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The Automorphisms of Generalized Witt Type Lie Algebras

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Abstract. We find the Lie automorphisms of generalized Witt type Lie algebras $W[x, e^x]$ and $W[x, e^{\pm x}]$.

1. Introduction

Simplicity of several generalized Witt type Lie algebras have been considered by many authors over a field F of characteristic zero. Kac [3] studied the generalized Witt algebra on the F-algebra in the formal power series algebra $F[[x_1, \cdots, x_n]]$. There exist many generalized Witt type simple Lie algebras using the algebras stable under the action of derivations ([1], [3], [4], [6]). We consider one-variable cases based on using the exponential functions. Let $\partial = \frac{d}{dx}$, $F[x^{\pm 1}, e^{\pm x}] = F[x, x^{-1}, e^x, e^{-x}]$, and let $F[a_1, ..., a_n]$ be a subalgebra of $F[x^{\pm 1}, e^{\pm x}]$ generated by $a_1, ..., a_n$. If $F[a_1, ..., a_n]$ is ∂-stable we put $W[a_1, ..., a_n] = \{f\partial \mid f \in F[a_1, ..., a_n]\}$. Then $W[a_1, ..., a_n]$ is a Lie algebra over F with the usual product

$$[f\partial, g\partial] = f\partial \circ g\partial - g\partial \circ f\partial = (f(\partial g) - (\partial f)g)\partial \quad (f, g \in F[a_1, ..., a_n]).$$

The Lie algebras $W[x]$, $W[x^{\pm 1}]$, $W[e^{\pm x}]$, $W[x, e^{\pm x}]$, and $W[x^{\pm 1}, e^{\pm x}]$ are simple, while $W[x, e^x]$ and $W[x^{\pm 1}, e^x]$ are not simple. The automorphisms of $W[x]$ is considered in [7] (cf. also [2]). The automorphisms of generalized Witt type Lie algebras of Laurent polynomials are considered in [1], [5]. In this paper we find the Lie automorphisms of $W[x, e^x]$ and $W[x, e^{\pm x}]$ containing polynomials and exponential functions. The automorphism group of $W[x, e^x]$ is isomorphic to $F^* \times F$, while the automorphism group of $W[x, e^{\pm x}]$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \ltimes (F^* \times F)$.

2. Preliminaries

Let $\mathbb{Z}$ be the set of integers, $\mathbb{Z}_+$ the set of positive integers, $\mathbb{Z}_-$ the set of negative integers, and $\mathbb{N}$ the set of non-negative integers. For the field F we denote by

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F^* the set of non-zero elements of F. Recall that $W[x,e^x] = \bigoplus_{n\in\mathbb{N}} W_n$ and $W[x,e^{\pm x}] = \bigoplus_{n\in\mathbb{Z}} W_n$ are graded Lie algebras, where $W_n = \{fe^{nx}\partial \mid f \in F[x]\}$ is a homogeneous component of degree n. Let $\alpha = \alpha_n + \alpha_{n-1} + \cdots + \alpha_m$, where $\alpha_i \in W_i$ and $\alpha_n, \alpha_m \neq 0$. Then we denote by $\overline{\alpha}$ the non-zero homogeneous component of α of highest degree α_n, and by $\underline{\alpha}$ the non-zero homogeneous component of lowest degree α_m. Hence $\alpha = \overline{\alpha} + \cdots + \underline{\alpha}$. Let $W_+ = \bigoplus_{n\in\mathbb{Z}_+} W_n$ and $W_- = \bigoplus_{n\in\mathbb{Z}_-} W_n$. Then $W[x,e^{\pm x}] = W_+ + W_0 + W_-$, and $\alpha = \alpha_+ + \alpha_0 + \alpha_-$ for some $\alpha_+ \in W_+$, $\alpha_0 \in W_0$, and $\alpha_- \in W_-$. For $\alpha, ..., \beta \in W[x,e^{\pm x}]$ we denote by $\langle \alpha, ..., \beta \rangle$ the subalgebra of $W[x,e^{\pm x}]$ generated by $\alpha, ..., \beta$. We denote by $sp\{\alpha, ..., \beta\}$ the subspace of $W[x,e^{\pm x}]$ spanned by $\alpha, ..., \beta$. Hence, $\langle\alpha\rangle = sp\{\alpha\} = F\alpha$. For $a \in F^*$, $b \in F$ we define

$$\begin{aligned} \varphi_a :\ & x^n e^{mx}\partial \longmapsto a^m x^n e^{mx}\partial, \\ \psi_b :\ & x^n e^{mx}\partial \longmapsto (x+b)^n e^{mx}\partial, \\ \tau :\ & x^n e^{mx}\partial \longmapsto (-1)^{n-1} x^n e^{-mx}\partial. \end{aligned} \tag{1}$$

Then it is easy to see that φ_a, $\psi_b \in \mathrm{Aut}_F\, W[x,e^x]$, and that φ_a, ψ_b, $\tau \in \mathrm{Aut}_F\, W[x,e^{\pm x}]$. Here we use the same symbols to denote the same type of the automorphisms in (1). Note that $W[x,e^x] = \langle \partial, x^3\partial,\ e^x\partial\rangle$ and $W[x,e^{\pm x}] = \langle \partial, x^3\partial,\ e^x\partial,\ e^{-x}\partial\rangle$.

Note 2.1. *Let φ be a Lie automorphism of $W[x,e^x]$ (resp. $W[x,e^{\pm x}]$). If $\varphi(x^n\partial) = x^n\partial$ $(n \in \mathbb{N})$ and $\varphi(e^{mx}\partial) = e^{mx}\partial$ $(m \in \mathbb{N}$ (resp. $\mathbb{Z}$)), then $\varphi = 1_{W[x,e^x]}$ (resp. $1_{W[x,e^{\pm x}]}$).* ■

Note 2.2. *The Lie algebra $W[x,e^{\pm x}]$ is self-centralizing, that is, if $[\alpha,\beta] = 0$ and α, β are non-zero elements of $W[x,e^{\pm x}]$, then $\langle\alpha\rangle = \langle\beta\rangle$.* ■

Note 2.3. *Let $\beta \in W[x,e^{\pm x}]$. If $[\partial,\beta] = a\beta$ for some $a \in F^*$, then $\beta \in \langle e^{ax}\partial\rangle$ where $a \in \mathbb{Z}$.* ■

Note 2.4. *Let I be one of $\mathbb{N}$, $\mathbb{N}\cup\{-1\}$, and $\mathbb{Z}$. Let $a_n \in F^*$ $(n \in I)$ satisfy the condition $a_{n+m} = a_n a_m$ for any $n \neq m$. Then $a_n = a_1^n$ for any $n \in I$.* ■

3. Stabilizers

We determine the elements α, β satisfying the condition $[\alpha,\beta] = \beta$ in some generalized Witt type Lie algebras.

Proposition 3.1. *Let α, β be non-zero elements of $W[x]$ such that $[\alpha,\beta] = \beta$. Then $\alpha - \frac{1}{n-1}(x+c)\partial$, $\beta \in \langle (x+c)^n\partial\rangle$ for some $c \in F$ and $n \in \mathbb{N}\setminus\{1\}$.*

Proof. Let $\alpha = f\partial$, $\beta = g\partial$ and let $f = a_m x^m + \cdots + a_0$, $g = b_n x^n + \cdots + b_0$, where $m, n \geq 0$ and $a_m, b_n \neq 0$. If $m \neq n$, then from $[\alpha,\beta] = \beta$ we have $m = 1$ and

$$f = \frac{1}{n-1}(x+c), \quad g = b_n(x+c)^n$$

for some $c \in F$. If $m = n$, then it follows by taking $h = f - ag$, where $a = \frac{a_n}{b_n} \neq 0$, that $f = a_n(x+c)^n + \frac{1}{n-1}(x+c)$, $g = b_n(x+c)^n$. ■

Proposition 3.2. *Let α, β be non-zero elements of $W[x, e^x]$ and $[\alpha, \beta] = \beta$. Then $\beta_+ = 0$ or $\beta_0 = 0$ and one of the following statements holds*: (1) $\alpha - \frac{1}{n-1}(x + c)\partial,\ \beta \in \langle (x+c)^n \partial \rangle$ *for some* $c \in F$ *and* $n \in \mathbb{N} \setminus \{1\}$, *or* (2) $\alpha - \frac{1}{n}\partial,\ \beta \in \langle e^{nx}\partial \rangle$ *for some* $n \in \mathbb{Z}_+$.

Proof. Let $\alpha = (f_m e^{mx} + \cdots + f_0)\partial,\ \beta = (g_n e^{nx} + \cdots + g_0)\partial$, where $f_m, ..., f_0$, $g_n, ..., g_0 \in F[x]$, $f_m, g_n \neq 0$, and $m, n \in \mathbb{N}$. If $m \neq n$, then by some computation we deduce from $[\alpha, \beta] = \beta$ that $m = 0,\ n > 0$ and that

$$\alpha = \frac{1}{n}\partial, \quad \beta = b_n e^{nx}\partial \quad (b_n \neq 0).$$

Let $m = n$. If $n = 0$, then we can apply Proposition 3.1. If $n > 0$, then we have $f_n g_n' - f_n' g_n = 0$, $(\frac{f_n}{g_n})' = 0$, and $g_n = c f_n$ for some constant $c \neq 0$, where we write simply f' instead of ∂f. From $[\alpha - \frac{1}{c}\beta, \beta] = \beta$ we have $\alpha = a e^{nx}\partial + \frac{1}{n}\partial,\ \beta = b e^{nx}\partial$ for some $a, b \in F$. ■

We continue to characterize the elements α, β satisfying the condition $[\alpha, \beta] = \beta$ in $W[x, e^{\pm x}]$.

Lemma 3.3. *Let α, β be non-zero elements of $W[x, e^{\pm x}]$ and $[\alpha, \beta] = \beta$. Then*
(1) *For $\overline{\alpha}$ and $\overline{\beta}$ we have either*
(i) $\overline{\alpha - k\beta} - \frac{1}{n-1}(x + c)\partial,\ \overline{\beta} \in \langle (x+c)^n \partial \rangle$ *for some* $k, c \in F$, $n \in \mathbb{N} \setminus \{1\}$, *or*
(ii) $\overline{\alpha - k\beta} = \frac{1}{n}\partial$ *and* $\overline{\beta} \in \langle e^{nx}\partial \rangle$ *for some* $k \in F$, $n \in \mathbb{Z} \setminus \{0\}$.
(2) *For $\underline{\alpha}$ and $\underline{\beta}$ we have either*
(i) $\underline{\alpha - l\beta} - \frac{1}{m-1}(x + d)\partial,\ \underline{\beta} \in \langle (x+d)^m \partial \rangle$ *for some* $l, d \in F$, $m \in \mathbb{N} \setminus \{1\}$, *or*
(ii) $\underline{\alpha - l\beta} = \frac{1}{m}\partial$ *and* $\underline{\beta} \in \langle e^{mx}\partial \rangle$ *for some* $l \in F$, $m \in \mathbb{Z} \setminus \{0\}$.

Proof. We show Case (1), since Case (2) will be proved similarly. Since $[\alpha - k\beta, \beta] = \beta$ for any $k \in F$, if necessary we can replace α with $\alpha - k\beta$. Hence we may assume $\langle \overline{\alpha} \rangle \neq \langle \overline{\beta} \rangle$. Then by Note 2.2 we have $[\overline{\alpha}, \overline{\beta}] \neq 0$. Therefore $[\overline{\alpha}, \overline{\beta}] = \overline{\beta}$ and $\overline{\alpha} \in W_0 = W[x]$ since $W[x, e^{\pm x}]$ is $\mathbb{Z}$-graded. We determine $\overline{\alpha}$ and $\overline{\beta}$. Apply the automorphism τ if necessary. Then by Proposition 3.2 we have $\overline{\alpha} - \frac{1}{n-1}(x + c)\partial,\ \overline{\beta} \in \langle (x+c)^n \partial \rangle$ for some $c \in F$, $n \in \mathbb{N} \setminus \{1\}$, or $\overline{\alpha} - \frac{1}{n}\partial,\ \overline{\beta} \in \langle e^{nx}\partial \rangle$ for some $n \in \mathbb{Z} \setminus \{0\}$. In the later case $\overline{\alpha} = \frac{1}{n}\partial + b e^{nx}\partial$ for some $b \in F$, and $\overline{\alpha} = \frac{1}{n}\partial$ since $\overline{\alpha}$ is homogeneous. ■

Lemma 3.4. *Let α, β be non-zero elements of $W[x, e^{\pm x}]$ and $[\alpha, \beta] = \beta$. Then we have the following statments*:
(1) *If $\alpha_+ \neq 0$, then $\beta_+ \neq 0$, $\langle \overline{\alpha} \rangle = \langle \overline{\beta} \rangle \subseteq W_n$ for some $n \in \mathbb{Z}_+$, and also $\beta = \frac{1}{k}\alpha_+ + \frac{1}{k}\left(\alpha_0 - \frac{1}{n}\partial\right) + \beta_-$ for some $k \in F^*$.*
(2) *If $\alpha_- \neq 0$, then $\beta_- \neq 0$, $\langle \underline{\alpha} \rangle = \langle \underline{\beta} \rangle \subseteq W_m$ for some $m \in \mathbb{Z}_-$, and $\beta = \beta_+ + \frac{1}{l}\left(\alpha_0 - \frac{1}{m}\partial\right) + \frac{1}{l}\alpha_-$ for some $l \in F^*$.*
(3) *If $\alpha_+, \alpha_- \neq 0$, then $\beta \in sp\{\alpha_+,\ \alpha_-,\ \alpha_0,\ \partial\}$.*

Proof. (1) Let $\alpha_+ \neq 0$. Then $\overline{\alpha} \in W_n$ for some $n \in \mathbb{Z}_+$. Assume that $[\overline{\alpha}, \overline{\beta}] \neq 0$. Then $[\overline{\alpha}, \overline{\beta}] = \overline{\beta}$. If $\overline{\beta} \in W_m$, then $\overline{\beta} = [\overline{\alpha}, \overline{\beta}] \in W_{n+m}$, a contradiction. Hence $[\overline{\alpha}, \overline{\beta}] = 0$, and by Note 2.2 we have $\langle\overline{\alpha}\rangle = \langle\overline{\beta}\rangle$ and $\beta_+ \neq 0$. Hence $\overline{\beta} \in W_n$, and we have $\overline{\alpha - k\beta} = \frac{1}{n}\partial$ for some non-zero $k \in F$ by Lemma 3.3. Then $\alpha - k\beta = \frac{1}{n}\partial + \alpha_- - k\beta_-$ and $\beta = \frac{1}{k}\alpha_+ + \frac{1}{k}\left(\alpha_0 - \frac{1}{n}\partial\right) + \beta_-$.

(2) Let $\alpha_- \neq 0$. Then $\langle\underline{\alpha}\rangle \in W_m$ for some $m \in \mathbb{Z}_-$, and we have $\beta = \beta_+ + \frac{1}{l}\left(\alpha_0 - \frac{1}{m}\partial\right) + \frac{1}{l}\alpha_-$ for some $l \in F^*$.

(3) Let $\alpha_+, \alpha_- \neq 0$. Then from (1) and (2) we have

$$\beta = \frac{1}{k}\alpha_+ + \frac{1}{k}\left(\alpha_0 - \frac{1}{n}\partial\right) + \beta_- = \beta_+ + \frac{1}{l}\left(\alpha_0 - \frac{1}{m}\partial\right) + \frac{1}{l}\alpha_-$$

for some $k, l \in F^*$, $n \in \mathbb{Z}_+$, $m \in \mathbb{Z}_-$. Thus $\beta \in sp\{\alpha_+,\ \alpha_-,\ \alpha_0,\ \partial\}$. ∎

Lemma 3.5. *Let α be a non-zero element of $W[x, e^{\pm x}]$, and let $\{\beta_i \mid i \in I\}$ be an infinite and linearly independent subset of $W[x, e^{\pm x}]$. If $[\alpha, \beta_i] = a_i\beta_i$ and $a_i \neq 0$ for any $i \in I$, then $\alpha_0 \neq 0$ and either $\alpha_+ = 0$ or $\alpha_- = 0$.*

Proof. Assume that $\alpha_+ \neq 0$ and $\alpha_- \neq 0$. Since $[\frac{1}{a_i}\alpha, \beta_i] = \beta_i$, by Lemma 3.4(3) the set $\{\beta_i \mid i \in I\}$ is contained in the finite dimensional subspace $sp\{\alpha_+,\ \alpha_-,\ \alpha_0,\ \partial\}$, a contradiction. Hence $\alpha_+ = 0$ or $\alpha_- = 0$. If both $\alpha_+ = 0$ and $\alpha_- = 0$, then clearly $\alpha_0 \neq 0$. Let $\beta = \beta_i$. If $\alpha_- \neq 0$, then we apply the automorphism τ. Hence we may assume that $\alpha = \alpha_+ + \alpha_0$. By Lemma 3.4 we have

$$\beta = \frac{1}{ka_i}\alpha_+ + \frac{1}{k}\left(\frac{1}{a_i}\alpha_0 - \frac{1}{n}\partial\right) + \beta_-$$

for some $k \in F^*$ and $n \in \mathbb{Z}_+$ such that $\overline{\beta} \in W_n$. Hence $\beta_+ \neq 0$. If $\beta_- = 0$, then by Proposition 3.2 we have $\frac{1}{a_i}\alpha_0 - \frac{1}{n}\partial = k\beta_0 = 0$ and $\alpha_0 \neq 0$. If $\beta_- \neq 0$, then $[\frac{1}{a_i}\underline{\alpha}, \underline{\beta}] = \underline{\beta}$ since $\langle\underline{\alpha}\rangle \neq \langle\underline{\beta}\rangle$. Hence $\alpha_0 = \underline{\alpha} \neq 0$. ∎

4. Automorphisms

We determine the automorphisms of $W[x, e^x]$ and $W[x, e^{\pm x}]$ in this section.

Lemma 4.1. *Let φ be an injective homomorphism of $W[x]$. Then $\varphi(x^n\partial) = a^{n-1}(x+b)^n\partial$ $(n \in \mathbb{N})$ for some $a \in F^*$, $b \in F$.*

Proof. Let φ be an injective homomorphism of $W[x]$. Since

$$[\varphi(x^m\partial), \varphi(x^n\partial)] = (n-m)\varphi(x^{m+n-1}\partial), \tag{2}$$

we have $[\frac{1}{n-1}\varphi(x\partial), \varphi(x^n\partial)] = \varphi(x^n\partial)$ $(n \neq 1)$. Since $\varphi(x^n\partial)$ $(n \in \mathbb{N})$ are linearly independent it follows easily from Proposition 3.1 that

$$\varphi(x^n\partial) = a_{n-1}(x+b)^n\partial \quad (n \in \mathbb{N})$$

for some $a_{n-1} \in F^*$, $b \in F$. Then from (2) we have $a_{m-1}a_{n-1} = a_{m+n-2} = a_{m-1+n-1}$ $(n, m \in \mathbb{N},\ n \neq m)$, that is, $a_m a_n = a_{n+m}$ $(n, m \in \mathbb{N} \cup \{-1\},\ n \neq m)$. By Note 2.4, $a_n = a_1^n$ and $\varphi(x^n\partial) = a^{n-1}(x+b)^n\partial$, where $a = a_1 \in F^*$. ∎

Let $\rho_a : x^n\partial \longmapsto a^{n-1}x^n\partial$. Then it is easy to see that ρ_a $(a \in F^*)$ is an automorphism of $W[x]$. By Lemma 4.1 we note that the automorphism group of $W[x]$ is isomorphic to $F^* \ltimes F$, where F^* is the multiplicative group and F is the additive group (cf. [2],[7]).

Proposition 4.2. *Let φ be an automorphism of $W[x, e^x]$ or $W[x, e^{\pm x}]$. Then $\varphi(W[x]) \subseteq W[x]$.*

Proof. It holds that $[\varphi(x\partial), \varphi(x^n\partial)] = (n-1)\varphi(x^n\partial)$ $(n \in \mathbb{N})$. Let $\alpha = \varphi(x\partial)$. Then by Lemma 3.5 we have $\alpha_0 \neq 0$, and $\alpha_+ = 0$ or $\alpha_- = 0$. Let $\beta = \varphi(\partial)$. Then similary from $[\varphi(\partial), \varphi(e^{mx}\partial)] = m\varphi(e^{mx}\partial)$ $(m \in \mathbb{N})$ we have $\beta_0 \neq 0$, and $\beta_+ = 0$ or $\beta_- = 0$. Assume that $\alpha_+ \neq 0$. Then from $[-\alpha, \beta] = \beta$ and Lemma 3.4(1) we have $\beta_+ \neq 0$ and $\alpha, \beta \in W[x, e^x]$. Hence $\beta_0 \neq 0$ and $\beta_+ \neq 0$, but this contradicts to Proposition 3.2. Assume that $\alpha_- \neq 0$. Then applying τ we have a contradiction similar to the above. Therefore $\alpha = \alpha_0 \in W[x]$. Then the case $\beta_+ \neq 0$ and the case $\beta_- \neq 0$ cause similar contradictions. Thus $\beta = \beta_0 \in W[x]$. From $[\beta, \varphi(x^n\partial)] = (n-1)\varphi(x^{n-1}\partial)$ we have $\varphi(x^n\partial) \in W[x]$ $(n \in \mathbb{N})$ by induction. ∎

Theorem 4.3. *Let φ be an automorphism of $W[x, e^x]$. Then φ is a product of φ_a and ψ_b for some $a \in F^*$, $b \in F$.*

Proof. Let φ be an automorphism of $W[x, e^x]$. Then by Proposition 4.2 and Lemma 4.1 we have $\varphi(\partial) = a^{-1}\partial$ and $\varphi(x^n\partial) = a^{n-1}(x+b)^n\partial$ for some $a \in F^*$, $b \in F$. Since $[\varphi(\partial), \varphi(e^{mx}\partial)] = m\varphi(e^{mx}\partial)$, we have $[\partial, \varphi(e^{mx}\partial)] = am\varphi(e^{mx}\partial)$. Then by Note 2.3 we have $\varphi(e^{mx}\partial) \in \langle e^{amx}\partial\rangle$ and $am \in \mathbb{N}$. Since φ is surjective, it follows that $a = 1$, $\varphi(x^n\partial) = (x+b)^n\partial$ $(n \in \mathbb{N})$, and $\varphi(e^{mx}\partial) = c_m e^{mx}\partial$ $(m \in \mathbb{N})$. Then from $[\varphi(e^{mx}\partial), \varphi(e^{kx}\partial)] = (k-m)\varphi(e^{(m+k)x}\partial)$ $(m, k \in \mathbb{N})$ and Note 2.4, we have $c_m = c^m$ for some $c \in F^*$. Thus $\varphi(e^{mx}\partial) = c^m e^{mx}\partial$. Hence $(\varphi_c \circ \psi_b)^{-1} \circ \varphi = 1_{W[x,e^x]}$ by Note 2.1, and therefore $\varphi = \varphi_c \circ \psi_b$. ∎

Corollary 4.4. *The automorphism group of $W[x, e^x]$ is isomorphic to $F^* \times F$.*

Proof. This is clear from $\varphi_a \circ \psi_b = \psi_b \circ \varphi_a$ for any $a \in F^*$, $b \in F$. ∎

Theorem 4.5. *An automorphism of $W[x, e^{\pm x}]$ is a product of φ_a, ψ_b, and τ for some $a \in F^*$, $b \in F$.*

Proof. Let φ be an automorphism of $W[x, e^{\pm x}]$. Then as in the proof of Theorem 4.3 $\varphi(x^n\partial) = a^{n-1}(x+b)^n\partial$ $(n \in \mathbb{N})$ and $\varphi(e^{mx}\partial) = c_m e^{amx}\partial$ $(m \in \mathbb{Z})$. Since φ is surjective, we have $a = \pm 1$, and applying τ if necessary we may assume $a = 1$. Then it follows that $\varphi(e^{mx}\partial) = c^m e^{mx}\partial$ for some $c \in F^*$ and that $(\varphi_c \circ \psi_b)^{-1} \circ \varphi = 1_{W[x,e^{\pm x}]}$. ∎

Corollary 4.6. *The automorphism group of $W[x, e^{\pm x}]$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \ltimes (F^* \times F)$.*

Proof. This is clear from $\varphi_a \circ \psi_b = \psi_b \circ \varphi_a$, $\tau \circ \varphi_a \circ \tau = \varphi_{a^{-1}}$, and $\tau \circ \psi_b \circ \tau = \psi_{-b}$. ∎

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