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Local Integrating Factors

Dedicated to D. Pumplün on the occasion of his seventieth birthday

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Abstract. This is an investigation of local (analytic or formal) integrating factors near certain degenerate stationary points of plane analytic vector fields. The method is to use blow-ups and then apply known results from the non-degenerate case. The main result is that in general there is no formal integrating factor which is algebraic over the quotient field of the formal power series ring.

1. Introduction and preliminaries

Nonlinear symmetries of vector fields are hard to approach and to investigate, but they are relevant both for the structural and the qualitative understanding of ordinary differential equations. In dimension two, the situation is a little easier to understand, due to the correspondence between infinitesimal symmetries and integrating factors. The purpose of this article is to discuss integrating factors of analytic vector fields near degenerate stationary points, continuing the work in [13] and [14]. We note that another motivation for the investigation of integrating factors comes from differential algebra; see Prelle and Singer [10].

We will investigate degenerate stationary points using blow-up techniques. Thus we obtain detailed results on the possible structure of integrating factors in various cases, and as a consequence we find that in general there are not even formal integrating factors.

Let us introduce some notation and review known facts. (References to be consulted here are Lie [7], Olver [8, 9], Prelle and Singer [10], and [13], among others.) All functions and vector fields under consideration will be analytic, unless specified otherwise.

To a given differential equation $\dot{x} = f(x)$ on an open, nonempty subset U of $\mathbf{C}^2$, one associates the derivation L_f which assigns to a function $\phi : U \to \mathbf{C}$ its *Lie derivative* $L_f(\phi)$, with $L_f(\phi)(x) := D\phi(x)\, f(x)$. A function ψ is called a

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first integral, resp. a *semi-invariant*, of $\dot{x} = f(x)$ (or, briefly, of f) if $L_f(\psi) = 0$, resp. $L_f(\psi) = \mu\psi$ for some analytic function μ on U. The set of zeros of a semi-invariant, as well as any level set of a first integral, is an invariant set for $\dot{x} = f(x)$.

We call the vector field f *divergence-free* if $\mathrm{div}(f) := \mathrm{tr}\, Df = 0$. (This definition depends on the choice of coordinates.) To a scalar-valued function ϕ one assigns its *Hamiltonian vector field* $q_\phi := \left(\begin{smallmatrix} -\partial\phi/\partial x_2 \\ \partial\phi/\partial x_1 \end{smallmatrix}\right)$, which is obviously divergence-free, and has the first integral ϕ by construction. Locally, a two-dimensional divergence-free vector field is equal to the Hamiltonian vector field of some function, whence finding a first integral of a divergence-free vector field reduces to a quadrature problem. One directly verifies that $\mathrm{div}(\psi f) = L_f(\psi) + \mathrm{div}(f) \cdot \psi$, so ψf will be divergence-free if and only if $L_f(\psi) + \mathrm{div}(f) \cdot \psi = 0$. If ψ is nonzero, and this equation is satisfied, then we call ψ an *integrating factor* of $\dot{x} = f(x)$. Another function $\tilde{\psi}$ is an integrating factor if and only if $\tilde{\psi}/\psi$ is a first integral or constant. (It should be noted here that there are other versions of the differential equation in question, e.g. $f_2\, dx_1 - f_1\, dx_2 = 0$, which correspond to other, equivalent, notions of integrating factor.)

There is a well-known correspondence between integrating factors and infinitesimal orbital symmetries in dimension two: If g is a vector field such that $[g, f] = \lambda f$ for some analytic λ (with the Lie bracket defined by $[g, f](x) := Df(x) \cdot g(x) - Dg(x) \cdot f(x)$, as usual), then either $\theta(x) := \det(f(x), g(x))$ is identically zero, or θ^{-1} is an integrating factor for f. Conversely, one can construct infinitesimal orbital symmetries from integrating factors, cf. [13], for instance.

By $\mathbf{C}[[x_1, x_2]]_c$ (resp. $\mathbf{C}[[x_1, x_2]]$) we denote the algebra of power series with a nontrivial region of convergence (resp. the algebra of formal power series); see Ruiz [11] for an account of the algebraic properties of these rings. We recall that both are noetherian and unique factorization domains, and that the invertible elements are precisely those with a nonzero constant term. Moreover, by $\mathbf{C}((x_1, x_2))_c$, resp. $\mathbf{C}((x_1, x_2))$ we denote the respective quotient fields. We are interested in integrating factors of local analytic (or formal) vector fields that are algebraic over $\mathbf{C}((x_1, x_2))_c$, resp. over $\mathbf{C}((x_1, x_2))$. As for one motivation, we recall the following classical result of Prelle and Singer ([10], Props. 1 and 2, specialized to our context):

Theorem1.1. *Let $\dot{x} = f(x)$ be given, with f an analytic or formal vector field.*
(a) If this equation has a first integral that is elementary (in the sense of differential algebra) over $\mathbf{C}((x_1, x_2))_c$ (resp. $\mathbf{C}((x_1, x_2))$) then it admits an integrating factor that is algebraic over $\mathbf{C}((x_1, x_2))_c$ (resp. $\mathbf{C}((x_1, x_2))$).
(b) If the equation admits an integrating factor that is algebraic over $\mathbf{C}((x_1, x_2))_c$ (resp. $\mathbf{C}((x_1, x_2))$) then it also admits an integrating factor of the specific form

$$(\varphi_1^{d_1} \cdots \varphi_r^{d_r})^{-1},$$

with the φ_i non-invertible and irreducible semi-invariants, and rational numbers d_i. (The possibility $r = 0$ is included, with integrating factor 1.) ■

In the local theory it is sensible to reserve the notion "semi-invariant" for non-invertible series, and we will do so in the following. We will sometimes employ coordinate transformations in order to simplify a problem. Then an integrating factor $\phi_1^{d_1}\cdots\phi_r^{d_r}\exp(\mu)$ will be changed to the integrating factor $(\phi_1\circ\Psi)^{d_1}\cdots(\phi_r\circ\Psi)^{d_r}\exp(\nu)$ by a coordinate transformation Ψ, with $\exp(\nu)$ also incorporating the functional determinant of Ψ; see [14], for instance.

2. Degenerate stationary points

We assume that $f(0)=0$, thus $f=\sum_{i\geq 1} f^{(i)}$, with each $f^{(i)}$ a homogeneous polynomial of degree i, and abbreviate $B:=f^{(1)}=Df(0)$. Let $B=B_s+B_n$ be the decomposition into semisimple and nilpotent part, and α_1, α_2 the eigenvalues of B. We call the stationary point 0 of the differential equation *nondegenerate* if $B_s\neq 0$ (equivalently, not both α_i are zero), and *degenerate* otherwise. For nondegenerate stationary points we have the Poincaré-Dulac normal form at our disposal (see Bibikov [3], Bruno [4], and [12]): There is an invertible formal power series $\Psi=\mathrm{id}+\ldots$ that is solution-preserving from $\dot{x}=f(x)$ to $\dot{x}=\tilde{f}(x)$, with $\tilde{f}=B+\ldots$ a formal power series vector field in normal form, thus satisfying $[B_s,\tilde{f}]=0$. For easy reference we quote some parts of a result recorded in [14], Thm. 2.3.

Lemma2.1. *Let $B_s=\mathrm{diag}(\alpha_1,\alpha_2)$, with $\alpha_1\neq 0$. Let f be a (formal power series) vector field in normal form.*

(a) If α_2/α_1 is not a rational number then x_1 and x_2 are, up to multiplication with invertible series, the only irreducible semi-invariants of $f=B_s$ in $\mathbf{C}[[x_1,x_2]]$. There is no first integral that is algebraic over $\mathbf{C}((x_1,x_2))$, and the only integrating factor algebraic over $\mathbf{C}((x_1,x_2))$, up to multiplication by scalars, is $(x_1x_2)^{-1}$.

(b) Let $\alpha_2/\alpha_1=-q/p$, with p and q relatively prime positive integers, $p\leq q$, and $f=B+\sum_{j\geq 1}\gamma^j(\sigma_j\mathrm{id}+\tau_jB)$, where $\sigma_j,\tau_j\in\mathbf{C}$, and $\gamma(x):=x_1^qx_2^p$. Then x_1 and x_2 are, up to multiplication with invertible series, the only irreducible semi-invariants of f in $\mathbf{C}[[x_1,x_2]]$. If not all the σ_j are equal to zero then there exists no first integral that is algebraic over $\mathbf{C}((x_1,x_2))$. Up to multiplication by constants, the only integrating factor that is algebraic over $\mathbf{C}((x_1,x_2))$ is

$$\left(x_1^{1+lq}x_2^{1+lp}\rho(x)\right)^{-1}=\left(x_1x_2\gamma(x)^l\rho(x)\right)^{-1},$$

with l the smallest index such that $\sigma_l\neq 0$, and

$$x_1x_2\rho(x)=\det(B(x),\,\sigma_lx+\sum\gamma(x)^i\sigma_{l+i}(x)).$$

(c) Let $\alpha_2=0$, and $f=B+\sum_{j\geq 1}\gamma^j(\sigma_j\mathrm{id}+\tau_jB)$, where $\sigma_j,\tau_j\in\mathbf{C}$, and $\gamma(x):=x_2$. Then x_1 and x_2 are, up to multiplication with invertible series, the only irreducible semi-invariants of f in $\mathbf{C}[[x_1,x_2]]$. If not all the σ_j are equal to zero then there exists no first integral that is algebraic over $\mathbf{C}((x_1,x_2))$. Up

to multiplication by constants, the only integrating factor that is algebraic over $\mathbf{C}((x_1, x_2))$ *is*

$$\left(x_1 x_2^{1+l}\rho(x)\right)^{-1} = \left(x_1 x_2 \gamma(x)^l \rho(x)\right)^{-1},$$

with l *the smallest index such that* $\sigma_l \neq 0$, *and* ρ *is defined similarly to (b).* ∎

The remaining case, with α_2/α_1 rational and positive, is just the case of a dicritical nondegenerate stationary point, which is characterized by the existence of infinitely many irreducible semi-invariants; see [14], loc. cit., for details.

In [14], applications of (2.1) to polynomial vector fields were presented. Here we discuss an application to degenerate stationary points of an analytic differential equation $\dot{x} = f(x)$. Degeneracy then means that $B = Df(0)$ is nilpotent. The general properties of degenerate stationary points are not as well understood as those of nondegenerate ones. A relatively recent result by Camacho and Sad [5] shows that there is always an analytic semi-invariant. Little seems to be known about integrating factors.

A familiar strategy to deal with degenerate stationary points is to use blow-ups (σ-processes), and thus obtain stationary points of simpler type. One method in the real setting is to use polar coordinates (see Guckenheimer and Holmes [6], Ch. 7) to blow up a stationary point into an invariant circle that contains less degenerate stationary points. An other approach employs rational functions, and in essence yields a blow-up of a stationary point to a projective line; see Arnold [2], Ch. 1, §2, and also Anosov et al. [1], Ch. 5, §1, for details. According to theorems by Bendixson and Seidenberg (see [1], loc. cit.), every stationary point can be transformed to a collection of nondegenerate stationary points by a finite series of blow-ups. We will use the second version in an algebraic incarnation, which amounts to employing what Bruno [4] calls "power transformations". In the following, a number of properties of such blow-ups, most of them well-known, will be listed, and ad-hoc proofs are provided for the sake of completeness. (There is no intention to state the results in full generality.)

Lemma2.2. *Define* $\Phi : \mathbf{C}^2 \to \mathbf{C}^2$ *by*

$$\Phi(x) = \begin{pmatrix} x_1 \\ x_1 x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_1(x_2 - a) + x_1 a \end{pmatrix},$$

with $a \in \mathbf{C}$ *arbitrary.*

(a) For any a *the map* Φ *induces a morphism from* $\mathbf{C}[[x_1, x_2]]$ *to* $\mathbf{C}[[x_1, x_2 - a]]$ *via* $\psi \mapsto \psi \circ \Phi$, *and convergent series are mapped to convergent series.*

(b) For any a *there is a unique analytic vector field* f^* *near* $(0, a)^t$ *such that the identity* $D\Phi(x) f^*(x) = f(\Phi(x))$ *holds; thus* Φ *maps parameterized solutions of* $\dot{x} = f^*(x)$ *to parameterized solutions of* $\dot{x} = f(x)$. *There is a unique integer* $s \geq 0$ *such that* $f^* = x_1^s \cdot \tilde{f}$, *and* x_1 *does not divide both entries of* $\tilde{f}$.

(c) Let ψ *be a formal or analytic semi-invariant of* f, *and let* q *be a nonnegative integer and* $\tilde{\psi}$ *a (formal resp. convergent) series such that (at the point* $(0, a)^t$*) one has* $\psi \circ \Phi = x_1^q \cdot \tilde{\psi}$, *with* x_1 *and* $\tilde{\psi}$ *relatively prime. Then* $\tilde{\psi}$ *is invertible or a semi-invariant of* $\tilde{f}$.

(d) Let ρ *be an irreducible local analytic semi-invariant of* $\tilde{f}$ *at* $(0, a)^t$, *with* x_1 *and* ρ *relatively prime, and assume that locally the zero set* Z *of* ρ *is the*

image of some non-constant analytic map $\mu : U \to \mathbf{C}^2$, with $U \subseteq \mathbf{C}$ open and nonempty. Then there is a semi-invariant σ of f such that $\Phi(Z)$ is contained in the zero set of σ.

Proof. (a) For $\psi = \sum c_{ik} x_1^i x_2^k$, the definitions and the binomial formula yield

$$\psi \circ \Phi = \sum_{i,k} c_{ik} \sum_{l=0}^{k} \binom{k}{l} a^l x_1^{i+k} (x_2 - a)^l,$$

and clearly this series is summable, i.e., each monomial $x_1^m (x_2 - a)^n$ occurs only finitely many times with nonzero coefficient.
(b) Since $f(0) = 0$, each entry of $f \circ \Phi$ is divided by x_1, whence the factor x_1^{-1} in the product with $D\Phi(x)^{-1}$ cancels. The second assertion is obvious.
(c) By a general argument, $\psi^* := \psi \circ \Phi$ is a semi-invariant of f^*. Since products and (non-invertible) divisors of semi-invariants are again semi-invariants, and x_1 is a semi-invariant of f^*, $\tilde{\psi}$ is also a semi-invariant of f^*. Thus $x_1^s L_{\tilde{f}}(\psi) = \nu \cdot \psi$ for some analytic ν, and x_1^s must divide ν. But this implies that $\tilde{\psi}$ is a semi-invariant for $\tilde{f}$.
(d) The image $\Phi(Z) = \Phi(\mu(U))$ is contained in some one-dimensional local analytic set, which coincides with the local zero set of some analytic σ; cf. Ruiz [11]. Since Z is locally invariant for $\tilde{f}$ and for f^*, $\Phi(Z)$ is locally invariant for $\dot{x} = f(x)$, whence σ is a semi-invariant of f. ∎

Remark2.3. (a) Obviously one gets analogous results for $\Theta(x) = \begin{pmatrix} x_1 x_2 \\ x_2 \end{pmatrix}$, and for compositions of Φ and Θ.
(b) In the situation of (2.2)(c), $\tilde{\psi}$ can only be a semi-invariant (i.e., non-invertible) if $(0, a)^t$ is a stationary point of $\tilde{f}$. If a nondegenerate stationary point of $\tilde{f}$ admits a linearization of a type as enumerated in (2.1), then the hypothesis of (2.2)(d) is satisfied, due to smoothness of the zero set of the semi-invariant and the implicit function theorem.

Let us now return to $\dot{x} = f(x)$, with $Df(0) = B$. If $B \neq 0$ then we may assume that $B = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, and this case will be discussed next. We will use a suitable normalization, and for our purposes the Arnold-Bogdanov normal form seems the most convenient. The first statement of the following lemma is well-known (see, for instance, Guckenheimer and Holmes [6], 7.3), while the second results from an elementary computation.

Lemma2.4. *Let $B = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, and $f(x) = B(x) + \sum_{j \geq 2} f^{(j)}(x)$, with $f^{(j)}$ homogeneous of degree j, a formal power series vector field. Then there exists an invertible formal transformation into Arnold-Bogdanov normal form*

$$f^*(x) = B(x) + \sum_{j \geq 2} \begin{pmatrix} 0 \\ \alpha_j x_1^{j-1} x_2 + \beta_j x_1^j \end{pmatrix}.$$

If f is analytic then for any integer $k \geq 2$ there is an analytic transformation into a vector field which is in normal form up to degree k. For

$$f(x) = B(x) + \begin{pmatrix} \rho_1 x_1^2 + \rho_2 x_1 x_2 + \rho_3 x_2^2 \\ \sigma_1 x_1^2 + \sigma_2 x_1 x_2 + \sigma_3 x_2^2 \end{pmatrix} + \text{t. h. o.},$$

(with "t.h.o." standing for "terms of higher order"), one has

$$f^*(x) = B(x) + \begin{pmatrix} 0 \\ \sigma_1 x_1^2 + (\sigma_2 + 2\rho_1)x_1x_2 \end{pmatrix} + \text{t. h. o.}$$

■

Theorem2.5. *The analytic equation*

$$\dot{x} = f(x) := \begin{pmatrix} x_2 \\ 0 \end{pmatrix} + \sum_{j=2}^{k} \begin{pmatrix} 0 \\ \alpha_j x_1^{j-1} x_2 + \beta_j x_1^j \end{pmatrix} + \text{t. h. o.},$$

with $\alpha_2 \neq 0$ and $\beta_2 \neq 0$, and $k \geq 4$, admits a unique irreducible analytic semi-invariant $\phi(x) = x_2^2 + \text{t. h. o.} \in \mathbf{C}[[x_1, x_2]]$, up to multiplication by an invertible series. Furthermore, if there is an integrating factor that is algebraic over $\mathbf{C}((x_1, x_2))$ then it is equal to $\phi^{-7/6}\exp(\mu)$, for some series μ which is unique up to an additive constant.

Proof. (i) To show the assertion about semi-invariants, we use a blow-up. We adopt the procedure in Bruno [4], Ch. II, 3.5. The map

$$\Psi(x) = \begin{pmatrix} x_1^2 x_2 \\ x_1^3 x_2^2 \end{pmatrix}$$

is solution-preserving from $\dot{x} = x_1 \tilde{f}(x)$, with

$$\tilde{f}(x) = \begin{pmatrix} -\beta_2 x_1 + 2x_1x_2 - \alpha_2 x_1^2 x_2 + \dots \\ 2\beta_2 x_2 - 3x_2^2 + 2\alpha_2 x_1 x_2^2 + \dots \end{pmatrix},$$

to $\dot{x} = f(x)$. (In the notation of (2.2) and (2.3) one has $\Psi = \Phi \circ \Theta \circ \Phi$, hence the results above are applicable.) The inverse image of $\{0\}$ under Ψ is the union Y of the coordinate axes. Therefore we have induced maps $\Psi^* = \Psi^*_{(a_1,\,0)} : \mathbf{C}[[x_1, x_2]] \to \mathbf{C}[[x_1 - a_1, x_2]]$, $\sigma \mapsto \sigma \circ \Psi$, and $\Psi^* = \Psi^*_{(0,\,a_2)} : \mathbf{C}[[x_1, x_2]] \to \mathbf{C}[[x_1, x_2 - a_2]]$, $\sigma \mapsto \sigma \circ \Psi$, for all $a_1, a_2 \in \mathbf{C}$. Also note that x_1 and x_2 are semi-invariants of $\tilde{f}$, or invertible, near any point of Y.

Let ψ be a semi-invariant of f. Then $\psi \circ \Psi$ is a semi-invariant of $\tilde{f}$ or invertible at any point of Y, and the same holds for the series $\tilde{\psi}$ obtained from $\psi \circ \Psi$ by dividing out maximal powers of x_1 and x_2. Given a point $(a_1, a_2)^t$ on Y (so $a_1 = 0$ or $a_2 = 0$) such that the corresponding $\tilde{\psi}$ is not invertible, this point must be stationary for $\tilde{f}$. On Y there are the two stationary points $(0, 0)$ and $(0, \frac{2\beta_2}{3})$. (The higher order terms have no influence on this.) From $D\tilde{f}(0, 0) = \mathrm{diag}(-\beta_2, 2\beta_2)$ and (2.1) follows that x_1 and x_2 are the only irreducible semi-invariants for $\tilde{f}$ at $(0, 0)$. Their vanishing sets are contained in Y, and therefore $\tilde{\rho}$ (taken at $(0, 0)$) is invertible for any semi-invariant ρ of f. The eigenvalues of $D\tilde{f}(0, \frac{2\beta_2}{3})$ are $\frac{\beta_2}{3}$ and $-2\beta_2$; hence there are exactly two irreducible semi-invariants, x_1 being one of them. The other one is easily computed as

$$\tilde{\phi} := 8\alpha_2\beta_2 x_1 - 21(x_2 - \frac{2\beta_2}{3}) + \text{t. h. o.}$$

Thus, by (2.2)(c) there is only one irreducible (formal and analytic) semi-invariant ϕ for $\dot{x} = f(x)$ at 0. To get its initial terms explicitly, one may proceed as in Bruno [4] , loc. cit., and use (2.2)(d). Computing the eigenvector of $D\tilde{f}(0,\, 2\beta_2/3)$ yields a local analytic parameterization

$$v(s) = \begin{pmatrix} 21s + \dots \\ 2\beta_2/3 + 8\alpha_2\beta_2 s + \dots \end{pmatrix}$$

of the zero set of $\tilde{\phi}$, and thus

$$\Psi(v(s)) = \begin{pmatrix} 2\beta_2/3 \cdot 21^2 s^2 + \dots \\ (2\beta_2/3)^2 \cdot 21^3 s^3 \dots \end{pmatrix}$$

is a local analytic parameterization of the zero set of ϕ. This leads to

$$\phi(x) = x_2^2 - \frac{2}{3}\beta_2 x_1^3 + \dots .$$

(ii) Consider the equation $L_f(\phi) = d \cdot \operatorname{div}(f) \cdot \phi$, for some rational d. With $\operatorname{div}(f) = \alpha_2 x_1 + \dots$, and $\phi = x_2^2 + \phi_3 + \dots$ one obtains by comparing terms of degree 3 that

$$x_2 \frac{\partial \phi_3}{\partial x_1} + (\alpha_2 x_1 x_2 + \beta_2 x_1^2) \cdot 2x_2 = d \cdot \alpha_2 x_1 x_2^2,$$

which yields

$$\phi_3 = -\frac{2}{3}\beta_2 x_1^3 + \left(\frac{d}{2} - 1\right) \alpha_2 x_1^2 x_2 + \rho_3 x_2^3.$$

Comparing homogeneous terms of degree 4 yields an equation $x_2 \frac{\partial \phi_4}{\partial x_1} = \dots$, and the condition that the coefficient of x_1^4 must vanish in the remaining terms turns out to force $\left(1 - \frac{d}{2}\right)\alpha_2\beta_2 = \frac{2}{3} d \alpha_2 \beta_2$, hence $d = \frac{6}{7}$.
(iii) The equation admits no (nonconstant) first integral in $\mathbf{C}((x_1,\, x_2))$. Such a first integral σ would be a product of powers of irreducible semi-invariants and an invertible series. Since ϕ is the only irreducible semi-invariant, this would imply that $\sigma = \phi^k \exp(\nu)$. One may assume that $k = 1$ (take some power of σ, which is also a first integral, and note that $\sigma(0) = 0$ w.l.o.g.). But now the result of (ii), with $d = 0$, yields a contradiction. As a consequence, there is no first integral which is algebraic over $\mathbf{C}((x_1,\, x_2))$, since the coefficients of its minimum polynomial would themselves be first integrals or constant.

From (ii) it also follows that $L_f(\phi \exp(\mu))^{1/d} = \operatorname{div}(f)(\phi \exp(\mu))^{1/d}$ forces $d = \frac{6}{7}$, and the integrating factor is unique according to the previous paragraph. ∎

Corollary2.6. *(a) If the vector field*

$$f(x) = B(x) + \begin{pmatrix} \rho_1 x_1^2 + \rho_2 x_1 x_2 + \rho_3 x_2^2 \\ \sigma_1 x_1^2 + \sigma_2 x_1 x_2 + \sigma_3 x_2^2 \end{pmatrix} + \text{t. h. o.},$$

with $\sigma_1 \neq 0$*, and* $2\rho_1 + \sigma_2 \neq 0$*, admits an integrating factor that is algebraic over* $\mathbf{C}((x_1, x_2))$ *then it is unique and of the form* $(x_2^2 + \text{t. h. o.})^{-7/6}$*.*
(b) In the setting of (2.5) there exists an integrating factor only if a certain polynomial in the α_i *and* β_i*, with* $i \leq 3$*, vanishes. Thus, in general there will be no algebraic integrating factor.*

Proof. (*Sketch.*) Part (a) is immediate from (2.4) and (2.5), while part (b) follows from a straightforward investigation of the degree 5 and degree 6 terms of the recursion in part (ii) of the proof of (2.5). ∎

Remark2.7. As (2.6) shows, computation of a normal form with respect to a nilpotent linear part will not generally produce a vector field that is integrable in an elementary manner. Note that this fact does not depend on the specific choice of a normal form. This stands in marked contrast to the case of two-dimensional normal forms with respect to a semisimple linear part.
However, there do exist vector fields which admit an integrating factor as in (2.6). For instance, the equation

$$\dot{x} = \begin{pmatrix} x_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 2x_1^2 \\ \frac{3}{2}x_1^2 + 7x_1x_2 \end{pmatrix}$$

(which was constructed using [13], Prop. (3.4), with $B = \text{diag}(2, 3)$), admits the integrating factor $(x_1^3 - x_2^2)^{-7/6}$. It is worth noting that the irreducible semi-invariant of an equation as in (2.6), if it exists, is formally equivalent to $x_2^2 + \tau x_1^3$, for some $\tau \neq 0$. (This follows from the expression for ϕ_3 in the proof of (2.4), and a Newton polygon argument; see, for instance, Bruno [4], Ch. I.) Therefore [13] essentially allows the construction of all the equations in (2.6) which admit an algebraic integrating factor.

Let us next discuss degenerate stationary points with vanishing linearization. We do not attempt to investigate these completely, but we will present some results showing that generically there will be no integrating factor which is algebraic over $\mathbf{C}((x_1, x_2))$. Thus, let $\dot{x} = f(x)$ be given, with $f \neq 0$, $f(0) = 0$, and $Df(0) = 0$. Then there is an integer $r \geq 2$ such that $f = \sum_{k \geq r} f^{(k)}$, with homogeneous polynomials $f^{(k)}$, and $f^{(r)} \neq 0$. Let us collect a few well-known facts about homogeneous systems. (One source of complete proofs is [15].)

Lemma2.8. *For a given homogeneous polynomial vector field* $f^{(r)}$*, define* $\psi(x) := \det(f^{(r)}(x), x)$*. Then* $\psi \neq 0$ *unless there is a polynomial* μ *such that* $f(x) = \mu(x) \cdot x$ *for all* x*.*

If $\psi \neq 0$ *then* $L_{f^{(r)}}(\psi) = \text{div}(f^{(r)})\psi$*, hence* ψ^{-1} *is an integrating factor. Moreover,* $\psi = \phi_1 \cdots \phi_{r+1}$ *is a product of linear forms, and there are homogeneous polynomials* λ_i*, of degree* $r - 1$*, such that* $L_{f^{(r)}}(\phi_i) = \lambda_i \phi_i$*, for* $1 \leq i \leq r + 1$*.*

The vector field $f^{(r)}$ *admits a (homogeneous) rational first integral if and only if the* λ_i *are linearly dependent over the field of rational numbers.*

Proof. (*Sketch.*) It follows from [13], Prop. 1.1, that ψ^{-1} is an integrating factor. Since the only irreducible complex homogeneous polynomials in two variables are those of degree 1, one gets a prime factorization with linear forms. Since ψ is semi-invariant, so are all the ϕ_i.

As to the last assertion, note that if there is a rational first integral then there is a homogeneous rational first integral, which must be of the form $\phi_1^{k_1}\cdots\phi_{r+1}^{k_{r+1}}$. (This follows from simple computations; see [15].) The observation

$$L_{f^{(r)}}(\phi_1^{k_1}\cdots\phi_{r+1}^{k_{r+1}}) = (k_1\lambda_1+\ldots+k_{r+1}\lambda_{r+1})(\phi_1^{k_1}\cdots\phi_{r+1}^{k_{r+1}})$$

then finishes the proof. ∎

For the detailed investigation of the stationary point a blow-up is once again useful.

Lemma2.9. *Let the analytic differential equation $\dot{x} = f(x)$ be given, with $f = \sum_{k\geq r} f^{(k)}$, and*

$$f^{(r)} = \begin{pmatrix} \sum_{i=0}^{r} \rho_i x_1^{r-i} x_2^i \\ \sum_{i=0}^{r} \sigma_i x_1^{r-i} x_2^i \end{pmatrix}.$$

Assume that $\rho_r \neq 0$ (whence $\psi(x) := \det(x, f^{(r)}(x))$ is not identically zero), and that ψ has pairwise relatively prime linear factors $\phi_1, \ldots, \phi_{r+1}$.

Then $\Phi(x) = \binom{x_1}{x_1x_2}$ is solution-preserving from an analytic equation $\dot{x} = x_1^{r-1}\tilde{f}(x)$ to $\dot{x} = f(x)$, with

$$\tilde{f}(x) = \begin{pmatrix} \rho_0 x_1 + \rho_1 x_1 x_2 + x_1^2(\ldots) \\ \nu(x_2) + x_1(\ldots) \end{pmatrix},$$

and $\nu(x_2) = \sigma_0 + (\sigma_1-\rho_0)x_2 + \ldots + (\sigma_r - \rho_{r-1})x_2^r - \rho_r x_2^{r+1}$.

The vector field $\tilde{f}$ has exactly $r+1$ stationary points on the invariant line $x_1 = 0$, and these are nondegenerate. If none of these is dicritical then f has precisely $r+1$ irreducible analytic (or formal) semi-invariants $\psi_i = \phi_i + \mathrm{t.h.o.}$ in 0.

Proof. This follows immediately from (2.2) and (2.3), since by (2.1) every stationary point on $x_1 = 0$ yields precisely one nontrivial semi-invariant. ∎

Theorem2.10. *Let the notation and hypotheses be as in (2.9). Assume that none of the stationary points of $\tilde{f}$ on the line $x_1 = 0$ is dicritical, and that $f^{(r)}$ has no nonconstant rational first integral.*
(a) If $\dot{x} = f(x)$ admits an integrating factor that is algebraic over $\mathbf{C}((x_1, x_2))$ then it is necessarily of the form $(\phi_1\cdots\phi_{r+1} + \mathrm{t.h.o.})^{-1}$.
(b) Given $f^{(r)}$, the vector field $f^{(r+1)}$ can be chosen in such a way that $\dot{x} = f(x)$ admits no algebraic integrating factor.

Proof. (a) Assume that f admits the integrating factor

$$(\psi_1^{d_1}\cdots\psi_{r+1}^{d_{r+1}})^{-1}\exp(\mu).$$

Then a simple computation shows that $f^{(r)}$ admits the integrating factor

$$(\phi_1^{d_1}\cdots\phi_{r+1}^{d_{r+1}})^{-1},$$

and (2.7) shows that $d_1 = \ldots = d_{r+1} = 1$.
(b) Let γ^{-1} be an integrating factor for f. Then $\gamma = \sum_{i\geq r+1}\gamma_i$, with γ_i a homogeneous polynomial of degree i, and $\gamma_{r+1} \neq 0$ is a product of pairwise different primes, according to part (a). The integrating factor condition yields

$$(*) \qquad L_{f^{(r)}}(\gamma_{r+2}) - \operatorname{div}(f^{(r)})\gamma_{r+2} = -L_{f^{(r+1)}}(\gamma_{r+1}) + \operatorname{div}(f^{(r+1)})\gamma_{r+1}.$$

Denote by $\mathcal{S}_j$ the vector space of homogeneous polynomials of degree j (which has dimension $j+1$), and by $\mathcal{P}_j$ the space of homogeneous vector fields with entries of degree j.

The image of the linear map $\Delta : \mathcal{S}_{r+2} \to \mathcal{S}_{2r+1}$, $\mu \mapsto L_{f^{(r)}}(\mu) - \operatorname{div}(f^{(r)})\mu$, has dimension at most equal to $r+3 < 2r+2$, and therefore Δ is not onto.

On the other hand, the linear map $\Gamma : \mathcal{P}_{r+1} \to \mathcal{S}_{2r+1}$, $p \mapsto L_p(\gamma_{r+1}) - \operatorname{div}(p)\gamma_{r+1}$, is onto: Note that $p \in \ker(\Gamma)$ if and only if γ_{r+1}^{-1} is an integrating factor for p. Since the entries of $\gamma_{r+1}^{-1}p$ are homogeneous of degree 0, it follows from the proof of [13], Prop. 3.4, that this is the case if and only if

$$p = \rho \begin{pmatrix} -\partial\gamma_{r+1}/\partial x_2 \\ \partial\gamma_{r+1}/\partial x_1 \end{pmatrix} + \lambda x,$$

with a linear form ρ, and λ (and therefore p) uniquely determined by ρ. This shows that $\dim(\ker(\Gamma)) = \dim(\mathcal{S}_1) = 2$, and comparing dimensions proves the surjectivity of Γ.

Therefore, the equation $\Delta(\gamma_{r+2}) = -\Gamma(f^{(r+1)})$ (which is just $(*)$ rewritten) has a solution only if $f^{(r+1)}$ lies in some fixed proper subspace of $\mathcal{P}_{r+1}$. ∎

Note that the question of existence of rational first integrals of $f^{(r)}$ can be decided by (2.8). One also knows that there can be no rational first integral of $f^{(r)}$ if one of the stationary points of $\tilde{f}$ on the line $x_1 = 0$ has a non-rational eigenvalue ratio; see (2.1).

If $f^{(r)}$ is given then an elementary computation yields the conditions on the coefficients of $f^{(r+1)}$. Moreover, there will be additional conditions from higher degree terms. For example, the equation

$$\dot{x} = \begin{pmatrix} -2x_1^2 + x_1x_2 + 2x_2^2 \\ -x_1x_2 + x_2^2 \end{pmatrix} + \begin{pmatrix} x_1x_2^2 \\ 0 \end{pmatrix}$$

admits no integrating factor that is algebraic over $\mathbf{C}((x_1,\, x_2))$.

In view of the theorems of Bendixson and Seidenberg, the strategy of blowing up and reducing to the nondegenerate case is clearly applicable to all degenerate stationary points in the plane. But with the currently available methods one seems to be restricted to a case-by-case analysis of various special classes.

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