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On the Multiplier of a Lie Algebra

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Abstract. Under fairly general conditions, we extend the 5-sequence of cohomology for nilpotent Lie algebras a step further. We then derive some consequences of the construction.
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1. Introduction

We begin with the following definitions from [2].

Definition 1.1. A pair of Lie algebras (K, M) is a defining pair for L if

1. $L \cong K/M$
2. $M \subset Z(K) \cap K^2$

Definition 1.2. (K,M) is a maximal defining pair if the dimension of K is maximal. For these maximal defining pairs, K is called a cover for L and M, is called the multiplier for L and is denoted M/L.

In her dissertation on the Lie algebra version of the Schur multiplier, Peggy Batten showed $M(L) \cong H^2(L, F)$ and extended, under certain conditions, the 5-sequence of cohomology for Lie algebras a step further. In particular, if L is a Lie algebra over a field F, H is an ideal in L and F is a trivial L-module, the 5-sequence is:

$$0 \longrightarrow \mathrm{Hom}\,(L/H, F) \xrightarrow{\mathrm{Inf}} \mathrm{Hom}(L, F) \xrightarrow{\mathrm{Res}}$$

$$\mathrm{Hom}(H, F) \xrightarrow{\mathrm{Tra}} H^2\,(L/H, F) \xrightarrow{\mathrm{Inf}} H^2(L, F)$$

As is known and is shown in [1], $H^2(L, F)$ is the multiplier of L. When H is central and L is nilpotent, the exact sequence extends by $\delta : H^2(L, F) \to L/L^2 \otimes H$. This

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result has been used in [5] in further investigations of the multiplier. Using group results [8] as a guide, another possible extension should exist when L is nilpotent of class n and $\delta : H^2(L, F) \to L/\mathcal{Z}_{n-1} \otimes L^n$ where $\mathcal{Z}_j$ and L^j are the jth terms in the upper and lower central series of L. Showing that this is the case is one of the objectives of this paper. In fact, we show a Theorem which contains both of the results as consequences. We then obtain a number of corollaries which are in the form of inequalities. Finally, we record an example which shows that several of the inequalities are the best possible. Further discussions of multipliers are found in [1], [2], [3], [5] and [7].

2. The Main Result

Theorem 2.1. *Let L be a nilpotent Lie algebra over a field F. Let A and B be ideals of L with $L^2 \subseteq A$ and $B \subseteq \mathcal{Z}(L)$. If $f(A, B) = 0$ for all $f \in \mathcal{Z}^2(L, F)$, then there exists a homomorphism δ, defined below, such that*

$$H^2(L/B, F) \xrightarrow{\text{Inf}} H^2(L, F) \xrightarrow{\delta} Bil(L/A, B : F)$$

is exact.

Proof. Let $x \in L, y \in B$ and $f' \in \mathcal{Z}^2(L, F)$. Define $f'' \in Bil(L/A, B : F)$ by $f''(x + A, y) = f'(x, y)$. Since $f'(A, B) = 0$, it follows that f'' is well defined. Let $\delta' : \mathcal{Z}^2(L, F) \to Bil(L/A, B : F)$ be defined by $\delta'(f') = f''$.

If $f' \in B^2(L, F)$, then there exists a linear function $g : L \to F$ such that $f'(x, y) = -g([x, y])$. For $y \in B$, $f''(x + A, y) = f'(x, y) = -g([x, y]) = 0$. Hence, $\delta'(f') = 0$. Therefore, δ' induces $\delta : H^2(L, F) \to Bil(L/A, B : F)$ by $\delta(f' + B^2(L, F)) = f''$.

We now show exactness. For $f \in \mathcal{Z}^2(L/B, F)$, let $f'(x, y) = f(x + B, y + B)$. Recall that Inf: $H^2(L/B, F) \to H^2(L, F)$ is given by $\text{Inf}(f + B^2(L/B, F)) = f' + B^2(L, F)$ and is induced by $I : \mathcal{Z}^2(L/B, F) \to \mathcal{Z}^2(L, F)$ where $I(f) = f'$.

For $x \in L, y \in B$, it follows that $f(x + B, y + B) = f'(x, y) = f''(x + A, y)$ where $f \in \mathcal{Z}^2(L/B, F)$ and f' and f'' are induced. Now $\delta(\text{Inf}(f + B^2(L/B, F))) = \delta(f + B^2(L, F)) = 0$ with $f''(x + A, y) = f(x + B, y + B) = 0$. Hence, $f'' = 0$ and $\text{Image}(\text{Inf}) \subseteq \text{Ker}(\delta)$.

Now let $f' \in \mathcal{Z}^2(L, F)$ be such that $f' + B^2(L, F) \in \text{Ker}(\delta)$. Then $\delta(f' + B^2(L, F)) = f'' = 0$. Let $x \in L, y \in B$. Then $0 = f''(x + A, y) = f'(x, y)$. By anti-symmetry, it follows that $f'(y, x) = 0$. Set $g(x + B, y + B) = f'(x, y)$. In order to see that g is well defined, let $x + B = x_1 + B$ and $y + B = y_1 + B$. Then $x_1 = x + b, y_1 = y + c$ for $b, c \in B$. Then

$$g(x_1 + B, y_1 + B) = f'(x_1, y_1) = f'(x + b, y + c) =$$

$$f'(x, y) + f'(b, y) + f'(x, c) + f'(b, c) = f'(x, y) = g(x + B, y + B).$$

Hence, g is well defined. Furthermore, g satisfies the cocycle condition since f' does and $g(x + B, y + B) = f'(x, y)$. Therefore, $g \in \mathcal{Z}^2(L/B, F)$. By the definition of I, $I(g) = f'$ and, therefore, $\text{Inf}(g + B^2(L/B, F)) = f' + B^2(L, F)$. Thus, $\text{Ker}(\delta) \subseteq \text{Image}(\text{Inf})$ and the sequence is exact. ∎

Since $\text{Bil}(L/A, B : F) \simeq L/A \otimes B$, the extension can be written as $H^2(L, F) \xrightarrow{\delta} L/A \otimes B$ with exactness at $H^2(L, F)$.

3. Applications

If we let $A = L^2$ and $B = Z \subseteq Z(L)$, then Theorem 2.1 provides an alternate way to prove Theorem 4.1 on p. 45 in [1].

Proposition 3.1. *Let L be nilpotent and $\mathcal{Z} \subseteq \mathcal{Z}(L)$. Then*

$$H^2(L/\mathcal{Z}, F) \xrightarrow{\text{Inf}} H^2(L, F) \xrightarrow{\delta} L/L^2 \otimes \mathcal{Z}$$

is exact.

Proof. All that needs to be verified is that $f(L^2, \mathcal{Z}) = 0$ for all $f \in \mathcal{Z}^2(L, F)$. But $f(L^2, \mathcal{Z}) = f([L, \mathcal{Z}], L) + f([\mathcal{Z}, L], L) = 0$. ■

It should be noted that Proposition 3.1 was used to show [5], Proposition 2.

Corollary 3.2. *If L is nilpotent, finite dimensional with $\mathcal{Z} \subseteq \mathcal{Z}(L) \cap L^2$ and $\dim \mathcal{Z} = 1$, then*

$$\dim H^2(L, F) + 1 \leq \dim H^2(L/\mathcal{Z}, F) + \dim(L/L^2)$$

The next result is an analogue of a theorem of Vermani's in group theory [8].

Theorem 3.3. *Let L be nilpotent of class n. Then*

$$H^2(L/L^n, F) \xrightarrow{\text{Inf}} H^2(L, F) \xrightarrow{\delta} L/\mathcal{Z}_{n-1} \otimes L^n$$

is exact.

Proof. Since $L^n \subseteq /Z(L)$ and $L^2 \subseteq \mathcal{Z}_{n-1}$, it remains to verify the cocycle condition in Theorem 2.1.

First, note that $[L^s, \mathcal{Z}_i] \subseteq \mathcal{Z}_{i-s}$. This holds for $s = 1$ and is shown generally by induction on s.

To obtain that $f(\mathcal{Z}_{s-1}, L^s) = 0$ for all s, we use induction. For $s = 1, f(\mathcal{Z}_0, L) = 0$. Suppose that the result holds for $s = k$. Now,

$$\begin{aligned} f(\mathcal{Z}_k, L^{k+1}) &= f([L, L^k], \mathcal{Z}_k) \\ &= f([L^k, \mathcal{Z}_k], L) + f([\mathcal{Z}_k, L], L^k) \\ &= f(\mathcal{Z}_0, L) + f(\mathcal{Z}_{k-1}, L^k) \\ &= 0 \end{aligned}$$

Therefore, Theorem 3.3 follows from Theorem 2.1. ■

Corollary 3.4. *Let L be nilpotent of class n and finite dimensional. Then*

$$\dim H^2(L,F) \leq \dim H^2(L/L^n,F) + \dim L^n \dim(L/\mathcal{Z}_{n-1}) - \dim L^n \ .$$

Proof. Set $q = \dim \mathrm{Hom}(L^n,F), r = \dim H^2(L/L^n,F), s = \dim H^2(L,F)$ and $t = \dim(L/\mathcal{Z}_{n-1} \otimes L^n)$ where the terms are from the extended 5-sequence:

$$\mathrm{Hom}(L,F) \xrightarrow{\mathrm{Res}} \mathrm{Hom}(L^n,F) \xrightarrow{\mathrm{Tra}} H^2(L/L^n,F) \xrightarrow{\mathrm{Inf}} H^2(L,F) \xrightarrow{\delta} L/\mathcal{Z}_{n-1} \otimes L^n$$

Since $F^n = 0$ for all $n \geq 2$, for any $f \in \mathrm{Hom}(L,F), f : L^n \to F^n = 0$. Thus, $\mathrm{Res}(f) = 0 = \mathrm{Ker(Tra)}$. Hence, $q = \dim(\mathrm{Image(Tra)}) = \dim(\mathrm{Ker(Inf)})$. Then $r - q = \dim H^2(L/L^n,F) - \dim(\mathrm{Ker(Inf)}) = \dim(\mathrm{Image(Inf)}) \leq s$.
Also, $\dim H^2(L,F) - \dim(Ker(\delta)) = \dim(\mathrm{Image}(\delta)) \leq t$.
Therefore, $s - \dim(\mathrm{Ker}(\delta)) \leq t$. Since $r - q = \dim(\mathrm{Image(Inf)}) = \dim(\mathrm{Ker}(\delta))$, it follows that $s - (r-q) \leq t$. Hence,

$$\dim H^2(L,F) \leq \dim H^2(L/L^n,F)$$

$$+ \dim(L/\mathcal{Z}_{n-1} \otimes L^n) - \dim(\mathrm{Hom}(L^n,F))$$

$$= \dim H^2(L/L^n,F) + \dim(L/\mathcal{Z}_{n-1}) \dim L^n - \dim L^n \ .$$

■

The next corollary is a Lie algebra analogue of a result of Gaschütz, Neubüser and Yen [4].

Corollary 3.5. *Let L be nilpotent and finite dimensional. Then*

$$\dim H^2(L,F) \leq \dim H^2(L/L^2,F) + \dim L^2[\dim L/Z(L) - \dim(L/Z(L))^2 - 1] \ .$$

Proof. We use induction on the class of L. If L is of class 1, then $L^2 = 0$ and the result holds. Assume the result for nilpotent Lie algebras of class less than n and let L have class n. Note that $L^n \subseteq \mathcal{Z}(L), L^2 \subseteq \mathcal{Z}_{n-1}(L), (L/L^n)^2 = L^2/L^n$ and $\mathcal{Z}(L)/L^n \subseteq \mathcal{Z}(L/L^n)$. For convenience, let $A = (L/L^n)/\mathcal{Z}(L/L^n)$ and $B = L/\mathcal{Z}(L) = (L/L^n)/(\mathcal{Z}(L)/L^n)$. Since A is a homomorphic image of B, it follows that $\dim A/A^2 \leq \dim B/B^2$. By induction,

$$\begin{aligned}
\dim H^2(L/L^n,F) &\leq \dim H^2((L/L^n)/(L/L^n)^2,F) + \dim(L/L^n)^2[\dim(A/A^2) - 1] \\
&\leq \dim H^2(L/L^2,F) + \dim(L^2/L^n)[\dim(B/B^2) - 1].
\end{aligned}$$

By the last corollary,

$$\dim H^2(L,F) \leq \dim H^2(L/L^n,F) + \dim(L/\mathcal{Z}_{n-1}) \dim L^n - \dim L^n \ .$$

Now,

$$\dim(L/\mathcal{Z}_{n-1}) \leq \dim(L/(L^2 + \mathcal{Z}(L))) = \dim B/B^2 \ .$$

Therefore,

$$\dim H^2(L,F) \leq \dim H^2(L/L^2,F)$$

$$+ \dim L^2[\dim(B/B^2) - 1] - \dim L^n[\dim(B/B^2) - 1]$$

$$+ \dim(B/B^2) \dim L^n - \dim L^n$$

which gives the result. ■

Since $\dim B/B^2 \leq \dim L/L^2$, we obtain

Corollary 3.6. $\dim H^2(L,F) \leq \dim H^2(L/L^2,F) + \dim L^2[\dim(L/L^2) - 1]$.

We also write this another way.

Corollary 3.7. *Let* $\dim L = n$ *and* $\dim L/L^2 = d$. *Then*

$$\dim H^2(L,F) \leq \frac{-d^2 + d + 2dn - 2n}{2}$$

Proof. This follows from Corollary 3.6 and the observation that

$$\dim H^2(L/L^2,F) = \frac{1}{2}d(d-1).$$

■

For example, if $d = 2$, and in particular if L is of maximal class, this becomes

Corollary 3.8. *If* $\dim L = n$ *and* $d = 2$, *then* $\dim H^2(L,F) \leq n - 1$.

4. Example

Let L be the nilpotent Lie algebra of dimension $n+1$ with basis $r, x_i, \dots, x_n$ and multiplication

$$\begin{aligned}
&[r, x_i] = x_{i+1} \quad i = 1, \dots, n-1\\
&[r, x_n] = 0\\
&[x_i, x_j] = 0
\end{aligned}$$

It is shown in [7] that the multiplier of L has dimension

a. $\frac{n}{2} + 1$ when n is even

b. $\frac{n+1}{2}$ when n is odd

and the construction provides an explicit basis. Considering Corollary 3.2 in conjunction with this example yields, when n is even: $\dim H^2(L,F) = \frac{n}{2} + 1$, $\dim \mathcal{Z}(L) = 1$, $\dim H^2(L/\mathcal{Z}(L),F) = \frac{n}{2}$ and $\dim(L/L^2) = 2$. Hence, equality holds. Considering Corollary 3.4, we note further that: $\dim(L/\mathcal{Z}_{n-1}) = 2$, $\dim L^n = 1$ and $\dim H^2(L/L^n,F) = \frac{n}{2}$. Hence, equality also holds in Corollary 3.4. Note that when n is odd, the inequalities in both corollaries are strict. Note that when $n = 2$, the inequalities in the other corollaries are actually equalities.

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