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Determination of the Topological Structure of an Orbifold by its Group of Orbifold Diffeomorphisms

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Abstract. We show that the topological structure of a compact, locally smooth orbifold is determined by its orbifold diffeomorphism group. Let $\mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O})$ denote the C^r orbifold diffeomorphisms of an orbifold $\mathcal{O}$. Suppose that $\Phi\colon \mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O}_1) \to \mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O}_2)$ is a group isomorphism between the the orbifold diffeomorphism groups of two orbifolds $\mathcal{O}_1$ and $\mathcal{O}_2$. We show that Φ is induced by a homeomorphism $h\colon X_{\mathcal{O}_1} \to X_{\mathcal{O}_2}$, where $X_{\mathcal{O}}$ denotes the underlying topological space of $\mathcal{O}$. That is, $\Phi(f) = hfh^{-1}$ for all $f \in \mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O}_1)$. Furthermore, if $r > 0$, then h is a C^r manifold diffeomorphism when restricted to the complement of the singular set of each stratum.

1. Introduction

Given a topological space X with some geometric structure (including topological structures, differentiable structures, symplectic structures and contact structures) and the group of transformations that preserve these structures (the group of homeomorphisms, diffeomorphisms, symplectic diffeomorphisms and contact diffeomorphisms), one can ask whether these groups of structure preserving transformations determine the corresponding structures. The topological case has been studied by Gerstenhaber [10],[11], Fine and Schweigert [9], Rubin [15], Wechsler [19], and Whittaker [20]. The differentiable case has been studied by Banyaga [1], [2], Filipkiewicz [8], Rubin [15], Rybicki [16] and Takens [17]. The symplectic and contact cases have been studied by Banyaga [3],[4], [5]. Rubin [15] has also studied many other variants of this question including the PL, Lipschitz and quasiconformal cases. A key ingredient in our proof in the orbifold case will be a theorem of Rubin:

Theorem 1.1. *(Rubin [15]) Let $X_i, (i = 1, 2)$ be locally compact Hausdorff spaces and G_i subgroups of the group of homeomorphisms of X_i such that for*

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every open set $T \subset X_i$ *and* $x \in T$ *the set* $\{g(x) \mid g \in G_i \text{ and } g\mid_{(X_i - T)} = \mathrm{Id}\}$ *is somewhere dense. Then if* $\Phi : G_1 \to G_2$ *is a group isomorphism, then there is a homeomorphism* h *between* X_1 *and* X_2 *such that for every* $g \in G_1$, $\Phi(g) = hgh^{-1}$.

Recall that a subset S of a topological space X is called *somewhere dense* if the interior of its closure is nonempty. That is, $\mathrm{int}(\mathrm{cl}(S)) \neq \emptyset$. Our theorem is the following:

Theorem 1.2. *Let* $\mathcal{O}_1$ *and* $\mathcal{O}_2$ *be two compact, locally smooth orbifolds. Fix* $r \geq 0$. *Suppose that* $\Phi : \mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O}_1) \to \mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O}_2)$ *is a group isomorphism. Then* Φ *is induced by a homeomorphism* $h : X_{\mathcal{O}_1} \to X_{\mathcal{O}_2}$. *That is,* $\Phi(f) = hfh^{-1}$ *for all* $f \in \mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O}_1)$. *Furthermore, if* $r > 0$, h *is a* C^r *manifold diffeomorphism when restricted to the complement of the singular set of each stratum.*

Here, $\mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O})$ denotes the C^r orbifold diffeomorphism group and $X_{\mathcal{O}}$ the underlying topological space of an orbifold $\mathcal{O}$. We review the definitions of these notions in the next few sections. The restriction in Theorem 1.2 to compact orbifolds cannot be removed as the following example shows.

Example 1.3. Let $\mathcal{O}_1 = (0,1)$ and $\mathcal{O}_2 = [0,1]$, the open and closed unit intervals. These orbifolds have the same homeomorphism group, but are clearly not homeomorphic spaces.

In general, the homeomorphism h in Theorem 1.2 is not necessarily an orbifold homeomorphism. To see this, consider the following

Example 1.4. Let $\mathcal{O}_i, (i = 1,2)$ be two so–called $\mathbb{Z}_{p_i}$–teardrops (see Example 4.5) with $p_1 \neq p_2$. It is clear that the homeomorphism groups of $\mathcal{O}_i$ are each isomorphic to the subgroup of the homeomorphism group of the 2–sphere S^2 which fix the north pole. To see this, just observe that any homeomorphism of S^2 that fixes the north pole can be locally lifted to a p_i–fold covering of a neighborhood of the north pole. Note, however that the orbifolds themselves are not *orbifold* homeomorphic, even though their underlying spaces $X_{\mathcal{O}_i} = S^2$, are topologically homeomorphic.

In light of Theorem 1.2 and Example 1.4, it is natural to ask what happens if we fix an orbifold $\mathcal{O}$ and consider a group automorphism $\Phi : \mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O}) \to \mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O})$. Theorem 1.2 guarantees that there exists a topological homeomorphism $h : X_{\mathcal{O}} \to X_{\mathcal{O}}$ such that $\Phi(f) = hfh^{-1}$ for all $f \in \mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O})$. The following theorem shows that, in general, $h \notin \mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O})$, and thus it is possible that some automorphisms of $\mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O})$ may not be inner automorphisms.

Proposition 1.5. *For each* $n > 1$ *there exists a compact connected orbifold* $\mathcal{O}$ *of dimension* n, *such that the group of automorphisms* $\mathrm{Aut}(\mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O})) \neq \mathrm{Inn}(\mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O}))$, *the group of inner automorphisms.*

Proof. Parametrize S^2 with spherical coordinates (θ, ϕ), $0 \leq \theta < 2\pi$, $-\pi/2 \leq \phi \leq \pi/2$. Let $A = (\theta, -\pi/2)$ be the north pole and $B = (\theta, \pi/2)$ be the south pole. Give S^2 the structure of a (p,q)–football orbifold $\mathcal{F}$ (see Example 4.5) with $p \neq q$ so the singular set $= \{A\} \cup \{B\}$. It is not hard to see that $\mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{F})$ is isomorphic to the group of C^r diffeomorphisms of S^2 which fix A and B

pointwise. Consider the group automorphism $\Phi : \mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{F}) \to \mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{F})$ defined by $(\Phi(f)) = g \circ f \circ g^{-1}$ where $g(\theta, \phi) = (\theta, -\phi)$. $\Phi \notin \mathrm{Inn}(\mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{F}))$. To see this, suppose f is an orbifold diffeomorphism with support in a neighborhood $\mathcal{U}$ of A and $B \in \mathrm{int}(\mathcal{O} - \mathcal{U})$ and Ψ is an inner automorphism. $\Psi(f)$ has support in a neighborhood of A and $B \in \mathrm{int}(\mathcal{O} - \mathrm{supp}(\Psi(f))$. However $\Phi(f)$ has support in a neighborhood of B, hence Φ cannot be an inner automorphism. Higher dimensional examples can be constructed by considering products with spheres $\mathcal{F} \times S^n$. ■

Remark 1.6. The work of [5], [8], [15], [16] collectively show that such examples do not exist in the topological (with or without boundary), differentiable, PL, Lipschitz, symplectic and contact categories. In addition, Theorem 1.2 implies that one–dimensional orbifold examples do not exist. This follows since the only non-trivial orbifolds are closed rays and closed intervals, and a topological homeomorphism of such a 1–orbifold is also an orbifold homeomorphism.

If one considers two non–homeomorphic Riemannian manifolds with trivial isometry group one is easily convinced that if the automorphism group of a particular structure is not rich enough then the underlying topological structure cannot be determined at all. In fact, in our case, it is a very interesting question to determine the conditions necessary to guarantee that h is an orbifold homeomorphism. It turns out that, in order to guarantee that h is an orbifold homeomorphism, one must introduce a more general notion of orbifold diffeomorphism group, the *unreduced* orbifold diffeomorphism group $\mathrm{Diff}^r_{\mathrm{unred}}\,\mathcal{O}$. There is a natural projection $\mathrm{Diff}^r_{\mathrm{unred}}\,\mathcal{O} \to \mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O})$. The details can be found in [6]. Before proceeding with the proof of our theorem we need to review some definitions involving the orbifold category.

2. Orbifold Preliminaries

Orbifolds. Our definition is modeled on the definition in Thurston [18]. A (topological) *orbifold* $\mathcal{O}$, consists of a paracompact, Hausdorff topological space $X_{\mathcal{O}}$ called the *underlying space*, with the following local structure. For each $x \in X_{\mathcal{O}}$ and neighborhood U of x, there is a neighborhood $U_x \subset U$, an open set $\tilde{U}_x \cong \mathbb{R}^n$, a finite group Γ_x acting continuously and effectively on $\tilde{U}_x$ which fixes $0 \in \tilde{U}_x$, and a homeomorphism $\phi_x : \tilde{U}_x/\Gamma_x \to U_x$ with $\phi_x(0) = x$. These actions are subject to the condition that for a neighborhood $U_z \subset U_x$ with corresponding $\tilde{U}_z \cong \mathbb{R}^n$, group Γ_z and homeomorphism $\phi_z : \tilde{U}_z/\Gamma_z \to U_z$, there is an embedding $\tilde{\psi} : \tilde{U}_z \to \tilde{U}_x$ and an injective homomorphism $f : \Gamma_z \to \Gamma_x$ so that $\tilde{\psi}$ is equivariant with respect to the f $\big($that is, for $\gamma \in \Gamma_z, \tilde{\psi}(\gamma y) = f(\gamma)\tilde{\psi}(y)$ for all $y \in \tilde{U}_z\big)$, such that the following diagram commutes:

$$\begin{tikzcd}
\tilde{U}_z \arrow[r, "\tilde{\psi}"] \arrow[d] & \tilde{U}_x \arrow[d] \\
\tilde{U}_z/\Gamma_z \arrow[r, "\psi=\tilde{\psi}/\Gamma_z"] \arrow[dd, "\phi_z"] & U_x/f(\Gamma_z) \arrow[d] \\
 & \tilde{U}_x/\Gamma_x \arrow[d, "\phi_x"] \\
U_z \arrow[r] & U_z
\end{tikzcd}$$

The covering $\{U_x\}$ of $X_{\mathcal{O}}$ is not an intrinsic part of the orbifold structure. We regard two coverings to give the same orbifold structure if they can be combined to give a larger covering still satisfying the definitions.

Let $0 \le r \le \infty$. An orbifold $\mathcal{O}$ is a C^r *orbifold* if each Γ_x acts C^r–smoothly and the embedding $\tilde{\psi}$ is C^r.

Locally Smooth Orbifolds. We say that an orbifold $\mathcal{O}$ is *locally smooth* if the action of Γ_x on $\tilde{U}_x \cong \mathbb{R}^n$ is an *orthogonal* action for all $x \in \mathcal{O}$. That is, for each $x \in \mathcal{O}$, there exists a representation $L : \Gamma_x \to \mathbb{O}(n)$ such that if $\gamma \cdot y$ denotes the Γ_x action on $\tilde{U}_x$, then we have $\gamma \cdot y = L(\gamma)y$ for all $y \in \tilde{U}_x$.

Orbifold Strata and Isotropy Groups. Let $\mathcal{O}$ be a connected n-dimensional locally smooth orbifold. Given a point $x \in \mathcal{O}$, there is a neighborhood U_x of x which is homeomorphic to a quotient $\tilde{U}_x/\Gamma_x$ where $\tilde{U}_x$ is homeomorphic to $\mathbb{R}^n$ and Γ_x is a finite group acting orthogonally on $\mathbb{R}^n$. The definition of orbifold implies that the germ of this action in a neighborhood of the origin of $\mathbb{R}^n$ is unique. We define the *isotropy group of* x to be the group Γ_x. The *singular set* of $\mathcal{O}$ is the set of points $x \in \mathcal{O}$ with $\Gamma_x \neq \{1\}$. Denote the singular set of $\mathcal{O}$ by Σ_1. Then Σ_1 is also a (possibly disjoint) union $\bigcup_{l_1} \Sigma^{(l_1)}$ of connected locally smooth orbifolds of strictly lower dimension (though different components may have different dimensions). See the section of examples. Each of the orbifolds $\Sigma_1^{(l_1)}$ has a singular set $\bigcup_{l_2} \Sigma_1^{(l_1)(l_2)}$. Define the singular set of Σ_1 to be $\Sigma_2 = \bigcup_{(l_1)(l_2)} \Sigma_1^{(l_1)(l_2)}$. Proceeding inductively, we get a stratification of $\mathcal{O}$:

$$\mathcal{O} = \Sigma_0 \supset \Sigma_1 \supset \Sigma_2 \supset \cdots \Sigma_{k-1} \supset \Sigma_k = \emptyset \text{ for some } k \le n+1$$

By a result of M.H.A Newman [7], the singular set of a topological orbifold is a closed nowhere dense set. In the locally smooth case, the proof is much easier. See Proposition 3.1 and the remark that follows.

3. Elementary Properties of Locally Smooth Orbifolds

The two results that we shall need involving locally smooth orbifolds are the following:

Proposition 3.1. *If $\mathcal{O}$ is locally smooth then in each local orbifold chart $\tilde{U}_x$ the fixed point set $S_x = \{y \in \tilde{U}_x \mid \Gamma_x \cdot y = y\}$ is a topological submanifold of $\tilde{U}_x$.*

Proof. Let $\tilde{U}$ be an orbifold chart for x and Γ_x its isotropy group. Since $\mathcal{O}$ is locally smooth there is a Γ_x-equivariant embedding $F : \tilde{U} \to \mathbb{R}^n_L$, where $\mathbb{R}^n_L$ denotes $\mathbb{R}^n$ with the orthogonal Γ_x-action L. Since we can regard L as a representation $L : \Gamma_x \to \mathbb{O}(n)$, we will use the notation $L(\gamma) = L_\gamma$. Thus, we have $F(\gamma \cdot x) = L_\gamma(F(x))$. If $y \in S_x$, and $z = F(y)$ then we have that $z = L_\gamma(z)$, hence $F(S_x) \subset \bigcap_{\gamma\in\Gamma_x} \ker(L_\gamma - I)$. Let $W = \bigcap_{\gamma\in\Gamma_x} \ker(L_\gamma - I)$ and let $w \in W$, with $F(v) = w$ for some $v \in \tilde{U}$. Then

$$v = F^{-1}(w) = F^{-1}L_\gamma(w) = F^{-1}L_\gamma F(v) = F^{-1}F(\gamma \cdot v) = \gamma \cdot v$$

for all $\gamma \in \Gamma_x$, hence $v \in S_x$. Thus we have shown $F(S_x) = W$. Since W is a subspace, we have that $S_x = F^{-1}(W)$ is a submanifold of $\tilde{U}$. This completes the proof. ∎

Remark 3.2. It follows easily from this result, that the singular set of a locally smooth orbifold is closed and nowhere dense. This is because the intersection of the singular set with a fundamental chart is closed and nowhere dense by Proposition 3.1, and $\mathcal{O}$ is a Baire space ($\mathcal{O}$ is locally compact Hausdorff).

Proposition 3.3. *If $\mathcal{O}$ is a smooth C^r orbifold with $r > 0$, then it is locally smooth.*

Proof. Let Γ_x be the isotropy group of x and B a neighborhood of x with a neighborhood $\tilde{B}$ of 0 in $\mathbb{R}^n$ together with a homeomorphism $\phi_x : \tilde{B}/\Gamma_x \to B$ where Γ_x acts C^r-smoothly on $\tilde{B}$. We denote the action of Γ_x by $(\gamma, y) \to \gamma \cdot y$ for all $\gamma \in \Gamma_x$ and $y \in \tilde{B}$. Without loss of generality we may assume that $\phi_x(0) = x$ and thus $\Gamma_x \cdot 0 = 0$. Let $L_\gamma : T_0\tilde{B} \to T_0\tilde{B}$ be the linearization at 0 of $y \to \gamma \cdot y$. Note that L_γ, being the linearization *at* 0, is a *fixed* linear map, and is therefore C^∞. Define $F : \tilde{B} \to \mathbb{R}^n$ by

$$F(y) = \frac{1}{|\Gamma_x|} \sum_{\eta\in\Gamma_x} L_\eta(\eta^{-1} \cdot y)$$

Then F is C^r since L_η is C^∞ and the action of Γ_x is C^r. Also, $dF(0) = \mathrm{Id}$ and $F(\gamma \cdot y) = L_\gamma(F(y))$. To see the last statement, note that

$$\begin{aligned} F(\gamma \cdot y) &= \frac{1}{|\Gamma_x|} \sum_{\eta\in\Gamma_x} L_\eta(\eta^{-1}\gamma \cdot y) = \frac{1}{|\Gamma_x|} \sum_{\eta\in\Gamma_x} L_\eta((\gamma^{-1}\eta)^{-1} \cdot y) \\ &= \frac{1}{|\Gamma_x|} \sum_{\mu\in\Gamma_x} L_{\gamma\mu}(\mu^{-1} \cdot y) \qquad \text{where } \mu = \gamma^{-1}\eta \\ &= \frac{1}{|\Gamma_x|} \sum_{\mu\in\Gamma_x} L_\gamma(L_\mu(\mu^{-1} \cdot y)) = L_\gamma\Big(\frac{1}{|\Gamma_x|} \sum_{\mu\in\Gamma_x} L_\mu(\mu^{-1} \cdot y)\Big) = L_\gamma(F(y)) \end{aligned}$$

So by the inverse function theorem, there is a neighborhood $\tilde{C}$ of 0 in $\tilde{B}$ on which F is an equivariant C^r diffeomorphism. F conjugates the action of Γ_x to the linear action L_γ. Since the linear action L_γ is linearly conjugate to an orthogonal action, the proof is complete. ∎

4. Examples

Example 4.1. (Hemisphere) Let $\mathcal{O} = (S^n, \mathrm{can})/G$, $n > 1$, be the n-dimensional hemisphere of constant curvature 1 (topologically $\mathcal{O}$ is just the closed n–disk D^n). $G = \mathbb{Z}_2 \subset \mathbb{O}(n+1)$ is the group generated by reflection through an equatorial $(n-1)$–sphere. In this case Σ_1 is the equatorial $(n-1)$–sphere. More generally, one might consider the quotient of S^n by the group generated by reflections in all coordinate n-planes of $\mathbb{R}^{n+1}$, getting an orbifold whose topological structure is that of a so-called manifold with corners.

Example 4.2. (Football) Let $\mathcal{O}$ be a $\mathbb{Z}_p$–football. $\mathcal{O} = (S^2, \mathrm{can})/G$, where $G \subset \mathbb{O}(3)$ is rotation around the z–axis in $\mathbb{R}^3$, through an angle of $2\pi/p$. Here $\Sigma_1 = \{\text{north pole}\} \cup \{\text{south pole}\}$.

Example 4.3. ($\mathbb{Z}_p$–hemisphere) Let $\mathcal{O}$ be a $\mathbb{Z}_p$–football$/G$, where G is reflection in the equator of the football that does not contain the singular points. Topologically, $\mathcal{O}$ is D^2. Note that the singular set $\Sigma_1 = \{\text{equator}\} \cup \{\text{point}\}$, thus it is possible for different components of the singular set to have different dimensions.

Example 4.4. (Pillow) Let $\mathcal{O} = \mathbb{R}^2/G$, where G is the crystallographic group generated by reflecting an equilateral triangle or square in each of its sides to produce a tiling of $\mathbb{R}^2$. Then $\mathcal{O}$ is just the closed triangle or square, with singular set the boundary of the tiling region. The stratification of $\mathcal{O}$ is as follows:

$$\mathcal{O} = \Sigma_0 \supset \Sigma_1 = \{\text{the boundary of the triangle or square}\} \supset$$
$$\Sigma_2 = \{\text{the vertices}\} \supset \Sigma_3 = \emptyset$$

Here, Σ_1 is union of the closed line segments making up the boundary of the triangle or square and each of these line segments is a 1–dimensional orbifold with 2 singular points. One should observe that Σ_1 is not a 1–dimensional orbifold but a *union* of 1–dimensional orbifolds. The lowest dimensional stratum has dimension 0. Note that the manifold $\Sigma_1 - \Sigma_2$ is a union of open line segments. If one only quotients out by the index 2 subgroup G_0 of orientation preserving elements of G then $\mathcal{O}$ becomes topologically a 2–sphere. The complement of the singular set is topologically $\mathbb{R}^2 - \{2 \text{ points or } 3 \text{ points}\}$.

Example 4.5. (Teardrop and (p,q)-footballs) Let $\mathcal{O}$ be a $\mathbb{Z}_p$–teardrop. The underlying space of this orbifold is S^2 with a single conical singularity of order p at the north pole. One may also construct a (p,q)-football whose underlying space is also S^2 with two conical singularities, one of order p at the north pole and the other of order $q \neq p$ at the south pole.

Example 4.6. Consider the group $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ generated by rotations of π about the three coordinate axes of $\mathbb{R}^3$. If we consider the quotient of the 2–sphere S^2/G, we get a 2–dimensional orbifold $\mathcal{O}$ whose underlying space is topologically the 2–sphere with 3 singular points. The sin–suspension $\Sigma_{\sin}\mathcal{O} = S^3/\Sigma G$ is an orientable 3–dimensional orbifold. ΣG denotes the suspension of the action on S^2 to S^3. In this case, Σ_1 is the union of the 3 line segments joining the suspension points and passing through one of the singular points of $\mathcal{O}$. Σ_2 is just the two suspension points.

Example 4.7. Let $L_p = S^3/G$ be a 3–dimensional lens space. Suspend the action of G to an action ΣG on the 4–sphere S^4. Let $\mathcal{O} = S^4/\Sigma G$. Then the underlying space of $\mathcal{O}$ is *not* a manifold (or manifold with boundary).

5. Maps Between Orbifolds

Orbifold Maps. A map $f : \mathcal{O} \to \mathcal{O}'$ of C^r orbifolds is a C^r *orbifold map* if for each $x \in X_{\mathcal{O}}$ there are open neighborhoods U_x and V_y of x and $y = f(x)$ in $\mathcal{O}$ and $\mathcal{O}'$ respectively, open sets $\tilde{U}_x$ and $\tilde{V}_y$ in $\mathbb{R}^n$ with finite groups Γ_x and Γ_y acting C^r on $\tilde{U}_x$ and $\tilde{V}_y$ respectively, a homomorphism $\Theta : \Gamma_x \to \Gamma_y$ and a C^r map $\tilde{f} : \tilde{U}_x \to \tilde{V}_y$ equivariant with respect to Θ, (that is, the following diagram commutes:

$$\begin{tikzcd}
\tilde{U}_x \arrow[r, "\tilde{f}"] \arrow[d] & \tilde{V}_y \arrow[d] \\
\tilde{U}_x/\Gamma_x \arrow[r] \arrow[dd] & \tilde{V}_y/\Theta(\Gamma_x) \arrow[d] \\
 & \tilde{V}_y/\Gamma_y \arrow[d] \\
U_x \arrow[r, "f"] & V_y
\end{tikzcd}$$

We will write $C^r_{\mathrm{Orb}}(\mathcal{O}, \mathcal{O}')$ for the set of C^r orbifold maps.

It is immediate from the definition that composition of C^r orbifold maps gives another C^r orbifold map. We may therefore define the category C^r Orb with objects the C^r orbifolds and morphisms the C^r maps between them, although we will not use this terminology. We now define the objects in the automorphism group of the orbifold structure.

Orbifold Homeomorphisms and Diffeomorphisms. For any topological space X, let $\mathrm{H}(X)$ denote its group of homeomorphisms. For a topological orbifold $\mathcal{O}$, the group of *orbifold homeomorphisms*, $\mathrm{H}_{\mathrm{Orb}}(\mathcal{O})$ will be the subgroup of $\mathrm{H}(X_{\mathcal{O}})$ so that $f, f^{-1} \in C^0_{\mathrm{Orb}}(X_{\mathcal{O}}, X_{\mathcal{O}})$. If $\mathcal{O}$ is a C^r orbifold, $\mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O})$ is the subgroup of $\mathrm{H}_{\mathrm{Orb}}(\mathcal{O})$ with $f, f^{-1} \in C^r_{\mathrm{Orb}}(\mathcal{O})$. We will also use $\mathrm{Diff}^0_{\mathrm{Orb}}(\mathcal{O})$ for $\mathrm{H}_{\mathrm{Orb}}(\mathcal{O})$. Let

$$\mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O}, \Sigma_m) = \{f \in \mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O}) \mid f(x) = x \text{ for all } x \in \Sigma_m\}$$

be the subgroup of $\mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O})$ fixing the entire stratum Σ_m pointwise.

Lemma 5.1. *Any element of* $\mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O})$ *leaves* Σ_i *invariant (as a set), where* Σ_i *is any substratum of* $\mathcal{O}$.

Proof. We show first that Σ_1 is invariant. Since $\Sigma_1 = \{x \in \mathcal{O} \mid \Gamma_x \neq \{1\}\}$, just note that if f is an orbifold diffeomorphism then by definition there is an isomorphism of the isotropy subgroups Γ_x and $\Gamma_{f(x)}$. Hence singular points get sent to singular points. To see the general case, it suffices to show the invariance of Σ_2. Let $y \in \Sigma_2$ with isotropy subgroup Γ_y. Then there is an orbifold chart U

about y which contains $x \in \Sigma_1 - \Sigma_2$ with $\Gamma_x \subsetneq \Gamma_y$. Denote the action of Γ_y on $\tilde{U}$ by α, and the action of $\Theta(\Gamma_y)$ on $\tilde{f}(\tilde{U})$ by α^Θ. Note that the equivariance of $\tilde{f}$ on $\tilde{U}$ implies that $\tilde{f} \circ \alpha_\gamma \circ \tilde{f}^{-1} = \alpha^\Theta_{\Theta(\gamma)}$. Thus, the action of $\Theta(\Gamma_x)$ on $\tilde{f}(\tilde{U})$ is the restriction of α^Θ to $\Theta(\Gamma_x)$. Since f preserves Σ_1 and $\Theta(\Gamma_x)$ is a proper subgroup of $\Theta(\Gamma_y)$, we see that the singular set Σ_2 of Σ_1 is preserved. This completes the proof. ∎

Using this lemma it is easily verified that

$$\mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O}) \rhd \mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O}, \Sigma_{k-1}) \rhd \cdots \rhd \mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O}, \Sigma_2) \rhd \mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O}, \Sigma_1)$$

where $G \rhd H$ means that H is normal subgroup of G.

6. Extending Orbifold Diffeomorphisms

For any subgroup G of the homeomorphism group $\mathrm{H}(X)$ of a topological space X, let $G_c \subset G$ denote those elements of G with compact support in X. Let G_0 be the subgroup of G_c whose elements are isotopic to the identity through elements of G with compactly supported isotopy. For any self-map $f : X \to X$ of a topological space X, let the support $\mathrm{supp}(f) = \mathrm{cl}\{x \in X \mid f(x) \neq x\}$ where $\mathrm{cl}(S)$ denotes the closure of the set S. By compactly supported isotopy we mean an isotopy $f : [0,1] \times X \to [0,1] \times X$, such that $\mathrm{supp}(f) \subset [0,1] \times X$ is compact.

Proposition 6.1. *(Extension of Orbifold Diffeomorphisms) The group* $\mathrm{Diff}^r(\mathcal{O} - \Sigma)_c$ *is a subgroup of* $\mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O})$ *for any* C^r *orbifold* $\mathcal{O}$, $0 \le r \le \infty$. *Moreover, if* $1 \le r \le \infty$, *then for each component* $\mathcal{A} = \Sigma_m^{(l_1)(l_2)\cdots(l_m)}$ *of* Σ_m, *each* $f \in \mathrm{Diff}^r(\mathcal{A} - \Sigma_{\mathcal{A}})_0$, *and open neighborhood of* $\mathrm{supp}(f)$ *in* $\mathcal{O}$, *there is an extension* $g \in \mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O})_0$ *such that* $\mathrm{supp}(g) \subset U$. *Here,* $\Sigma_{\mathcal{A}}$ *denotes the singular set of* $\mathcal{A}$

Proof. Note that $\mathrm{Diff}^r(\mathcal{O} - \Sigma)_c$ is the group of compactly supported diffeomorphisms of the *manifold* $X_{\mathcal{O}} - \Sigma$. If $f \in \mathrm{Diff}^r(\mathcal{O} - \Sigma)_c$, then $\mathrm{supp}(f)$ is a compact subset of $X_{\mathcal{O}} - \Sigma$ disjoint from Σ and so in any sufficiently small neighborhood U of Σ, $f(x) = x$ for all $x \in U - \Sigma$. Therefore we can define an extension $\overline{f}$ of f to $X_{\mathcal{O}}$ by

$$\overline{f}(x) = \begin{cases} f(x) & \text{if } x \in X_{\mathcal{O}} - \Sigma \\ x & \text{if } x \in \Sigma \end{cases}$$

It is clear that $\overline{f} \in \mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O})$.

Since the rest of the argument is fairly technical, we will first give an outline of the proof that follows. Many of the ideas that we use here are the same ones used to show that for a smooth manifold M, the group, $\mathrm{Diff}(M)_0$, of diffeomorphisms isotopic to the identity through compactly supported isotopies, has the so-called *fragmentation property* [5]. Let $f \in \mathrm{Diff}^r(\mathcal{A} - \Sigma_{\mathcal{A}})_0$. In the final part of the proof we use the isotopy of f to the identity to show that it is sufficient to find an extension g for f close to the identity. To obtain g, we first construct a manifold $\tilde{M}$, and a map from $\tilde{M}$ to an appropriate neighborhood of $\mathcal{A}$. By construction, the manifold $\tilde{M}$ contains a copy the stratum. Using the fact that a local chart at the identity in the group of compactly supported diffeomorphisms is given by the vector fields with compact support, we extend the vector field on $\mathcal{A}$ that corresponds to f to a compactly supported equivariant vector field on $\tilde{M}$. This enables us to extend f to an equivariant diffeomorphism on $\tilde{M}$, which projects to an extension of f to $\mathcal{O}$. The details now follow.

Let $N(\Sigma_{\mathcal{A}})$ be a (small) closed neighborhood of $\Sigma_{\mathcal{A}}$ in $\mathcal{O}$. Let $\{V_i\}_{i\in I}$ be a covering of $\mathcal{A} - N(\Sigma_{\mathcal{A}})$ by orbifold charts such that $V_i \cap \Sigma_{\mathcal{A}} = \emptyset$ which are themselves covered by $\{\tilde{V}_i\}_{i\in I}$ equipped with group actions Γ_i and projections $\pi_i : \tilde{V}_i \to V_i$. For each $i \in I$ choose a Dirichlet fundamental domain $\tilde{D}_i$ and let $\iota_i : V_i \to \tilde{D}_i$ be the function assigning the unique $\tilde{x} \in \tilde{D}_i$ with $\pi_i(\tilde{x}) = x$ for each $x \in V_i$. Since $\Gamma_x = \Gamma_y$ for all x, y in $\mathcal{A} - \Sigma_{\mathcal{A}}$, we let $\Gamma = \Gamma_x$. Define a C^r manifold $\tilde{M}$ with a C^r Γ action via

$$\tilde{M} = \bigcup_{i\in I} \tilde{V}_i / \sim$$

where $\sim$ is the equivalence relation we now define. For $\tilde{x}_i \in \tilde{V}_i$, $\tilde{x}_j \in \tilde{V}_j$,

1. If $i = j$, $\tilde{x}_i \sim \tilde{x}_i$
2. If $i \neq j$, $\tilde{x}_i \sim \tilde{x}_j$ if and only if $\pi_i(\tilde{x}_i) = \pi_j(\tilde{x}_j)$, $\Psi_i(\gamma) \cdot \tilde{x}_i \in \tilde{D}_i$ and $\Psi_j(\gamma) \cdot \tilde{x}_j \in \tilde{D}_j$ for some fixed $\gamma \in \Gamma$, where $\Psi_i : \Gamma \to \Gamma_i$, $\Psi_j : \Gamma \to \Gamma_j$ are the natural identification homomorphisms.

The action of Γ on $\tilde{M}$ is given by the action of $\Psi_i(\Gamma)$ on $\tilde{V}_i$. Let $M = \bigcup_{i\in I} V_i$. The functions ι_i, by definition, glue together to give a function

$$\iota : M \to \tilde{M}$$

which restricts to a C^r embedding ι of $\mathcal{A} - N(\Sigma_{\mathcal{A}})$ into $\tilde{M}$. To see this, just note that by construction, $\tilde{x} \in \iota(\mathcal{A} - N(\Sigma_{\mathcal{A}})) \Leftrightarrow \Gamma\tilde{x} = \tilde{x}$. Without loss of generality, we can assume that $\tilde{M}$ is an equivariant tubular neighborhood of $\iota(\mathcal{A} - N(\Sigma_{\mathcal{A}}))$ with projection $\rho : \tilde{M} \to \iota(\mathcal{A} - N(\Sigma_{\mathcal{A}}))$. Moreover, $\tilde{M}/\Gamma = M$ is an orbifold that is homeomorphic to a neighborhood of $\mathcal{A} - N(\Sigma_{\mathcal{A}})$ in $\mathcal{O}$. Identify M with this neighborhood and let $\pi : \tilde{M} \to M$ be the quotient map.

Fix $1 \leq r \leq \infty$. By results in [12], we may assume that $\tilde{M}$ carries a C^∞ structure $\tilde{M}^\infty$ which is C^r diffeomorphic to $\tilde{M}$. That is, there exists a C^r diffeomorphism $\Delta : \tilde{M} \to \tilde{M}^\infty$. In addition, the paper [14] allows us to assume that Γ is equivalent to a C^∞ Γ action on $\tilde{M}^\infty$. Endow $\tilde{M}^\infty$ with a C^∞ Riemannian metric which is equivariant with respect to the induced orthogonal Γ action on $T\tilde{M}^\infty$. This makes $\rho : \tilde{M}^\infty \to \iota(\mathcal{A} - N(\Sigma_{\mathcal{A}}))$ a Riemannian submersion and $\iota(\mathcal{A} - N(\Sigma_{\mathcal{A}}))$ a totally geodesic submanifold of $\tilde{M}^\infty$ (being the fixed point set of Γ acting by isometries). In what follows we will identify $\tilde{M}$ with $\tilde{M}^\infty$ via Δ and $\mathcal{A} - N(\Sigma_{\mathcal{A}})$ with its image $\iota(\mathcal{A} - N(\Sigma_{\mathcal{A}}))$ in $\tilde{M}$.

It is well known, see for example [5], that for a manifold N the group $\mathrm{Diff}^r(N)_0$ carries the structure of an infinite dimensional manifold whose local model $T_f \,\mathrm{Diff}^r(N)_0$ is the vector space $\boldsymbol{\chi}^r(N)$ of C^r vector fields on N with compact support in the uniform C^r topology. In addition, for each $f \in \mathrm{Diff}^r(N)_0$ there is a neighborhood $\mathcal{U}_f$ of $0 \in T_f \,\mathrm{Diff}^r(N)_0 \cong \boldsymbol{\chi}^r(N)$ and a smooth open embedding

$$\overline{\exp}_f : \mathcal{U}_f \subset T_f \,\mathrm{Diff}^r(N)_0 \to \mathrm{Diff}^r(N)_0$$

defined by $\big(\overline{\exp}_f(\nu)\big)(x) = \exp_{f(x)}(\nu(x))$ for $\nu \in \mathcal{U}_f$, where exp is the Riemannian exponential map.

Let $f \in \mathrm{Diff}^r(\mathcal{A} - \Sigma_{\mathcal{A}})_0$ and $f_t(x) = f(t,x)$, $t \in [0,1]$ a C^r compactly supported isotopy with $f_0(x) = \mathrm{Id}$ and $f_1(x) = f(x)$. Choose $N(\Sigma_{\mathcal{A}})$ so that $\mathrm{supp}(f_t) \subset [0,1] \times (\mathcal{A} - N(\Sigma_{\mathcal{A}}))$. We can regard f_t as a C^r path in $\mathrm{Diff}^r(\mathcal{A}-\Sigma_{\mathcal{A}})_0$ joining Id to f. Let $\mathcal{U}_0$ be a neighborhood of 0 in $\boldsymbol{\chi}^r(\mathcal{A} - N(\Sigma_{\mathcal{A}}))$ small enough so that $\overline{\exp}_{\mathrm{Id}_{\mathcal{A}-N(\Sigma_{\mathcal{A}})}}\big|_{\mathcal{U}_0}$ is an embedding. Denote by $\boldsymbol{\chi}^r_{\mathcal{A}-N(\Sigma_{\mathcal{A}})}(\tilde{M})$ the C^r sections of $T_{\mathcal{A}-N(\Sigma_{\mathcal{A}})}(\tilde{M})$ with compact support equipped with the uniform C^r topology on $\mathcal{A} - N(\Sigma_{\mathcal{A}})$, and let $\mathcal{V}_0$ be a neighborhood of 0 in $\boldsymbol{\chi}^r_{\mathcal{A}-N(\Sigma_{\mathcal{A}})}(\tilde{M})$ so that $\mathcal{V}_0 \cap \boldsymbol{\chi}^r(\mathcal{A} - N(\Sigma_{\mathcal{A}})) = \mathcal{U}_0$. Let $\tilde{\Omega}_0 \subset \mathrm{Diff}^r(\tilde{M})_0$ be a neighborhood of $\mathrm{Id}_{\tilde{M}}$ so that

$$\big(\overline{\exp}_{\mathrm{Id}_{\tilde{M}}}\big)^{-1}(\tilde{\Omega}_0)\big|_{\mathcal{A}-N(\Sigma_{\mathcal{A}})} = \mathcal{V}_0$$

Let

$$\Omega_t = \overline{\exp}_{\mathrm{Id}_{\mathcal{A}-N(\Sigma_{\mathcal{A}})}}(\mathcal{U}_0) \circ f_t$$

The collection Ω_t, $t \in [0,1]$ forms an open cover of the curve $f_{[0,1]} \subset \mathrm{Diff}^r(\mathcal{A} - N(\Sigma_{\mathcal{A}}))_0$. Since $f_{[0,1]}$ is compact and the isotopy f_t is continuous in t, there exists a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ of $[0,1]$ so that $\bigcup_i \Omega_{t_i}$ is a cover of $f_{[0,1]}$ with $f_{t_{i+1}} \circ f_{t_i}^{-1} \in \Omega_0$ for $i = 0, \ldots, n-1$. Thus,

$$\big(\overline{\exp}_{\mathrm{Id}_{\mathcal{A}-N(\Sigma_{\mathcal{A}})}}\big)^{-1}(f_{t_{i+1}} \circ f_{t_i}^{-1}) \in \mathcal{U}_0,\ i = 0, \ldots, n-1$$

Let $g_i = f_{t_{i+1}} \circ f_{t_i}^{-1}$, then $f_1 = g_{n-1} \circ g_{n-2} \circ \cdots \circ g_0$ and let

$$\nu_i = \big(\overline{\exp}_{\mathrm{Id}_{\mathcal{A}-N(\Sigma_{\mathcal{A}})}}\big)^{-1}(g_i)$$

Define $\overline{\nu}_i(y)$ to be the unique horizontal lift of $\nu_i(\rho(y))$ to $T_y\tilde{M}$ for $y \in \tilde{M}$ where the lift is taken with respect to the Γ equivariant Riemannian submersion $\rho : \tilde{M} \to \mathcal{A} - N(\Sigma_{\mathcal{A}})$. By construction, $\overline{\nu}_i(y) \in \boldsymbol{\chi}^r(\tilde{M})$ is Γ equivariant and $\rho_*\overline{\nu}_i = \nu_i$.

Now, let $\eta_i : \tilde{M} \to [0,1]$ be C^r, Γ equivariant functions with $\eta_i|_{\mathcal{A}-N(\Sigma_{\mathcal{A}})} = 1$ and $\eta_i = 0$ outside some compact neighborhood of $\mathrm{supp}(g_i)$ in $\tilde{M}$. By decreasing the mesh of the partition if necessary, we may ensure that $\tilde{\nu}_i = \eta_i\overline{\nu}_i \in \mathcal{U}_0$ for $i = 0, \ldots, n-1$. Then $\tilde{g}_i = \overline{\exp}_{\mathrm{Id}_{\tilde{M}}}(\tilde{\nu}_i)$ is a Γ equivariant extension of g_i to a relatively compact neighborhood of $\mathrm{supp}(g_i) \subset \mathcal{A} - N(\Sigma_{\mathcal{A}})$ in $\tilde{M}$ and $\tilde{f}_1 = \tilde{g}_{n-1} \circ \tilde{g}_{n-2} \circ \cdots \circ \tilde{g}_0$, defines the required extension of f_1. This completes the proof. ■

7. Local Contractions

The proof of the main theorem will require that there are enough local orbifold diffeomorphisms whose behavior under the group isomorphism can be controlled. For a locally compact Hausdorff space X, a subgroup $G \subset \mathrm{H}(X)$ and $x \in X$, we say that $g_x \in G$ is a *local contraction about* x if:

1. $x \in \mathrm{supp}(g_x)$ and $\mathrm{supp}(g_x)$ is compact

2. for all open neighborhoods V and W of x in $\operatorname{supp}(g_x)$ with $\overline{W} \subset V \subset \overline{V} \subset \operatorname{int}(\operatorname{supp}(g_x))$ there is an $N \in \mathbb{N}$ so that $g_x^n(\overline{V}) \subset W$ for all $n > N$.

3. $g_x(x) = x$

While it is not hard to show that (3) is a consequence of (1) and (2), we wanted to make this explicit.

Proposition 7.1. *If $\mathcal{O}$ is locally smooth, then for each $x \in \mathcal{O}$ and neighborhood U of x there is a local contraction about x with support in U.*

Proof. The result is clearly local so there is no loss in generality if U is assumed to be in an orbifold chart. As $\mathcal{O}$ is locally smooth, there is an orthogonal action L of Γ_x on $\mathbb{R}^n$, and an orbifold homeomorphism $f : \mathbb{R}^n/\Gamma_x \to U$ sending 0 to x. Let $\chi : [0, \infty) \to [0, 1]$ be a smooth, decreasing function with $\chi|_{[0,r/2)} = 1$ and $\chi|_{[r,\infty)} = 0$ where $r > 0$ is such that $B_r(0) \subset (f \circ \pi)^{-1}(U)$, $\pi : \mathbb{R}^n_L \to \mathbb{R}^n_L/\Gamma_x$ is the projection and $B_r(0)$ is the ball of radius r about 0 with respect to the Euclidean metric. Define the vector field

$$\nu(x) = -\chi(|x|)x$$

Since Γ_x is an orthogonal action, ν is a Γ_x invariant vector field on $\mathbb{R}^n$ and so the flow g_t generated by ν will also be Γ_x invariant. g_1 is clearly a local contraction about 0 supported in $(f \circ \pi)^{-1}(U)$ and may be extended outside of this set to all of $\mathbb{R}^n$ by the identity. Since g_1 is Γ_x equivariant, $\pi \circ g_1 \circ \pi^{-1}$ is well defined. Thus $g_x = f \circ \pi \circ g_1 \circ (f \circ \pi)^{-1}$ is an orbifold homeomorphism of U. Extending g_x outside of U by the identity gives the required local contraction. This completes the proof. ∎

8. Proof Of Theorem 1.2

Before the main part of the proof, we need to record two observations that will be used throughout.

Remark 8.1. 1. For any open subset U of an orbifold $\mathcal{O}$, $x \in U$ if and only if there is a neighborhood V of x so that $V - \Sigma \subset U - \Sigma$.

2. For an open subset U as above, $x \in \operatorname{cl}(U)$ if and only if $(V - \Sigma) \cap (U - \Sigma) \neq \emptyset$ for all neighborhoods V of x.

These follow trivially from the fact that the singular set Σ of an orbifold $\mathcal{O}$ is nowhere dense.

Proof. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two compact, locally smooth orbifolds and let $\Phi : \operatorname{Diff}^r_{\text{Orb}}(\mathcal{O}_1) \to \operatorname{Diff}^r_{\text{Orb}}(\mathcal{O}_2)$ be a group isomorphism. Lemma 5.1 implies that $\operatorname{Diff}^r_{\text{Orb}}(\mathcal{O}_i)|_{\mathcal{O}_i - \Sigma_i}$ is a subgroup of $\operatorname{Diff}^r(\mathcal{O}_i - \Sigma_i)$. Let T be an open subset of $\mathcal{O}_i - \Sigma_i$ and assume $x \in T$. Since $\mathcal{O}$ is locally compact, we may choose a compact neighborhood $U \subset T$ of x. Proposition 6.1 implies that $\operatorname{Diff}^r(\mathcal{O}_i - \Sigma_i)_c \subset \operatorname{Diff}^r_{\text{Orb}}(\mathcal{O}_i)$. Thus, given $y \in U$ with $y \neq x$, there is $g \in \operatorname{Diff}^r(\mathcal{O}_i - \Sigma_i)_c$ with support in U so that $g(x) = y$. This follows from the local homogeneity of the manifold $\mathcal{O}_i - \Sigma_i$. Thus, we conclude that the groups $\mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O}_i)$ satisfy the hypotheses of Rubin's theorem and so we have a homeomorphism $h : \mathcal{O}_1 - \Sigma_1 \to \mathcal{O}_2 - \Sigma_2$ such that for every $f \in \mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O}_1)$ we have $\Phi(f) = hfh^{-1}$. Note that this implies that the singular sets of $\mathcal{O}_i$ are either both empty or are both non-empty. To see this, suppose $\Sigma_1 \neq \emptyset$ and that $\Sigma_2 = \emptyset$. Then $\mathcal{O}_2 = \mathcal{O}_2 - \Sigma_2$ is a closed manifold. $\mathcal{O}_1 - \Sigma_1$, however, is a non–compact manifold, and this contradicts the existence of a homeomorphism $h : \mathcal{O}_1 - \Sigma_1 \to \mathcal{O}_2 - \Sigma_2$ guaranteed by Rubin's theorem. Since Rubin's theorem implies Theorem 1.2 when $\Sigma_1 = \Sigma_2 = \emptyset$ (the manifold case), we need only concern ourselves with case when Σ_1 and Σ_2 are non–empty.

Since we are about to embark on a long technical argument, we first give an outline of the rest of the proof. We use the existence of local contractions to extend the homeomorphism $h : \mathcal{O}_1 - \Sigma_1 \to \mathcal{O}_2 - \Sigma_2$ to a bijection $\overline{h} : \mathcal{O}_1 \to \mathcal{O}_2$ which induces the group isomorphism $\Phi : \mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O}_1) \to \mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O}_2)$. This is actually a delicate argument. If we knew *a priori* that a local contraction g_x at x was sent to a local contraction under Φ, then we could easily define our required extension by sending $x \in \Sigma_1$ to the unique fixed point of $\Phi(g_x)$. The problem with doing this is that the behavior of $\Phi(g_x)$ on the singular set is not well determined by knowledge of $h \circ g_x \circ h^{-1}$, which is only defined on the complement of the singular set. This means that until we establish the existence of an appropriate extension of the homeomorphism h, we do not know that local contractions are sent to local contractions by Φ. To define the extension of the homomorphism h, we show that $\Phi(g_x)$ possesses a unique fixed point $y \in \mathrm{int}(\mathrm{supp}(\Phi(g_x)))$, and then define an extension $\overline{h}$ via $\overline{h}(x) = y$. We then show that this extension is independent of the choice of local contraction g_x. Next, we verify that our extension is continuous and has continuous inverse, and thus is a homeomorphism which induces the group isomorphism Φ. Lastly, we show $\overline{h}$ and $(\overline{h})^{-1}$ are smooth on the non-singular part of each stratum.

To extend the homeomorphism $h : \mathcal{O}_1 - \Sigma_1 \to \mathcal{O}_2 - \Sigma_2$ to a bijection $\overline{h} : \mathcal{O}_1 \to \mathcal{O}_2$ which induces the group isomorphism $\Phi : \mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O}_1) \to \mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O}_2)$, let $x \in \Sigma_1$, and let U_x be a relatively compact open neighborhood of x in $\mathcal{O}_1$. By Proposition 7.1, there exists a $g_x \in \mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O})$ which is a local contraction about x with support in U_x. Let $\hat{g}_x = \Phi(g_x)$, and $\hat{U}_x = \mathrm{int}\big(\mathrm{cl}(h(U_x - \Sigma_1))\big)$. Note that by Rubin's theorem, we have that $\hat{g}_x = hg_xh^{-1}$ on $\mathcal{O}_2 - \Sigma_2$. It follows that $\mathrm{supp}(\hat{g}_x) \subset \mathrm{cl}(\hat{U}_x)$. Because h is not defined on all of $\mathcal{O}_2$, this statement needs justification. Consider the set $S = \{z \in \mathcal{O}_2 - \Sigma_2 \mid hg_xh^{-1}z \neq z\}$. It's not hard to see that $S \subset h\big(\mathrm{supp}(g_x) - \Sigma_1\big)$. Thus,

$$\mathrm{cl}(S) \subset \mathrm{cl}\left(h(\mathrm{supp}(g_x) - \Sigma_1)\right) \subset \mathrm{cl}\left(h(U_x - \Sigma_1)\right) = \mathrm{cl}(\hat{U}_x)$$

By definition, we have that

$$\mathrm{supp}(\hat{g}_x) - \Sigma_2 = \mathrm{supp}(hg_xh^{-1}) - \Sigma_2 \subset \mathrm{cl}(S)$$

Since Σ_2 is nowhere dense we have that $\mathrm{supp}(\hat{g}_x) = \mathrm{cl}(\mathrm{supp}(\hat{g}_x) - \Sigma_2)$, from which it follows that $\mathrm{supp}(\hat{g}_x) \subset \mathrm{cl}(\hat{U}_x)$.

We now extend h by showing that $\hat{g}_x$ possesses a unique fixed point $y \in \mathrm{int}(\mathrm{supp}(\hat{g}_x))$ and then define an extension $\overline{h}$ via $\overline{h}(x) = y$.

Since $\mathcal{O}_2$ is locally compact, we may choose a relatively compact open subset $\hat{W} \subset \hat{U}_x$ of $\mathcal{O}_2$ with

$$x \in \operatorname{int}\big(\operatorname{cl}(h^{-1}(\hat{W} - \Sigma_2))\big)$$

For any neighborhood V of x with $\operatorname{cl}(V) - \Sigma_1 \subset h^{-1}(\hat{W} - \Sigma_2)$ there is an $m > 0$ so that

$$g_x^m\big(h^{-1}(\hat{W} - \Sigma_2)\big) \subset V \subset \operatorname{int}\big(\operatorname{cl}(h^{-1}(\hat{W} - \Sigma_2))\big)$$

since g_x is a local contraction about x. Thus, for any $N > 0$,

$$x \in \bigcap_{n<N} g_x^{mn}\Big(\operatorname{cl}\big(h^{-1}(\hat{W} - \Sigma_2)\big)\Big) \neq \emptyset$$

which implies

$$\bigcap_{n<N} \operatorname{cl}\Big(g_x^{mn}\big(h^{-1}(\hat{W} - \Sigma_2)\big)\Big) \neq \emptyset$$

and so by definition of $\hat{g}_x$ and h,

$$\bigcap_{n<N} \operatorname{cl}\Big(h^{-1}\big(\hat{g}_x^{mn}(\hat{W} - \Sigma_2)\big)\Big) \neq \emptyset$$

which in turn implies,

$$\bigcap_{n<N} h^{-1}\big(\hat{g}_x^{mn}(\hat{W}) - \Sigma_2\big) \neq \emptyset$$

It now follows that

$$\emptyset \neq \bigcap_{n<N} h \circ h^{-1}\big(\hat{g}_x^{mn}(\hat{W}) - \Sigma_2\big) \subset \bigcap_{n<N} \hat{g}_x^{mn}(\hat{W})$$

and so

$$\bigcap_{n<N} \hat{g}_x^{mn}(\operatorname{cl}(\hat{W})) \neq \emptyset$$

Then the collection of closed sets $\big\{\hat{g}_x^{mn}(\operatorname{cl}(\hat{W}))\big\}$ has the finite intersection property, and so by compactness of $\mathcal{O}_2$ we have

$$Y_x = \bigcap_{n>0} \hat{g}_x^{mn}(\operatorname{cl}(\hat{W})) \neq \emptyset.$$

By construction, $Y_x = \bigcap_{m>0} \hat{g}_x^{mn}(\operatorname{cl}(\hat{W}))$ is a compact, $\hat{g}_x$ invariant set. We claim that Y_x is independent of g_x and the subset $\hat{W}$. To see this, suppose that g'_x is another local contraction with fixed point x, and $\hat{W}' \subset \mathcal{O}_2$ is a compact subset of $\operatorname{int}\big(\operatorname{supp}(\Phi(g'_x))\big)$ satisfying the same requirement of $\hat{W}$ as above. As both g_x and g'_x are local contractions, for any $n > 0$ there is an $m > 0$ so that:

$$g_x^m\big(\operatorname{int}(\operatorname{cl}(h^{-1}(\hat{W} - \Sigma_2)))\big) \subset g_x'^n\big(\operatorname{int}(\operatorname{cl}(h^{-1}(\hat{W}' - \Sigma_2)))\big)$$

and for any $m > 0$ there is an $n > 0$ so that:

$$g_x'^n\big(\operatorname{int}(\operatorname{cl}(h^{-1}(\hat{W}' - \Sigma_2)))\big) \subset g_x^m\big(\operatorname{int}(\operatorname{cl}(h^{-1}(\hat{W} - \Sigma_2)))\big)$$

Therefore $\bigcap_{n>0} \hat{g}_x^n(\hat{W}) \subset \bigcap_{m>0} \hat{g}_x'^m(\hat{W}') \subset \bigcap_{n>0} \hat{g}_x^n(\hat{W})$ which shows the independence of Y_x on the local contraction.

The next step in the proof is to show that if $x \neq x'$ then $Y_x \cap Y_{x'} = \emptyset$. Let g_x and $g'_{x'}$ be local contractions about x and x' respectively with disjoint supports such that $\operatorname{supp}(g_x) \subset U$ and $\operatorname{supp}(g'_{x'}) \subset U'$ where U and U' are open sets with $U \cap U' = \emptyset$, $U = \operatorname{int}(\operatorname{cl}(U))$ and $U' = \operatorname{int}(\operatorname{cl}(U'))$. Therefore $h(U - \Sigma_1) \cap h(U' - \Sigma_1) = \emptyset$ and by the remark above, if $z \in \operatorname{int}\big(\operatorname{cl}(h(U - \Sigma_1))\big)$, then there is a neighborhood V of z so that $V - \Sigma_2 \subset \operatorname{int}\big(\operatorname{cl}(h(U - \Sigma_1))\big) - \Sigma_2 = h(U - \Sigma_1)$. Therefore $z \notin \operatorname{int}\big(\operatorname{cl}(h(U' - \Sigma_1))\big)$. Thus,

$$\operatorname{int}\big(\operatorname{cl}(h(U - \Sigma_1))\big) \bigcap \operatorname{int}\big(\operatorname{cl}(h(U' - \Sigma_1))\big) = \emptyset$$

Since $Y_x \subset \operatorname{int}\big(\operatorname{cl}(h(U - \Sigma_1))\big)$ and $Y_{x'} \subset \operatorname{int}\big(\operatorname{cl}(h(U' - \Sigma_1))\big)$, $Y_x \cap Y_{x'} = \emptyset$. Therefore for any two such subsets Y_x and $Y_{x'}$ of $\mathcal{O}_2$, if $Y_x \cap Y_{x'} \neq \emptyset$ then $Y_x = Y_{x'}$ and $x = x'$.

Given a $k \in \operatorname{Diff}^r_{\text{Orb}}(\mathcal{O}_1)$, $x \in \Sigma_1$ and a local contraction g_x about x, the orbifold diffeomorphism $k \circ g_x \circ k^{-1}$ is a local contraction about $k(x)$. Hence $\Phi(k \circ g_x \circ k^{-1})$ will have invariant set $Y_{k(x)}$. Since Φ is a group isomorphism between $\operatorname{Diff}^r_{\text{Orb}}(\mathcal{O}_1)$ and $\operatorname{Diff}^r_{\text{Orb}}(\mathcal{O}_2)$, the invariant set of $\Phi(k \circ g_x \circ k^{-1})$ will be $\Phi(k)(Y_x)$. Therefore $\Phi(k)(Y_x) = Y_{k(x)}$ for all $x \in \Sigma_1$. We will use this below to prove that the sets Y_x consist of a single point.

To show this last assertion, let $y \in Y_x$, and $\hat{g}_y \in \operatorname{Diff}^r_{\text{Orb}}(\mathcal{O}_2)$ be a local contraction about y. Let $g_y = \Phi^{-1}(\hat{g}_y)$ and then by definition $y \in \hat{g}_y^n(Y_x) = Y_{g_y^n(x)}$ for all $n \geq 0$. Hence $Y_x \cap \hat{g}_y^n(Y_x) \neq \emptyset$ for all $n \geq 0$ and so $Y_x = \hat{g}_y^n(Y_x)$ for all $n \geq 0$. If $z \in Y_x \cap \operatorname{supp}(\hat{g}_y)$ then for any neighborhood V of y in $\mathcal{O}_2$, there is an $n > 0$ so that $\hat{g}_y^n(z) \in V$ which implies that $Y_x \cap \operatorname{int}(\operatorname{supp}(\hat{g}_y)) = \{y\}$. Since $\hat{g}_y$ was essentially arbitrary, this implies that $Y_x = \{y\}$, that is, the invariant set Y_x of g_x consists of a single point.

We now define the extension $\overline{h}$ of h to all of $\mathcal{O}_1$ by the following:

$$\overline{h}(x) = \begin{cases} h(x), & \text{if } x \in \mathcal{O}_1 - \Sigma_1 \\ Y_x, & \text{if } x \in \Sigma_1 \end{cases}$$

By construction, $\overline{h}$ is an bijection inducing the group isomorphism. Similarly we can construct an bijection $\overline{h^{-1}}$. Continuity of $\overline{h}$ follows from the following. Given $x \in \mathcal{O}_1$ and a neighborhood U_x of x, then there is a local contraction g_x about x with support in U_x (by Proposition 7.1). By construction, $x \in \operatorname{int}(\operatorname{supp}(g_x))$ and so the collection

$$\mathcal{B} = \bigcup_{x \in \mathcal{O}_1} \bigcup_{U_x \ni x} \big\{\operatorname{int}(\operatorname{supp}(g_x)) \,\big|\, \operatorname{int}(\operatorname{supp}(g_x)) \subset U_x\big\}$$

forms a base for the topology of $\mathcal{O}_1$. Let $\operatorname{Fix}(f) = \{x \in \mathcal{O} \mid f(x) = x\}$. Note that:

$$\operatorname{Fix}\big(\Phi(f)\big) = \overline{h}\big(\operatorname{Fix}(f)\big)$$

and that for any local contraction g_x, $\operatorname{Fix}(g_x) = \big(\mathcal{O}_1 - \operatorname{int}(\operatorname{supp}(g_x))\big) \cup \{x\}$. Thus

$$\overline{h}\big((\mathcal{O}_1 - \operatorname{int}(\operatorname{supp}(g_x))) \cup \{x\}\big) = \overline{h}\big(\operatorname{Fix}(g_x)\big) = \operatorname{Fix}\big(\Phi(g_x)\big)$$

$$= \big(\mathcal{O}_2 - \operatorname{int}(\operatorname{supp}(\Phi(g_x)))\big) \cup \{\overline{h}(x)\}$$

so

$$\overline{h}\big(\mathcal{O}_1 - \operatorname{int}(\operatorname{supp}(g_x))\big) = \mathcal{O}_2 - \operatorname{int}\big(\operatorname{supp}(\Phi(g_x))\big)$$

and therefore

$$\overline{h}\big(\operatorname{int}(\operatorname{supp}(g_x))\big) = \operatorname{int}\big(\operatorname{supp}(\Phi(g_x))\big)$$

and so $\overline{h}$ maps basic open sets to basic open sets and so $\overline{h}$ is continuous.

Similarly, $\overline{h^{-1}}$ is continuous. Note that by construction

$$\overline{h} \circ \overline{h^{-1}} = \mathrm{Id} \text{ on } \mathcal{O}_2 - \Sigma_2$$

and

$$\overline{h^{-1}} \circ \overline{h} = \mathrm{Id} \text{ on } \mathcal{O}_1 - \Sigma_1.$$

Since $\mathcal{O}_2 - \Sigma_2$ is dense in $\mathcal{O}_2$ and $\mathcal{O}_1 - \Sigma_1$ is dense in $\mathcal{O}_1$, we have that $\overline{h} \circ \overline{h^{-1}} = \mathrm{Id}$ on $\mathcal{O}_2$ and $\overline{h^{-1}} \circ \overline{h} = \mathrm{Id}$ on $\mathcal{O}_1$. Hence $\overline{h^{-1}} = (\overline{h})^{-1}$ and so $\overline{h}$ is a homeomorphism that induces the group isomorphism Φ.

We note that it is only at this stage of the proof that we know that $\Phi(g_x)$ is a local contraction if g_x is a local contraction. We will now proceed to the smoothness assertions of Theorem 1.2.

To prove that $\overline{h}$ and $(\overline{h})^{-1}$ are smooth on the non-singular part of each stratum, we follow closely the argument given in [1]. Note that it is enough to show for any C^r function f on $\mathcal{O}_2$, and connected component $\mathcal{A} = \Sigma_k^{(l_1)\cdots(l_k)}$ of some stratum of $\mathcal{O}_1$ that

$$f \circ \overline{h}|_{\mathcal{A}-\Sigma_{\mathcal{A}}} \in C^r(\mathcal{A} - \Sigma_{\mathcal{A}}).$$

We will use the notation as in the proof of Proposition 6.1. Note that $\mathcal{A} - \Sigma_{\mathcal{A}}$ is *a priori* a connected manifold. Let ζ be any C^r vector field on $\mathcal{A} - \Sigma_{\mathcal{A}}$ with compact support contained in $\mathcal{A} - N(\Sigma_{\mathcal{A}})$. Let $\overline{\zeta}$ be the Γ equivariant horizontal lift of $i_*\zeta$ to $\tilde{M}$. By choosing an appropriate equivariant cutoff function whose support is contained in a relatively compact neighborhood of $\mathcal{A} - N(\Sigma_{\mathcal{A}})$ in $\tilde{M}$, we obtain a Γ equivariant vector field $\tilde{\zeta}$ on $\tilde{M}$. Let $\tilde{z}_t$ be the 1–parameter group of diffeomorphisms generated by $\tilde{\zeta}$, and let $z_t = \pi \circ \tilde{z}_t$. For each t, $\Phi(z_t) = \overline{h} \circ z_t \circ (\overline{h})^{-1} \in \mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O}_2)$ and the map:

$$(t, x) \mapsto \overline{h} \circ z_t \circ (\overline{h})^{-1}$$

is continuous. Moreover, when restricted to $\overline{h}(\mathcal{A} - \Sigma_{\mathcal{A}})$, for fixed t, $\Phi(z_t) \in \mathrm{Diff}^r(\overline{h}(\mathcal{A} - \Sigma_{\mathcal{A}}))_0$. Hence, we have a continuous action of $\mathbb{R}$ on $\overline{h}(\mathcal{A} - \Sigma_{\mathcal{A}})$ by C^r diffeomorphisms. By Montgomery-Zippin [13, p. 208 - 214], it follows that $\overline{h} \circ z_t \circ (\overline{h})^{-1}$ is C^r in both t and x. Therefore the restriction of $\overline{h} \circ z_t \circ (\overline{h})^{-1}$ to $\overline{h}(\mathcal{A} - \Sigma_{\mathcal{A}})$ is a 1-parameter group of diffeomorphisms, which has an infinitesimal generator $\xi_{\overline{h}}$ defined by:

$$\frac{d}{dt}\Big(\overline{h} \circ z_t \circ (\overline{h})^{-1}\Big)\Big|_{\overline{h}(\mathcal{A}-\Sigma_{\mathcal{A}})} = \xi_{\overline{h}}\Big(\overline{h} \circ z_t \circ (\overline{h})^{-1}\Big|_{\overline{h}(\mathcal{A}-\Sigma_{\mathcal{A}})}\Big)$$

By construction of the vector field $\xi_{\overline{h}}$ it is easily seen that

$$\begin{aligned}
\xi(f\circ\overline{h})\big|_{\mathcal{A}-\Sigma_{\mathcal{A}}} &= \frac{d}{dt}\Big|_{t=0}(f\circ\overline{h})\circ z_t\big|_{\mathcal{A}-\Sigma_{\mathcal{A}}}\\
&= \lim_{t\to 0}\frac{f\circ\overline{h}\circ z_t\circ(\overline{h})^{-1}\circ\overline{h}-f\circ\overline{h}}{t}\Big|_{\mathcal{A}-\Sigma_{\mathcal{A}}}\\
&= \lim_{t\to 0}\frac{f\circ\Phi(z_t)\circ\overline{h}-f\circ\overline{h}}{t}\Big|_{\mathcal{A}-\Sigma_{\mathcal{A}}}\\
&= \Big(\lim_{t\to 0}\frac{f\circ\Phi(z_t)-f}{t}\Big)\circ\overline{h}\big|_{\mathcal{A}-\Sigma_{\mathcal{A}}}\\
&= \Big(\frac{d}{dt}\Big|_{t=0}(f\circ\Phi(z_t))\Big)\circ\overline{h}\big|_{\mathcal{A}-\Sigma_{\mathcal{A}}}\\
&= \xi_{\overline{h}}(f)\circ\overline{h}\big|_{\mathcal{A}-\Sigma_{\mathcal{A}}}
\end{aligned}$$

To compute higher derivatives, we can iterate this formula:

$$\begin{aligned}
\xi_2(\xi_1(f\circ\overline{h}))\big|_{\mathcal{A}-\Sigma_{\mathcal{A}}} &= \xi_2\big(\xi_{1,\overline{h}}(f)\circ\overline{h}\big)\big|_{\mathcal{A}-\Sigma_{\mathcal{A}}}\\
&= \big(\xi_{2,\overline{h}}(\xi_{1,\overline{h}}(f)\big)\circ\overline{h}\big|_{\mathcal{A}-\Sigma_{\mathcal{A}}}
\end{aligned}$$

Let $x\in\Sigma_1$ and $\tilde{U}_x\to U_x$ an orbifold chart around x. Then there is a connected component $\mathcal{A}=\Sigma_k^{(l_1)\cdots(l_k)}$ for which $x\in\mathcal{A}-\Sigma_{\mathcal{A}}$. Since $\mathcal{A}-\Sigma_{\mathcal{A}}$ is a manifold, we can choose ξ_i to be vector fields which agree with the coordinate vector fields in the neighborhood of x (for local coordinates in $\mathcal{A}-\Sigma_{\mathcal{A}}$ around x). Thus, we can obtain continuous partial derivatives up to order r of $f\circ\overline{h}$. Therefore, $\overline{h}$ is C^r when restricted to the component $\mathcal{A}-\Sigma_{\mathcal{A}}$ of the singular set. This completes the proof of the main theorem. ∎

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