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A Note on the Linear Cycle Space for Groups of Hermitian Type

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Abstract. Let G_0 be a simple Lie group of hermitian type and let B denote the corresponding hermitian symmetric space. The linear cycle space for any nonholomorphic type flag domain of G_0 is biholomorphic to $B \times \overline{B}$. When G_0 is a classical group this was proved by the authors in a paper published several years ago. Here we show that the result follows for arbitrary groups of hermitian type. This is done without case by case arguments by combining results from that paper with recent results of A. T. Huckleberry and the first author.

1. Introduction

Let G be a complex semisimple Lie group with noncompact real form G_0 of hermitian type. Let K_0 be a maximal compact subgroup of G_0 and K its complexification. Denote by Q a parabolic subgroup of G and set $Z = G/Q$, the corresponding flag variety. There is a one to one correspondence between K-orbits and G_0-orbits on Z ([2], [8], [9]). Under this correspondence open G_0-orbits correspond to closed K-orbits. Following [13, page 529] we let D be an open G_0-orbit in Z and Y the corresponding closed K-orbit. The closed K-orbit Y is a maximal compact subvariety of D. The *linear cycle space* is the connected component of

$$\{g \cdot Y : g \in G \text{ and } g \cdot Y \subset D\}$$

containing Y and is denoted by M_D. Since G_0 is noncompact and of hermitian type, it is not transitive on Z [12, Corollary 1.7], and M_D is not reduced to a point. Now there are two types of open orbits D, holomorphic type where it is known that M_D is biholomorphic to B or to $\overline{B}$, and nonholomorphic type where we conjectured (and proved in many cases) that M_D is biholomorphic to $B \times \overline{B}$. See [13, Definition 1.2] for the definitions and [13, Theorem 1.3] for a number of equivalent conditions.

As M_D sits in a holomorphic double fibration with D, it plays a role in the study of representations occurring in cohomology on D. Therefore it is of

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interest to understand the precise structure of M_D. Theorem 1 gives an explicit description of M_D.

2. The structure theorem

Let B denote the hermitian symmetric space G_0/K_0. In the above setting the complete theorem on the structure of the linear cycle space is the following.

Theorem 1. *If G_0 is an arbitrary simple Lie group of hermitian type then*

(1) *if D is of nonholomorphic type then M_D is biholomorphic to $B \times \overline{B}$,*

(2) *if D is of holomorphic type then M_D is biholomorphic to B or $\overline{B}$.*

Proof. Statement (2) is known by two different general arguments, [11] and [13].

Statement (1) was proved in [13] for classical groups as follows. The inclusion $B \times \overline{B} \subset M_D$ is proved in [13] for the classical groups by a case by case argument. The other inclusion is proved in [13] (see Section 3 below) for arbitrary hermitian type groups. Now we observe that the other inclusion, $B \times \overline{B} \subset M_D$ for arbitrary hermitian type groups, is essentially contained in the literature.

The containment is fairly direct in [7]: A certain domain $\Omega_S(D)$, based on Schubert incidence theory, was introduced in [6], and it was recently proved that $M_D = \Omega_S(D)$ [7, Corollary 3.4].

The containment is slightly less direct in [4]. A. T. Huckleberry observed that one can obtain the desired inclusion $B \times \overline{B} \subset M_D$ from a careful look at [4, Proposition 2.4], though it is not actually stated in [4]. $\square$

Corollary 2. *For G_0 a simple Lie group of hermitian type and D any nonholomorphic type orbit $\mathcal{A} \simeq M_D$.*

In this note, D is measurable in the sense of [10] and [11] because G_0 is of hermitian type; it has been known for some time [11] that if D is measurable then M_D is Stein. The Schubert domain considerations [7], used in our proof of Theorem 1, show that M_D is Stein without any measurability condition on D.

3. Remark

It is announced in [4] that [13] contains a gap in the proof of $M_D \subset B \times \overline{B}$ (Theorem 3.8) for general hermitian type groups. At the very end of our proof of Case 1 in the proof of that Theorem 3.8, we made a few inaccurate statements which tend to obscure the argument. Referring to [13], the following adjustments should clarify that argument.

1. Omit '$c_\Gamma =$' in line 2 of page 537.

2. Omit the sentence 'In particular $c_\Gamma(z) \notin D$.' in line 3, page 537. This is true by Lemma 3.6, but is irrelevant in view of (1) above.

3. In view of (1) above, we must change 'c_Γ' to '$c_{\Gamma \cap \Delta(\mathfrak{r}_+)}$' in line 4, page 537. The corresponding change of g_0 in the same line to $g_0' = \prod_{\psi \in \Delta(\mathfrak{r}_+) \setminus \Gamma} g_{0,\psi}$ is also necessary.

Now make the following simple observation. Let $\psi \in \Psi^{\mathfrak{g}} \subset \Delta(\mathfrak{s}_+)$, and suppose that $g \in G[\psi]$ has decomposition $g = \exp(\xi_+)k\exp(\xi_-)$, $\xi_\pm \in \mathfrak{g}_{\pm\psi}$ and $k \in K \cap G[\psi]$. Then $k \in H \subset Q$ and
(1) if $\psi \in \Delta(\mathfrak{q}) = \Delta(\mathfrak{l} + \mathfrak{r}_-)$ then $\xi_+ \in \mathfrak{g}_\psi \subset \mathfrak{q}$ so $\exp(\xi_+)(z) = z$, and

(2) if $\psi \in \Delta(\mathfrak{r}_+)$ then $\xi_- \in \mathfrak{g}_{-\psi} \subset \mathfrak{r}_-$ so $\exp(\xi_+)(z) = \exp(\xi_+)k\exp(\xi_-)(z) = g(z)$.

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