.

4. Endomorphisms and self-extensions of principal series

In this section we will prove the main result concerning endomorphism rings and extensions. A first step in this direction is the indecomposability of the principal series. Although this seems to be a well-known result, it was not possible to find a reference for it. Moreover the proof is very general and therefore interesting in itself:

Lemma 4.1. *All principal series modules $\mathcal{P}_{x,y}$ (where x, $y \in W$) are indecomposable.*

Proof. For x, $y \in W$ and a simple reflection s such that $ys > y$ we consider (see (8)) the short exact sequence

$$\mathcal{L}(\Delta(x\cdot 0),\nabla(ys\cdot 0)) \longrightarrow \theta_s\mathcal{L}(\Delta(x\cdot 0),\nabla(y\cdot 0)) \longrightarrow \mathcal{L}(\Delta(x\cdot 0),\nabla(y\cdot 0)), \qquad (10)$$

For $y = e$, the shortest element in the Weyl group, (and $x \in W$ arbitrary) the bimodule $\mathcal{L}(\Delta(x\cdot 0),\nabla(y\cdot 0)) \otimes_{\mathcal{U}} \Delta(0)$ is by property (P3) a dual Verma module and so it is indecomposable. This is the starting point for an induction argument. We have to show, that the indecomposability of the quotient in (10) implies the indecomposability of the submodule on the left hand side.
More generally, we consider an exact sequence in $\mathcal{O}$ of the form $A \hookrightarrow \theta_s A \twoheadrightarrow B$. We show that if A is decomposable, then so is B: let $A = C \oplus D$. The canonical inclusion in the exact sequence corresponds to the identity after translating onto the wall. Hence, the direct sum decomposition of A gives rise to two exact sequences of the form

$$\begin{aligned} C &\hookrightarrow \theta_s C \twoheadrightarrow \operatorname{coker}_1\\ D &\hookrightarrow \theta_s D \twoheadrightarrow \operatorname{coker}_2\end{aligned}$$

such that $\operatorname{coker}_1 \oplus \operatorname{coker}_2 \cong B$. Assume B to be indecomposable and let $\operatorname{coker}_1 = 0$. There are the following two possibilities:

I.) There exists no $x \in W$ such that $xs > x$ and $[C : L(xs \cdot 0)] \neq 0$. This implies $\theta_s C = 0$ (see [13, 4.12 (3)]) and therefore a contradiction.

II.) There exists an $x \in W$ such that $xs > x$ and $[C : L(xs \cdot 0)] \neq 0$. For simplicity we choose x maximal. By [13, 4.12 (3) and 4.13 (3')] this implies $[\theta_s C : L(xs \cdot 0)] = 2[C : L(xs \cdot 0)]$, which is also a contradiction.

Hence B is decomposable. ∎

A stronger result than the previous theorem is the Endomorphism Theorem for which we need the following key lemma.

Lemma 4.2. *Let α be a simple reflection. Let $f : M \to N$ be a nontrivial homomorphism in $\mathcal{O}_0$, where M is α-cofree. Then the induced homomorphism*

$$\mathbf{C}_\alpha f : \mathbf{C}_\alpha M \to \mathbf{C}_\alpha N$$

is also not trivial.

Proof. The completion functor is left exact. Hence the exact sequence

$$0 \to \ker f \hookrightarrow M \twoheadrightarrow \operatorname{im} f \to 0$$

leads to an exact sequence

$$0 \to \mathbf{C}_\alpha \ker f \hookrightarrow \mathbf{C}_\alpha M \xrightarrow{\mathbf{C}_\alpha f} \mathbf{C}_\alpha \operatorname{im} f \twoheadrightarrow X \to 0.$$

We have to show that $\mathbf{C}_\alpha \ker f \neq \mathbf{C}_\alpha M$ Assume equality, namely

$$\mathbf{C}_\alpha \ker f = \mathbf{C}_\alpha M, \tag{11}$$

and consider the following two four-step exact sequences (see [16, 3.2]):

$$\begin{array}{ccccccccc} 0 \to & \mathbf{C}_\alpha \ker f & \hookrightarrow & \mathbf{C}_\alpha^2 \ker f & \longrightarrow & \mathbf{D}_\alpha^+ \ker f \twoheadrightarrow \mathbf{D}_\alpha \ker f & \to 0 \\ 0 \to & \mathbf{C}_\alpha M & \hookrightarrow & \mathbf{C}_\alpha^2 M & \longrightarrow & \mathbf{D}_\alpha^+ M \twoheadrightarrow \mathbf{D}_\alpha M & \to 0. \end{array}$$

Here the functor $\mathbf{D}_\alpha$ is the composition of the functors $\mathbf{D}_\alpha^+ \mathbf{C}_\alpha$ ([15, 3.6]). The assumption (11) implies the equality $\mathbf{D}_\alpha^+ \ker f = \mathbf{D}_\alpha^+ M$. On the other hand M is α-cofree, hence by definition $M = \mathbf{D}_\alpha^+ M$. By definition $\mathbf{D}_\alpha^+ \ker f$ is also a subset of $\ker f$. Since f is nontrivial, this subset is not the whole of M. This gives the desired contradiction; hence $\mathbf{C}_\alpha f$ is not the zero map. ∎

The Lemma 3.2 ensures the existence of 'enough' modules which are α-cofree; because of this the previous lemma is a strong tool. Now we are ready to prove the main theorem, which indicates that principal series modules behave in some sense like Verma modules.

Theorem 4.1. Endomorphism Theorem

1.) All principal series have one-dimensional endomorphism rings, i.e.

$$\operatorname{End}_{\mathcal{H}} \big(\mathcal{L}\big(\Delta(x \cdot 0), \nabla(y \cdot 0)\big) \cong \mathbb{C}$$

for all $x \in W$.

2.) Let x, $y \in W$ and let s be a simple reflection such that $y > ys$. Let

$$\begin{aligned} A &:= \mathcal{L}\big(\Delta(x \cdot 0), \nabla(y \cdot 0)\big), \\ B &:= \mathcal{L}\big(\Delta(x \cdot 0), \nabla(ys \cdot 0)\big). \end{aligned}$$

Then the following hold:

a.) $\operatorname{Hom}_{\mathcal{H}}(A, B) = 0$.
In particular, if $x = w_o$ then this is just $\operatorname{Hom}_{\mathfrak{g}}(\Delta(w_o y \cdot 0), \Delta(w_o ys \cdot 0)) = 0$.

b.) $\operatorname{Hom}_{\mathcal{H}}(A, \theta_s A) \cong \operatorname{Hom}_{\mathcal{H}}(A, \theta_s B) \cong \operatorname{Hom}_{\mathcal{H}}(B, \theta_s A) = \operatorname{Hom}_{\mathcal{H}}(B, \theta_s B) \cong \mathbb{C}$.

c.) $\dim \operatorname{End}_{\mathcal{H}}(\theta_s A) = \dim \operatorname{End}_{\mathcal{H}}(\theta_s B) = 2$.

d.) $\operatorname{Hom}_{\mathcal{H}}(B, A) \cong \mathbb{C}$.
In particular, if $x = w_o$ this is just $\operatorname{Hom}_{\mathfrak{g}}(\Delta(w_o ys \cdot 0), \Delta(w_o y \cdot 0)) \cong \mathbb{C}$.

e.) The sequence

$$0 \to A \overset{\text{can}}{\to} \theta_s A \overset{\text{can}}{\to} B \to 0 \tag{12}$$

(see (8)) does not split in $\mathcal{H}$.

f.) Assume $tx < x$ for some simple reflection t. Then $\operatorname{Hom}_{\mathcal{H}}(\mathcal{P}_x, \mathcal{P}_{tx}) = \mathbb{C}$.

Proof. For $x = w_o$, the bimodule $A = \mathcal{L}\big(\Delta(x \cdot 0), \nabla(y \cdot 0)\big)$ corresponds to a Verma module in $\mathcal{O}$ by the properties (P2), (P3) and (P5) in section 2. So the statement is well-known. For $x \neq w_0$ let s_α be a simple reflection such that $xs_\alpha > x$. By Lemma 3.2 the module $\tilde{A} = A \otimes_{\mathcal{U}} \Delta(0)$ is α-cofree. Every nontrivial endomorphism f of $\tilde{A}$ gives by Lemma 4.2 a nontrivial endomorphism of $\mathbf{C}_\alpha \tilde{A}$, hence a nontrivial endomorphism of $\mathcal{L}(\Delta(xs \cdot 0), \nabla(y \cdot 0))$. An iterated use of these two Lemmas implies that we have an inclusion

$$\begin{aligned} \operatorname{End}_{\mathcal{H}}\left(\mathcal{L}(\Delta(x \cdot 0), \nabla(y \cdot 0))\right) &\hookrightarrow \operatorname{End}_{\mathcal{H}}\left(\mathcal{L}(\Delta(w_o \cdot 0), \nabla(y \cdot 0))\right) \\ &= \operatorname{End}_{\mathcal{O}}(\Delta(w_o y \cdot 0)) = \mathbb{C} \end{aligned}$$

for all $y \in W$. This proves the first part.

a.) Assume the assertion is false. We choose a simple reflection s_α such that $xs_\alpha > x$. So $\tilde{A} = A \otimes_{\mathcal{U}} \Delta(0)$ is α-cofree by Lemma 3.2; Lemma 4.2 yields a nontrivial map from $\mathbf{C}_\alpha \tilde{A}$ to $\mathbf{C}_\alpha(B \otimes_{\mathcal{U}} \Delta(0))$. Repeating this argument we get (with (3)) a nontrivial morphism from $\Delta(w_o y \cdot 0)$ to the Verma module $\Delta(w_o ys \cdot 0)$. Since $ys < y$, we have $w_o y < w_o ys$, so this is a contradiction. Therefore, the space of homomorphisms in question is trivial.

b.) The exact sequence (12) gives an exact sequence of the form

$$0 \to \underbrace{\operatorname{End}_{\mathcal{H}}(A)}_{\cong \mathbb{C}} \longrightarrow \operatorname{Hom}_{\mathcal{H}}(A, \theta_s A) \longrightarrow \underbrace{\operatorname{Hom}_{\mathcal{H}}(A, B)}_{=0},$$

where we already know the outer terms. This implies the first statement. The others follow directly from the fact that $\theta_s A \cong \theta_s B$ by the selfadjointness of θ_s.

c.) This is obvious, since θ_s is self-adjoint and has the property $\theta_s^2 \cong \theta_s \oplus \theta_s$.

d.) Take a (unique up to a scalar) homomorphism

$$f \in \mathrm{Hom}_{\mathcal{H}}(\mathcal{P}_{(e,ys)}, \mathcal{P}_{(e,y)}) = \mathrm{Hom}_{\mathfrak{g}}(\nabla(ys \cdot 0), \nabla(y \cdot 0)).$$

By Lemma 4.2 it induces a nontrivial morphism $\mathbf{C}_{x^{-1}} f$ from B to A. On the other hand, the sequence (12) gives an inclusion

$$\mathrm{Hom}_{\mathcal{H}}(B, A) \hookrightarrow \mathrm{Hom}_{\mathcal{H}}(B, \theta_s A);$$

therefore, $\dim \mathrm{Hom}_{\mathcal{H}}(B, A) = 1$.

e.) Applying the functor $\mathrm{Hom}_{\mathcal{H}}(B, \bullet)$ to the sequence (12) gives rise to an exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{H}}(B, A) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{H}}(B, \theta_s A) \xrightarrow{\mathrm{can} \circ} \mathrm{Hom}_{\mathcal{H}}(B, B).$$

Hence, the identity $\mathrm{id} \in \mathrm{Hom}_{\mathcal{H}}(B, B)$ has no preimage and consequently the sequence does not split.

f.) If $x = w_o$, the domain corresponds to a projective Verma module; hence the statement follows from character formulas. If $x \neq w_o$ a nontrivial morphism $f \in \mathrm{Hom}_{\mathcal{H}}(\mathcal{P}_x, \mathcal{P}_{tx})$ induces a nontrivial element of $\mathrm{Hom}_{\mathcal{H}}(\mathcal{P}_{(xz,x)}, \mathcal{P}_{(txz,tx)})$ for all simple reflections z with the property $xz > x$. Interchanging the right and the left action of $\mathfrak{g}$, gives rise to an element $f_1 \in \mathrm{Hom}_{\mathcal{H}}(\mathcal{P}_{(x,xz)}, \mathcal{P}_{(tx,txz)})$ which is not the zero map.
Continuing in this way, one finally ends up with an inclusion

$$\mathrm{Hom}_{\mathcal{H}}(\mathcal{P}_x, \mathcal{P}_{tx}) \hookrightarrow \mathrm{Hom}_{\mathcal{H}}(\mathcal{P}_{w_o}, \mathcal{P}_{w_o a}) = \mathrm{Hom}_{\mathfrak{g}}(\Delta(0), \mathbf{C}_{aw_o} \nabla(w_o a \cdot 0)) = \mathbb{C}$$

for some simple reflection a. The existence of at least one nontrivial element in the space of morphisms in question is well-known ([15, 4.7]).

■

Remark 4.3. Let $x \in W$ and let s be a simple reflection such that $sx > x$. There is an exact four-step sequence of the form

$$0 \longrightarrow \mathcal{P}_{(x,sx)} \longrightarrow \mathcal{P}_{(sx,sx)} \xrightarrow{f_{sx,x}} \mathcal{P}_{(x,x)} \longrightarrow \mathcal{P}_{(x,sx)} \longrightarrow 0$$

where the outer maps are the canonical ones (see [15]). This is the so-called *Duflo-Zhelobenko* exact sequence. Since the image of a $f_{sx,x}$ contains the simple module corresponding to the trivial weight, we get the following nontrivial map

$$\psi_x = f_{s_{l+1}x,x} \circ \cdots \circ f_{s_1 w_o, s_2 s_1 w_o} \circ f_{w_o, s_1 w_o},$$

for $w_o = s_r s_{r-1} \cdot \ldots \cdot s_1$ and $x = s_l \cdot \ldots \cdot s_1$ some reduced expressions. Up to a scalar this map must be the Duflo-map from the principal series module $\mathcal{P}_{w_o}$ corresponding to the dominant Verma module into the principal series $\mathcal{P}_x$. On the other hand, the Endomorphism Theorem shows that the map in the middle of the Duflo-Zhelobenko sequence is unique up to a scalar. This is important for the definition of a graded version of this sequence, defined in [22], which describes the

composition factors of the image of the Duflo-map.
We proved that all principal series have trivial endomorphism rings. Note that the 'converse' is not true: i.e. given a module M with the same character as a Verma module and having trivial endomorphism ring then it does not have to be a principal series module/twisted Verma module in general. An example can be given for type A_2.

The previous Theorem together with the next one indicates that principal series modules in general have similar properties as Verma modules. However, there are some differences. For example, their socle and radical filtrations do not coincide in general (see section 4.).

Theorem 4.2. Extensions of Principal Series

1.) All principal series have trivial self-extension, i.e.

$$Ext^1_{\mathcal{O}}\big(\mathcal{P}_{(x,y)} \otimes_{\mathcal{U}} \Delta(0), \mathcal{P}_{(x,y)} \otimes_{\mathcal{U}} \Delta(0)\big) = 0$$

for all x, $y \in W$.

2.) Let x and $y \in W$. Let $A = \mathcal{L}\big(\Delta(x\cdot 0), \nabla(y\cdot 0)\big)$ and $B = \mathcal{L}\big(\Delta(x\cdot 0), \nabla(ys\cdot 0)\big)$, where s is a simple reflection such that $y > ys$.
Then the following statements concerning extensions in $\mathcal{O}$ hold for $\tilde{A} := A \otimes_{\mathcal{U}} \Delta(0)$ and $\tilde{B} := B \otimes_{\mathcal{U}} \Delta(0)$:

a.) $\mathrm{Ext}^1(\tilde{A}, \tilde{B}) = 0$.

b.) $\mathrm{Ext}^1(\tilde{A}, \theta_s\tilde{A}) \cong \mathrm{Ext}^1(\tilde{B}, \theta_s\tilde{B}) = 0$.

c.) $\mathrm{Ext}^1(\tilde{B}, \tilde{A}) \cong \mathbb{C}$.

Remark 4.4. With $x = w_o$, these are well-known results about Verma modules.

Proof. a.) Let $A \xrightarrow{f} E \xrightarrow{g} A$ be an extension with trivial central character from the right. If $x = w_o$, the longest element of the Weyl group, then $\tilde{A}$ is a Verma module and the sequence splits. Assume $x \neq w_o$ and the assertion was true for all Weyl group elements having greater length. Let $t = t_\alpha$ be a simple reflection such that $xt > x$. Translation through the t-wall from the right hand side gives the following commuting diagram in $\mathcal{H}$:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] & E \arrow[r, "g", two heads] & A \arrow[r] & 0 \\
0 \arrow[r] & \theta_t^r A \arrow[r, "\theta_t^r f"] \arrow[u, two heads, "\mathrm{can}"] & \theta_t^r E \arrow[r, "\theta_t^r g", two heads] \arrow[u, two heads] & \theta_t^r A \arrow[r] \arrow[u, two heads, "\mathrm{can}"] & 0
\end{tikzcd}$$

The (canonical) map in the middle is surjective according due to the Five Lemma. The Snake Lemma yields the kernel sequence

$$C := \mathcal{L}(\Delta(xt\cdot 0), \nabla(y\cdot 0)) \hookrightarrow E' \twoheadrightarrow \mathcal{L}(\Delta(xt\cdot 0), \nabla(y\cdot 0)).$$

By [15, Proposition 3.2]) E' is isomorphic to the bimodule corresponding to the object in $\mathcal{O}$ which arises as the α-completion of the module in $\mathcal{O}$ corresponding to E. In particular, E' has trivial central character from the right and $\theta_t^r E' \cong \theta_t^r E$.
First of all we determine dimensions of some homomorphism spaces. The kernel sequence splits by inductive assertion hence,

$$\begin{aligned}
\dim \mathrm{Hom}_{\mathcal{H}}(\theta_t^r A, E) &= \dim \mathrm{Hom}_{\mathcal{H}}(\theta_t^r C, E)\\
&= \dim \mathrm{Hom}_{\mathcal{H}}(C, \theta_t^r E) = \dim \mathrm{Hom}_{\mathcal{H}}(C, \theta_t^r E')\\
&= \dim \mathrm{Hom}_{\mathcal{H}}(\theta_t^r C, E') = \dim \mathrm{Hom}_{\mathcal{H}}(\theta_t^r C, C \oplus C)\\
&= \dim \mathrm{Hom}_{\mathcal{H}}(\theta_t \eta(C), \eta(C) \oplus \eta(C)) = 2
\end{aligned}$$

by Theorem 4.1 2b.
On the other hand $\mathrm{Hom}_{\mathcal{H}}(C, E) = 0$, since a nontrivial morphism would either have its image contained in the image of f, or composition with g would be non-zero. Both situations would imply the existence of a nontrivial homomorphism from A to C. Applying η this contradicts Theorem 4.1 2a.
The following exact sequence

$$\mathrm{Hom}_{\mathcal{H}}(A, E) \hookrightarrow \mathrm{Hom}_{\mathcal{H}}(\theta_t^r A, E) \to \mathrm{Hom}_{\mathcal{H}}(C, E),$$

gives $\dim \mathrm{Hom}_{\mathcal{H}}(A, E) = 2$.
The exact sequence we started with gives rise to an exact sequence of the form

$$0 \to \mathrm{Hom}_{\mathcal{H}}(A, A) \xrightarrow{f\circ} \mathrm{Hom}_{\mathcal{H}}(A, E) \xrightarrow{g\circ} \mathrm{Hom}_{\mathcal{H}}(A, A).$$

Considering the dimensions (1-2-1) yields the surjectivity of the outer right map. Hence, a preimage of the identity gives a splitting. Therefore, the statement follows.

b.) Let

$$B \xrightarrow{f} E \xrightarrow{g} A \tag{13}$$

be an extension with trivial central character from the right. Let $0 \neq h \in \mathrm{Hom}_{\mathcal{H}}(A, E)$. If $\operatorname{im} h \subseteq \operatorname{im} f$ then this contradicts Theorem 4.1 2a; hence $g \circ h \neq 0$. Theorem 4.1 1. shows that h has to be the desired splitting up to a scalar. So we have to find a reason why such an h should exist. Let t be a simple reflection such that $xt > t$. Applying θ_t^r to (13) yields a kernel sequence

$$B' \hookrightarrow E' \twoheadrightarrow A',$$

which splits. (E' has trivial central character from the right, since it corresponds to $\mathbf{C}_\alpha(E \otimes_{\mathcal{U}} \Delta(0))$ by [15, 3.2]). Choosing some splitting $\phi \in \mathrm{Hom}_{\mathcal{H}}(A', E')$ yields by functorality a map $\theta_t^r \phi \in \mathrm{Hom}_{\mathcal{H}}(\theta_t^r A', \theta_t^r E')$. A similar diagram to the one in a.) gives a nontrivial element of $\mathrm{Hom}_{\mathfrak{g}}(A, E)$. This is just a morphism h such as we were looking for.

c.) The exact sequence (12) gives an exact sequence

$$\ldots \;\rightarrow\; \operatorname{Ext}^1_{\mathcal{O}}(\tilde{A},\tilde{A}) \rightarrow \operatorname{Ext}^1_{\mathcal{O}}(\tilde{A},\theta_s\tilde{A}) \rightarrow \operatorname{Ext}^1_{\mathcal{O}}(\tilde{A},\tilde{B}) \rightarrow \ldots .$$

We already know that the outer terms are trivial, so the one in the middle must be trivial as well. This proves the first statement. The second one follows directly from the isomorphism $\theta_s\tilde{A} \cong \theta_s\tilde{B}$ and the selfadjointness of θ_s.

d.) The exact sequence (12) gives rise to an exact sequence

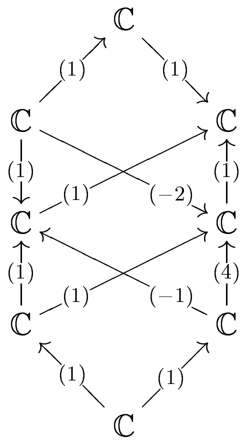
$$\begin{array}{ccccccc} 0 & \rightarrow & \operatorname{Hom}_{\mathcal{O}}(\tilde{B},\tilde{A}) & \rightarrow & \operatorname{Hom}_{\mathcal{O}}(\tilde{B},\theta_s\tilde{A}) & \rightarrow & \operatorname{Hom}_{\mathcal{O}}(\tilde{B},\tilde{B}) \\ & \rightarrow & \operatorname{Ext}^1_{\mathcal{O}}(\tilde{B},\tilde{A}) & \rightarrow & \operatorname{Ext}^1_{\mathcal{O}}(\tilde{B},\theta_s\tilde{B}) & \rightarrow & \ldots . \end{array}$$

Comparing the dimensions (1-1-1-?-0) implies that $\dim \operatorname{Ext}^1_{\mathcal{O}}(\tilde{B},\tilde{A}) = 1$. So we are done.

■

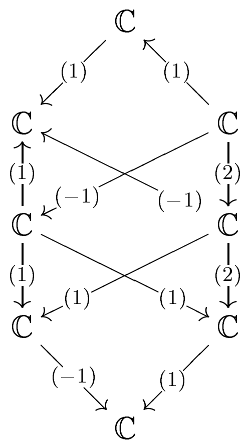
4.1. An explicit Example. In [21] the author computed quivers of the category $\mathcal{O}$. Using these results it is possible to compute quiver representations corresponding to the principal series for root systems of rank 2 (details can be found in [22]). If we consider type B_2, we get the following representations of principal series, where α is the long simple root.

B_2 **principal series** $\mathbf{C}_\alpha\nabla(s_\alpha\cdot 0)$



$L_3 < L_1, L_4, L_5$
$< L_2, L_6, L_7 < L_8$ (S,G)
$L_3 < L_4, L_5 < L_1$,
$L_6, L_7 < L_2, L_8$ (R)

and its dual $\mathbf{C}_\beta\mathbf{C}_\alpha\mathbf{C}_\beta\nabla(s_\beta s_\alpha s_\beta\cdot 0)$



$L_2, L_8 < L_1, L_6$,
$L_7 < L_4, L_5 < L_3$ (S)
$L_8 < L_2, L_6, L_7 <$
$< L_1, L_4, L_5 < L_3$ (R,G)

The arrows representing zero maps are omitted. Unless otherwise stated the map corresponding to an arrow is just the identity. Each vertex corresponds to a simple module which we number from 1 to 8 (from above and from the left). The vertices are arranged according to the highest weights of the corresponding simple modules. Without much effort, the socle (S) and radical (R) filtrations can be computed. They are listed below the pictures. The letter (G) indicates the 'weight filtration' defined in [6]. An algebraic approach to this filtration can be found in [22].

We can observe that the socle and the radical filtrations do not coincide in this case. We also see that the head of the first of the two representations above is a direct summand of *two* simples; hence, there are at least two linearly independent morphisms from this module to its dual.

5. An application: Twisted tilting modules

In this section we define some generalized tilting modules. These modules are indecomposable and have filtrations whose subquotients are principal series modules (or twisted Verma modules).
Denote by $\Delta^x(y)$ the object $\mathcal{L}(\Delta(x\cdot 0), \nabla(y\cdot 0)) \otimes_{\mathcal{U}} \Delta(0)$ in $\mathcal{O}$. We call these modules $w_o x$*-twisted* Verma modules. In the special case $x = w_o$, they are just Verma modules (twisted by the identity); the case $x = e$ gives dual Verma modules.

The basic property of twisted Verma modules, on which the existence of twisted tilting modules relies, is formulated in the following

Lemma 5.1. *Let x, y, $z \in W$. The following implications hold:*

1. $\mathrm{Hom}_{\mathfrak{g}}(\Delta^x(y), \Delta^x(z)) \neq 0 \Longrightarrow y \leq z$.
2. $\mathrm{Ext}^1_{\mathcal{O}}(\Delta^x(y), \Delta^x(z)) \neq 0 \Longrightarrow y < z$.

Proof. Assume there is a nontrivial morphism $f \in \mathrm{Hom}_{\mathcal{H}}\left(\Delta^x(y), \Delta^x(z)\right)$. It induces (as in the proof of Theorem 4.1) inductively a nontrivial morphism $g \in \mathrm{Hom}_{\mathcal{H}}\left(\Delta^{w_o}(y), \Delta^{w_o}(z)\right)$. On the other hand, the existence of such a morphism between Verma modules implies that the inequality $w_o y \geq w_o z$ holds; equivalently $y \leq z$. So the first statement is true.
To prove the second statement we can assume $y \not\leq z$ because we have already proven that there are no nontrivial self-extensions. When $x = e$ the statement is well-known. Given an exact sequence of the form

$$\Delta^x(z) \hookrightarrow E \twoheadrightarrow \Delta^x(y) \tag{14}$$

consider the exact sequence

$$0 \to \mathrm{Hom}_{\mathfrak{g}}(\Delta^x(y), \Delta^x(z)) \longrightarrow \mathrm{Hom}_{\mathfrak{g}}(\Delta^x(y), E) \longrightarrow \mathrm{Hom}_{\mathfrak{g}}(\Delta^x(y), \Delta^x(y)).$$

As we proved above, the first term is trivial and the outer right term is of dimension one. Therefore, it is sufficient to show that $\mathrm{Hom}_{\mathfrak{g}}(\Delta^x(y), E) \neq 0$, since then a preimage of the identity $\mathrm{id} \in \mathrm{End}_{\mathfrak{g}}(\Delta^x(y))$ gives a splitting of the sequence (14). Using similar arguments to those in Theorem 4.2 2a ensures the existence of such a nontrivial morphism. ■

Standard arguments to construct tilting modules (see e.g. [20]) imply the existence of twisted tilting modules

Theorem 5.1. Existence and Character of Twisted Tilting Modules

1. *For all $y \in W$, there exists an indecomposable module $T^x(y) \in \mathcal{O}_0$, unique up to isomorphism, with the following properties:*

a.) $\mathrm{Ext}^1(\Delta^x(z), T^x(y)) = 0$ *for all* $z \in W$

b.) $T^x(y) \in \mathcal{O}_0$ *has a* Δ^x*-flag, (i.e. a flag whose subquotients are isomorphic to some* $\Delta^x(z)$*) starting with* $\Delta^x(y) \subseteq T^x(y) \in \mathcal{O}_0$.

2. The characters of these modules are given by the following formula:

$$[T^x(y)] = \sum_{z \in W} [T(w_o y \cdot 0) \ : \ \Delta(w_o z \cdot 0)][\Delta^x(z)],$$

where $T^{w_o}(y) = T(w_o y \cdot 0)$ *denotes the 'usual' tilting module belonging to the weight* $w_o y \cdot 0$.

Proof. The first part is [20, Proposition 3.1].
For the second part we first construct modules with the desired character formulas and then we will show afterwards that they fulfill the conditions to be twisted tilting modules.
For $x = w_o$, there is nothing to do, since in this case we have just the 'usual' tilting modules and the character formula follows directly from the definitions. Given $x \in W$, let s be a simple reflection such that $xs < x$. We assume the character formula holds for T^x.
The exact sequence

$$\Delta^x(y) \hookrightarrow T^x(y) \twoheadrightarrow \mathrm{coker}$$

gives rise to an exact sequence of the form

$$\begin{tikzcd}
0 \arrow[r] & \Delta^x(y) \arrow[r, hook, "f"] \arrow[d, "\mathrm{can}"] & T^x(y) \arrow[r, two heads, "g"] \arrow[d, "\mathrm{can}"] & \mathrm{coker} \arrow[r] \arrow[d, "\mathrm{can}"] & 0 \\
0 \arrow[r] & \theta_s^r \Delta^x(y) \arrow[r, hook, "\theta_t^r f"] & \theta_s^r T^x(y) \arrow[r, two heads, "\theta_t^r g"] & \theta_s^r \,\mathrm{coker} \arrow[r] & 0.
\end{tikzcd}$$

The vertical maps are all inclusions: the left hand one is obvious and the right hand one follows by induction on the length of a flag using the Five Lemma. Hence, the map in the middle is also an inclusion. The cokernel sequence is of the form

$$\Delta^{xs}(y) \hookrightarrow T \twoheadrightarrow K.$$

It is easy to see that K has a Δ^{xs}-flag: Given a Δ^x-flag $0 \subset F_1 \subset \ldots \subset F_r = \mathrm{coker}$ of the cokernel we can define a filtration $0 \subset G_1 \subset \ldots \subset G_r = K$ of K where G_i is the cokernel of the canonical map $F_i \hookrightarrow \theta_s^r F_i$.
The argumentation in the proof of Lemma 4.1 shows, that the indecomposability of $T^x(y)$ implies the indecomposability of T.
We now show, that property 1a. holds for T.
Assume that there is a nontrivial extension

$$0 \to T \xrightarrow{f} E \xrightarrow{g} \Delta^{xs}(z) \to 0. \tag{15}$$

Translation (from the right) through the s-wall gives a kernel sequence of the form

$$0 \to T^x(y) \longrightarrow E' \twoheadrightarrow \Delta^x(z) \to 0. \tag{16}$$

Assume that E is indecomposable, then E' is also indecomposable (see proof of Lemma 4.1). This is a contradiction to T^x being an x-twisted tilting module, i.e.

(16) splits. Hence, $E \cong A \oplus B$ for some modules A and B in $\mathcal{O}_0$. Since T is indecomposable, the image of f is contained in one of these two summands; say $f(T) \subseteq A$. This implies that $A/f(T) \oplus B \cong \Delta^{xs}(z)$. Due to the indecomposability of the principal series $f(T)$ has to be A and $B \cong \Delta^{xs}(z)$. By virtue of this last isomorphism we can construct a nontrivial morphism from $\Delta^{xs}(z)$ to E whose image has trivial intersection with $f(T)$. This means, it is also nontrivial after composition with g. Since the endomorphism ring of $\Delta^{xs}(z)$ is one-dimensional, we have constructed, up to a scalar, a splitting of g.
Therefore, $\mathrm{Ext}^1_{\mathcal{O}}(\Delta^{xs}(z \cdot 0), T) = 0$. Altogether we showed that T is a module having the properties characterizing $T^{xs}(y)$.
Inductively, the construction of T gives the desired character formula

$$\begin{aligned}[T(w_o y) : \Delta(w_o z)] &= [T^{w_o}(y) : \Delta^{w_o}(z)] \\ &= [T^{w_o s}(y) : \Delta^{w_o s}(z)] \\ &= [T^x(y) : \Delta^x(z)].\end{aligned}$$

■

Remark 5.2.

- The antidominant projective module is an x-twisted tilting module for all $x \in W$. This follows from the fact that this module becomes a direct sum of copies of itself after translating through the wall (see [15, Lemma 3.16]). In particular, the antidominant projective module comes up with a lot of different filtrations. This result is also contained in [11, Theorem 4.1].

- These twisted tilting modules do not necessarily have (in contrast to "usual" tilting modules) a dual Δ^x-flag (or $\Delta^{w_o x}$-flag).

- Independently of the type of the Lie algebra $T^x(e) = \Delta^x(0) \cong \nabla(x^{-1} \cdot 0)$ holds.

- For $\mathfrak{sl}_2$ the "usual" tilting modules are $T^s(s) = T(0) = P(s \cdot 0)$ and $T^s(e) = T(s \cdot 0) = \Delta(s \cdot 0)$. On the other hand there are the s-twisted tilting modules $T^e(e) = \nabla(0)$ and the antidominant projective equipped with the flag $\Delta^e(s) = \Delta(s \cdot 0) \hookrightarrow T^e(s) \twoheadrightarrow \nabla(0)$.
  Without difficulty we can check the character formula

$$\begin{aligned}T^e(e) &= \sum_{z \in W} [T(s \cdot 0) : \Delta(w_o z \cdot 0)][\Delta^e(z)] \\ &= [\Delta(s \cdot 0) : \Delta(s \cdot 0)][\Delta^e(e)] \\ &= [\nabla(0)].\end{aligned}$$

  And the one for the second 'non-usual' tilting module

$$\begin{aligned}T^e(s) &= \sum_{z \in W} [T(0) : \Delta(w_o z \cdot 0)][\Delta^e(z)] \\ &= \sum_{z \in W} [P(s \cdot 0) : \Delta(w_o z \cdot 0)][\Delta^e(z)] \\ &= [\Delta^e(e)] + [\Delta^e(s)] \\ &= [P(s \cdot 0)].\end{aligned}$$

- The flag of the antidominant projective considered as an x-twisted tilting module ends with $\nabla(x^{-1}w_o\cdot 0)$ and starts with the Verma module $\Delta(x^{-1}\cdot 0)$.
- Just recently V. Mazorchuk ([18]) proved that all usual tilting modules have a x-twisted Verma flag for any $x \in W$.

References

[1] Andersen, H. and N. Lauritzen, *Twisted Verma modules*, arXiv.org Mathematics e-Print archive, math. QA/0105012.

[2] Arkhipov, S., *A new construction of the semi-infinite BGG-resolution*, arXiv.org Mathematics e-Print archive, q-alg/9605043.

[3] Bernstein, I., and S. I. Gelfand, *Tensor products of finite and infinite dimensional representations of semisimple Lie algebras*, Compositio math. **41** (1980), 245–285.

[4] Bernstein, I., I. M. Gelfand, and S. I. Gelfand, *A category of $\mathfrak{g}$-modules*, Funct. Anal. and Appl. **10** (1976), 87–92.

[5] Beilinson, A. and V. Ginzburg, *Wall-crossing functors and $\mathcal{D}$-Modules*, Represent. Theory **3** (1999), 1–31.

[6] Casian, L., and D. Collingwood, *Weight Filtrations for Induced Representations of Real Reductive Lie Groups*, Adv. Math. **73** (1989), 79–146.

[7] Dixmier, J., "Enveloping Algebras," Graduate Studies in Mathematics **11**, Amer. Math. Soc. 1996.

[8] Duflo, M., *Sur la classification des ideaux primitifs dans l'algebre enveloppante d'une algebre de Lie semi-simple*, Ann. Math. **105** (1977), 107–120.

[9] Enright, T., *On the fundamental series of a real semisimple Lie algebra their irreducibility, resolutions and multiplicity formula*, Ann. Math. **110** (1979), 1–82.

[10] Irving, S., *The socle filtration of a Verma module*, Ann. scient. Ec. Norm. Sup. **21** (1988), 47–65.

[11] Irving, S., *Shuffled Verma modules and principal series modules over complex semisimple Lie algebras*, J. London Math. Soc. 2 **48** (1993), 263–277.

[12] Jantzen, J. C., „Moduln mit einem höchsten Gewicht", Springer-Verlag, Berlin etc., 1979.

[13] —, „Einhüllende Algebren halbeinfacher Liealgebren", Springer-Verlag, Berlin etc., 1983.

[14] —, "Representations of algebraic groups," Pure and Applied Mathematics, **131**, Academic Press 1987.

[15] Joseph, A., *The Enright Functor on the Bernstein-Gelfand-Gelfand Category $\mathcal{O}$*, Invent. Math. **67** (1982), 423–445.

[16] —, *Completion Functors in the $\mathcal{O}$ Category*, In: Non Commutative Harmonic Analysis and Lie Groups, Springer Lecture Notes **1020** (1983), 80–106.

[17] Knapp, A., "Representations of semisimple Lie groups," IAS/Park City Mathematics Series **8**, Amer. Math. Soc., 2000.

[18] Mazorchuk, V., *Twisted and shuffled filtrations on tilting modules*, Preprint.

[19] Soergel, W., *The combinatorics of Harish-Chandra bimodules*, Journal Reine Angew. Math. **429** (1992), 49–74.

[20] —, *Character formulas for tilting modules over quantum groups at a root of one*, Current developments in mathematics, 1997 (Cambridge, MA), 161–172, Int. Press, Boston, MA, 1999.

[21] Stroppel, C., *Category $\mathcal{O}$: Quivers and Endomorphism Rings of Projectives*, Preprint.

[22] —, „Der Kombinatorikfunktor $\mathbb{V}$: Graduierte Kategorie $\mathcal{O}$, Hauptserien und primitive Ideale“, Dissertation, Universität Freiburg i. Br., 2001.

[23] Varadarajan, V. S., “An Introduction to Harmonic Analysis on Semisimple Lie Groups,” Cambridge Studies in Adv. Math. **16**, Cambridge University Press 1989.

[24] Wallach, N., “Real reductive groups 1,” Pure and Applied Mathematics **132**, Academic Press 1988.

[25] —, “Real reductive groups 2,” Pure and Applied Mathematics **132-II**, Academic Press 1992.

Catharina Stroppel
University of Leicester
University Road
Leicester LE1 7RH (England)
cs93@le.ac.uk

Received January 28, 2002
and in final form May 02, 2002