Journal of Lie Theory
Volume **13** (2003) 307–309

A Connected Lie Group Equals the Square of the Exponential Image

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Communicated by J. Hilgert

Abstract. It is shown that in every connnected real Lie group every element is the product of two elements in the exponential image.

More than one hundred years ago, F. Engel and E. Study considered the question of the surjectivity of the exponential function of Lie groups ([2],[3], compare also the remarks in [14]). Interestingly, a general solution of this problem is not found yet though there exist equivalent criteria for the surjectivity of special classes of Lie groups (compare [12]). M. Moskowitz and R. Sacksteder first formulated the fact that every real connected Lie group is equal to $(\exp \mathfrak{g})^2$ ([7]) and provided a proof in the final version of their paper using control theory like methods. Using partly well-known decomposition results on Lie groups, we shall provide a short proof for this result.

In the whole article, G denotes a real connected Lie group. A subgroup $K \subseteq G$ is called *compactly embedded* if $\mathrm{Ad}(K)$ is relatively compact in $\mathrm{GL}(\mathfrak{g})$. It is shown in [10] that there exist maximal compactly embedded subgroups and that all maximal compactly embedded subgroups are closed and connected. The last property ensures that all maximal compactly embedded subgroups are conjugate to each other. Moreover, it is well-known that every compactly embedded connected subgroup K has surjective exponential function: Indeed, $K = K'Z_0$, where Z_0 denotes the identity component of the center. Moreover, K' is compact, hence has surjective exponential function. This implies the surjectivity of the exponential function of K.

First, let us assume that G is semisimple. The Iwasawa decomposition says that every semisimple Lie group can be written as KAM where K is maximal compactly embedded, A is abelian connected and consists of Ad-semisimple elements with $\operatorname{spec}\operatorname{ad}(a) \subseteq \mathbb{R}$ for every $a \in \mathfrak{a}$, M is nilpotent connected, and AM is solvable ([4]). By Dixmier's and Saito's Theorem ([1],[9]), $S := AM = \exp(\mathfrak{a}+\mathfrak{m})$. Thus, $G = \exp \mathfrak{k} \exp \mathfrak{s}$.

Now we recall the Levi decomposition: If G is a real connected Lie group, then for every maximal semisimple subgroup (called *Levi factor*) L we have $G = L\,\mathrm{Rad}(G)$ and $L \cap \mathrm{Rad}(G)$ discrete. All Levi factors are conjugate to each other.

ISSN 0949–5932 / \$2.50

This follows directly from the Levi decomposition of real finite-dimensional Lie algebras (see e.g. Korollar II.4.8 of [5]).

A *Cartan subalgebra* $\mathfrak{h} \subset \mathfrak{g}$ of a real finite dimensional Lie algebra $\mathfrak{g}$ is a nilpotent subalgebra which equals its own normalizer $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})$. Cartan subgroups are defined in various ways which in case of connected Lie groups are all equivalent (compare [8]). We will use the following definition: $H \subseteq G$ is a Cartan subgroup if $\mathfrak{h} := \mathbf{L}(H)$ is a Cartan subalgebra and for all $g \in H$ and all $x \in \mathfrak{h}$ we have $\mathrm{Ad}(g)\,\mathrm{ad}(x)_s = \mathrm{ad}(x)_s\,\mathrm{Ad}(g)$ where $\mathrm{ad}(x)_s$ is the semisimple part of the Jordan decomposition of $\mathrm{ad}(x)$.

Theorem 1.9 (ii) of [13] states that for all Cartan subgroups H the intersection $H \cap \mathrm{Rad}(G)$ is connected. Since $H \cap \mathrm{Rad}(G)$ is nilpotent, we have $H \cap \mathrm{Rad}(G) = \exp(\mathfrak{h} \cap \mathrm{rad}(\mathfrak{g}))$. Theorem 1.11 of [13] says that for each Cartan subgroup H of G there exists a Levi factor L such that $H \cap L$ is a Cartan subgroup of L, $H = (H \cap L)(H \cap \mathrm{Rad}(G))$, and $H \cap \mathrm{Rad}(G) \subseteq Z_G(L)$. This leads to the following:

Lemma 1. *If $\mathfrak{h}$ is a Cartan subalgebra of a real finite-dimensional Lie algebra $\mathfrak{g}$ and $\mathfrak{n}$ the nilradical of $\mathfrak{g}$, then $\mathrm{rad}(\mathfrak{g}) = (\mathfrak{h} \cap \mathrm{rad}(\mathfrak{g})) + \mathfrak{n}$. If H is a Cartan subgroup of a real Lie group G and N the nilradical, then $\mathrm{Rad}(G) = (H \cap \mathrm{Rad}(G))N$.*

Proof. We choose $\mathfrak{l}$ such that $\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{l}) \oplus (\mathfrak{h} \cap \mathrm{rad}(\mathfrak{g}))$ (compare Theorem 1.8 of [13]). We observe that $\mathfrak{g}' \subseteq \mathfrak{l} + \mathfrak{n}$. Moreover, $\mathfrak{g} = \mathfrak{h} + \mathfrak{g}' = (\mathfrak{h} \cap \mathrm{rad}(\mathfrak{g})) + \mathfrak{n} + \mathfrak{l}$, hence $\mathrm{rad}(\mathfrak{g}) = (\mathfrak{h} \cap \mathrm{rad}(\mathfrak{g})) + \mathfrak{n}$. The proof for Lie groups works similarly. ■

Thus, with the above notation we obtain $G = KS(H \cap \mathrm{Rad}(G))N = K(H \cap \mathrm{Rad}(G))SN = \exp\mathfrak{k}\exp(\mathfrak{h} \cap \mathrm{rad}(\mathfrak{g}))\exp\mathfrak{s}\exp\mathfrak{n}$. Since $[\mathfrak{k}, \mathfrak{h} \cap \mathrm{rad}(\mathfrak{g})] = \{0\}$, we get $\exp\mathfrak{k}\exp(\mathfrak{h} \cap \mathrm{rad}(\mathfrak{g})) = \exp(\mathfrak{k} + (\mathfrak{h} \cap \mathrm{rad}(\mathfrak{g})))$. Moreover, SN is a solvable connected Lie group and $\operatorname{spec}\mathrm{ad}_{\mathfrak{g}}(s) \subseteq \mathbb{R}$ for each $s \in \mathfrak{s}$ by Proposition 3.7 of [11]. Again by Dixmier's and Saito's Theorem we get $SN = \exp(\mathfrak{s} + \mathfrak{n})$ because by Proposition 3.7 of [11], every element $x \in \mathfrak{s} + \mathfrak{n}$ satisfies $\operatorname{spec}\mathrm{ad}(x) \subseteq \mathbb{R}$.

Definition 2. An element x of a real finite-dimensional Lie algebra is called exp-*regular* if $\operatorname{spec}\mathrm{ad}\,x \cap 2\pi i\mathbb{Z} = \{0\}$. The set of all exp-regular elements of $\mathfrak{g}$ is denoted by $\operatorname{reg}\exp$.

Taken together, we have proved the following:

Theorem 3. *If G is a real connected Lie group, then $G = (\exp\mathfrak{g})^2$. Moreover, for every element $g \in G$ there is an $\exp$-regular element x and an element y in $\mathfrak{g}$ with $g = \exp x \exp y$.*

Theorem 4. *If G is a connected real Lie group and for every $g \in G$ there is an exp-regular $x \in \mathfrak{g}$ and a $y \in \mathfrak{g}$ with $g = \exp x \exp y$, then there are exp-regular $u, w \in \mathfrak{g}$ with $g = \exp u \exp w$.*

Proof. Since $\operatorname{reg}\exp$ is dense in $\mathfrak{g}$ (for example by Lemma 2 of [6]), the image $\exp(\operatorname{reg}\exp)$ is dense in $\exp\mathfrak{g}$. Since $\exp$ is regular at each $x \in \operatorname{reg}\exp$, the set $\exp(\operatorname{reg}\exp)$ is open in G, hence also in $\exp\mathfrak{g}$. The openess of $\exp(\operatorname{reg}\exp)$ implies that there is a symmetric 1-neighborhood U such that $\exp x \cdot U \subseteq \exp(\operatorname{reg}\exp)$. The density implies that $\exp y \cdot U \cap \exp\operatorname{reg}\exp \neq \emptyset$. So there is a $u \in U$ such that $(\exp x)u^{-1}$ and $u \exp y$ are in $\exp(\operatorname{reg}\exp)$. This implies the assertion. ■

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Received July 25, 2002
and in final form August 9, 2002